A QUBO Formulation for the Generalized LinkedIn Queens and Takuzu/Tango Game

Alejandro Mata Ali^{1,2} and Edgar Mencia^{2,3}

In this paper, we present a QUBO formulation designed to solve a series of generalisations of the LinkedIn queens game, a version of the N-queens problem, for the Takuzu game (or Binairo), for the most recent LinkedIn game, Tango, and for its generalizations. We adapt this formulation for several particular cases of the problem, as Tents & Trees, by trying to optimise the number of variables and interactions, improving the possibility of applying it on quantum hardware by means of Quantum Annealing or the Quantum Approximated Optimization Algorithm (QAOA). We also present two new types of problems, the Coloured Chess Piece Problem and the Max Chess Pieces Problem, with their corresponding QUBO formulations.

1 Introduction

Combinatorial games have always been an inspiration for developing and perfecting new mathematical and computational problem-solving techniques, as well as for benchmarking. Combinatorial problems are known for their practical application in academic and industrial contexts, but also for their usefulness in creating entertaining games. Some of these are the Akari, the KenKen, the Takuzu or the N-queens, which led to different combinatorial games, among which is the LinkedIn Queens.

The N-queens problem [1] is a problem that has attracted enormous attention since its proposal in 1848 by the chess player Max Bezzel,

Alejandro Mata Ali: alejandro.mata@itcl.es Edgar Mencia: edmenciab@gmail.com leading many mathematicians, including Gauss, to try to solve and generalise it. This problem can be solved efficiently, while the problem of determining how many solutions exist becomes much more complicated. Such a problem has no applications, but it is a good problem on which to test methods of solving logically constrained problems.

LinkedIn recently published its Queens game (which we will call LQueens), based on N-queens problem, but with some changes in the restrictions to be carried out and the initial conditions. In this new version, the restriction on diagonals is eased, but a restriction on regions is imposed. Following the success that this game has had among the community, LinkedIn announced its new game called Tango, a version of the Takuzu game (also named Binairo or 0h h1) for a 6×6 board, which consists of placing a set of suns and moons on a board following a series of constraints. This game can be visualized as a single-player version of tic-tac-toe, and has some applications such as the modeling of quantum agents [2]. This problem has been solved previously using genetic algorithms [3].

In recent years, quantum computing has gained great popularity due to its different computational capabilities compared to classical computing, making it possible to tackle certain problems more efficiently. Some of these problems are combinatorial optimisation problems, in which quantum annealing using quantum annealers [4, 5] such as those of Dwave and the Quantum Approximated Optimization Algorithm (QAOA) [6] using digital quantum computers stand out. In both cases, the formulation of the problem comes through a quadratic unconstrained optimisation problem (QUBO) [7], which contains the interac-

¹Instituto Tecnológico de Castilla y León (ITCL), Spain

²Quantum Information and Quantum Computing Group, QuantumQuipu, National University of San Marcos, Av. Germán Amézaga s/n. Ciudad Universitaria, Lima, Perú

³Thinkcomm SRL, (1527) Asunción, Paraguay

tions to be optimised between the different variables, represented by logical qubits, minimising the energy of the system to find the lowest cost combination. These methods can be adapted to cases with constraints by adding terms that increase the energy substantially if the constraints are not respected. An example can be the QUBO formulation of the Sudoku game [8].

Recently, the N-queens problem has been solved with a quantum algorithm in a quantum simulator [9], with two quantum algorithms based on the W-states and backtracking [10], and has also been solved with several QUBO formulations [11, 12]. However, these solutions are tailored and restricted to the N-queens, and not to all generalizations we may have of the problem.

Motivated by this new version of the game and the quantum computing algorithms, we present a series of generalisations of the N-queens problem and a QUBO formulation associated with each of them so that they can be solved by quantum means, such as quantum annealing or the Quantum Approximated Optimization Algorithm (QAOA). With these generalizations, we can define two new types of problems, which take advantage of the definitions and terms we created for the N-queens generalizations. Due to the popularity of Takuzu as a combinatorial problem and its connection to quantum computing, it is also interesting to propose a QUBO formulation for Takuzu, Tango and generalizations of it that we want to formulate.

2 LinkedIn's Queens and N-Queens problem

2.1 Problem definition

The original problem to be solved in the N-queens consists of placing N queens on a chessboard of dimensions $N \times N$ in such a way that none of them threatens the others. That is, no queen can be in the same row or column as another queen, nor be diagonal to another queen. The constraints are shown in Fig. 1.

To formalize it mathematically, we say that the state of the board is determined by a binary matrix x whose elements x_{ij} determine the state of the cell in row i and column j, so that if it is 0 there is no queen in that cell, and if it is 1 there is. Therefore, the restrictions are

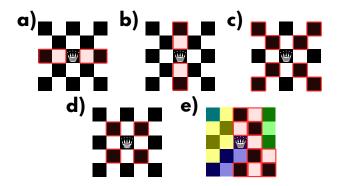


Figure 1: Constraints on a 5×5 problem, such that no queens can be placed in any of the cells marked in red. a) Column constraint, b) Row constraint, c) Diagonal constraint on N-queens, d) Diagonal constraint on LQueens, e) Region constraint.

• Rows condition: there can only be one queen per row (Fig. 1 a):

$$\sum_{j=1}^{N} x_{ij} = 1 \quad \forall i \in [1, N].$$

• Columns condition: there can only be one queen per column (Fig. 1 b):

$$\sum_{i=1}^{N} x_{ij} = 1 \quad \forall j \in [1, N].$$

• Diagonal condition: there cannot be two queens in the same diagonal (Fig. 1 c):

$$\sum_{(k,l) \in D_{i,j}} x_{kl} \leq 1 \quad \forall i,j \in [1,N],$$

being D_{ij} the set of cells on the diagonals of cell (i, j), including it.

The condition that there be N queens is implicit in the first two restrictions.

LinkedIn's version of the game is slightly different. While it retains the first two restrictions, the third restriction becomes softer, making it so that queens can now not be diagonal only in the case where they are adjacent, which translates to

• Diagonal condition: there cannot be more than one queen in the same adjacent diagonal (Fig. 1 d): $x_{ij} + X_{i+1,j+1} + X_{i-1,j+1} + X_{i+1,j-1} + X_{i-1,j-1} \le 1 \quad \forall i,j \in [1,N]$, being $X_{kl} = x_{kl}$ if $k,l \in [1,N]$ and 0 otherwise.

Another modification in the LinkedIn version is the region restriction, which imposes on us that

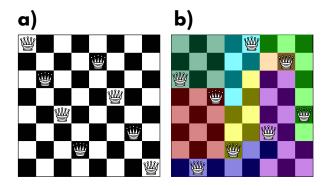


Figure 2: a) Solution of an 8-queen problem, b) Solution of an 8-queen problem from LinkedIn

we have to place one and only one queen for each one of the $N_R=N$ colored regions, which translates to

• Region condition: there can only be one queen per region (Fig. 1 e): $\sum_{(i,j)\in R_k} x_{ij} = 1 \quad \forall k \in [1, N_R]$, being R_k the cells belonging to the k-th region.

An additional condition in this case is that there are a number N_I of queens already placed in cells A_I , as an initial condition. This can be addressed by forcing the variables associated with these cells to be forcibly 1, which leads to all other adjacent variables in the same row, in the same column and in the same region to be forcibly 0. This translates to

- Initial condition: there are N_I pre-placed queens:
 - Cell initialized: $x_{lm} = 1 \quad \forall (l, m) \in A_I$
 - Row: $x_{lj} = 0 \quad \forall j \in [1, N] \text{ if } j \neq m \text{ and } \forall (l, m) \in A_I$
 - Column: $x_{im} = 0 \quad \forall i \in [1, N] \text{ if } i \neq l$ and $\forall (l, m) \in A_I$
 - Region: $x_{ij} = 0 \quad \forall (i, j) \in R_k \text{ if } (i, j) \neq (l, m) \text{ and } (l, m) \in R_k \ \forall (l, m) \in A_I$

In Fig. 2 a) we can see an example solution in the original N-queens, while in Fig. 2 b) we can see an example solution of the LinkedIn Queens.

2.2 QUBO resolution

For the QUBO formulation of the problem, we need to be able to put the constraints as a quadratic form, making the constraints work for

the variables in pairs. We first tackle the standard N-queens and then the LinkedIn queens.

For the N-queens we need all the N^2 variables, so we need N^2 logical qubits. The first constraint is an equality, which we can impose with a quadratic term of the form

$$Q_r = \sum_{i=1}^{N} \left(1 - \sum_{j=1}^{N} x_{ij} \right)^2.$$
 (1)

This term has its minimum, equal to zero, when for each row there is only one variable equal to 1, so that all solutions that do not meet this restriction have a higher cost.

For the column condition, the term is analogous, interchanging the summands in i and in j

$$Q_c = \sum_{i=1}^{N} \left(1 - \sum_{i=1}^{N} x_{ij} \right)^2.$$
 (2)

For the restriction of the diagonals we use a term of the type

$$Q_d = \sum_{i,j}^{N-1,N} \sum_{(k,l) \in D'_{i,j}} x_{ij} x_{kl},$$
 (3)

so that, for each occupied cell (i, j), if another cell on its diagonal is occupied, this term sums to 1, otherwise it sums to 0. In this case we define $D'_{i,j}$ as the cells that are on the same diagonals as cell (i, j) such that their row is greater than i. In this way we avoid adding twice over the same pair of cells and we also avoid the term $x_{ij}^2 = x_{ij}$, which would give us 1 whenever that cell was occupied. As only cells in rows below the cell where there is a queen are computed, we do not need to include the condition in the last row for i.

The total QUBO formulation is

$$Q = Q_r + Q_c + Q_d =$$

$$= \sum_{i=1}^{N} \left(1 - \sum_{j=1}^{N} x_{ij} \right)^2 + \sum_{j=1}^{N} \left(1 - \sum_{i=1}^{N} x_{ij} \right)^2 +$$

$$+ \sum_{i,j} \sum_{(k,l) \in D'_{i,j}} x_{ij} x_{kl}.$$
(4)

All valid solutions to this problem have a cost Q = 0, while all solutions that do not satisfy the constraints have a cost Q > 0.

For the LQueens the formulation is similar. The first adjustment we have to make is to consider what are the fixed variables due to the initial condition. These are removed directly from the optimisation problem, but we keep the notation of the celles for clarity. That is, if we have the (3,3) cell, the variable referring to the (4,3) cell will still be $x_{4,3}$, since we know that there really is no $x_{3,3}$. So we will have a number of variables equal to N^2 if there are no queens at the beginning, and less if we add queens in the initial condition.

Let's define for simplicity the sets of cells:

- A_C : contains the cells that have not been fixed by the initial condition (all variables to be optimised).
- A'_C : contains the cells of A_C that are not in the last row of A_r .
- A_R : contains the indexes of all but one of the regions that have at least one cell in A_C .

and the sets of indexes:

- A_r : contains all rows i containing at least one cell in A_C .
- A_c : contains all columns j containing at least one cell in A_C
- $A_c(i)$: the set of j such that $(i, j) \in A_C$.
- $A_r(j)$: the set of i such that $(i,j) \in A_C$.

Thus, the term of the rows is

$$Q_r = \sum_{i \in A_r} \left(1 - \sum_{j \in A_r(i)} x_{ij} \right)^2.$$
 (5)

The term of the columns is

$$Q_c = \sum_{j \in A_c} \left(1 - \sum_{i \in A_r(j)} x_{ij} \right)^2.$$
 (6)

In both cases, the only modification with respect to the N-queens has been to eliminate from the summations the rows and columns without cells to be determined and the cells already fixed.

For the diagonal condition in this case, we have the same term, redefining in this case D'_{ij} as the diagonal cells adjacent to cell (i, j), such that their row is greater than i, instead of the whole diagonals, and which have not already been fixed by the initial condition. Because of the initial condition, we have to construct the term as

$$Q_d = \sum_{(i,j)\in A'_C} \sum_{(k,l)\in D'_{i,j}} x_{ij} x_{kl},$$
 (7)

In this case we only consider cells that are not fixed

Now, for the region condition, we add a term

$$Q_g = \sum_{k \in A_R} \left(1 - \sum_{(i,j) \in R_k | (i,j) \in A_C} x_{ij} \right)^2. \tag{8}$$

ensuring that the minimum value 0 in the case that each region has only one queen in each region. We consider only $N - N_I - 1$ regions, since if there is a queen in each of these, the remaining queen must necessarily be in the remaining region, and it can only be one, because of the above restrictions.

Therefore, the total QUBO formulation in this case is

$$Q = Q_r + Q_c + Q_d + Q_g =$$

$$= \sum_{i \in A_r} \left(1 - \sum_{j \in A_c(i)} x_{ij} \right)^2 +$$

$$+ \sum_{j \in A_c} \left(1 - \sum_{i \in A_r(j)} x_{ij} \right)^2 +$$

$$+ \sum_{(i,j) \in A'_C(k,l) \in D'_{i,j}} x_{ij} x_{kl} +$$

$$+ \sum_{k \in A_R} \left(1 - \sum_{(i,j) \in R_k \mid (i,j) \in A_C} x_{ij} \right)^2. \tag{9}$$

As before, all valid solutions to this problem have a cost Q = 0, while all solutions that do not satisfy the constraints have a cost Q > 0.

2.3 General cases

In the previous section we have solved the particular cases of N-queens and the LQueens. However, we can generalise this formulation to solve more complex problems. As we have seen, each constraint has an associated QUBO term, and the initial condition determines the number of variables we have in the problem. Therefore, let's determine the terms of the QUBO problem for each constraint. Since the constraints may be incompatible with each other, this formulation may

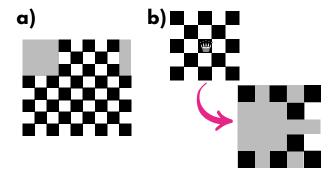


Figure 3: a) Irregular shaped board. b) Regular board with initial condition and its equivalent irregular board without initial condition. The cells in grey are squares that are not on the board.

also be used to test whether a certain generalized problem has a solution that satisfies its constraints.

As the initial condition only fixes the set of variables of the problem, in order to make the notation clear and without loss of generality, we will not add it. We will consider a rectangular board of dimensions $N \times M$. If we wanted irregular shapes, we would only have to create a rectangular board such that it can contain all the cells of the wanted one, and fix all cells outside the desired board to zero. In the same way, we can reduce a problem with an initial condition to a problem without an initial condition with a board with only the cells that are not fixed by the initialization, and adjusting the number of queens in each region according to the number of queens already existing in that region by the initial condition. We can see an example of an irregular board in Fig. 3 a), and how to translate a problem with initial condition to one without initial condition by changing the shape of the board in Fig. 3 b).

2.3.1 Number of queens per row/column

The first generalised constraint to consider is the following. Given a board of dimension $N \times M$, there can only be r_i queens in the i-th row. As can be understood, the row restriction in the cases studied above is the particular case where $r_i = 1$ for all rows.

The term that implements this constraint is a

generalisation of (1)

$$Q_r = \sum_{i=1}^{N} \left(r_i - \sum_{j=1}^{M} x_{ij} \right)^2.$$
 (10)

The next constraint to implement is, given a board of dimension $N \times M$, there can only be c_j queens in the j-th column. Once again, the column restriction in the cases studied above is the particular case where $c_j = 1$ for all columns.

The term that implements this constraint is a generalisation of (2)

$$Q_c = \sum_{j=1}^{M} \left(c_j - \sum_{i=1}^{N} x_{ij} \right)^2.$$
 (11)

In the case of an initial condition, these terms have to be modified both by eliminating the cells in which there can no longer be any queens (because their column and row already contain the required number of queens) and by reducing the number of queens required in the row and column in which we have already placed a queen. That is, if in row 3 we have placed a queen by initial condition, the number of queens available for the rest of cells in the row will be $r_3 - 1$, so in the QUBO we will have to reduce each r_i and c_j with the number of queens we initially placed in row i and column j, respectively.

2.3.2 Distant diagonals constraint

In the case of N-queens, we required that if there was a queen on a cell, there could not be any more queens on any of its diagonals until the end of the board, whereas in LQueens it is only required for cells on the adjacent diagonal. To generalise this case, we say that we have a diagonal condition of distance d_{ij} if in case there is a queen on the cell (i, j) there cannot be any queen on any cell located on any of its diagonals at a distance equal to or less than d_{ij} cells. In this way, not only do we allow diagonals of different lengths to be considered, but also that this depends on the cell.

The case of the N-queens is the particular case of $d_{ij} = \max(N, M) \quad \forall i \in [1, N], j \in [1, M], \text{ and }$ LQueens that of $d_{ij} = 1 \quad \forall i \in [1, N], j \in [1, M].$

The term that implements this is similar to the ones shown above in (3), as d_{ij} only change the set $D_{i,j}$. In this case, because of the non-symmetry of the required diagonal distance between cells,

D'' has to be used in the summation, instead of D'. D''_{ij} contains all the cells of D_{ij} but (i, j).

$$Q_d = \sum_{i,j}^{N,M} \sum_{(k,l) \in D_{i,j}''} x_{ij} x_{kl}, \tag{12}$$

Given this abstract formulation, this condition can be generalized so that the 4 diagonals do not have the same distance, that instead of being continuous lines they can be skipped lines, that instead of being 4 diagonals they are n_d lines with different slopes, or that they are directly abstract regions, since the only thing that change is the definition of the D_{ij} over which to sum for each cell.

2.3.3 Number of queens per region

In the case of LQueens we imposed that, for a set of N non-overlapping regions, in each of them there should be one and only one queen. However, we can generalize it so that regions can overlap, since it will only affect the definition of the R_k , and that instead of there having to be 1 queen per region, there must be q_k queens in the k-th region.

This term will be a generalization of (8) for a number N_R of regions.

$$Q_g = \sum_{k=1}^{N_R} \left(q_k - \sum_{(i,j) \in R_k} x_{ij} \right)^2.$$
 (13)

As can be seen, this term is a generalization of both the row term and the column term, since we can make one region per row and one region per column, and it would result in the same terms.

If we had an initial condition, this term would have to be corrected by subtracting from q_k the number of queens we have placed at the beginning in the k-th region, eliminating the cells that have already been fixed or that are in regions in which there can be no more queens.

Therefore, a generalized queens problem with a board of dimension $N \times M$ with no initial condition can be expressed by the two terms

$$Q = Q_g + Q_d = \sum_{k=1}^{N_R + N + M} \left(q_k - \sum_{(i,j) \in R'_k} x_{ij} \right)^2 + \sum_{i,j} \sum_{k,l \in D_i, i} x_{ij} x_{kl}$$

$$(14)$$

where the set of regions R' is contains the N_R regions R_k , the N rows and the M columns.

2.3.4 Two possible numbers of queens per region

A final generalization we can make is to create a constraint that is: given a board of dimension $N \times M$, in which there are N_V regions that can be overlapping, there must be a number v_k or $v_k + 1$ of queens in region k for each region. This constraint can be imposed by using a fractional term [13] that places the minimum cost in the middle of both values

$$Q_v = \sum_{k=1}^{N_V} \left(v_k + \frac{1}{2} - \sum_{(i,j) \in R_k} x_{ij} \right)^2.$$
 (15)

If we had an initial condition, we would have to apply the same correction as in the previous case.

As can be seen, the diagonal constraints are only particular cases of this constraint in which there is one region for each cell, which would be its diagonals. The value v_k would be 0 for all the regions, since the number of queens must be 0 or 1. We can rewrite the eq (14), with N_R regions R_k with a single possible number of queens and N_V regions V_k with two consecutive possible numbers of queens, as

$$Q = Q_g + Q_v = \sum_{k=1}^{N_R} \left(q_k - \sum_{(i,j) \in R_k} x_{ij} \right)^2 + \sum_{k=1}^{N_V} \left(v_k + \frac{1}{2} - \sum_{(i,j) \in V_k} x_{ij} \right)^2.$$
 (16)

Finally, if we want to write the whole cost function as a single term, for N_R regions R_k , we can define a value t_k , which will be 0 if the region k can only have a single number of queens q_k and 1 if the region k can have two numbers of queens q_k or $q_k + 1$, such that

$$Q = \sum_{k=1}^{N_T} \left(q_k + \frac{t_k}{2} - \sum_{(i,j) \in R_k} x_{ij} \right)^2.$$
 (17)

2.3.5 Toroidal board

An interesting generalization is to consider a toroidal board, so that the pieces that pass through the upper part of the board exit through the lower part, those that pass through the right part exit through the left part and vice versa. To

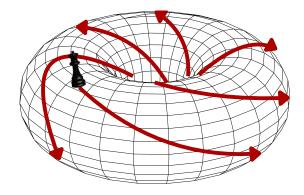


Figure 4: Toroidal board with a queen, where we mark in red what would be the squares that this queen could attack from only one of its 4 diagonals.

consider this problem, we must first impose that the side of the board has a minimum prime factor of 5, otherwise it will have no solution. We can also approach the case of a cylindrical board, in which this cyclic behavior will only occur when crossing the left and right sides.

To take into account this generalization, the terms of rows and columns will be exactly the same, as well as the term of the regions. The only term that will change will be that of the diagonals, since this, according to the length and shapes of the diagonals, will have to consider the cells reached after curving the boundary as many times as appropriate. Therefore, we will only have to change the definition of the D_{ij}' to include all the necessary cells.

2.4 Definition of generalised N-queens

With the terms and generalisations we have created for the N-queens, we can define a fully general N-queens, and its associated QUBO, and particularise it to a number of interesting families of N-queens problems.

The general N-queens problem is defined as:

Given a board of arbitrary shape, of maximum height N and maximum width M, on which there are placed N_I initial queens in positions given by the set A_I , and divided into N_R regions, which can overlap each other, such that in k-th region there must be q_k queens if $t_k = 0$ and q_k or $q_k + 1$ queens if $t_k = 1$, and we have initially placed p_k queens, the objective will be to place all the possible queens on the board such that the given restrictions are satisfied.

The QUBO formulation of this problem is given

in general form as

$$Q = \sum_{k=1}^{N_R} \left(q_k - p_k + \frac{t_k}{2} - \sum_{(i,j) \in R_k} x_{ij} \right)^2, \quad (18)$$

where R_k is the set of cells included in the k-th region that have not been set in the initial condition and that do not belong to a region where the number of queens available after the initial queens have been placed is 0.

We will propose the form of two particular versions of this problem.

2.4.1 N-queens with regions

In this case the problem is a combination of the original N-queens with the LQueens, adding to the N-queens the N_R regions in which there can only be one queen.

In this case, the QUBO formulation is

$$Q = Q_r + Q_c + Q_d + Q_g =$$

$$= \sum_{i=1}^{N} \left(1 - \sum_{j=1}^{N} x_{ij} \right)^2 + \sum_{j=1}^{N} \left(1 - \sum_{i=1}^{N} x_{ij} \right)^2 +$$

$$+ \sum_{i,j}^{N-1,N} \sum_{(k,l) \in D'_{i,j}} x_{ij} x_{kl} +$$

$$+ \sum_{k=1}^{N_R} \left(1 - \sum_{(i,j) \in R_k} x_{ij} \right)^2.$$
(19)

2.4.2 N-queens with soft regions

This is the case of N-queens with regions, but allowing each region to have a queen or not.

In this case, the QUBO formulation is

$$Q = Q_r + Q_c + Q_d + Q_v =$$

$$= \sum_{i=1}^{N} \left(1 - \sum_{j=1}^{N} x_{ij} \right)^2 + \sum_{j=1}^{N} \left(1 - \sum_{i=1}^{N} x_{ij} \right)^2 +$$

$$+ \sum_{i,j}^{N-1,N} \sum_{(k,l) \in D'_{i,j}} x_{ij} x_{kl} +$$

$$+ \sum_{k=1}^{N_R} \left(\frac{1}{2} - \sum_{(i,j) \in R_k} x_{ij} \right)^2.$$
(20)

Another equivalent formulation would be to change the last term to one similar to that of the diagonals

$$Q = \sum_{i=1}^{N} \left(1 - \sum_{j=1}^{N} x_{ij} \right)^{2} + \sum_{j=1}^{N} \left(1 - \sum_{i=1}^{N} x_{ij} \right)^{2} + \sum_{i,j}^{N-1,N} \sum_{(k,l) \in D'_{i,j}} x_{ij} x_{kl} + \sum_{k=1}^{N_{R}} \sum_{(i,j) \in R_{k}} \sum_{(l,m) \in R_{k} | (l,m) \neq (i,j)} x_{ij} x_{lm}.$$
 (21)

2.5 Tents & Trees

An interesting game related to these generalizations is Tents & Trees. This game is formulated as follows: Given a board of dimensions $N \times N$ cells, with N_t trees placed at positions A_t , the objective is to place a series of tents such that the following constraints are satisfied:

- Each tree will have an associated tent, in a cell vertically or horizontally adjacent to it.
- No two tents can be adjacent to each other, not even diagonally.
- Each row i and column j has to have a number $N_{r,i}$ and $N_{c,j}$ of associated tents.

We can translate this problem to our formalism with the following considerations:

- Each tree cell is no longer considered as an optimizing variable and is fixed to 0.
- The k-th tree creates a region R_k , given by the 4 cells it has adjacent to it, such that in this region there must be at least 1 tent. In the cells not included by any of the regions, we fix the variables to 0.
- The tents act as queens, but can only attack the 8 adjacent cells.
- Each row i and column j has to have a number $N_{r,i}$ and $N_{c,j}$ of associated tents.

Our variables will be a set of binary x_{ij} indicating that in row cell i and column j there is no tent if equal to 0 and there is if equal to 1.

The terms imposing the number of tents per row and column are exactly as we posed in the general case in (10) and (11). For the non-contact constraint between tents, we are going to use the term of (12), but making $D'_{i,j}$, in addition to taking into account the adjacent diagonals, take into account the vertical and horizontal adjacencies. To reduce the number of terms, we will not take into account the cells above the tent, because if a tent has another one above it, the one above it will see it below it.

For the regions given by the trees, we will have to use a term of type (15) with $v_k = 1$. This is because if there were 3 or 4 tents surrounding the tree, there would necessarily be 2 of them in diagonal contact. Therefore, there can only be 1 or 2 tents in that region.

Thus, the QUBO of the problem is

$$Q = Q_r + Q_c + Q_d + Q_g =$$

$$= \sum_{i=1}^{N} \left(N_{r,i} - \sum_{j=1}^{N} x_{ij} \right)^2 + \sum_{j=1}^{N} \left(N_{c,j} - \sum_{i=1}^{N} x_{ij} \right)^2 +$$

$$+ \sum_{i,j}^{N,N} \sum_{(k,l) \in D'_{i,j}} x_{ij} x_{kl} +$$

$$+ \sum_{k=1}^{N_t} \left(\frac{3}{2} - \sum_{(i,j) \in R_k} x_{ij} \right)^2.$$
(22)

As in the previous cases, the minimum energy will be of $\frac{N_t}{4}$.

2.6 Chess Piece Problems

As can be seen, this formulation allows, not only to solve the case for the movement of queens, but also for the movement of any type of piece.

Therefore, we can generalise the problem to the chess piece problem, which will consist of trying to place as many pieces as possible on a board of maximum dimensions $N \times M$ so that no piece threatens another, with the extra restriction that there can only be one piece per coloured region and each cell can only be empty or have a specific type of piece, which can vary between cells.

The Coloured Chess Piece Problem is:

Given a board of arbitrary shape, of maximum height N and maximum width M, on which there are placed a number N_p of initial pieces in positions given by the set A_I , and divided into N_R regions, which cannot overlap each other, such that in each region there must be only one piece, the objective will be to place all the possible pieces on the board such that no piece threatens another.

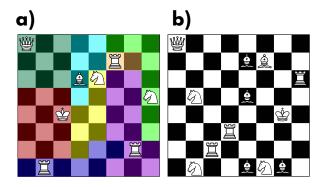


Figure 5: a) Solution of a rectangular Coloured Chess Piece Problem, b) Solution of a rectangular Max Chess Pieces Problem

Each cell (i, j) can be empty or have the piece of type P_{ij} , with possible moves $M_{P_{ij}}$, and can reach cells $C\left(M_{P_{ij}}\right)_{ij}$. A solution of this problem is represented in Fig. 5 a.

As in the case of the queens, we will only have to eliminate the cells in which we have set the variables when initializing. That is, in which we have placed the pieces $((i,j) \in A_I)$, in the regions in which we have placed them $((l,m) \in R_k | (i,j) \in R_k, \forall (i,j) \in A_I)$ and in the cells threatened by the placed pieces $((l,m) \in C(M_{P_{ij}})_{ij} \forall (i,j) \in A_I)$. These cells can only be empty, so their variables will not be considered in the optimization, although we can place them in the QUBO formulation, since such values of 0 will not alter the summations. We will define the R'_k regions as the N'_R regions in which no pieces have been placed at the beginning.

This problem has the restrictions:

• Coloured Regions: There can be only one piece per coloured region:

$$\sum_{(i,j)\in R'_{L}} x_{ij} = 1 \quad \forall k \in [0, N'_{R} - 1]$$

• No threat: No piece can be threatening another piece:

$$\sum_{(l,m)\in C\left(M_{P_{ij}}\right)_{ij}} x_{lm} = 0 \quad \forall (i,j) | x_{ij} = 1$$

To implement these restrictions, we only need a coloured term

$$Q_g = \sum_{k=0}^{N_R'-1} \left(1 - \sum_{(i,j) \in R_k'} x_{ij}\right)^2 \tag{23}$$

and a threat term

$$Q_t = \sum_{i,j=0}^{N-1,M-1} \sum_{(l,m)\in C(M_{P_{ij}})_{ij}} x_{ij} x_{lm}.$$
 (24)

The total QUBO formulation is

$$Q = Q_g + Q_t = \sum_{k=0}^{N_R'-1} \left(1 - \sum_{(i,j) \in R_k'} x_{ij} \right)^2 + \sum_{i,j=0}^{N-1,M-1} \sum_{(l,m) \in C(M_{P_{ij}})_{ij}} x_{ij} x_{lm}.$$
 (25)

Thus, a board that meets the constraints will be one with zero energy.

The other problem we can solve is the Max Chess Pieces Problem, which consists of placing as many pieces as possible on the board so that none of them threatens another, and there can be only one type per cell. We will define it as:

Given a board of arbitrary shape, of maximum height N and maximum width M, on which there are placed a number N_p of initial pieces in positions given by the set A_I , the objective will be to place the maximum number possible of pieces on the board such that no piece threatens another. Each cell (i,j) can be empty or have the piece of type P_{ij} , with possible moves $M_{P_{ij}}$, and can reach cells $C\left(M_{P_{ij}}\right)_{ij}$. A solution of this problem is represented in Fig. 5 b.

Since we will only keep the threat constraint, we can eliminate the coloured term, and we have to include a new term that reduces the energy the more tiles there are on the board. This counting term is

$$Q_n = -\sum_{i,j=1}^{N-1,M-1} x_{ij}$$
 (26)

so that the energy will decrease by one unit for each piece added to the board. This formulation even allows a different weight to be associated with each type of piece, so that it is more important to place some types of pieces than others.

To get the total QUBO, if we add directly the terms Q_t and Q_m we can have cases in which, by adding a single piece that threatens only one piece, we will have the same energy. Therefore, to make sure that the constraint is met, we will multiply the Q_t term by a constant λ large enough so that it is not convenient to add pieces that do not

meet the constraint. Therefore, the total QUBO is

$$Q = Q_n + \lambda Q_t = -\sum_{i,j=1}^{N-1,M-1} x_{ij} + \lambda \sum_{i,j=0}^{N-1,M-1} \sum_{(l,m)\in C(M_{P_{ij}})_{ij}} x_{ij} x_{lm}.$$
 (27)

3 Takuzu and LinkedIn's Tango Problems

3.1 Problem definition

The problem to be solved in Takuzu is the following:

Given a square board of dimension $N \times M$ cells, where each cell can contain only one 0 or 1, initially with N_{I0} zeros placed at positions A_{I0} and N_{I1} ones placed at positions A_{I1} , we have to determine for each cell whether there should be a zero or a one. The constraints are as follows:

- In each row and column there must be as many zeros as ones.
- There cannot be more than two zeros or ones in a row vertically or horizontally.
- No two rows and no two columns can have the same combination of zeros and ones.

A solution to this problem can be seen in Fig. 6 a.

The Tango problem is very similar, but adding additional constraints and removing the restriction of repeating rows or columns. It is expressed as follows:

Given a square board of dimension 6×6 cells, where each cell can contain only one sun icon or one moon icon, initially with N_{IS} suns placed at positions A_{IS} and N_{IM} moons placed at positions A_{IM} , we have to determine for each cell whether there should be a sun or a moon. The constraints are as follows:

- In each row and column there must be as many moons as suns.
- There cannot be more than two suns or moons in a row vertically or horizontally.

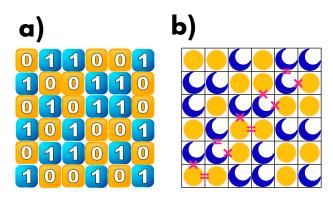


Figure 6: Solution for a problem a) Takuzu/0h h1 6×6 , b) Tango

- There are cells between which there is an '=' symbol, which indicates that in both cells there must be the same icon.
- There are cells between which there is an 'x' symbol, which indicates that there must be a different icon in both cells.

A solution to this problem can be seen in Fig. 6 b.

In this case, we can see that if we replace the suns by zeros and the moons by ones, and allow the board to be $N \times M$, we will have a generalization of Takuzu. We call this problem the Takuzu/Tango Problem (TTP).

To formulate this problem, the state of a board is given by a set of variables x_{ij} , which indicate the value in the cell of row i and column j. Therefore, the constraints of the problem can be expressed as:

• Row regularity: there must be as many zeros as ones in each row (Fig. 7 c):

$$\sum_{i=1}^{M} x_{ij} = \frac{M}{2} \quad \forall i \in [1, N].$$
 (28)

• Column regularity: there must be as many zeros as ones in each column (Fig. 7 c):

$$\sum_{i=1}^{N} x_{ij} = \frac{N}{2} \quad \forall j \in [1, M].$$
 (29)

• Vertical repetition condition: there cannot be more than two zeros or ones in a row vertically. Therefore, for each group of 3 vertically consecutive cells, there can only be 1 or 2 ones, which implies that their associated variables can only add up to 1 or 2 (Fig. 7 a):

$$\sum_{k=i}^{i+2} x_{kj} \in \{1, 2\} \quad \forall i \in [1, N-2], j \in [1, M].$$
(30)

• Horizontal repetition condition: there cannot be more than two zeros or ones in a row horizontally. Therefore, for each group of 3 horizontally consecutive cells, there can only be 1 or 2 ones, which implies that their associated variables can only add up to 1 or 2 (Fig. 7 b):

$$\sum_{k=j}^{j+2} x_{ik} \in \{1, 2\}$$

$$\forall i \in [1, N], j \in [1, M-2].$$
(31)

• Equal sign condition: If there is an '=' symbol between two cells, both cells must have the same value. These symbols can be between two cells horizontally or between two cells vertically (Fig. 7 d). There are N_{He} horizontal symbols and N_{Ve} vertical symbols. The k-th horizontal '=' is between positions (I_k^{He}, J_k^{He}) and $(I_k^{He}, J_k^{He} + 1)$, while the k-th vertical '=' is between positions (I_k^{Ve}, J_k^{Ve}) and $(I_k^{Ve} + 1, J_k^{Ve})$. Therefore, for the horizontals, the following is satisfied

$$x_{I_k^{He},J_k^{He}} = x_{I_k^{He},J_k^{He}+1} \ \forall k \in [1,N_{He}], \ \ (32)$$
 and for the verticals

 $x_{IV^e,JV^e} = x_{IV^e+1,JV^e} \quad \forall k \in [1, N_{Ve}].$ (33)

• Cross sign condition: If there is an 'x' symbol between two cells, both cells must have different value. These symbols can be between two cells horizontally or between two cells vertically (Fig. 7 e). There are N_{Hc} horizontal symbols and N_{Vc} vertical symbols. The k-th horizontal 'x' is between positions (I_k^{Hc}, J_k^{Hc}) and $(I_k^{Hc}, J_k^{Hc} + 1)$, while the k-th vertical 'x' is between positions (I_k^{Vc}, J_k^{Vc}) and $(I_k^{Vc} + 1, J_k^{Vc})$. Therefore, for the horizontals, the following is satisfied

$$x_{I_k^{Hc}, J_k^{Hc}} \neq x_{I_k^{Hc}, J_k^{Hc} + 1} \ \forall k \in [1, N_{Hc}], \ (34)$$

and for the verticals

$$x_{I_k^{Vc},J_k^{Vc}} \neq x_{I_k^{Vc}+1,J_k^{Vc}} \quad \forall k \in [1, N_{Vc}].$$
 (35)

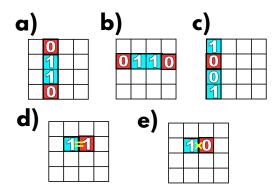


Figure 7: Boards with restrictions. The blue cells have the values previously set, while the red cells have their values determined by the constraints. a) Vertical repetition, b) Horizontal repetition, c) Regularity, d) Symbol '=', e) Symbol 'x'.

• Global non-repetition constraint: no two rows can have the same combination of zeros and ones. As the positions of the ones condition those of the zeros, we will only have to consider those of the ones. If two cells in the same row are at one, the product of their values will be one. In any other case it will be zero. Therefore, all the elements of two rows will coincide if the sum of their products is equal to the number of ones in each row.

$$\sum_{j=1}^{M} x_{aj} x_{bj} < \frac{M}{2} \ \forall a, b \in [1, N] \ | \ a \neq b. \ (36)$$

Neither can there be two columns with the same combination of zeros and ones.

$$\sum_{i=1}^{N} x_{ia} x_{ib} < \frac{N}{2} \ \forall a, b \in [1, M] \ | \ a \neq b. \ (37)$$

3.2 QUBO Formulation

The QUBO formulation of this problem is composed of different terms that are responsible for imposing each of the constraints. Before starting to formulate each term, we will discuss how to optimize the number of variables to be determined by fixing trivial values for some variables given an initialization. This way, some variables will have a fixed value in the QUBO terms, but they will not be introduced as variables to be optimized in algorithms such as QAOA or Quantum Annealers, but will be part of the coefficients of the cost matrix. This make it possible to reduce

the resources, e.g. qubits, to be used in quantum devices.

If we have a set of zeros and ones placed in their positions A_{I0} and A_{I1} , the variables associated with those cells do not need to be determined. This will reduce the number of variables to optimize to $NM - N_{I0} - N_{I1}$.

In case any of these already fixed variables is adjacent to an '=' symbol, we will trivially fix the adjacent variable to its same value. If the adjacent symbol is an 'x', we will fix the adjacent variable to the opposite value. This allow us to remove those variables from the optimization in the same way.

In case a row or column has already fixed as many zeros or ones as it should have by the regularity condition, the rest of the variables are set to the opposite value, as this is the only way to satisfy this constraint. With this we can eliminate even more variables trivially.

In case there are 2 adjacent variables fixed with the same value, we will automatically know that the ones adjacent to both of them that follow in their line have to have the opposite value, which allows us to fix some more variables.

Once the number of variables to be determined has been optimized, we proceed with the creation of the QUBO terms. The first term we need to create is the horizontal repetition term, which is

$$Q_{HR} = \sum_{i,j=1}^{N,M-2} \left(\frac{3}{2} - \sum_{k=j}^{j+2} x_{ik}\right)^2$$
 (38)

where we have that the minimum is centered between 1 and 2 using a fractional QUBO term [13], so that in each horizontal group of 3 variables the minimum value of the energy $(\frac{1}{4})$ is given when 1 or 2 variables have the value 1. For the vertical condition, we have an analogous term

$$Q_{VR} = \sum_{i,j=1}^{N-2,M} \left(\frac{3}{2} - \sum_{k=i}^{i+2} x_{kj}\right)^2.$$
 (39)

For the regularization conditions, we only need a term that imposes that the sum of the variables per row or column results in a specific value. These terms are for each case

$$Q_{Hr} = \sum_{i=1}^{N} \left(\frac{M}{2} - \sum_{j=1}^{M} x_{ij} \right)^{2}, \tag{40}$$

$$Q_{Vr} = \sum_{j=1}^{M} \left(\frac{N}{2} - \sum_{i=1}^{N} x_{ij} \right)^{2}.$$
 (41)

Finally, we will have the 4 terms necessary to impose the constraints of '=' horizontal, '=' vertical, 'x' horizontal and 'x' vertical. For the '=' terms, we need the energy when they are equal to be zero. We achieve this by knowing that, for a binary case x(1-y)+(1-x)y is null if x=y and 1 if $x\neq y$. We need the second summand so that there is no null if x=0 and y=1. Thus, the terms for equals are

$$Q_{He} = \sum_{k=1}^{N_{He}} \left(x_{I_k^{He}, J_k^{He}} \left(1 - x_{I_k^{He}, J_k^{He} + 1} \right) + \left(1 - x_{I_k^{He}, J_k^{He}} \right) x_{I_k^{He}, J_k^{He} + 1} \right), \quad (42)$$

$$Q_{Ve} = \sum_{k=1}^{N_{Ve}} \left(x_{I_k^{Ve}, J_k^{Ve}} \left(1 - x_{I_k^{Ve} + 1, J_k^{Ve}} \right) + \left(1 - x_{I_k^{Ve}, J_k^{Ve}} \right) x_{I_k^{Ve} + 1, J_k^{Ve}} \right). \tag{43}$$

For the case 'x' we need a term that is null only when both values are different. To do so, we will take advantage of the fact that xy + (1-x)(1-y) is equal to 0 if $x \neq y$ and 1 if x = y. Thus, the terms are

$$Q_{Hc} = \sum_{k=1}^{N_{Hc}} \left(x_{I_k^{Hc}, J_k^{Hc}} x_{I_k^{Hc}, J_k^{Hc}+1} + \left(1 - x_{I_k^{Hc}, J_k^{Hc}} \right) \left(1 - x_{I_k^{Hc}, J_k^{Hc}+1} \right) \right),$$

$$(44)$$

$$\begin{aligned} Q_{Vc} &= \sum_{k=1}^{N_{Vc}} \left(x_{I_k^{Vc}, J_k^{Vc}} x_{I_k^{Vc} + 1, J_k^{Vc}} + \right. \\ &+ \left. \left(1 - x_{I_k^{Vc}, J_k^{Vc}} \right) \left(1 - x_{I_k^{Vc} + 1, J_k^{Vc}} \right) \right). \end{aligned} \tag{45}$$

Given this formulation, we can perform the following optimization of variables by taking advantage of the symbols '=' and 'x'. As we know, if there is one of these symbols between two cells, both variables must have the same or different value. In this case in which we can only have two possible values, this implies that these two variables are going to be completely dependent, so that if we determine one, the other is automatically determined. Therefore, these last four terms of the formulation are unnecessary if we reduce the number of variables as follows.

In our optimization, we optimize only one of the variables of the pair affected by the symbols, and in case we have a chain of variables all affected by each other, we optimize only one of them. The rule we follow is that in each horizontal pair, we optimize the variable on the left, and in each vertical pair, the one on top. In case of having whole regions affected, we choose the variable that is more to the left, and in case of having several in the same column, the one that is higher among them.

To take this dependence into account, we will make the following substitution in the first four terms:

• In the case of having an '=' between (i, j) and (i, j + 1)

$$x_{i,j+1} = x_{i,j}.$$
 (46)

• In the case of having an '=' between (i, j) and (i + 1, j)

$$x_{i+1,j} = x_{i,j}. (47)$$

• In the case of having an 'x' between (i, j) and (i, j + 1)

$$x_{i,j+1} = (1 - x_{i,j}). (48)$$

• In the case of having an 'x' between (i, j) and (i + 1, j)

$$x_{i+1,j} = (1 - x_{i,j}). (49)$$

In this way, we are able to impose this constraint automatically, without having to optimize it, reducing the number of variables to optimize. Thus, the number of variables to be optimized will be, at the most

$$NM - N_{I0} - N_{I1} - N_{He} - N_{Hc} - N_{Ve} - N_{Vc}$$
.

Finally, the global non-repetition condition cannot be written as a QUBO term, so we will have to be confident that the solutions to the problem do not conflict with this constraint.

Therefore, the QUBO formulation of the TTP is

$$Q = Q_{HR} + Q_{VR} + Q_{Hr} + Q_{Vr} =$$

$$= \sum_{i,j=1}^{N,M-2} \left(\frac{3}{2} - \sum_{k=j}^{j+2} x_{ik}\right)^{2} +$$

$$+ \sum_{i,j=1}^{N-2,M} \left(\frac{3}{2} - \sum_{k=i}^{i+2} x_{kj}\right)^{2} +$$

$$+ \sum_{i=1}^{N} \left(\frac{M}{2} - \sum_{j=1}^{M} x_{ij}\right)^{2} +$$

$$+ \sum_{j=1}^{M} \left(\frac{N}{2} - \sum_{i=1}^{N} x_{ij}\right)^{2}.$$
 (50)

Thus, a solution to the problem that satisfies the constraints will have an energy of $\frac{1}{2}(NM - N - M)$.

3.3 Generalizations of the TTP

Having solved the original problem, we will present several generalizations of TTP.

3.3.1 Unbalanced regularity condition

The first generalization we propose is to allow for each row or column to have a different number of zeros than ones. Since we know that the number of ones determines the number of zeros, we can say that if there are N cells in that direction, we want there to be N_1 ones and $N-N_1$ zeros. This can also change between rows and columns. We say that in row i we want $M_{1,i}$ ones, and in column j we want $N_{1,j}$ ones. These generalized terms, for both horizontal and vertical, are

$$Q_{Hr} = \sum_{i=1}^{N} \left(M_{1,i} - \sum_{j=1}^{N} x_{ij} \right)^{2}, \quad (51)$$

$$Q_{Vr} = \sum_{j=1}^{M} \left(N_{1,j} - \sum_{i=1}^{N} x_{ij} \right)^{2}.$$
 (52)

3.3.2 Diagonal repetition

The second generalization we propose is that, in addition to not being able to have more than two ones or zeros in a row vertically or horizontally, it should not be possible to have more than two ones or zeros diagonally. For this, we define 2 new terms, exactly the same as the original ones

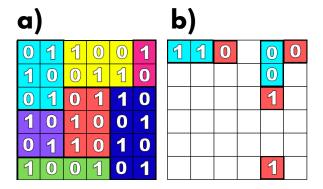


Figure 8: Generalized cases. a) Takuzu with colored regions, b) toroidal Takuzu, where the blue cells have fixed initial values and the red cells are a consequence of the repetition constraint.

for this, but changing the second summation, so that they evaluate the lower diagonals for each cell. We have Q_{DLR} for the left diagonals and Q_{DRR} for the right diagonals:

$$Q_{DLR} = \sum_{i,j=1,3}^{N-2,M} \left(\frac{3}{2} - \sum_{k=0}^{2} x_{i+k,j-k}\right)^{2}, \quad (53)$$

$$Q_{DRR} = \sum_{i,j=1}^{N-2,M-2} \left(\frac{3}{2} - \sum_{k=0}^{2} x_{i+k,j+k}\right)^{2}, \quad (54)$$

being $x_{a,b} = 0 \quad \forall b < 1, a \in [1, N]$, and $\forall b > M, a \in [0, N-1]$.

3.3.3 Regions condition

Another possible generalization is the imposition of a specific number of ones in certain colored regions. That is, for a set of N_R regions, in the k-th region R_k there must be m_k ones, as we can see in Fig. 8 a. This condition is imposed with a term

$$\sum_{k=1}^{N_R} \left(m_k - \sum_{(i,j) \in R_k} x_{ij} \right)^2. \tag{55}$$

3.3.4 Toroidal board

An interesting generalization is to consider a toroidal board, so that the condition that there are no more than 2 ones or zeros in a row vertically or horizontally is cyclically extended to the other end of the board, as can be seen in Fig. 8 b

The only term that will change will be the repetition constraint, so that we will have to perform

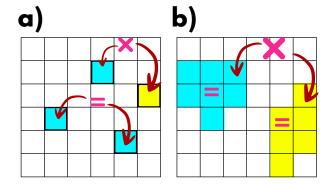


Figure 9: a) Symbols that act over long distances. b) Regions of equality that are different from each other. All cells in the blue region have the same value, all cells in the yellow region have the same value, and the blue cells all have different values than the yellow cells.

the summations with all rows and columns, making $x_{a,b} = x_{a-N,b}$ if a > N and $x_{a,b} = x_{a,b-M}$ if b > M. In this way we will take into account the toroid shape. In case we consider a cylinder instead of a toroid, we will only have to keep one of these changes, in the direction that is cyclic.

3.3.5 Long distance equal/cross

In Tango's original problem, we have that there can only be '=' and 'x' symbols acting between vertically and horizontally adjacent cells. We extend this idea to mean that both symbols can act between any two cells, as can be seen in Fig. 9 a. To implement this generalization, we need only allow that the variable changes exhibited in (46), (47), (48) and (49) can be made between two non-adjacent variables. The rest of the formulation is the same.

On the other hand, we also generalize the possibility of defining regions in which all their variables have to be equal (equality regions), and separately defining that a pair of equality regions have to have different values for their variables from each other, as in Fig. 9 b. To do this, just impose a set of '=' symbols between the cells of the region, so that they form a chain of equalities with each other, and then impose a single 'x' symbol between any pair of cells of one and the other region. In this way, since all cells in each region have to be equal, if a cell in one is different from a cell in the other, all cells in one region will be opposite cells in the other region.

4 Conclusions

We have presented several generalizations of the N-queens problem, including the case of LinkedIn, presenting a general QUBO formulation to solve them, and interesting particular cases with their corresponding formulations, as the Tents & Trees. We have also presented two new interesting types of problems, associated with N-queens, and their respective QUBO formulations. We have also presented a mathematical formulation for the Takuzu and the LinkedIn Tango games, shown a generalization of it, and created the QUBO formulation that solves it. We have seen that this formulation requires at most $NM - N_{I0} - N_{I1} - N_{He} - N_{Hc} - N_{Ve} - N_{Vc}$ variables, being greatly optimized thanks to the initial conditions and equal/cross terms. We have also presented several generalizations of the problem and their corresponding QUBO formulations.

Future lines of research may include its implementation in various types of quantum hardware, benchmarking them using these problems, its extension to the case in which we allow more than two consecutive values for the number of queens in each region, the creation of a Grover's [14] oracle to solve the problem, its transformation to QUDO or HOBO formulation or addressing how to efficiently enforce the row and column non-repetition constraint.

Acknowledgment

This work is framed within the Quipucamayocs, the activities of the QuantumQuipu community.

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