DFT

$$f : \mathbb{C} \to \mathbb{C}$$

$$(x_0, x_1, x_2 \dots x_{N-1})^T \to (y_0, y_1, y_2 \dots y_{n-1})^T$$

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{\frac{2i\pi jk}{N}} x_j$$

$$y_k = \frac{1}{\sqrt{N}} \left(e^{\frac{2i\pi 0.k}{N}} x_0 + e^{\frac{2i\pi 1 k}{N}} x_1 + \dots + e^{\frac{2i\pi (N-1)k}{N}} x_{N-1} \right)$$

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad N = 2$$

$$y_0 = \frac{1}{\sqrt{2}} \left(\cancel{e^{\frac{2i\pi 0.0}{2}}}^{1} 1 + \cancel{e^{\frac{2i\pi 1.0}{N}}}^{1} . 2 \right) = \frac{1}{\sqrt{2}} . 3 \to \frac{3}{\sqrt{2}}$$

$$y_1 = \frac{1}{\sqrt{2}} \left(\cancel{e^{\frac{2i\pi 0.1}{N}}}^{1} 1 + e^{\frac{2i\pi 1}{N}} . 2 \right) = \frac{-1}{\sqrt{2}}$$

$$y = \left(3/\sqrt{2} \quad -1/\sqrt{2} \right)^T$$

QFT

$$|\psi\rangle \to DFT\,|\psi\rangle = QFT$$

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} e^{\frac{i2\pi jk}{N}} x_j |k\rangle$$

for basis state $\rightarrow \frac{1}{\sqrt{N}} \sum_{K=0}^{N-1} e^{\frac{i\pi 2 y k}{N}} |K\rangle$

one $x_y = 1$
and all the
other bits are 0

$$|y\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{QFT} \frac{1}{\sqrt{N}} \begin{pmatrix} \overset{1\cdot 0}{\cancel{w^{00}}} + & \overset{1\cdot 0}{\cancel{w^{01}}} + & \overset{1\cdot 1}{\cancel{w^{02}}} + & \overset{1\cdot 0}{\cancel{w^{03}}} \\ \cancel{w^{10}} & w^{11} & w^{12} & w^{13} \\ \cancel{w^{20}} & w^{21} & w^{22} & w^{21} \\ \cancel{w^{30}} & w^{31} & w^{32} & w^{33} \end{pmatrix}$$

$w = e^{\frac{i\pi 2}{N}}$

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ w^{12} \\ w^{22} \\ w^{32} \end{pmatrix} \rightarrow \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ w^{y} \\ w^{2y} \\ w^{3y} \end{pmatrix}$$

$$QFT(|10\rangle) \rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ w^2 \\ w^4 \\ w^6 \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ e^{\frac{i\pi 4}{4}} \\ e^{\frac{i\pi 8}{4}} \\ e^{\frac{i\pi 12}{4}} \end{pmatrix} \rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{N}} \begin{pmatrix} \overset{1\cdot 0}{\cancel{w^{00}}} + & \overset{1\cdot 1}{\cancel{w^{01}}} + & \overset{1\cdot 1}{\cancel{w^{02}}} + & \overset{1\cdot 0}{\cancel{w^{03}}} \\ \cancel{w^{10}} & w^{11} & w^{12} & w^{13} \\ \cancel{w^{20}} & w^{21} & w^{22} & w^{21} \\ \cancel{w^{30}} & w^{31} & w^{32} & w^{33} \end{pmatrix}$$

$$\frac{1}{\sqrt{4}} \cdot \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 2 \\ w + w^2 \\ w^2 + w^4 \\ w^3 + w^6 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ e^{\frac{i2\pi}{4}} + \cancel{e^{\frac{i4\pi}{4}}}\ -1 \\ e^{\frac{i4\pi}{4}} + \cancel{e^{\frac{i8\pi}{4}}}\ 1 \\ e^{\frac{i6\pi}{4}} + \cancel{e^{\frac{i12\pi}{4}}}\ -1 \end{pmatrix}$$

$$\frac{1}{2\sqrt{2}} \left(2|00\rangle + e^{i\frac{\pi}{2}}|01\rangle - |01\rangle \cancel{-|10\rangle} \cancel{+|10\rangle} + e^{i\pi\frac{3}{2}}|11\rangle - |11\rangle \right.$$

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{(i-1)}{2\sqrt{2}}|01\rangle - \frac{(i+1)}{2\sqrt{2}}|11\rangle$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$QFT\,|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & w^{1\cdot 1} \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1.\alpha + 1.\beta \\ 1.\alpha - 1\beta \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}$$

Inverse QFT

QFT^{-1} or $QFT^{\dagger}$

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & w^{-1} & w^{-2} & w^{-3} & \cdots \\ 1 & w^{-2} & w^{-4} & w^{-6} & \cdots \\ 1 & w^{-3} & w^{-6} & w^{-9} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Linear Shift

$$QFT \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{N-1} \end{pmatrix} \qquad QFT \begin{pmatrix} \alpha_{N-1} \\ \alpha_0 \\ \vdots \\ \alpha_{N-2} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \omega \beta_1 \\ \vdots \\ \omega^{N-1} \beta_{N-1} \end{pmatrix}$$

$$QFT \begin{pmatrix} \alpha_0 \\ \omega \alpha_1 \\ \vdots \\ \omega^{N-1} \alpha^{N-1} \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_0 \end{pmatrix}$$

$$QFT\,|11\rangle \rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad QFT\,|11\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\omega \\ \omega^2 \\ -\omega^3 \end{pmatrix}$$

$$= \frac{1}{2} \left(\begin{matrix} 1 \\ -i \\ -1 \\ i \end{matrix} \right.$$

QPE

$$\underbrace{A \cdot v = \overset{\swarrow \text{eigenvalue}}{\lambda}}_{\text{eigenvector}} v$$

$X|+\rangle = |+\rangle$ eigenvector $= \{|+\rangle, |-\rangle\}$

$X|-\rangle = -|-\rangle$ eigenvalue $= \{1, -1\}$

$$|\psi\rangle = \frac{|0\rangle + (-1)^x|1\rangle}{\sqrt{2}}$$ x is either 0 or 1

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + (-1)^x|-\rangle\right)$$

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + (-1)^x \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$\frac{1}{2}\left(|0\rangle + |1\rangle + (-1)^x|0\rangle - (-1)^x|1\rangle\right)$$

$x = 0$ $\quad |0\rangle + \cancel{|1\rangle} + |0\rangle - \cancel{|1\rangle} \rightarrow |0\rangle$

$x = 1$ $\quad \cancel{|0\rangle} + |1\rangle - \cancel{|0\rangle} + |1\rangle \rightarrow |1\rangle$

ϕ is in binary as:

$\phi 2^0 \rightarrow 0, \varphi_1 \varphi_2 \dots \varphi_n \varphi_{n+1} \dots$

$\phi 2^1 \rightarrow \varphi_1, \varphi_2 \dots \varphi_n \varphi_{n+1} \dots$

$\phi 2^{n-1} \rightarrow \varphi_1 \varphi_2 \varphi_3 \dots \dots$