We have an oracle that maps $\{0,1\}^n \to \{0,1\}$ and we want to find a string $x \in \{0,1\}^n$ which $f(x)=1$ (marked item)

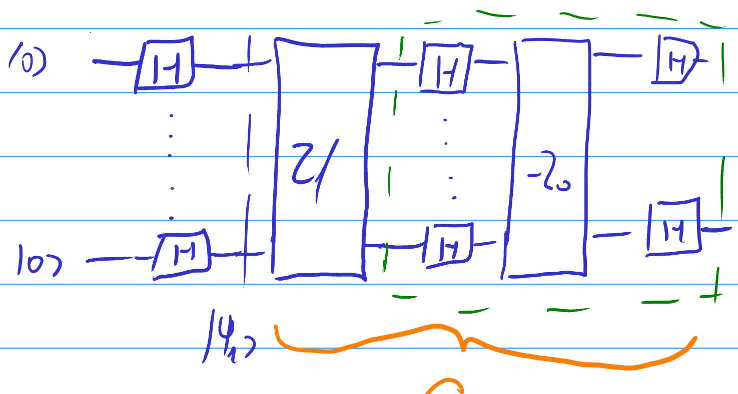
$N = 2^n$ → total possible values for x

classically → $\Omega(N)$ ← lower bound ← brute force search
Quantum → $\Omega(\sqrt{N})$ ← the best possible

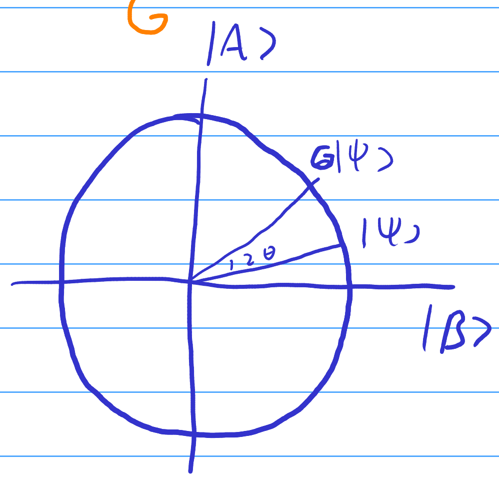
→ Grover algorithm works applying interference repeatedly

→ Grover's algorithm

$$G = -H^{\otimes n} Z_0 H^{\otimes n} Z_1 \quad \leftarrow \text{operator}$$



→ apply G k times



$f(x)=1$ $|A\rangle$ = Good state
$f(x)=0$ $|B\rangle$ = Bad state

$A = \{x \in \{0,1\}^n \mid f(x)=1\}$
$B = \{x \in \{0,1\}^n \mid f(x)=0\}$

$a = |A|$ → n° of good things

$b = |B|$ → number of bad things

$$|A\rangle = \frac{1}{\sqrt{a}} \sum_{x \in A} |x\rangle$$

$$|B\rangle = \frac{1}{\sqrt{b}} \sum_{x \in B} |x\rangle$$

the whole circuit is just a superposition of state $|A\rangle$ and $|B\rangle$

$\alpha|A\rangle + \beta|B\rangle$

$$|h\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle$$

← all possible states $|A\rangle$ & $|B\rangle$

$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle = |h\rangle$$

$$|h\rangle = \sqrt{\frac{a}{N}}\,|A\rangle + \sqrt{\frac{b}{N}}\,|B\rangle$$

Z_f is the oracle so $Z_f|\psi\rangle = (-1)^{f(x)}|\psi\rangle$

$$Z_0 = \begin{cases} -|x\rangle, & \text{when } x = 0^{\otimes n} \\ |x\rangle, & \text{when } x \neq 0^{\otimes n} \end{cases}$$

so $$Z_0 = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \end{pmatrix}$$

$$Z_0 = \mathbb{I} - 2|0^n\rangle\langle 0^n|$$

$$G|A\rangle \Rightarrow (-H^{\otimes n} Z_0 H^{\otimes n} \underbrace{Z_f}) |A\rangle \qquad f(x) \text{ in } |A\rangle = 1$$

$$(-H^{\otimes n} Z_0 H^{\otimes n}) (-1)^{f(x)} |A\rangle$$

$$(\cancel{-}H^{\otimes n} Z_0 H^{\otimes n}) \cancel{-}1 |A\rangle$$

$$(\underbrace{H^{\otimes n} Z_0} H^{\otimes n}) |A\rangle$$

$$H^{\otimes n} (\mathbb{I} - 2|0^n\rangle\langle 0^n|) H^{\otimes n}$$

$$\mathbb{I} - 2\underbrace{H^{\otimes n}|0^n\rangle}_{|h\rangle}\underbrace{\langle 0^n| H^{\otimes n}}_{\langle h|}$$

$$(\mathbb{I} - 2|h\rangle\langle h|)|A\rangle$$

$$\left(\mathbb{I} - 2\left(\sqrt{\frac{a}{N}}|A\rangle + \sqrt{\frac{b}{N}}|B\rangle\right)\left(\sqrt{\frac{a}{N}}\langle A| + \sqrt{\frac{b}{N}}\langle B|\right)\right)|A\rangle$$

$$\left(\mathbb{I} - 2\left(\frac{a}{N}|A\rangle\langle A| + \frac{\sqrt{ab}}{N}|A\rangle\langle B| + \frac{\sqrt{ab}}{N}|B\rangle\langle A| + \frac{b}{N}|B\rangle\langle B|\right)\right)|A\rangle$$

$$|A\rangle - 2\left(\frac{a}{N}|A\rangle\cancel{\langle A|A\rangle}^{1} + \frac{\sqrt{ab}}{N}|A\rangle\cancel{\langle B|A\rangle}^{0} + \frac{\sqrt{ab}}{N}|B\rangle\cancel{\langle A|A\rangle}^{1} + \frac{b}{N}|B\rangle\cancel{\langle B|A\rangle}^{0}\right)$$

$$|A\rangle - 2\left(\frac{a}{N}|A\rangle + \frac{\sqrt{ab}}{N}|B\rangle\right)$$

$$|A\rangle - \frac{2a}{N}|A\rangle - \frac{2\sqrt{ab}}{N}|B\rangle$$

$$G|B\rangle = \frac{2\sqrt{cb}}{N}|A\rangle - |B\rangle - \frac{2b}{N}|B\rangle$$

$$M = \begin{array}{ccc} & |B\rangle & |A\rangle \\ \begin{array}{c}|B\rangle \\ |A\rangle\end{array} & \multicolumn{2}{c}{\begin{pmatrix} -\left(1-\frac{2b}{N}\right) & -\frac{2\sqrt{cb}}{N} \\ \frac{2\sqrt{cb}}{N} & \left(\frac{1-2c}{N}\right) \end{pmatrix}} \end{array}$$

$$= \underbrace{\begin{pmatrix} \sqrt{\frac{b}{N}} & -\sqrt{\frac{c}{N}} \\ \sqrt{\frac{c}{N}} & \sqrt{\frac{b}{N}} \end{pmatrix}}_{\text{rotation matrix, can be seen as: } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{2} \rightarrow \text{do the rotation twice}$$

$$\sin\theta = \sqrt{\frac{c}{N}} \qquad \cos\theta = \sqrt{\frac{b}{N}}$$

$$|h\rangle = \cos\theta\,|B\rangle + \sin\theta\,|A\rangle$$

after applying G k times we have

$$|h\rangle = \cos(2k+1)\theta\,|B\rangle + \sin(2k+1)\theta\,|A\rangle$$

we want to approx. $\sin(k+1)\theta \approx 1$

$$(2k+1)\theta \rightarrow 2k\theta + \theta$$

↑ add θ and rotate twice at each iteration (k)

$$\sin (2k+1)\theta \approx 1 \rightarrow (2k+1)\theta \approx \frac{\pi}{2}$$

$$(2k+1)\theta = \frac{\pi}{2} \rightarrow 2k+1 = \frac{\pi}{2\theta}$$

$$2k = \frac{\pi}{2\theta} - 1 \rightarrow k = \frac{\frac{\pi}{2\theta} - \frac{1}{1}}{\frac{2}{1}}$$

$$k = \frac{\frac{\pi - 2\theta}{2\theta}}{\frac{2}{1}} \rightarrow k = \frac{\pi - 2\theta}{4\theta} \rightarrow \frac{\pi}{4\theta} - \frac{\cancel{2\theta}}{\cancel{4\theta}}$$

$$k = \frac{\pi}{4\theta} - \frac{1}{2}$$

$$\sin\theta = \sqrt{\frac{a}{N}} \qquad \theta = \sin^{-1}\left(\sqrt{\frac{a}{N}}\right)$$

$$a = 1 \rightarrow \theta \approx \frac{1}{\sqrt{N}}$$

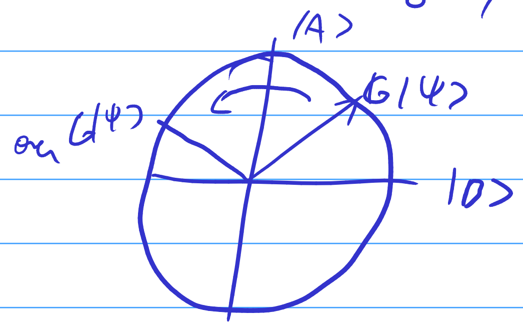
$$k = \frac{\pi}{4 \cdot \frac{1}{\sqrt{N}}} - \frac{1}{2} \rightarrow k = \frac{\pi}{\frac{4}{\frac{\sqrt{N}}{1}}} - \frac{1}{2}$$

$$k = \frac{\pi\sqrt{N}}{4} - \frac{1}{2} \quad \text{or just} \quad k = \frac{\pi\sqrt{N}}{4}$$

$O(\sqrt{N})$

so every time that we need to apply is the Grover's $O(1)$

If you apply apply too many times (overshot) it's possible to from the $|A\rangle$ state, and actually start to go far away from it.



→ Inversion around the mean

$$U = -H^{\otimes n} Z_0 H^{\otimes n} = 2|h\rangle\langle h| - \mathbb{I}$$
$$G = U Z_f$$

$$U = \frac{2}{N}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \ddots & 1\,1 & \cdots & 1 \\ 1 & & \ddots & & 1 \\ 1 & & & \ddots & 1 \\ 1 & & \cdots & & 1 \end{pmatrix} - \mathbb{I}$$

$$U\left(\sum_{x\in\{0,1\}^n} \alpha_x |x\rangle\right)$$ → U applied on a superposition

$$\sum_x \alpha_x U|x\rangle$$

will be the mean values

will be $|x\rangle$

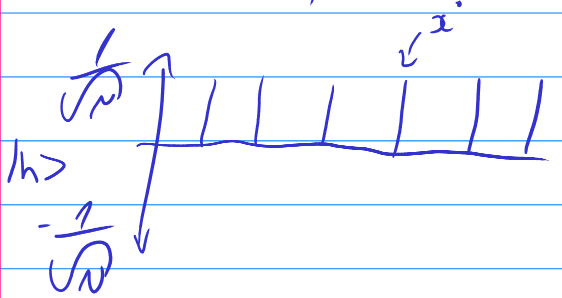
$$= \sum_x \alpha_x \left(\frac{2}{N}\begin{pmatrix} 1 & \cdots & 1 \\ 1 & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} - I\right)|x\rangle$$

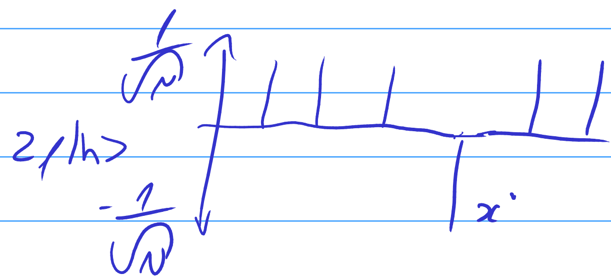
$$\sum_x (2\mu - \alpha_x)|x\rangle$$

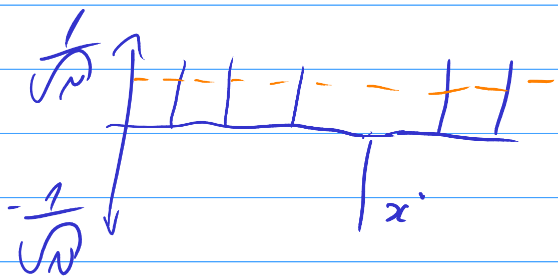
how the amplitudes change after applying U

mean of amplitudes

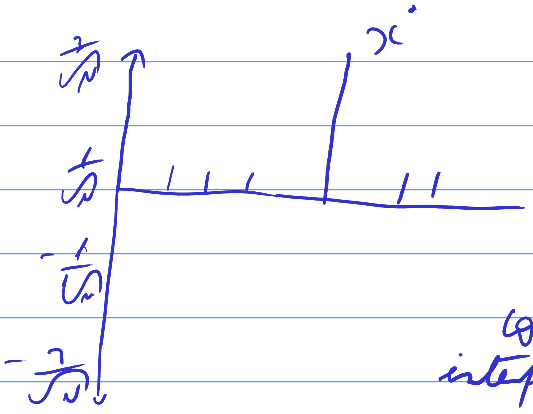
$$\mu = \frac{1}{N}\sum_x \alpha_x$$







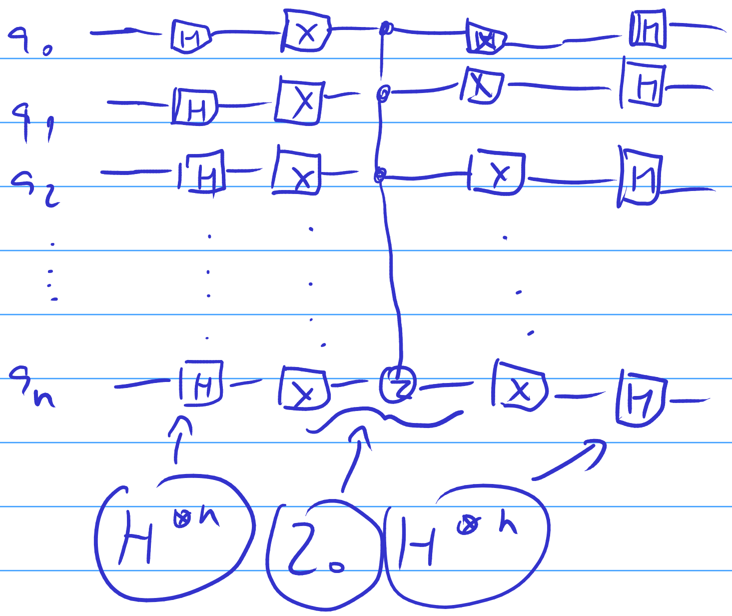
new mean amplitude ($\bar{\alpha}$)



$$\leftarrow \quad 2\mu - x$$

$$2\left(\frac{1}{\sqrt{N}}\right) - \left(-\frac{1}{\sqrt{N}}\right)$$

constructive interference $\rightarrow$ $2\left(\frac{1}{\sqrt{N}}\right) + \left(\frac{1}{\sqrt{N}}\right) = \frac{3}{\sqrt{N}}$



Z_0 just apply -1 if $q_{0\ldots n} = |0^{\otimes n}\rangle$