

## Sorting in linear time

### Counting Sort

Sort in linear time an array  $A$  of non-negative integers if  $\max_k(A) = O(\underbrace{|A|}_n)$

Example

10 10000 5

$|A| = 3$

$k = 10000$

Target time complexity is  $O(n + k)$  time

Any idea?

Pos 1 2 3 4 5 6

A 1 3 5 (4) 6 5

C 0 1 0 1 (1) 2 1  
0 1 2 3 (4) 5 6  
"k"

$c[A[i]] += 1$

$A_s$  1 3 4 5 5 6

Counting Sort\* ( $A, k$ )

let  $C[0 \dots k]$  be a new array

for  $i = 0$  to  $k$  //  $\Theta(k)$   
     $C[i] = 0$

for  $i = 1$  to  $A.length$  //  $\Theta(n)$   
     $C[A[i]] += 1$

$j = 1$

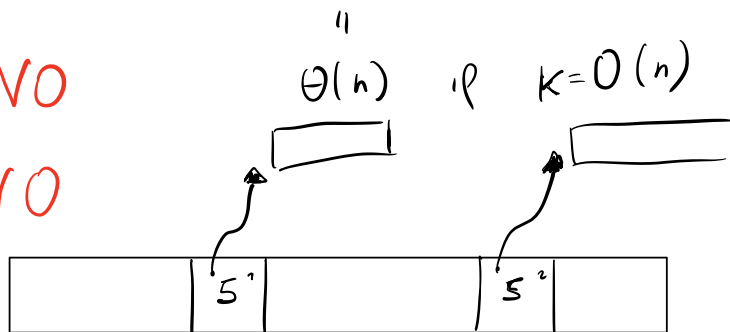
for  $i = 0$  to  $k$   
    for  $p = 1$  to  $C[i]$  //  $\Theta(C[i])$   
         $A[j] = i$   
         $j++$

$$\Theta\left(\sum_{i=0}^k (1 + C[i])\right) \\ = \Theta(k + n)$$

Overall running time is  $\Theta(k + n)$  time

In place? NO

stable? NO



## Stable Counting Sort

Counting Sort ( $A, k$ )

1 let  $C[0 \dots k]$  be a new array

2 for  $i = 0$  to  $k$   $// \Theta(k)$

3  $C[i] = 0$

4 for  $i = 1$  to  $A.length$   $// \Theta(n)$

5  $C[A[i]] += 1$

6 for  $i = 1$  to  $k$   $C[i] = \sum_{j=0}^i C[j]$

7  $C[i] = C[i] + C[i-1]$   $// \text{prefix sums}$   
 $// \Theta(k)$

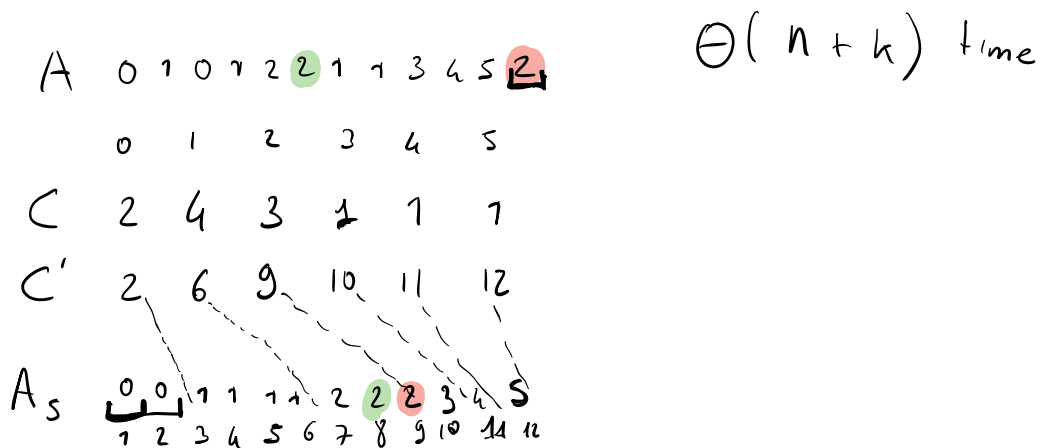
$// C[i]$  is the number of elements  $\leq i$  in  $A$

$// \Rightarrow$  values  $i$  goes in  $A_S[C[i-1] + 1 \dots C[i]]$

8 for  $i = A.length$  down to 1

9  $B[C[A[i]] - 1] = A[i]$   $// \Theta(n)$

10  $C[A[i]] -= 1$



A    4   0   3<sup>1</sup>   2   3<sup>2</sup>   6

C    1   0   1   ~~1~~<sup>2</sup>   1   0   ~~1~~  
       0   1   2   3   4   5   6

C    ~~1~~<sup>0</sup>   1   ~~2~~<sup>1</sup>   ~~4~~<sup>2</sup>   5   5   ~~6~~<sup>5</sup>  
       1   2   3   4   5   6

B    0   2   3<sup>1</sup>   3<sup>2</sup>   4   6  
       1   2   3   4   5   6

$$B[C[C[A[i]]]] = A[i]$$

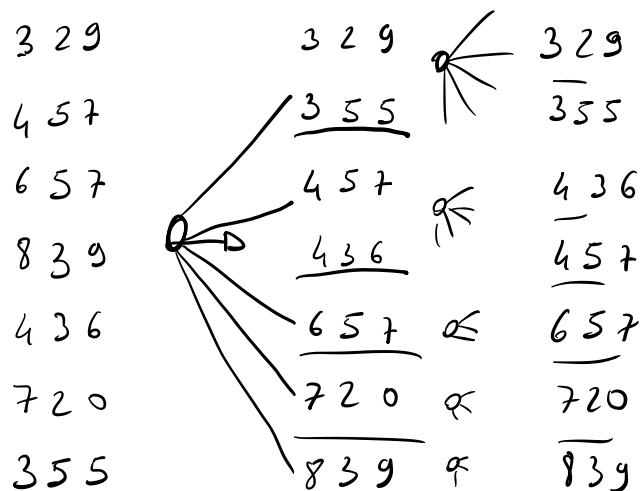
## Radix Sort

CS sorts in  $\Theta(n)$  if  $k = O(n)$

BS sorts in  $\Theta(n)$  if  $k = O(n^c)$

for some constant  $c$

	CS	BS
$n$	1000	1000
$k$	3000	$(1000)^4 = 1,000,000,000,000$
process digits from most significant one		



A lot of groups

$$10 + 10^2 + 10^3 + \dots + 10^d$$

process numbers from the least significant digit.

- no groups
- big surprise: it works!

3 2 9		7 2 0		7 2 0		3 2 9
4 5 7		3 5 5		3 2 9		3 5 5
6 5 7		4 3 6		4 3 6		4 3 6
8 3 9	$\Rightarrow$	4 5 7	$\Rightarrow$	8 3 9	$\Rightarrow$	4 5 7
4 3 6		6 5 7		3 5 5		6 5 7
7 2 0		3 2 9		4 5 7		7 2 0
3 5 5		8 3 9		6 5 7		8 3 9

correctness: proof by induction on  $d$

Radix Sort ( $A, d$ )

for  $i = 1$  to  $d$  (i.e. counting sort)

use a stable sorting algorithm

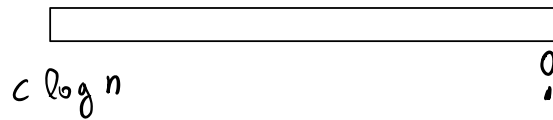
to sort  $A$  on digit  $i$

BS (with counting sort)  $\Theta(d(n+k))$  time

where  $k$  is the largest value for a digit

$$k = \cancel{O(n^c)} \leq n^c$$

a number in A in binary



$\log n^c = c \log n$  bits  
to represent any number  
up to  $n^c$

$$\Theta(d(n+k)) \text{ time}$$

- if digits are bits,

$$d = c \log n \quad k = 2$$

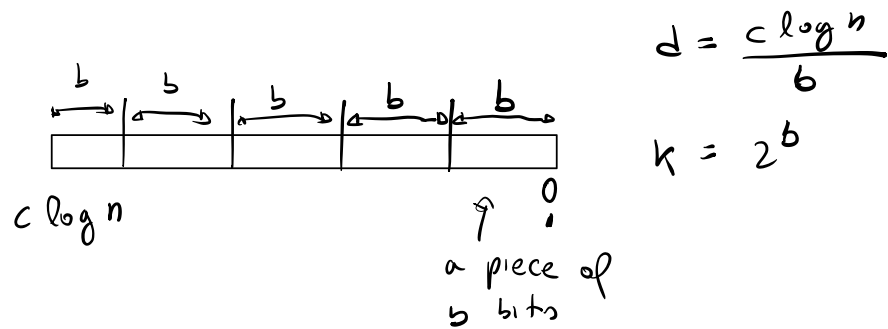
$$\begin{aligned} \text{RS runs } \Theta(c \log n (n+2)) \\ = \Theta(n \log n) \text{ time} \end{aligned}$$

- if  $d = 1$ ,

$$k = n^c$$

$$\text{RS runs } \Theta(1 \cdot (n + n^c)) = \Theta(n^c)$$

- Let's optimize choices of  $d$  and  $k$

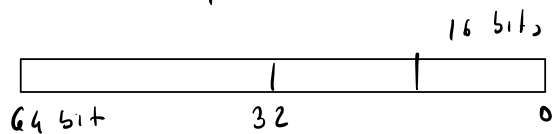


RS runs  $\Theta\left(\frac{c \log n}{b} (n + 2^b)\right)$  time

if  $b = \log n$ ,  $\Theta\left(\frac{c \log n}{\log n} (n + 2^{\log n})\right)$

$\approx \Theta(2c n) \approx \Theta(n)$  time

Real life



- 4 rounds of CS
- array  $C$  has size  $2^{16} = 65536$

RS runs in  $\Theta(n + \underline{65536})$