B.Sc. I Semester Examination, Mathematics

Time: 2 Hours Maximum Marks: 75

Paper: Differential and Integral Calculus

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Note: Attempt all questions in Section A and one question from each Part of Sections B and C.

Section A (Short Answer Type Questions)
Each question carries 3 marks. Attempt all.

(a) Evaluate the limit

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}.$$

(b) Test the convergence of

$$\int_1^\infty \frac{dx}{x^{1/3}\,(1+x^{1/2})}.$$

(c) Show that

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right) = 2\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = 4\int_0^\infty \frac{x^2\,dx}{1+x^4} = \pi\sqrt{2}.$$

(d) Show that the whole length of arc of the cardioid

$$r = a(1+\cos\theta)$$

(e) Show that

$$\mathbf{F} = (\sin y + z)\mathbf{i} + (x\cos y - z)\mathbf{j} + (x-y)\mathbf{k}$$

is a conservative field, and find a function ϕ such that $\mathbf{F} = \nabla\phi$.

(f) Evaluate the following limits:

$$\lim_{x\to a}\frac{x^m - a^m}{x-a}.$$

(g) Show that the following function is continuous but not differentiable at $x = 0$:

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

(h) Find the area of the curve $y = \sqrt{x}$ bounded by $x = 0$ and $x = 4$.

(i) For $\vec{F} = (x^2y)\hat{i} + (y^2z)\hat{j} + (z^2x)\hat{k}$, find $\nabla\cdot\vec{F}$.

Section B (Descriptive Type Questions)

Each question carries 12 marks. Attempt one question from each Part.

Part A

(Q1) (i) Expand $x^3 - 3x^2 + 5x - 7$ in powers of $(x + 2)$.

(ii) If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}.$$

(Q2) (i) Define limit, continuity and differentiability. Examine the continuity of $f(x) = |x|$ at $x = 0$.

(ii) State and prove Borel's theorem.

Part B

(Q3) (i) Prove Lagrange's Mean Value Theorem.

(ii) Expand $\sin x$ in Maclaurin's series up to the term containing x^5.

(Q4) (i) Find the radius of curvature at the point (x, y) curve: $y^2 = 4ax$.

(ii) Find the envelope of the straight lines

$$x \cos\alpha + y \sin\alpha = l \sin\alpha \cos\alpha,$$

α being the parameter.

Section C (Long Answer Type Questions)
Each question carries 12 marks. Attempt one question from each Part.

Part A

(Q5) (i) Find the asymptotes of the curve

$$x^3 + y^3 - 3axy = 0.$$

(ii) Find the points of inflexion of the curve

$$y = 3x^4 - 4x^3 + 1.$$

(Q6) (i) Find the volume of the solid generated by the revolution of the tractrix

$$x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2}, \quad y = a\sin t$$

about its asymptote.

(ii) The cardioid

$$r = a(1+\cos\theta)$$

revolves about the initial line. Find the volume of the solid thus generated.

Part B

(Q7) (i) Evaluate

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}.$$

(ii) Evaluate the following triple integrals:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\,dy\,dz}{(x+y+z+1)^3}.$$

(Q8) (i) Test for convergence or divergence of the series:

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \ldots + \frac{n+1}{n^p} + \ldots$$

(ii) Test the convergence of the series:

$$1 + a + \frac{a(a+1)}{1\cdot 2} + \frac{a(a+1)(a+2)}{1\cdot 2\cdot 3} + \ldots$$

— End of Question Paper —