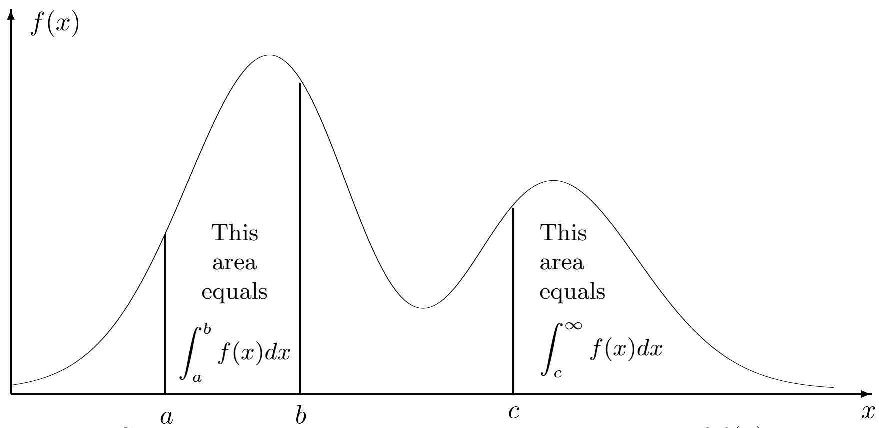

FIGURE 12.2: *Integrals are areas under the graph of $f(x)$.*

12.5 Matrices and linear systems

A **matrix** is a rectangular chart with numbers written in rows and columns,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1c} \\ A_{21} & A_{22} & \cdots & A_{2c} \\ \cdots & \cdots & \cdots & \cdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{pmatrix}$$

where r is the number of rows and c is the number of columns. Every element of matrix A is denoted by A_{ij}, where $i \in [1, r]$ is the row number and $j \in [1, c]$ is the column number. It is referred to as an "$r \times c$ matrix."

Multiplying a row by a column

A row can only be multiplied by a column of the same length. The product of a row A and a column B is a number computed as

$$(A_1, \ldots, A_n) \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix} = \sum_{i=1}^{n} A_i B_i.$$

Example 12.2 (Measurement conversion). To convert, say, 3 hours 25 minutes 45 seconds into seconds, one may use a formula

$$(3\ 25\ 45) \begin{pmatrix} 3600 \\ 60 \\ 1 \end{pmatrix} = 12345 \text{ (sec)}.$$

◊

Multiplying matrices

Matrix A may be multiplied by matrix B only if the number of columns in A equals the number of rows in B.

If A is a $k \times m$ matrix and B is an $m \times n$ matrix, then their product $AB = C$ is a $k \times n$ matrix. Each element of C is computed as

$$C_{ij} = \sum_{s=1}^{m} A_{is} B_{sj} = \begin{pmatrix} i^{\text{th}} \text{ row} \\ \text{of } A \end{pmatrix} \begin{pmatrix} j^{\text{th}} \text{ column} \\ \text{of } B \end{pmatrix}.$$

Each element of AB is obtained as a product of the corresponding row of A and column of B.

Example 12.3. The following product of two matrices is computed as

$$\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 9 & -3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} (2)(9)+(6)(-3), & (2)(-3)+(6)(1) \\ (1)(9)+(3)(-3), & (1)(-3)+(3)(1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\Diamond$

In the last example, the result was a zero matrix "accidentally." This is not always the case. However, we can notice that matrices do not always obey the usual rules of arithmetics. In particular, a product of two non-zero matrices may equal a 0 matrix.

Also, in this regard, matrices *do not commute*, that is, $AB \neq BA$, in general.

Transposition

Transposition is reflecting the entire matrix about its main diagonal.

<u>Notation</u> ‖ A^T = transposed matrix A ‖

Rows become columns, and columns become rows. That is,

$$A^T_{ij} = A_{ji}.$$

For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 7 \\ 2 & 8 \\ 3 & 9 \end{pmatrix}.$$

The transposed product of matrices is

$$\boxed{(AB)^T = B^T A^T}$$

Solving systems of equations

In Chapters 6 and 7, we often solve systems of n linear equations with n unknowns and find a *steady-state distribution*. There are several ways of doing so.

One method to solve such a system is by **variable elimination**. Express one variable in terms of the others from one equation, then substitute it into the unused equations. You will get a system of $(n-1)$ equations with $(n-1)$ unknowns. Proceeding in the same way, we reduce the number of unknowns until we end up with 1 equation and 1 unknown. We find this unknown, then go back and find all the other unknowns.

Example 12.4 (Linear system). Solve the system

$$\left\{\begin{array}{rcrcrcr} 2x & + & 2y & + & 5z & = & 12 \\ & & 3y & - & z & = & 0 \\ 4x & - & 7y & - & z & = & 2 \end{array}\right.$$

We don't have to start solving from the first equation. Start with the one that seems simple. From the second equation, we see that

$$z = 3y.$$

Substituting $(3y)$ for z in the other equations, we obtain

$$\left\{\begin{array}{rcrcr} 2x & + & 17y & = & 12 \\ 4x & - & 10y & = & 2 \end{array}\right.$$

We are down by one equation and one unknown. Next, express x from the first equation,

$$x = \frac{12 - 17y}{2} = 6 - 8.5y$$

and substitute into the last equation,

$$4(6 - 8.5y) - 10y = 2.$$

Simplifying, we get $44y = 22$, hence $y = 0.5$. Now, go back and recover the other variables,

$$x = 6 - 8.5y = 6 - (8.5)(0.5) = 1.75; \quad z = 3y = 1.5.$$

The answer is $\underline{x = 1.75,\ y = 0.5,\ z = 1.5}$.

We can check the answer by substituting the result into the initial system,

$$\left\{\begin{array}{rcrcrcr} 2(1.75) & + & 2(0.5) & + & 5(1.5) & = & 12 \\ & & 3(0.5) & - & 1.5 & = & 0 \\ 4(1.75) & - & 7(0.5) & - & 1.5 & = & 2 \end{array}\right.$$

$\Diamond$

We can also eliminate variables by multiplying entire equations by suitable coefficients, adding and subtracting them. Here is an illustration of that.

Example 12.5 (Another method). Here is a shorter solution of Example 12.4. Double the first equation,

$$4x + 4y + 10z = 24,$$

and subtract the third equation from it,

$$11y + 11z = 22, \quad \text{or} \quad y + z = 2.$$

This way, we eliminated x. Then, adding $(y + z = 2)$ and $(3y - z = 0)$, we get $4y = 2$, and again, $y = 0.5$. Other variables, x and z, can now be obtained from y, as in Example 12.4. $\Diamond$

The system of equations in this example can be written in a matrix form as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 2 & 3 & -7 \\ 5 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 2 \end{pmatrix},$$

or, equivalently,

$$\begin{pmatrix} 2 & 2 & 5 \\ 0 & 3 & -1 \\ 4 & -7 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 & 0 & 2 \end{pmatrix}.$$

Inverse matrix

Matrix B is the **inverse matrix** of A if

$$AB = BA = I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where I is the *identity matrix*. It has 1s on the diagonal and 0s elsewhere. Matrices A and B have to have the same number of rows and columns.

Notation $\|$ $A^{-1} \quad = \quad$ inverse of matrix A $\|$

Inverse of a product can be computed as

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

To find the inverse matrix A^{-1} by hand, write matrices A and I next to each other. Multiplying rows of A by constant coefficients, adding and interchanging them, convert matrix A to the identity matrix I. The same operations convert matrix I to A^{-1},

$$\left(A \,\middle|\, I \right) \longrightarrow \left(I \,\middle|\, A^{-1} \right).$$

Example 12.6. Linear system in Example 12.4 is given by matrix

$$A = \begin{pmatrix} 2 & 2 & 5 \\ 0 & 3 & -1 \\ 4 & -7 & -1 \end{pmatrix}.$$

Repeating the row operations from this example, we can find the inverse matrix A^{-1},

$$\left(\begin{array}{ccc|ccc} 2 & 2 & 5 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & -7 & -1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 4 & 4 & 10 & 2 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & -7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 0 & 11 & 11 & 2 & 0 & -1 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & -7 & -1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 2/11 & 0 & -1/11 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & -7 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 0 & 4 & 0 & 2/11 & 1 & -1/11 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & -7 & -1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/22 & 1/4 & -1/44 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & -10 & 0 & 0 & -1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/22 & 1/4 & -1/44 \\ 0 & 0 & -1 & -3/22 & 1/4 & 3/44 \\ 4 & 0 & 0 & 10/22 & 3/2 & 34/44 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/44 & 3/8 & 17/88 \\ 0 & 1 & 0 & 1/22 & 1/4 & -1/44 \\ 0 & 0 & 1 & 3/22 & -1/4 & -3/44 \end{array}\right)$$

The inverse matrix is found,

$$A^{-1} = \begin{pmatrix} 5/44 & 3/8 & 17/88 \\ 1/22 & 1/4 & -1/44 \\ 3/22 & -1/4 & -3/44 \end{pmatrix}.$$

You can verify the result by multiplying $A^{-1}A$ or AA^{-1}. ◊

For a 2×2 matrix, the formula for the inverse is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Matrix operations in R

```
x <- c(1,8,0,3,3,-3,5,0,-1)  # Define a 1×9 vector and converting it...
A <- matrix(x,3,3)           # ... into a 3×3 matrix column by column
t(A)                         # Transposed matrix
B <- solve(A)                # Inverse matrix
A + B                        # Addition
A %*% B                      # Matrix multiplication
C <- A * B                   # Multiplying element by element, C_ij = A_ij B_ij
diag(n)                      # n×n identity matrix
matrix(rep(0,m*n),m,n)       # m×n matrix of 0s
cbind(A,B)                   # Joining matrices side by side (as columns)
rbind(A,B)                   # Joining matrices below each other (as rows)
A[2:3,]                      # Sub-matrix: rows 2-3 and all columns of A

# Calculation of a power of a matrix is a part of R package 'expm'
install.packages("expm")
library(expm)
A %^% 3                      # This calculates A^3 = A·A·A
solve(A %^% 3)               # The result is A^-3 = (A^3)^-1
```