11

Price Spikes and Real Options: Transmission Valuation

Michael Rosenberg, Nikolai Sidorenko and Joseph D. Bryngelson; Michael Baron*

TXU; University of Texas at Dallas

INTRODUCTION

Energy companies frequently exchange a commodity at one location for the same commodity at another location. An example of this kind of transaction is an exchange of gas from a location where a company has purchased or stored it for gas at a location where the company needs it, such as a power plant or a city. Another frequent transaction is the transportation, or in the case of power, transmission of a commodity from one location to another. Crude oil often must be transported to the refinery through a pipeline before it is processed. A third kind of situation is a conversion of one commodity into another. Gas or other fuels have to be converted into electricity at the power plants; this can be considered as an exchange of one kind of commodity for another.

If the commodity products involved in the exchange can be standardized, then an option on this exchange can be traded as a standard product on a commodity exchange, such as the New York Mercantile Exchange (NYMEX), or the Chicago Board of Trade (CBOT). This exchange instrument is known as a spread option, deriving its name from the price spread between two commodities

*The authors would like to thank Bob Logue for numerous helpful discussions, Patrick Coon and Leon Ricaud for providing technical details of ERCOT protocols, and Ehud Ronn for his insightful comments.

or commodity types. For example, the NYMEX spread options are traded on the differences (or spreads) between heating oil and crude oil, and between gasoline and crude oil.

Often, physical risk can be translated into the price of financial instruments that are traded to mitigate this risk. Therefore, we will concentrate on the description of financial derivatives as proxies for physical risks. The underlying assumption for this approach is that of liquidity, ie, that there is always a price at which the commodity is available. This is a good assumption for oil and gas but has some limitations for power. In the latter case, the penalties associated with delivery interruptions can be used to quantify the value of liquidity.

We can widely subdivide the energy delivery systems into transportation and pipeline types. For example, oil tankers collect oil at the place of exploration and transport it to the location of distribution. The pipeline operation is a distribution network with a continuous flow of commodity, such as a pipeline for oil or gas or a transmission system for power. What is common to both delivery systems is that the commodity is moved from the place of relative abundance, and lower price, to the place of scarcity and higher price. This price difference makes the operation of transporting the commodity profitable. Moreover, the operation itself is not a zero-sum game and is often worth a lot of money for its potential to continue the delivery and turn profit. The value of this operation depends upon the costs of delivery and the difference between the prices at two locations: the pick-up location and the final destination location. Once the payoff function is established, the methods of this chapter can be used to price the financial option on the spread, and hence establish a value for the physical operation that includes the transportation asset.

This chapter is organised as follows. First, we proceed to review models of commodity prices. The most widely used is the single factor model. One possible extension is into the multi-factor family of models, and we briefly describe it. Next we introduce spikes into the commodity prices. This issue is most acute in power prices where the probability of surges in prices occasionally changes the dynamics of the marketplace. After the model for the price process is established, we proceed to describe the valuation methods, starting with simplifying approximations and proceeding towards the exact solution. The section closes with the incorporation of spikes into the solution. The last section is devoted to the numerical solutions for methods just introduced. The simplest approaches have closed form solutions and thus can be evaluated in the standard functions. The exact solution requires numerical integration and should be done with proper controls to insure convergence. One of the most intuitive numerical schemes is a binomial tree approach that builds on the similar approach for valuations of derivatives contingent on one underlying. Finally, the Monte Carlo method provides a simulation-based approach to the valuation of spread options. Binomial trees and Monte Carlo methods can value models and options that cannot be reduced to closed-form solutions or numerical integration.

BEHAVIOUR OF COMMODITY PRICES

Single factor models

The single-factor model for energy prices is mathematically the simplest approach to modelling energy prices. As the name for this approach suggests, there is only one source of the uncertainty in the prices that can be modelled as a geometric mean-reverting process of the log type (see Schwartz, 1997) with time varying mean:

$$\mathrm{d}S = \eta(m(t) - \ln S)S\mathrm{d}t + \sigma_0 S\mathrm{d}z \tag{1}$$

This single source of uncertainty is the volatility of spot prices. (More information about properties and reasons for the choice of this process can be found in Appendix 1 at the end of this book.) It can be shown (see Clewlow and Srickland, 1999) that this assumption about the spot price, combined with the fact that in a risk-neutral world the future price of a commodity is equal to its expected spot price, leads to the following equation for the future commodity price at time T:

$$\mathrm{d}F(T) = F(T)\sigma(T)\mathrm{d}z \tag{2}$$

Equation (2) describes futures prices in the risk-neutral world. As discussed in the Section on Valuation Methods, risk-neutral price processes are the appropriate ones to use for valuing options.

We do not mean to imply that Equation (2) describes the observed behaviour of futures prices. Real futures prices may incorporate a market price of risk that would cause the futures price to have a non-zero drift over time. The correct pricing of options takes account of the extent to which this market risk can be hedged away. If it can be fully hedged, then the market price of risk, and therefore the non-zero drift in the futures price, is absent from the price of the option. The measurement of market prices of risk for commodities and their incorporation into option prices is an active and interesting area of research, which is beyond the scope of the present chapter. We shall therefore assume that all market risk can be hedged away, however, we shall also point out some situations where this assumption is likely to break down. Equation (2) is the right candidate for the description of future prices. The geometric Brownian motion with zero drift means that there is no preference as to the direction of the change of future prices. In other words, all currently available information was absorbed by the market into the current price of the futures. Again, we must distinguish between the risk-neutral distribution under which valuation takes place and the true/objective/statistical distribution. That true/objective/statistical distribution may have a non-zero drift owing to the presence of a market price of risk.

The variable that determines a particular form of future $F(T)$ in Equation (2) is the volatility term structure $\sigma(T)$. One of the simplest assumptions, consistent with empirical observations for a certain period of time is the exponentially decaying volatility term structure:

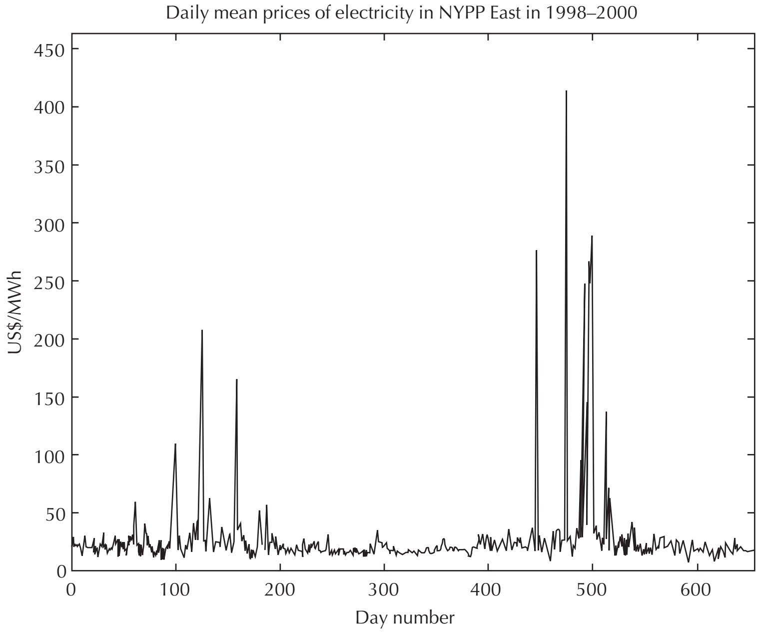
$$\sigma(T) = \sigma_0 e^{-\eta T} \tag{3}$$

Moreover, the assumption of geometric mean-reverting process in Equation (1) requires that the rate of decay is equal to the mean-reverting parameter, and in the limit $T \to 0$, the volatility of the future converges to the volatility of the spot (see Clewlow and Srickland, 1999).

Incorporation of spikes into commodity prices

Prices of certain types of commodities may pass through a sequence of different states. For example, a combination of various

Figure 1 Spikes in mean daily electricity prices



extreme factors may cause electricity prices to depart from their normal state and experience an abrupt *spike* (see Figure 1). In such situations, each state can be described by an equation of the type in Equation (1), however, it is difficult to describe the entire (unconditional) process by a single equation of this type. For example, in a mean reversion process with an additional Poissonian jump term (see Clewlow and Strickland, 1999), spikes start abruptly, at random times, after which the process gradually tends to its normal state. Contrary to this, in the historical data (see Figure 1), the spikes start abruptly and end abruptly.

Multiple states can be incorporated into the model by considering a *transition probability matrix* (see Appendix 1) that governs probabilities of switching from one state to another. We consider a two-state model that belongs to the class of hidden Markov chains and that is applicable to the price of electricity. It can be generalised to any multiple-state model of power prices (see Baron, Rosenberg, Sidorenko, 2001; Baron, Rosenberg, Sidorenko, 2002). Examples of such models are also found in quality control; in-control and out-of-control modes (see Lai, 1995; Lai, 2001; and Montgomery, 1997); climatology; global trends like cooling and warming (see Calvin, 2002); developmental psychology; different phases during development, learning, and problem solving (see Piaget, 1970; Granott and Parziale, 2002); epidemiology; regular, pre-epidemic, epidemic, and post-epidemic periods (see Bridges *et al*, 2000; Baron, 2002); speech recognition (see Rabiner, 1989); biological sequences alignment (see Durbin *et al*, 1998; Cloth and Backofen, 2000); and other fields.

We describe the dynamics of two states of electricity prices using a 2×2 transition probability matrix:

$$P = \begin{pmatrix} 1-p(t) & p(t) \\ q & 1-q \end{pmatrix} \quad \textbf{(4)}$$

where $\{1 - p(t)\}$ is a probability that a day without spikes is followed by another day without spikes, and $\{1 - q\}$ is the probability that a spike extends for one more day. The probability for a spike to start, which we denote by $p(t)$, changes with the season, so $p(t)$ must be a function of time t. At the same time, duration of the spikes are essentially independent of the season, so q may be assumed to be constant.

Suppose that the distribution of prices on a certain spike-free day has density $\mathrm{f}(x)$; let $\mathrm{g}(x)$ be the density of prices during spikes. The density $\mathrm{f}(x)$ should be understood as a *conditional density* of power prices, given that there is no spike, and $\mathrm{g}(x)$ is a conditional density, given a spike. According to Equation (4), the distribution of prices on the next day is a *mixture distribution* with an unconditional density

$$\{1-p(t)\}\mathrm{f}(x)+p(t)\mathrm{g}(x) \quad \textbf{(5)}$$

If a price spike occurs on one day, then the mixing coefficients in Equation (5) should be taken from the second row of P, and the distribution of prices on the following day has a density:

$$q\mathrm{f}(x)+(1-q)\mathrm{g}(x) \quad \textbf{(6)}$$

which represents a different mixture of f and g.

The distribution of prices two days ahead is also a mixture, with mixing coefficients being two-step transition probabilities $P^{(2)}(i, j)$, where

$$P^{(2)} = P^2 = \begin{pmatrix} \{1-p(t)\}^2 + p(t)q & \{1-p(t)\}p(t) + p(t)(1-q) \\ q\{(1-p(t)\} + (1-q)q & (1-q)^2 + qp(t) \end{pmatrix}$$

A mean-reverting model of the type in Equation (1) can adequately describe the process *between* the spikes. *During* the spikes, the behaviour of prices changes noticeably. (This is one of the reasons why such multi-state processes cannot be described by a single equation.) A quick look at the series of electricity prices reveals that the prices during spikes are qualitatively larger than the prices during normal times; hence, spikes should not be assumed to be generated by the same distribution. Instead, we assume a *Bayesian model* that allows parameters of the distribution of prices to differ from one spike to another. According to this model, parameters of each spike are generated from a *prior* distribution. The electricity prices are then generated from a distribution with these parameters.

We model the spikes of electricity prices by a *lognormal* distribution with the density:

$$\mathrm{f}\left(x \mid \mu, \tau\right) = \frac{1}{x\tau\sqrt{2\pi}} \mathrm{e}^{-(\ln x - \mu)^2 / 2\tau^2}$$

with parameters μ and τ representing the mean and volatility, respectively. The mean μ is random, with a normal prior distribution

$$\pi\left(\mu \mid \theta, \eta\right) = \frac{1}{\eta\sqrt{2\pi}} \mathrm{e}^{-(\mu - \theta)^2 / 2\eta^2}$$

Volatility τ may either be assumed the same for all spikes or random, having a prior distribution π_τ.

This Bayesian model reflects the fact that electricity prices are *dependent* within the same spike (ie, conditioned on the same parameters μ and τ), but spikes may be assumed independent of one another.

Calibration of such multi-state models will generally start with the separation of different states for example, in this case we must separate the times when the prices follow a normal price process from the times when price spikes occur. Statistical *change-point detection* techniques (Basseville and Nikiforov, 1993) can identify segments of the process that correspond to each state. Then, standard statistical tools can be used to estimate the parameters of each state.

VALUATION METHODS

In this section we discuss methods for valuing options for energy production and delivery systems. Given the variety of options and models for energy commodities, such an undertaking could easily become unwieldy. We shall solve this problem by focusing on a specific kind of option, viz, an option on the spread between two futures, and a specific kind of model for the futures prices, viz, a single-factor geometric Brownian motion. Instruments of this type are very important in valuing a number of real options, such as transmission congestion rights (see Panel 1 for a detailed description). In this section we will try to illustrate a comprehensive set of mathematical techniques that are useful for valuing *any* option on an energy commodity. We hope that the choice of a simple, but very important, option and a simple, but very flexible and useful, model for futures prices helps to show these useful valuation techniques in sharp relief. Given the practical emphasis of this chapter, we make no claims of mathematical rigor. Rather we shall try to present our valuation techniques so that the readers can understand the ideas behind applying them as easily as possible, so that they can apply the techniques to their own valuation problems.

PANEL 1 TEXAS STUDY: ERCOT TRANSMISSION CONGESTION RIGHTS

In the last three years the electricity market in the electric reliability council of Texas (ERCOT) area has undergone a full transformation from a regulated to a fully deregulated market setup.

Major steps of its transformation are presented below in the ERCOT Market Change Timeline.

May 1999	Senate Bill 7 (SB7) passed as deregulation legislation in Texas
Sep 1999	Stakeholders make the first public utility commission of Texas (PUCT) filing, outlining the proposed basic market structure

Jan 2000	ERCOT requests independent system operator (ISO) retail certification
Oct 2000	Protocol draft filed at PUCT for approval
Jan 2001	Retail energy provides certification; ERCOT begins new governance
Apr 2001	Mock retail tests
Jun 2001	Retail pilot and single control area operations begin
Jan 2002	Full retail competition

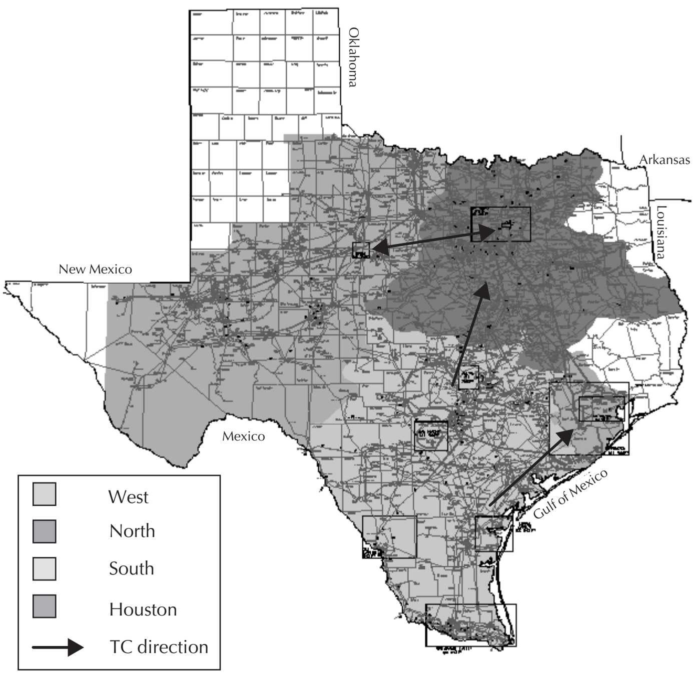
ERCOT data aggregation and settlement duties are performed by the ISO that balances the electricity load in most of Texas and the western part of Louisiana and ensures that supply meets demand. As a system operator, ERCOT ISO is revenue neutral, ie, it shares any profit or expense incurred from operations with all market participants (qualified scheduling entities (QSEs)) proportionally to their share in the total load. A QSE is any market participant that is certified by the ERCOT ISO to submit schedules and ancillary service bids. A QSE may be a load serving entity, retail energy provider, generating service provider, or power marketer, and may have a relevant legally recognised relationship with any combination of these entities.

The ERCOT coverage area is not a homogeneous distribution of energy production and consumption, but rather a heterogeneous clustering of power-consuming centers. Currently, ERCOT divides its territory into four congestion zones: North, Houston, South and West (see Figure 1).

Most of the generation is located in the West, and consumption is located in the North and Houston congestion zones. All zonal centers are connected by major transmission lines that ensure that energy can reach anywhere within ERCOT's territory. But even so, the demand in one area can become so great relative to the rest of the system that the line capacity to carry energy to that area is not sufficient. This situation is known as transmission congestion (TC). Rather than distribute the cost of TC among market participants in proportion to their load, or some other arbitrarily chosen measure, ERCOT adopted a new market approach where transmission congestion rights (TCR) are auctioned off based on the needs of market players.

For the purpose of the auction, ERCOT determines commercially significant constraints (CSCs) on which TCRs are auctioned off. These constraints correspond to the physical congestion of transmission between the four zones within its territory. ERCOT will reassess CSCs annually, based on changes to the ERCOT topology, and will define new congestion zones as needed by November 1 of each year. ERCOT analyses expected annual congestion cost, load flow data, and expected transmission system additions to determine expected operating limits and constraints to be used in the designation of CSCs in the forthcoming year. Immediately following ERCOT Board approval,

Figure 1 Major transmission network of interconnected systems in Texas (2002 congestion management zones)



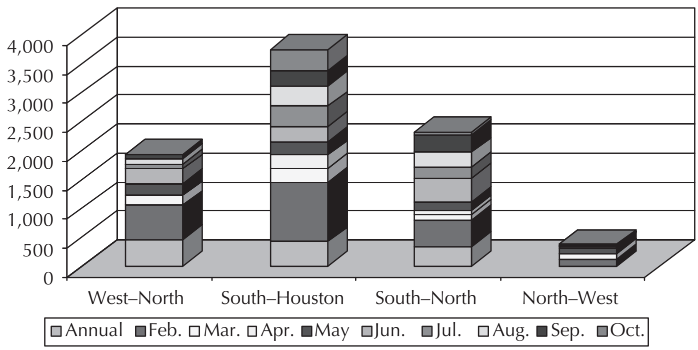
Source: ERCOT

changes in the CSCs and the resulting zonal boundaries are posted on the ERCOT's web site. This posting includes a bus-by-bus identification of relevant zones. ERCOT can also change any congestion zone definition.

Congestion Zones are determined using the following basic procedures.

- ❏ Shift factors (a fraction of load change to the incremental increase in load at an ERCOT reference bus, measured between two specific transmission points) are developed using a linearised direct current (DC) model for each transmission bus on each CSC.
- ❏ Statistical clustering aggregates transmission buses into zones based upon similar shift factors relative to all CSCs.
- ❏ Zonal shift factors for each CSC are created by averaging the individual bus shift factors, weighted by the megawatts on the bus (generator maximum ratings and bus loads).

Figure 2 2002 TCRs sold by CSC



Source: ERCOT

❏ ERCOT uses a stakeholder process to determine the number of CSC zones that balances the competing goals of minimising the number of zones and maximising the accuracy of the commercial model, that ERCOT employs to represent and forecast ERCOT physical system.

For 2002, ERCOT determined four CSCs: North to West, West to North, South to North and South to Houston. For 2003, the North to West CSC will be eliminated (see Figure 2, the broad arrows correspond to significant transmission constraints between the zones).

In 2002, the TCR auction is conducted as a single-round Dutch style with single clearing price, in 2003 it will switch to a more complex simultaneous combinatorial type. ERCOT conducts TCR auctions annually and monthly. TCRs are auctioned for each CSC, up to the maximum CSC rating. Each TCR is equal to 1 MW of impact on an individual CSC, and is applicable to the total number of hours during the period for which they are being auctioned (ie, annually or monthly). Each TCR will have a unique identification number that will allow ERCOT to account for its ownership. The annual auction releases 60% of the CSC's TCRs. The remaining 40% of the available transmission capability for any given CSC path is allocated to market participants based on the results of monthly auctions. TCRs are made available in the monthly auction for all hours of each day of each month for which the auction is being held. The auction process is open to all market participants, except ERCOT or its affiliates or any Transmission and Distribution Service Provider (TDSP). To participate in the auctions, a bidder must meet certain financial security requirements and must have access to the computer hardware, software and communications equipment required to participate. Sixty days prior to the annual auction and 10 days prior to the monthly auction, ERCOT posts:

- ❑ the zone-to-zone impact matrix (to determine the megawatt impact on each CSC);
- ❑ the number of TCRs to be issued for each CSC;
- ❑ the number of TCRs to be issued for each period (annual, monthly);
- ❑ deadline for bidders to satisfy financial requirements;
- ❑ specifications for the equipment necessary to participate in the auction;
- ❑ the date and time by which bids must be submitted;
- ❑ the bid format; and
- ❑ any other information of commercial significance to bidders.

Auction participants consider TCR as a partial financial hedge against their marginal costs. TCRs retain their full value even in the event of physical curtailment or flow limitation below the megawatt quantity for which TCRs were defined on any given CSC path. Settlement of TCRs will account for hourly TCR values in accordance with the TCR definition. TCRs will hedge against all hourly CSC Congestion costs, ie, full zonal clearing price differences and physical transmission costs.

Currently, each participant is restricted to no more than 25% of all auctioned TCRs. Market participants can exchange or sell TCRs in any secondary market. ERCOT accepts hourly TCRs for any electricity supply schedules. ERCOT auctions two types of TCRs: annual and monthly. ERCOT starts by auctioning off CSCs with the largest number of TCRs, first at the rate of two CSCs per day. Figure 2 shows the number of TCRs, both annual and monthly, that were auctioned off in 2002, and Table 1 contains the revenues per CSC gained from the auction. Figure 3 shows CSC clearing prices (annual and monthly). A quick look at these results clearly indicates that the South–North TC is the most significant, closely followed by the South–Houston. On the other hand, North–West is much less significant given the number of TCRs sold and the revenue its sale provided. As we mentioned before, this CSC will be eliminated in 2003. Figure 4 shows the actual TCR payouts that were realised during the same period. It is not surprising that the TCR auction clearing price averages for 2002 were sometimes significantly lower than the average shadow prices (eg, South–North May, 2002, contract). The degree of congestion speculated for May did not occur and thus the TCRs were overvalued. This could possibly be partially accounted for by the immaturity of historical market data for market participants to use to estimate their exposure to congestion.

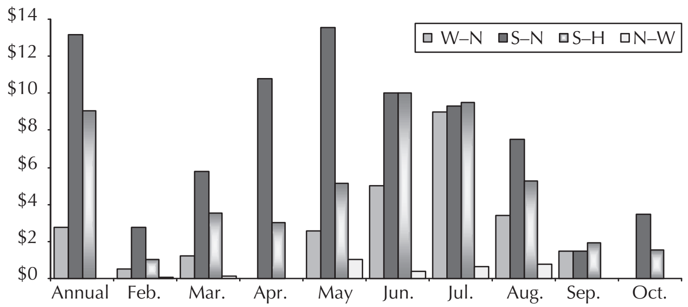
But how does an auction participant value the TCRs they are bidding for? There are two main factors that impact the TCR payoff. They are shadow price and probability of physical congestion on the line for

Table 1 ERCOT TCR auction results as of September 12

CSC	Revenues (US$)
S–H	36,756,754.92
S–N	41,701,370.38
W–N	11,465,402.40
N–W	113,463.36
Total revenues	90,036,991.06

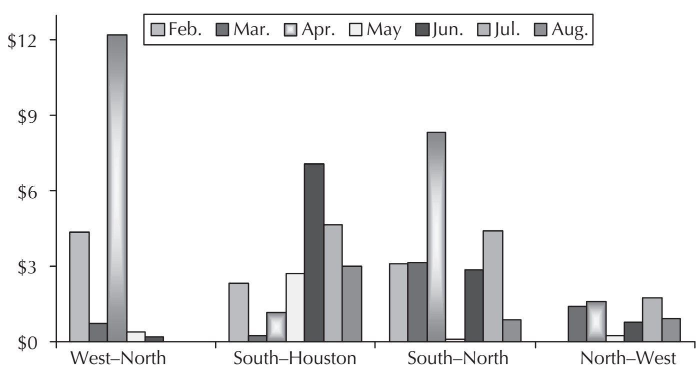
Source: ERCOT

Figure 3 CSC clearing prices by TCR auction



Source: ERCOT

Figure 4 Average CSC shadow prices



Source: ERCOT

which the TCR is held. In the case of just one congested line the shadow price is determined by the following equation:

$$SP_{1,2} = \frac{MCPE_2 - MCPE_1}{SF_{1|1,2} - SF_{2|1,2}}$$

where $MCPE_i$ is the market clearing price of power at location $i(i = 1, 2)$ and $SF_{k|i,j}$ is a shift factor for each node of the transmission line $(k = i, j)$. However, the *positive* price differential between the nodes is necessary, but not a sufficient condition for the payoff from TCR. The second factor that we mentioned is the probability of congestion, since the payment takes place only if the line is congested. *MCPE* is essentially the locational spot price that market participants should forecast on the day of the auction for the time period the TCR covers. Shift factors are supplied by ERCOT and are known for every month. We value the financial part of the TCR (first component) separately from the second physical component. The latter is assessed by market participants who use proprietary systems to model the ERCOT transmission system dynamics. The first factor is in fact a spread payoff, divided by a constant. There is a temptation to value this right as a simple difference of forward prices at two locations under the supplementary assumption that they are the best predictors of spot prices. However, this captures only the so-called intrinsic part of a TCR. The correct valuation of this right is given by the spread option, which can be calculated using the techniques presented in this chapter.

It is interesting to note that intrinsic value of a monthly TCR is a good approximation to the spread option value, when the price correlation between the congestion zones is very high. In fact, their difference was under 1% for 2002 monthly cases, when correlation was over 95%. With the decrease of correlations between congestion zones, the value of the spread option increases and vice versa. It is important to realise that an accurate estimation of correlation is the single most significant input into the model for capturing the correct value of TCRs. More broadly, as CSC historical data and congestion management experience matures, the market would be expected to improve its capability to better estimate the value of a TCR.

Deriving a differential equation

A versatile and conceptually straightforward method of valuing options consists of deriving and solving a differential equation for the option value. A conceptual advantage of this approach is that the derivation of the differential equation follows the methods that would be required to perfectly hedge the options, so that contact is kept with the concrete world of trading. Here we shall derive the differential equation for the value of an option on two futures contracts.

As mentioned above, we are modelling the behaviour of the futures prices by a simple, single-factor model. We denote the two futures prices by F_1 and F_2, and we assume that the futures prices have volatilities of σ_1 and σ_2 respectively. With these assumptions, the SDEs

$$dF_1 = \mu_1(F_1, F_2, t)dt + \sigma_1 F_1 dz_1 \quad (7)$$

$$dF_2 = \mu_2(F_1, F_2, t)dt + \sigma_2 F_2 dz_2 \quad (8)$$

where z_1 and z_2 represent standard Wiener processes, describe the behaviour of the futures prices. The relation between the two Weiner processes, z_1 and z_2 accounts for the correlation between the returns of the futures prices. To describe this correlation more precisely, define the operator $\mathbb{E}[x]$ to represent the expectation (average) of x; then the correlation coefficient, ρ, of the returns of the two futures is defined so that

$$\mathbb{E}[dz_1 dz_2] = \rho dt \quad (9)$$

This correlation coefficient is the usual correlation coefficient between price returns that trading organisations compute.

We denote by $V(F_1, F_2, t)$ the value of a financial instrument that depends on the values of the futures prices F_1 and F_2, and the time t. It is important to note that, although this section will focus on spread options, the differential equation derived in this section is applicable to *any* financial instrument with a value that depends only on the value of two futures prices and time. Consider a portfolio consisting of the financial instrument and short sales of Δ_1 futures contracts for the first future (which has price F_1), and Δ_2 futures contract for the second future (which has price F_2). Entering the futures contracts costs nothing, so the cost required to construct this portfolio is

$$\Pi = V \quad (10)$$

In a short time dt the value of the portfolio will change because of two effects: first the value of the financial instrument will change by dV, and second the short positions in the futures contracts will earn (or lose) an amount dS from the settlements of the futures contracts. The change in the value of the financial instrument follows from the multi-dimensional version of Ito's lemma (see Appendix 1 in this volume and Hull, 2002):

$$\mathrm{d}V = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 F_1^2 \frac{\partial^2 V}{\partial F_1^2} + \rho\sigma_1\sigma_2 F_1 F_2 \frac{\partial^2 V}{\partial F_1 \partial F_2} + \frac{1}{2}\sigma_2^2 F_2^2 \frac{\partial^2 V}{\partial F_2^2} \right) \mathrm{d}t + \frac{\partial V}{\partial F_1}\mathrm{d}F_1 + \frac{\partial V}{\partial F_2}\mathrm{d}F_2 \tag{11}$$

The amount the portfolio earns or loses from settlements of the short positions in the futures contracts during the time dt is given by

$$\mathrm{d}S = -\Delta_1 \mathrm{d}F_1 - \Delta_2 \mathrm{d}F_2 \tag{12}$$

The total change in the value of the portfolio in time dt is given by

$$\mathrm{d}\Pi = \mathrm{d}V + \mathrm{d}S \tag{13}$$

and substituting Equation (11) for the change in the value of the financial instrument, and Equation (12) for the earning arising from the settlements of the futures positions into Equation (13) for the change in the total value of the portfolio yields

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 F_1^2 \frac{\partial^2 V}{\partial F_1^2} + \rho\sigma_1\sigma_2 F_1 F_2 \frac{\partial^2 V}{\partial F_1 \partial F_2} + \frac{1}{2}\sigma_2^2 F_2^2 \frac{\partial^2 V}{\partial F_2^2} \right) \mathrm{d}t + \left(\frac{\partial V}{\partial F_1} - \Delta_1 \right)\mathrm{d}F_1 + \left(\frac{\partial V}{\partial F_2} - \Delta_2 \right)\mathrm{d}F_2 \tag{14}$$

The uncertainty in the portfolio can be hedged away by setting

$$\Delta_1 = \frac{\partial V}{\partial F_1} \tag{15}$$

$$\Delta_2 = \frac{\partial V}{\partial F_2} \tag{16}$$

which is a hedging strategy that can be (approximately) realised in practice once the function $V(F_1, F_2, t)$ is known, by buying the financial instrument and shorting the number of futures contracts given by Equations (15) and (16). After the uncertainty has been hedged away from the portfolio it can earn a constant, risk-free rate of return. To avoid the existence of an arbitrage opportunity the return on the portfolio must equal the risk-free interest rate r so that

$$d\Pi = r\Pi dt = rV dt \tag{17}$$

where we have used Equation (10) for the cost of setting up the portfolio. Taking the expression in Equation (14) for the change in the value of the portfolio, using Equations (15) and (16) to set Δ_1 and Δ_2 so that the uncertainty in the portfolio is hedged away, and equating this result to the change in the value of the portfolio, if it simply earned the risk-free interest rate yields, gives the differential equation that the value of the financial instrument, $V(F_1, F_2, t)$, must satisfy in order to avoid the existence of an arbitrage opportunity:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 F_1^2 \frac{\partial^2 V}{\partial F_1^2} + \rho\sigma_1\sigma_2 F_1 F_2 \frac{\partial^2 V}{\partial F_1 \partial F_2} + \frac{1}{2}\sigma_2^2 F_2^2 \frac{\partial^2 V}{\partial F_2^2} - rV = 0 \tag{18}$$

This equation will form the basis of most of what follows in this section.

So far the discussion has been very general in that it has concerned *any* financial instrument with a value that depends on the values of two futures prices and on time. What differentiates between the many possible financial instruments that have this property and therefore have a value that is described by Equation (18)? To answer this question we note that we have not completely specified the mathematical problem by showing that the financial instrument must satisfy Equation (18); to complete the specification of the mathematical problem we must also specify what happens at the boundaries of the regions where $V(F_1, F_2, t)$ is defined. These boundary conditions differentiate between different financial instruments that depend on the values of two futures prices and time. To make these ideas concrete, consider an option to exchange one futures contract for another at a given future time, which we denote by T. This time is the expiration time of the option. Such a contract would satisfy Equation (18) for the value of a general financial instrument, and in addition the value of the option at expiration is known to be

$$V(F_1, F_2, T) = \max[F_1(T) - F_2(T), 0] = P(F_1, F_2) \tag{19}$$

where we have introduced the notation $P(F_1, F_2)$ to denote the payoff of the option. This payoff at expiration is an example of the kind of boundary condition that distinguishes between the different financial instruments that satisfy Equation (18). Note that this payoff is the value of the option at expiration, ie, at the time boundary just before it ceases to exist.

Given the differential equation for the value of the option, Equation (18), and the payoff at expiration, Equation (19), we can use the risk-neutral valuation principle discussed in Appendix 1 to value the option. Applying the multi-variable version of the Feynman–Kac formula to this situation yields an expression for the value of the option as an expectation:

$$V(F_1, F_2, t) = \mathrm{e}^{-r(T-t)}\, \hat{\mathbb{E}}[\max(a_1 F_1 - a_2 F_2 - K, 0)] \tag{20}$$

where the expectation is that taken over values of the futures follow the risk-neutral stochastic processes

$$\mathrm{d}F_1 = \sigma_1 F_1 \mathrm{d}z_1 \tag{21}$$

$$\mathrm{d}F_2 = \sigma_2 F_2 \mathrm{d}z_2 \tag{22}$$

In the discussion below, we shall use both the differential equation for the option price, Equation (18) and the solution to that equation given by the risk-neutral expectation, Equation (20).

For future reference, also notice that the differential equation, Equation (18), is time homogeneous, by which we mean that it is not changed by the transformation $t \to t + \delta t$. The only time that matters in the value of the option is the time until expiration, $\tau \equiv T - t$. In the rest of this section we shall be concerned with European options. For these options the boundary conditions are given by the payoff of the options at expiration.

The option to exchange two futures, the Margrabe formula, and dimensional analysis

A simple extension of the option to exchange two futures contracts is the option to exchange a specified quantity of one futures contract for a possibly different quantity of another futures contract at the expiration of the option. More concretely, consider the option to obtain a_1 contracts of future 1, which has price F_1, in exchange for a_2 contracts of future 2, which has price F_2, and furthermore this option will become available at the option expiration, which occurs at a time T in the future. The value of this option at expiration, ie, the payoff of the option, is given by

$$V(F_1, F_2, T) = P(F_1, F_2) = \max[a_1 F_1(T) - a_2 F_2(T), 0] \tag{23}$$

The value of this option can be found by solving Equation (18) with the boundary condition in Equation (23). The work of solving this equation can be simplified by using dimensional analysis, a useful technique that we now describe. The basic idea behind dimensional analysis is that, in almost all models, the mathematical relationship between the original dimensional parameters of the model can be rewritten in terms of dimensionless parameters. Usually, the number of dimensionless parameters is smaller than the number of original parameters, and this difference can lead to a simplification of the mathematics of the model. The classic exposition of dimensional analysis is that of Bridgman (see Bridgman, 1931), which unfortunately seems to be out of print. Another outstanding exposition is that of Barenblatt (see Barenblatt, 1996). A clear and simple discussion of dimensional analysis is given in the useful book of Wilson (see Wilson, 1990), and the paper of Buckingham (see Buckingham, 1914) is the original reference for the formal mathematics used in dimensional analysis.

To illustrate how and why dimensional analysis works, consider finding the value of the aforementioned exchange option. As we have already noted, this value can be found by solving the differential equation, Equation (18), with the boundary condition, Equation (23). The only variables and parameters that occur in these expressions are $V, F_1, F_2, \sigma_1, \sigma_2, r, a_1, a_2$, and τ, where $\tau \equiv T - t$

is the time to the expiration of the option, so that the value of the option must satisfy the relation

$$V = G(F_1, F_2, \sigma_1, \sigma_2, r, a_1, a_2, \tau) \tag{24}$$

for some function G. The important point to notice is that the value of the option is in units of money, and is a function of two futures prices, which also have units of money, two volatilities, an interest rate, a time, and two dimensionless parameters, viz, a_1 and a_2. This rather general, abstract equation can be rewritten in the equally general and abstract form

$$\frac{V}{F_2} = H\left(F_2, \frac{F_1}{F_2}, \sigma_1, \sigma_2, r, a_1, a_2, \tau\right) \tag{25}$$

because, given the function H one can construct the function G. Three of the quantities in Equations (24) and (25) have the dimensions of money, viz, V, F_1, and F_2. A vitally important point is that the relation in Equation (25), and therefore Equation (24), must hold regardless of whether or not money is measured in units of dollars or pennies. More generally, Equation (25) must be invariant under the transformation that occurs when all quantities with dimensions of money are multiplied by α, because this transformation merely changes the units in which the money is measured, eg, setting $\alpha = 100$ changes from measuring money in dollars to measuring it in pennies. Implementing this transformation of units of money means making the transformations $V \to \alpha V$, $F_1 \to \alpha F_1$, and $F_2 \to \alpha F_2$. Transforming Equation (25) in this manner leads to the expression:

$$\frac{V}{F_2} = H\left(\alpha F_2, \frac{F_1}{F_2}, \sigma_1, \sigma_2, r, a_1, a_2, \tau\right) \tag{26}$$

The only way the value of the option can be independent of the units in which money is measured is for Equation (26) to be independent of α, and the only way that Equation (26) can be independent of α is if the function H is independent of its first argument, which means that the equation can be written as:

$$V = F_2 H_1\left(\frac{F_1}{F_2}, \sigma_1, \sigma_2, r, a_1, a_2, \tau\right) \tag{27}$$

Notice that the function H_1 depends on the futures prices only through the variable F_1/F_2. This argument shows how requiring the results to be independent of the units in which they are measured can lead to the restriction of the form of the solution of a well-defined mathematical problem. Dimensional analysis typically leads to such results. We could go on and consider the simplifications that would result from requiring that Equation (27) be independent of the units in which time is measured, however, we will not need this result and leave the analysis to the interested reader.

The result of the dimensional analysis, Equation (27), requires the function for the value of the exchange option, $V(F_1, F_2, t)$ to have the form

$$V(F_1, F_2, t) = F_2 h(z, t) \tag{28}$$

where we have defined

$$z = \frac{F_1}{F_2} \tag{29}$$

In terms of the new variable z the payoff of the exchange option, given in Equation (23), becomes

$$P(F_1, F_2) = a_1 F_2 \max\left[z - \frac{a_2}{a_1}, 0\right] = F_2 \varphi(z) \tag{30}$$

where we have defined

$$\varphi(z) = a_1 \max\left[z - \frac{a_2}{a_1}, 0\right] \tag{31}$$

and, using the chain rule of calculus, the differential equation, Equation (18), for the value of the option becomes a differential equation for $h(z)$

$$\frac{\partial h}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 z^2 \frac{\partial^2 h}{\partial z^2} - rh = 0 \tag{32}$$

where we have defined the effective volatility

$$\tilde{\sigma} = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \tag{33}$$

The boundary conditions for the function $h(z, t)$ follow from the boundary conditions for the value of the option, viz, the requirement that the value of the option equals the payout at the option expiration. This requirement is expressed in Equation (23). Substituting the expressions for the value of the option and the value of the payout in terms of the variable z, ie, Equations (28) and (30), into the boundary condition in Equation (23) leads to the desired boundary condition for the function $h(z, t)$

$$h(z, T) = \varphi(z) = a_1 \max\left[z - \frac{a_2}{a_1}, 0\right] \tag{34}$$

Now notice that the differential equation, Equation (32), for $h(z, t)$ together with the boundary condition, Equation (34), describes a_1 vanilla call options on a futures price z, with a strike price of a_2/a_1 and a volatility of $\tilde{\sigma}$ (see Hull, 2002), so the solution of the differential equation can be obtained directly from the Black (1976) formula:

$$h(z, t) = a_1 e^{-r\tau}\left[zN(d_1) - \left(\frac{a_2}{a_1}\right)N(d_2)\right] \tag{35}$$

where we have defined

$$d_1 = \frac{1}{\tilde{\sigma}\sqrt{\tau}} \log\left(\frac{a_1 z}{a_2}\right) + \frac{\tilde{\sigma}\sqrt{\tau}}{2} \tag{36}$$

$$d_2 = \frac{1}{\tilde{\sigma}\sqrt{\tau}} \log\left(\frac{a_1 z}{a_2}\right) - \frac{\tilde{\sigma}\sqrt{\tau}}{2} \tag{37}$$

and $N(x)$ represents the cumulative normal distribution, and we remind the reader again that $\tau = T - t$ represents the time until the option expires. Substituting this expression for $h(z, t)$ into

Equation (28) for $V(F_1, F_2, t)$ and using the definition in Equation (29) of z yields the desired value of an option to exchange two futures contracts:

$$V(F_1, F_2, t) = e^{-r\tau}[a_1 F_1 N(d_1') - a_2 F_2 N(d_2')] \qquad \textbf{(38)}$$

where

$$d_1' = \frac{1}{\tilde{\sigma}\sqrt{\tau}} \log\left(\frac{a_1 F_1}{a_2 F_2}\right) + \frac{\tilde{\sigma}\sqrt{\tau}}{2} \qquad \textbf{(39)}$$

$$d_2' = \frac{1}{\tilde{\sigma}\sqrt{\tau}} \log\left(\frac{a_1 F_1}{a_2 F_2}\right) - \frac{\tilde{\sigma}\sqrt{\tau}}{2} \qquad \textbf{(40)}$$

Equations (38) through (40) for the value of the exchange option are simply the Margrabe formula (Margrabe, 1978) modified so that it applies to futures contracts. We could have derived this result by postulating the form of Equation (28) for the value of the exchange option without motivation and showed how it solved the differential equation, like pulling a rabbit out of a hat. We hope that our exposition of how this form can be derived from dimensional analysis will prove useful to our readers in solving their own option pricing problems.

Putting in a fixed strike, the basket option approximation and new uses of the Ito lemma

The next level of complexity for exchange options is the option to exchange with a fixed strike. More concretely, consider the option to obtain a_1 contracts of future 1, which has price F_1, in exchange for a_2 contracts of future 2, which has price F_2 and a fixed strike payment of K, and furthermore this option will become available at the option expiration, which occurs at time T in the future. The value of this option at expiration, ie, the payoff of the option, is given by

$$V(F_1, F_2, T) = P(F_1, F_2) = \max[a_1 F_1(T) - a_2 F_2(T) - K, 0] \qquad \textbf{(41)}$$

The fixed strike, K, has units of money. In the previous section we used dimensional analysis methods to reduce the number of variables in the differential equation, Equation (18), for the asset value. This simplification occurred because all three of the variables or parameters in the problem that had dimensions of money were explicitly present in the differential equation. These three variables could combine to produce only two dimensionless variables, so rewriting the differential equation in terms of the two dimensionless variables reduced the number of variables in the equation. In an option with a fixed strike, there exists another parameter with the dimensions of money, viz, the fixed strike. This parameter occurs in the payoff, and hence the boundary conditions for the differential equation, but not in the differential equation itself, so that dimensional analysis cannot be used to reduce the number of variables in the differential equation.

As an exact reduction of the number of variables in the differential equation does not exist for the case of an exchange option with a fixed strike, we shall illustrate an approximate technique to reduce the number of variables in the differential equation. The simplest such approach, which we will discuss now, is to model the spread as a future, find an effective volatility of this future, and substitute the results into the Black (1976) formula. We do *not* advocate using this approach for spread options, for reasons that we will discuss below, however, it is a useful technique for valuing basket options. We include a discussion of this technique here to present the mathematics for a simple case so that the reader can use this approximation for options where it does work well. The basic idea is to model a linear combination of assets or futures as a single asset or future. To this end we define:

$$X = a_1 F_1 - a_2 F_2 \tag{42}$$

to represent the spread in the futures prices. We wish to model X as a futures price with the same statistical characteristics as the spread. To understand this statement better, recall that an option on a futures price X satisfies the Black (1976) differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_e^2 \, X^2 \, \frac{\partial^2 V}{\partial X^2} - rV = 0 \tag{43}$$

where σ_e represents the effective volatility of the spread (X) and the final condition of the differential equation is given by the payout.

In the present case, where we are modelling a call option on the spread, the final condition is

$$V(X,T) = \max(X-K, 0) \tag{44}$$

where T represents the expiration time of the option. From the Feynman–Kac formula, a representation of the solution of this differential equation is given by

$$V(X,t) = e^{-r(T-t)}\,\hat{\mathbb{E}}\left[\max\left(X-K, 0\right)\right] \tag{45}$$

where the expectation is taken over a stochastic variable X that follows the stochastic process

$$\mathrm{d}X = \sigma_e X \mathrm{d}z \tag{46}$$

where z represents a standard Wiener process. Readers should now notice a vitally important point, namely that the spread described by Equation (42) is a linear combination of futures, but it is not a future itself. Our modelling of this spread as a futures price is an approximate assumption. The real value of the spread option is given by the risk-neutral expectation expression of Equations (20), (21), and (22), which is not necessarily the same as the risk-neutral expectation expression of Equations (45) and (46) for any value of the parameter σ_e.

To understand why treating X as a future cannot hold exactly, note that if the spread in the futures prices is described by the SDE, Equation (46), then the probability density of the spread is lognormal. Similarly, if the behaviour of the individual futures prices is described by the SDEs, Equations (7) and (8), then the probability densities of the individual futures prices are also lognormal. These last two statements are inconsistent because X represents a linear combination of F_1 and F_2, but the probability density of a linear combination of variables, with lognormal probability densities, is not lognormal. In effect, we are approximating the probability density of X with a lognormal probability density, and the accuracy of this approximation determines the accuracy of this method of valuing the option. For the spread option, a more salient question involves the sign of the spread. The spread defined by X in Equation (42) can be negative and in practical cases of spread options the spreads frequently are negative. The SDE that we use to model X, Equation (46), implies that the probability density of X is lognormal, which means that X is never negative. Clearly this aspect of the model does not agree with reality! We will have more to say about this problem after we have illustrated the technique of calculating an effective volatility for a linear combination of futures prices.

Given the approximate assumption of using Equation (46) to model the behaviour of the spread, we must find some criterion for calculating as effective volatility, σ_e. Here we shall choose the effective volatility to match the variance of the spread. Calculating the required variances involves some new uses of Ito's lemma that are generally useful in valuing options on commodities. First, we calculate the first two moments of X, assuming that behaviour of X is described by the SDE, Equation (46). We define the two moments by:

$$M_1(t) = \mathbb{E}[X(t)] \tag{47}$$

$$M_2(t) = \mathbb{E}[X^2(t)] \tag{48}$$

The first moment, M_1, can be found by inspection. The SDE for X has no drift term. The Wiener term, $\sigma_e X\mathrm{d}z$, is as likely to push X in a positive direction as a negative direction, so the expected value of X at time t is the same as the initial value

$$M_1(t) = X(0) \tag{49}$$

To calculate $M_2(t)$ we define

$$y(t) = X^2(t) \tag{50}$$

so that

$$M_2(t) = \mathbb{E}[y(t)] \tag{51}$$

The SDE for $y(t)$ follows directly from the Ito lemma

$$\mathrm{d}y = \frac{1}{2}\sigma_e^2 X^2 \frac{\mathrm{d}^2 y}{\mathrm{d}X^2}\mathrm{d}t + \sigma_e X \frac{\mathrm{d}y}{\mathrm{d}X}\mathrm{d}z \tag{52}$$

which, after making the substitutions and calculating the derivatives, yields

$$\mathrm{d}y = \sigma_e^2 y \mathrm{d}t + 2\sigma_e y^{3/2} \mathrm{d}z \tag{53}$$

Taking the expectations of both sides of the equation leads to a differential equation for $M_2(t)$

$$\frac{\mathrm{d}M_2}{\mathrm{d}t} = \sigma_e M \tag{54}$$

with the initial condition that $M_2(0) = X^2(0)$. This differential equation has the solution

$$M_2(t) = X^2(0)\mathrm{e}^{\sigma_e^2 t} \tag{55}$$

Comparing the expressions for the first two moments of $X(t)$, Equations (49) and (55), leads to an expression for the effective volatility, σ_e, in terms of these moments

$$\sigma_e = \sqrt{\frac{1}{t}\log\left(\frac{M_2}{M_1^2}\right)} \tag{56}$$

Next, we calculate the first two moments of X using the definition of X in Equation (42) and the risk-neutral SDEs, Equations (21) and (22) for F_1 and F_2. We use the risk-neutral SDEs because it is the expectation of the risk-neutral processes that determines the value of the option in, for example, Equation (20). The same arguments that lead to Equation (49) for $M_1(t)$ above lead to:

$$M_1(t) = \mathbb{E}\left[a_1 F_1(t) - a_2 F_2(t)\right] = a_1 F_1(0) - a_2 F_2(0) \tag{57}$$

in this case. For the second moment

$$\begin{aligned} M_2(t) &= \mathbb{E}\left[\left(a_1 F_1(t) - a_2 F_2(t)\right)^2\right] \\ &= a_1^2 \mathbb{E}\left[F_1^2(t)\right] - 2a_1 a_2 \mathbb{E}\left[F_1(t)F_2(t)\right] + a_2^2 \mathbb{E}\left[F_1^2(t)\right] \end{aligned} \tag{58}$$

The same arguments that lead to Equation (55) for $\mathbb{E}[X^2(t)]$ above lead to

$$\mathbb{E}\left[F_1^2(t)\right] = F_1^2(0)\,e^{\sigma_1^2 t} \tag{59}$$

and

$$\mathbb{E}\left[F_2^2(t)\right] = F_2^2(0)\,e^{\sigma_2^2 t} \tag{60}$$

To complete the calculation of $M_2(t)$, we define

$$\gamma(t) = F_1(t)F_2(t) \tag{61}$$

and

$$C(t) = \mathbb{E}[\gamma(t)] = \mathbb{E}[F_1(t)F_2(t)] \tag{62}$$

The SDE for $\gamma(t)$ follows from the SDEs, Equations (7) and (8) for $F_1(t)$ and $F_2(t)$, from the multi-dimensional version of Ito's lemma, Equation (11), and from a small amount of algebra:

$$\mathrm{d}\gamma = \rho\sigma_1\sigma_2\gamma\mathrm{d}t + \sigma_1\gamma\mathrm{d}z_1 + \sigma_2\gamma\mathrm{d}z_2 \tag{63}$$

Taking the expectations of both sides of this equation leads to a differential equation for $C(t)$

$$\frac{\mathrm{d}C}{\mathrm{d}t} = \rho\sigma_1\sigma_2 C \tag{64}$$

with the initial condition $C(0) = F_1(0)\,F_2(0)$. This differential equation has the solution

$$C(t) = \mathbb{E}[F_1(t)\,F_2(t)] = F_1(0)\,F_2(0)e^{\rho\sigma_1\sigma_2 t} \tag{65}$$

Substituting the results of Equations (59), (60), and (65) into Equation (58) gives the desired expression for the second moment:

$$M_2(t) = a_1^2 F_1^2(0)\mathrm{e}^{\sigma_1^2 t} - 2a_1 a_2 F_1(0)F_2(0)\mathrm{e}^{\rho\sigma_1\sigma_2 t} + a_2^2 F_2^2(0)\mathrm{e}^{\sigma_2^2 t} \tag{66}$$

The expression for the effective volatility of the spread follows from substituting Equation (57) for $M_1(t)$ and Equation (66) for $M_2(t)$ into Equation (56) for the effective volatility to find:

$$\sigma_e = \sqrt{\frac{1}{t}\log\left[\frac{a_1^2 F_1^2(0)\mathrm{e}^{\sigma_1^2 t} - 2a_1 a_2 F_1(0)F_2(0)\mathrm{e}^{\rho\sigma_1\sigma_2 t} + a_2^2 F_2^2(0)\mathrm{e}^{\sigma_2^2 t}}{(a_1 F_1(0) - a_2 F_2(0))^2}\right]} \tag{67}$$

Now we can use this effective volatility of the spread to compute value for an exchange option with a fixed strike in the approximation where the spread is treated as a single future. Substituting the spread (Equation (42)) and the effective volatility (Equation (67)) into the Black (1976) formula yields the approximation:

$$V = \mathrm{e}^{-r\tau}\left[\left(a_1 F_1 - a_2 F_2\right) N\left(d_1^e\right) - KN\left(d_2^e\right)\right] \tag{68}$$

where

$$d_1^e = \frac{\log\left(\dfrac{a_1 F_1 - a_2 F}{K}\right)}{\sigma_e\sqrt{\tau}} + \frac{\sigma_e\sqrt{\tau}}{2} \tag{69}$$

and

$$d_2^e = \frac{\log\left(\dfrac{a_1 F_1 - a_2 F}{K}\right)}{\sigma_e\sqrt{\tau}} - \frac{\sigma_e\sqrt{\tau}}{2} \tag{70}$$

This approximation has the virtue of simplicity. It is based on fitting the option value to a well-known and oft used formula. The nagging question is that of the accuracy of the approximation. In this regard, notice that the expressions for d_1^e and d_2^e become meaningless if the spread is negative. This problem occurs because we are assuming that the probability density of the spread is lognormal, and the lognormal density is zero for negative values of the argument. In fact, the spread between two futures prices *can* become negative, which does not bode well for the validity of the approximation we have just discussed. This effective volatility approximation is more appropriate for valuing options on quantities that are always positive, eg, baskets of futures. Indeed, although we do not recommend using this technique for valuing spread options, it shows great promise as a fast technique for valuing basket options in cases where the value of the basket can never go negative. Clewlow and Strickland (1999) discusses the use of this technique for valuing basket options.

An exact solution of the exchange option with a fixed strike using risk-neutral pricing

Can the exchange option with a fixed strike be valued exactly? If we use the simple model of Equations and for the behaviour of the futures prices, then the answer is yes, however, the expressions are somewhat more complicated than those encountered so far. To find the exact value of the exchange option we use the risk-neutral expectation expression of Equation (20), together with the risk-neutral stochastic processes of Equations (21) and (22). As we showed in Appendix 1, if the stochastic variables F_1 and F_2 satisfy the SDEs, Equations (21) and (22), and have values of F_1 and F_2 at time t, then the probability density of these variables having values F_1' and F_2' at time $T > t$ is given by

$$P\left(F_1, F_2, t; F_1', F_2', T\right) = \frac{e^{-r\tau}}{2\pi\tau F_1' F_2' \sqrt{\det \mathbf{C}}} \exp\left[\frac{\mathbf{a} \cdot \mathbf{C}^{-1} \cdot \mathbf{a}}{2\tau}\right] \tag{71}$$

where $\tau \equiv T - t$, as before, $\mathbf{y}$ represents the two dimensional vector with components

$$[\mathbf{y}]_i = y_i = \log\left(\frac{F_i}{F_i'}\right) - \frac{\sigma_i^2 \tau}{2} \tag{72}$$

and $\mathbf{C}$ represents the covariance matrix,

$$\mathbf{C} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{73}$$

With this knowledge the expression for the value of the option in Equation (20) becomes

$$V\left(F_1, F_2, t\right) = \int_0^\infty \int_0^\infty P\left(F_1, F_2, t; F_1', F_2', T\right) \times \max\left(a_1 F_1' - a_2 F_2' - K, 0\right) \mathrm{d}\, F_1' \mathrm{d}\, F_2' \tag{74}$$

which, after substituting in Equation (71) for $P(F_1, F_2, t; F_1', F_2', T)$ in turn becomes

$$V\left(F_1, F_2, t\right) = \frac{\mathrm{e}^{-r\tau}}{2\pi\tau\sqrt{\det \mathbf{C}}} \int_0^\infty \int_0^\infty \exp\left[\frac{\mathbf{y} \cdot \mathbf{C}^{-1} \cdot \mathbf{y}}{2\tau}\right] \times \max\left(a_1 F_1' - a_2 F_2' - K, 0\right) \frac{\mathrm{d}F_1'}{F_1'} \frac{\mathrm{d}F_2'}{F_2'} \tag{75}$$

The determinant of the covariance matrix is

$$\det \mathbf{C} = \sigma_1^2 \sigma_2^2 \left(1 - \rho^2\right) \tag{76}$$

and the inverse of the covariance matrix is given by

$$\mathbf{C}^{-1} = \begin{pmatrix} \dfrac{1}{\sigma_1^2\left(1-\rho^2\right)} & \dfrac{-\rho}{\sigma_1\sigma_2\left(1-\rho^2\right)} \\ \dfrac{-\rho}{\sigma_1\sigma_2\left(1-\rho^2\right)} & \dfrac{1}{\sigma_2^2\left(1-\rho^2\right)} \end{pmatrix} \tag{77}$$

so that, after combining all of these expressions, the value of the option to exchange with a fixed strike is given by

$$V(F_1, F_2, \tau) = \frac{\mathrm{e}^{-r\tau}}{2\pi\sigma_1\sigma_2\tau\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \max\left[a_1 F_1' - a_2 F_2' - K, 0\right] \times \exp\left[-\frac{y_1^2}{2\sigma_1^2\tau(1-\rho^2)} - \frac{\rho y_1 y_2}{\sigma_1\sigma_2\tau(1-\rho^2)} - \frac{y_2^2}{2\sigma_2^2\tau(1-\rho^2)}\right] \frac{\mathrm{d}F_1'}{F_1'} \frac{\mathrm{d}F_2'}{F_2'} \tag{78}$$

where we have retained the notation $\tau = T - t$ and the definitions of y_1 and y_2 in order to keep the expression in Equation (78) from becoming unwieldy. Equation (78) is the desired *exact* expression for the value of the option to exchange with a fixed strike. It is in the form of a double integral that must be evaluated numerically, however, this numerical evaluation is straightforward, as we discuss below.

On a practical note, the double integral in Equation (78) for the option value can be reduced to a set of single integrals of a normal probability density multiplied by the cumulative normal distribution of a complicated function. This reduction makes the numerical evaluation of the value of the option much faster, easier, more accurate, and more reliable. The reduction involves completing the square in the exponential function several times.

Incorporation of spikes into the models

The Black–Scholes analysis assumes continuous dynamic replication of a portfolio. However, any such rebalancing incurs transaction costs. These costs force hedging to become discrete, and hence imperfect. Therefore, in practice, all portfolios bear some risk. This situation is exacerbated during the power price spikes when even approximate dynamic replication becomes impossible. This further limits the applications of the risk-neutrality assumption for option valuation. The valuation of options in cases where accurate hedging is impractical is a research topic that deserves (at least!) a chapter to itself. To make matters worse, transmission options are typically settled based on spot prices, but they are typically hedged with forward or futures contracts. Although spot prices of electricity have spiked in the past, the forward and futures prices of electricity have not shown spike behaviour, so the spot price can only be hedge with a financial instrument that is described by a qualitatively different stochastic process. This problem further adds to the challenge of calculating an accurate option value that includes our imperfect ability to hedge. We wish to show the effects of spikes on the value of transmission rights without delving into the complexities of valuing options that can not be perfectly Delta hedged, therefore, in this section we will value the *real* expected net present value of the payoff. More precisely, in the earlier sections of this chapter, we valued the option by calculating the expected net present value of the option using the risk-neutral probability distribution:

$$V\left(F_1, F_2, t\right) = \mathrm{e}^{-r(T-t)}\, \hat{\mathbb{E}}\left[\max\left(a_1 F_1 - a_2 F_2 - K, 0\right)\right] \tag{79}$$

where the notation $\hat{\mathbb{E}}$ represents an expectation over a risk-neutral probability distribution. We also explained that the risk-neutral probability distribution was *not* the real, observed probability distribution, but rather an artificial construction that is used to price options. In this section we calculate the net present value of the *real* expected payoff, which is given by

$$\tilde{V}(F_1, F_2, t) = \mathrm{e}^{-r(T-t)}\,\mathbb{E}[\max(a_1F_1 - a_2F_2 - K, 0)] \qquad \textbf{(80)}$$

where the notation $\mathbb{E}$ represents an expectation over the real, observed probability distribution and $\tilde{V}$ represents the net present value of the expected payoff. In the sections above we emphasised the difference between these two probability distributions. By taking the average over the real probability distribution values we calculate the net present value of the payoffs rather than the option prices, but we have the advantage of being able to calculate all the quantities we want by using standard statistical methods. In addition, much of the present section is devoted to examining the differences that spikes make. We expect that these differences will be similar for expected payoffs and for real option prices. Thus, the discussion of these differences for the expected payoffs should provide a reasonable qualitative guide to valuing options on price processes with and without spikes.

When commodity prices or futures prices F_1 and F_2 at one or both locations exhibit multi-state behaviour, their densities on any given day are given by mixtures similar to Equations (5) and (6). For a T-days-ahead price forecast, the mixing probabilities are elements of the T-step transition probability matrix $P^{(T)} = P^T$. On a given day, T days ahead from a day when prices were at state $i = 1, 2$, the unconditional density of prices will be

$$\mathrm{f}_T(x) = P_{i,1}^{(T)}\mathrm{f}(x) + P_{i,2}^{(T)}\mathrm{g}(x) \qquad \textbf{(81)}$$

The value $V(F_1, F_2, t)$ of an expected payoff that is based on both F_1 and F_2, eg, a spread option, can be expressed as the expectation of the payoff function $h_T(F_1, F_2)$ at the expiry date T. The expectation of any function with respect to a mixture density, Equation (81), is a weighted average of individual expectations, with the same mixing coefficients

$$E_T\left\{h(X)\right\} = \int h(x)\mathrm{f}_T(x)\mathrm{d}x = P_{i,1}^{(T)}\int h(x)\mathrm{f}(x)\mathrm{d}x + P_{i,2}^{(T)}\int h(x)\,\mathrm{g}(x)\mathrm{d}x \qquad \textbf{(82)}$$

where X=(F_1, F_2). That is

$$V_T(F_1, F_2, t) = P_{i,1}^{(T)}V_T^{(1)}(F_1, F_2, t) + P_{i,2}^{(T)}V_T^{(2)}(F_1, F_2, t)$$

where $\tilde{V}_T^{(1)}(F_1, F_2, t)$ and $\tilde{V}_T^{(2)}(F_1, F_2, t)$ are values of the expected values of the payoffs under single-mode processes, without spikes and with only spikes, respectively. Both components of the total expected value of the payoff can be computed in the same way as the expectation in Equation ■■ except that in Equation ■■ we are using risk-neutral probabilities, whereas here we are using real probabilities.

Suppose that there are n states at each of the two locations, each of the states denoting a different stochastic process. Then, in general, there are n^2 possible joint states, because each state in the first location may occur simultaneously with any state at the second location. In practice, however, transitions between the states at different locations are very strongly correlated. In particular, a spike of electricity prices in one city will typically cause an immediate spike in the other city. Thus, the spikes in two locations will always approximately coincide in time, so that the transition probability matrix of joint states in two locations is still given by Equation (4). Hence, Equation (82) applies to price spread between processes with spikes, where f(x) and g(x) are *bivariate lognormal* densities, with correlation coefficient ρ between the two locations. For example, the joint density of commodity prices during spikes in two locations, to be used in Equation (82), is given by

$$\mathrm{g}(x_1, x_2 \,|\, \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \frac{1}{2\pi x_1 x_2 \sigma_1 \sigma_2 (1-\rho^2)}\,\mathrm{e}^{-\frac{\left(\frac{\ln x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{\ln x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{\ln x_1-\mu_1}{\sigma_1}\right)\left(\frac{\ln x_2-\mu_2}{\sigma_2}\right)}{2(1-\rho^2)}} \qquad \textbf{(83)}$$

How does incorporation of spikes affect the value of the expected payoff of a price spread? To evaluate the strength of the spike effect, we fitted the introduced multi-state model to the two-year-long sequence of electricity prices in Pennsylvania, New Jersey and Maryland (PJM).

For the control-state distribution of de-trended log-prices, we used two correlated autoregressive processes (AR1 processes), Equation (1), processes:

$$X_{1t} = \phi_1 X_{1,t-1} + Z_{1t} \quad \text{and} \quad X_{2t} = \phi_2 X_{2,t-1} + Z_{2t}$$

where (Z_{1t}, Z_{2t}) is a bivariate white noise, $\text{var}(Z_{1t}) = \sigma_1^2$, $\text{var}(Z_{2t}) = \sigma_2^2$, $\text{cov}(Z_{1t}, Z_{2t}) = \rho\sigma_1\sigma_2$. Parameters σ_j, ϕ_j of each process X_{jt}, $j = 1, 2$, are estimated by standard methods (see Brockwell and Davis, 1991). To estimate the correlation coefficient ρ, we notice that

$$\text{cov}(X_{1t}, X_{2t}) = \text{cov}\left(\sum_{n=0}^{\infty} \phi_1^n Z_{1,t-n}, \sum_{n=0}^{\infty} \phi_2^n Z_{2,t-n}\right) = \frac{\rho\sigma_1\sigma_2}{1-\phi_1\phi_2}$$

from where the estimator of ρ is

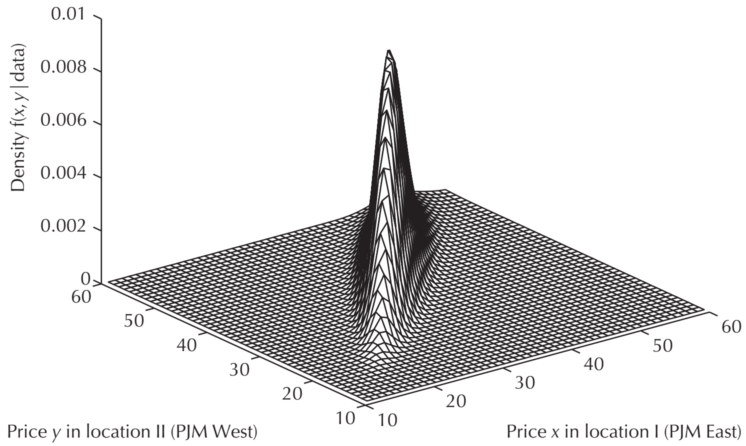
$$\hat{\rho} = \frac{(1-\hat{\phi}_1\hat{\phi}_2)\sum_t (X_{1t}-\bar{X}_1)(X_{2t}-\bar{X}_2)}{\hat{\sigma}_1\hat{\sigma}_2}$$

For PJM East and West regions, this correlation coefficient was found to be 0.9466.

During each spike, the mean vector (θ_1, θ_2) is generated from a bivariate normal distribution with (hyper-) parameters $(\mu_1, \mu_2, \tau_1, \tau_2, \rho_\mu)$. The correlation coefficient ρ_μ between the spike means, estimated by the sample correlation between observed spike price averages in PJM East and PJM West, equals 0.9967. Next, the vector of (de-trended log) prices for each spike for the two regions has a bivariate normal distribution with parameters $(\theta_1, \theta_2, \eta_1, \eta_2, \rho_{\text{spike}})$. Unconditionally on (θ_1, θ_2), we have

$$\begin{aligned}\text{cov}(X_{1t}, X_{2t}) &= \mathbb{E}\left[\text{cov}(X_{1t}, X_{2t} \mid \theta_1, \theta_2)\right] + \text{cov}\left\{\mathbb{E}\left[(X_{1t} \mid \theta_1)\right], \mathbb{E}\left[(X_{2t} \mid \theta_2)\right]\right\} \\ &= \rho_{\text{spikes}}\eta_1\eta_2 + \rho_\mu\tau_1\tau_2\end{aligned}$$

Figure 2 Bivariate predictive density of the power price for the given day



Then, the spike-mode distribution of de-trended log-prices in two locations has the form given in Equation (83), where the correlation coefficient is

$$\frac{\rho_{\text{spikes}}\eta_1\eta_2+\rho_\mu\tau_1\tau_2}{\sqrt{\left(\tau_1^2+\eta_1^2\right)\left(\tau_2^2+\eta_2^2\right)}}$$

This yields the joint density of electricity prices for the two regions for any given day (see Figure 2).

To evaluate the net present value of the expected payoff, we integrated the discounted payoff function:

$$h(F_1, F_2)\,\mathrm{e}^{-rt}$$

with this density and compared the results with values of the same options calculated from a no-spike model. The latter was computed by similar methods, but without the spike detection step. Two option values, for each month of the year ahead, are depicted in Figure 3.

As we see, the difference between the multi-state and no-spike models in valuing the expected payoff reaches 14 cents or 7% of the option value during the season of peak demand for the West to East spread that maximises the value of the expected payoff (see Figure 3a). During the shoulder months, the expected payoff values computed from the two models are practically indistinguishable. For comparison we also present the value of the expected payoff in the opposite direction (see Figure 3b). The value of the spread going against the positive average price differential is rather small but non-zero at any time due to a small probability to end up in the money. It also shows to be out of phase with the positive spread, as one would expect.

NUMERICAL SOLUTIONS

Numerical integration

The ability to express the value of an option as an integral, such as an equation, represents a significant step towards valuing and hedging the option. Although integral representations do not have the ease of use of closed form solutions, they are significantly easier to use and work with than binomial trees, finite difference solutions, or Monte Carlo solutions. When closed form solutions cannot be obtained, integral representations, when they can be obtained, have three main advantages over the other methods. First, because an integral representation is still, in essence, an analytical expression for the value of the option, it retains many of the advantages of closed-form solutions. For example, the standard procedures of differential calculus can be used to calculate integral representations of the Greeks. Second, there is a large literature on computing integrals numerically and many easy to implement, powerful, and well-understood algorithms are available. Almost every elementary numerical analysis text has a Section on numerical integration, such as, to take a very arbitrary and limited selection, the books of Dahlquist and Bjork (1974), Acton (1970), and Stoer and Bulirsch (1980). The book of Judd (1998) is oriented toward applications to economics and the truly amazing Press *et al* (1992), includes working computer code. One way to think about numerical integration is that it is the simplest differential equation, so it is reasonable that the algorithms for solving it should be simpler and more powerful, than the algorithms for more complex differential equations. Binomial trees and finite difference methods are essentially methods for

Figure 3a Spread option value between PJM East and PJM West

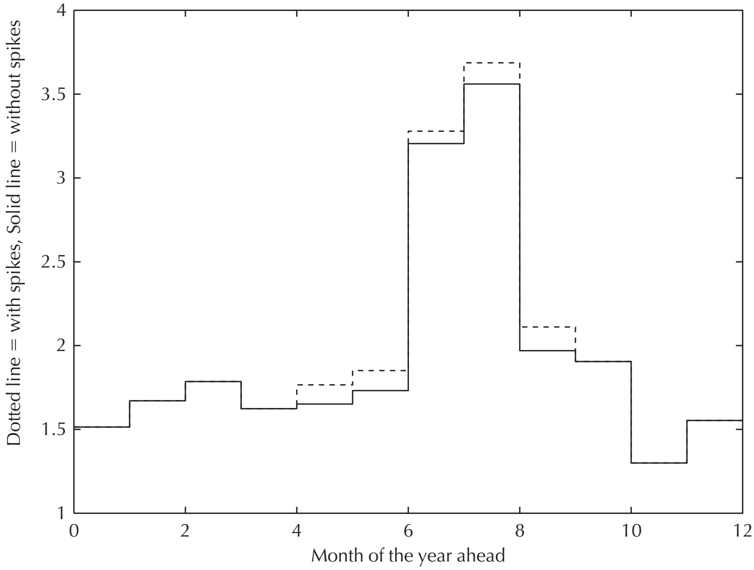
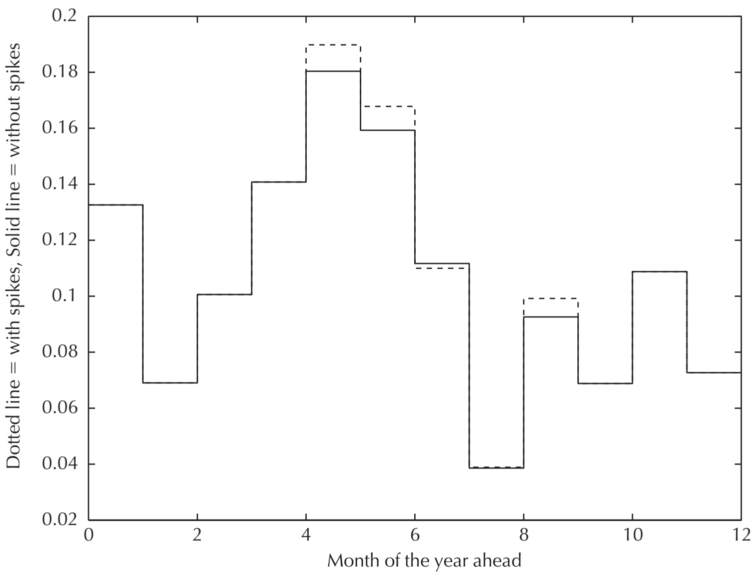


Figure 3b Spread option value between PJM West and PJM East



solving these more general kinds of differential equations. Third, it is relatively easy to write an algorithm that evaluates an integral to a given, pre-determined accuracy. The basic idea is that one evaluates the integral numerically using a certain number of points, checks for numerical convergence, and if the convergence is not reached, one just evaluates the integral with more points. A good, practical discussion of this procedure is given by Press *et al*, (1992), who give both a clear mathematical explanation and computer code.

Monte Carlo

❑ Monte Carlo simulation is a mathematical technique for numerically solving differential equations. One of the main uses of Monte Carlo simulation in Finance is for pricing options on multiple assets, like our spread option. Below we outline how the method of Monte Carlo simulation can be used to price such an option. Generate a large number of scenarios at the expiration date for the prices of two underlying assets. Do so in a manner that is consistent with the option volatilities, current prices and a correlation. Note that the joint distribution of assets prices at any time in the future can be fully described by their initial price levels, correlation and volatilities, under the assumption that they follow geometric Brownian motion processes.

❑ Evaluate option payoff at expiration for every simulated scenario. Each scenario consists of a possible, randomly drawn pair of assets prices.

❑ Calculate the fair value or expected, discounted option payoff by first taking the average of all future payoffs, calculated at the previous step, and then discounting it to the valuation date.

The spread option is an option on the difference between two commodity prices F_1, F_2, which follow geometric Brownian motion, described by the system of Equations (7) and (8) with instantaneous correlation ρ. In order to price this option via Monte Carlo simulation we need to derive a discrete time version of the system of Equation (2) (discussion on various discretising choices can be found in Kloeden and Platen (1994)). The following steps detail the

discritiation scheme:
Step 1: Apply Ito's Lemma to the logarithms of prices in Equations (7) and (8):

$$\begin{cases} \mathrm{d}\ln F_{1,T} = -0.5\sigma_1^2 T + \sigma_1 z_1^2 \sqrt{T} \\ \mathrm{d}\ln F_{2,T} = -0.5\sigma_2^2 T + \sigma_2 z_2^2 \sqrt{T} \end{cases} \tag{84}$$

where $F_{1,T}$ and $F_{2,T}$ are the assets' futures prices at time T. Random variables z_1 and z_2 come from the standard bivariate normal distribution with correlation ρ.
Step 2: Discretise the system of equations (84):

$$\begin{cases} \ln F_{1,t+\Delta t} - \ln F_{1,t} = -0.5\sigma_1^2 \Delta t + \sigma_1 z_1^2 \sqrt{\Delta t} \\ \ln F_{2,t+\Delta t} - \ln F_{2,t} = -0.5\sigma_2^2 \Delta t + \sigma_2 z_2^2 \sqrt{\Delta t} \end{cases} \tag{85}$$

Finally,

$$\begin{cases} F_{1,T} = F_1 \mathrm{e}^{-0.5\sigma_1^2 T + \sigma_1 z_1^2 \sqrt{T}} \\ F_{2,T} = F_2 \mathrm{e}^{-0.5\sigma_2^2 T + \sigma_2 z_2^2 \sqrt{T}} \end{cases} \tag{86}$$

Correlated random normal random variables z_1 and z_2 are derived by simulating and combining independent standard normal variables ε_1 and ε_2 as follows

$$z_1 = \varepsilon_1$$
$$z_2 = \rho\varepsilon_1 + \sqrt{1-\rho^2}\,\varepsilon_2$$

Note that the system of Equation (3) allows us to simulate the prices at expiry without simulating the entire price path. Finally, for each pair of simulated prices we evaluate the option's payoff and discount it to today ie, evaluate the following expression $\mathrm{e}^{-rT}\max(F_{2,T} - F_{1,T} - X, 0)$. The average of these numbers will be the fair value of the option V. Specifically

$$V = \mathrm{e}^{-rT} \frac{1}{N} \sum_{i=1}^{N} \max(F_{2,T} - F_{1,T} - X, 0)$$

Note that this solution yields an approximate price. By increasing the number of scenarios N the accuracy of the result could be improved. The main disadvantage of simple Monte Carlo simulation is that its accuracy increases only as the square root of the number of simulations.

There exist many techniques (like variance reduction and quasi-random techniques) that make it possible to improve the accuracy and decrease the computation time at the same time. Press *et al*, (1992), contains a good overview of both methods.

Binomial spread options with a strike

Another method for pricing spread options is the multidimensional binomial method (the last two option types cannot be easily valued by Monte Carlo). Kamrad and Ritchken (1991), were the first to suggest it. The binomial model assumes that the asset price follows a binomial process ie, at any step it can either go up or down with a given probability. Thus it assumes that the asset price has a binomial distribution. On the other hand, many financial models assume that the distribution of assets' returns is normal. The binomial option pricing models make use of the fact that as the number of observations/trials in the binomial distribution increases it approaches normal distribution.

There are many ways one can build a price evolution tree that preserves distribution properties of the assets. Computationally, an addition operation is more efficient than multiplication. In what follows we present a widely used additive binomial tree method. Since the price process is represented as an additive process in logarithms as

$$\log F_i(t+\Delta t) = \log F_i(t) + \xi_t$$

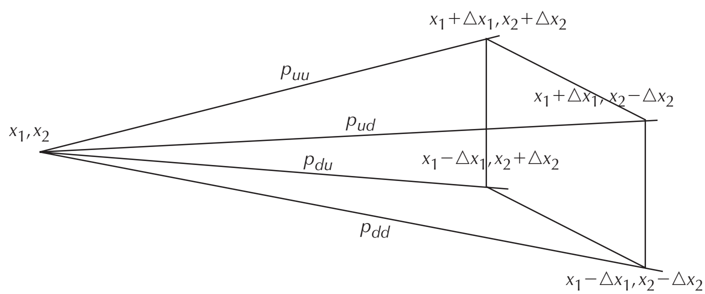
the tree is built for the logarithms of the assets' prices $(x_i = \log(F_i))$.

Applying Ito's lemma to this process (see Appendix 1), the risk-neutral process for x_i is $\mathrm{d}x_i = -0.5\sigma_i^2\,\mathrm{d}t + \sigma_i\,\mathrm{d}z_i$. The discrete version of the previous equation will be $\Delta x_i = -0.5\sigma_i^2\Delta t + \sigma_i\sqrt{\Delta t}$. The discrete time binomial model for x_1 and x_2 is illustrated in Table 1 and Figure 4. Assuming equal up and down jumps, each of the variables can either go up by Δx_i or down by $-\Delta x_i$. At any time step, the probability that logarithms of both prices go up is p_{uu},

Table 1 The joint normal random variable (***X*1(*t*), *X*2(*t*)**) is approximated by a pair of multinomial discrete normal variables having the following distribution

***X*1(*t*)**	***X*2(*t*)**	**Probability**
Δx_1	Δx_2	p_{uu}
Δx_1	$-\Delta x_2$	p_{ud}
$-\Delta x_1$	Δx_2	p_{du}
$-\Delta x_1$	$-\Delta x_2$	p_{dd}

Figure 4 One branch of the additive 2-variable binomial tree for logarithms of assets' prices



down is p_{dd}, first asset goes up, but second down p_{ud}, second goes up, but first goes down is p_{du}. The sum of these probabilities is one, as they contain all possible outcomes of price movements.

The next step is to establish jump sizes (Δx_1 and Δx_2) and four probabilities so that they match mean, variance and correlation of the bivariate normal distribution of assets' prices logarithms. For that we need to solve the system of Equation (86). It is an algebraic system of six equations with six unknowns:

$$\begin{cases} E(\Delta x_1) = \Delta x_1 (p_{uu} - p_{du} + p_{ud} - p_{dd}) = -0.5\sigma_1^2 \Delta t \\ \mathrm{var}(\Delta x_1) = \Delta x_1^2 (p_{uu} + p_{du} + p_{ud} + p_{dd}) = \sigma_1^2 \Delta t + O(\Delta t) \\ E(\Delta x_2) = \Delta x_2 (p_{uu} + p_{du} - p_{ud} - p_{dd}) = -0.5\sigma_2^2 \Delta t \\ \mathrm{var}(\Delta x_2) = \Delta x_2^2 (p_{uu} + p_{du} + p_{ud} + p_{dd}) = \sigma_2^2 \Delta t + O(\Delta t) \\ E(\Delta x_1, \Delta x_2) = \Delta x_1 \Delta x_2 (p_{uu} - p_{du} - p_{ud} + p_{dd}) = \rho \sigma_1 \sigma_2 \Delta t + O(\Delta t) \\ p_{uu} + p_{du} + p_{ud} + p_{dd} = 1 \end{cases} \tag{87}$$

The solution to this system of equations is:

$$\begin{cases} \Delta x_1 = \sigma_1 \sqrt{\Delta t} \\ \Delta x_2 = \sigma_2 \sqrt{\Delta t} \\ p_{uu} = \dfrac{\Delta x_1 \Delta x_2 - 0.5 \Delta x_2 \sigma_1^2 \Delta t - 0.5\, \Delta x_1 \sigma_2^2 \Delta t + \rho \sigma_1 \sigma_2 \Delta t \sigma_2^2}{4\, \Delta x_1 \Delta x_2} \\ p_{du} = \dfrac{\Delta x_1 \Delta x_2 + 0.5 \Delta x_2 \sigma_1^2 \Delta t - 0.5\, \Delta x_1 \sigma_2^2 \Delta t - \rho \sigma_1 \sigma_2 \Delta t \sigma_2^2}{4\, \Delta x_1 \Delta x_2} \\ p_{ud} = \dfrac{\Delta x_1 \Delta x_2 - 0.5 \Delta x_2 \sigma_1^2 \Delta t + 0.5\, \Delta x_1 \sigma_2^2 \Delta t - \rho \sigma_1 \sigma_2 \Delta t \sigma_2^2}{4\, \Delta x_1 \Delta x_2} \\ p_{dd} = \dfrac{\Delta x_1 \Delta x_2 + 0.5 \Delta x_2 \sigma_1^2 \Delta t + 0.5 \Delta x_1 \sigma_2^2 \Delta t + \rho \sigma_1 \sigma_2 \Delta t \sigma_2^2}{4\, \Delta x_1 \Delta x_2} \end{cases} \tag{88}$$

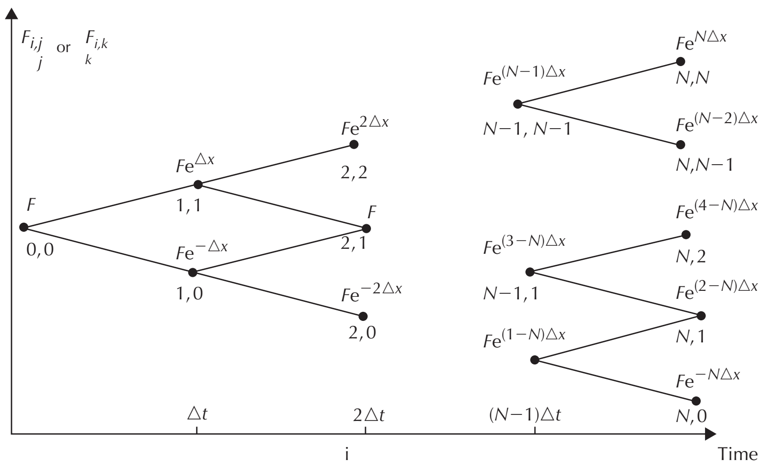
Now we have all the information necessary to build a two-dimensional tree and calculate the value of the option. The algorithm proceeds as follows.

A user sets up granularity of the tree N, ie, the number of steps between valuation and expiration dates of the option. The larger this number is the more accurate the solution will be. N equal to 200 is sufficient for most practical purposes. Thus, there will be N branching processes, at every time step $\Delta t = T/N$, where T is the time to expiry for a given option. Triplex $(i,\ j,\ k)$ refers to the nodes on the tree, where i is a time step, j is the first asset's price level, k is the second asset's price level. Since the tree is built for logarithms of assets' prices (x_1, x_2), we convert them to assets prices at each node of the tree via these simple transformations:

$$\begin{cases} F_{1,i,j,k} = F_1 \mathrm{e}^{(2j-i)\Delta x_1} \\ F_{2,i,j,k} = F_2 \mathrm{e}^{(2k-i)\Delta x_2} \end{cases}$$

Separately, each asset in the binomial tree follows a one-dimensional process that can be approximated by the one-dimensional tree (see Figure 5). At the same time their joint distribution characteristic – instantaneous correlation – reveals itself in their joint probabilities (see the last four equations in the system of Equation (88)).

Figure 5 Individual asset's prices at time node *i* and price level *j* for the first asset or *k* for the second



We initialise asset prices at expiration time (nodes (N, j, k) of the 2-variable binomial tree or (N, j) and (N, k) of the individual assets 1-dimensional trees).

At this step we estimate option values at expiration, utilising asset prices computed at the previous step. For all j and k of the final nodes (N, j, k) of the 2-variable binomial tree the option price $C(N, j, k)$ at expiration is $\max(F_2[k]-F_1[j]-X, 0)$.

Iteration stage

Now we calculate all option values $C(i, j, k)$ stepping back through the tree from the time node $(N-1)$ to 0. Option value at each time step i and price levels (j, k) is a discounted, weighted (by probabilities) average of the four option values at the nodes that are connected to a current node. Specifically,

$$C(i,j,k)=e^{-r\Delta t}\begin{pmatrix} p_{uu}C(i+1,j+1,k+1)+p_{du}C(i+1,j-1,k+1)+ \\ +p_{ud}C(i+1,j+1,k-1)+p_{dd}C(i+1,j-1,k-1) \end{pmatrix}$$

Fair value of the spread option at valuation date is given by the option value at the very first node of the tree ie, $C(0,0,0)$.

The above-described algorithm can be easily modified to accommodate American exercise. American options can be exercised at any time during their life. Thus at every time step a holder of the option has a choice to hold it to expiry (European feature) or to exercise it and receive an option payoff $\max(F_2[k]-F_1[j]-X,0)$ at every time node i (American feature). The last feature can easily be incorporated into our algorithm at the iteration stage 4. Specifically, American option price at time node i and price levels j and k becomes:

$$
\begin{aligned}
&C(i,j,k) \\
&\quad = \max\left\{ e^{-r\Delta t}\begin{pmatrix} p_{uu}C(i+1,j+1,k+1)+p_{du}C(i+1,j-1,k+1)+ \\ +p_{ud}C(i+1,j+1,k-1)+p_{dd}C(i+1,j-1,k-1)\end{pmatrix}, F_2[k]-F_1[j]-X \right\}
\end{aligned}
$$

All other steps remain unaffected. One can also improve the efficiency of the program by pre-multiplying probabilities by the one step discount factor $e^{-r\Delta t}$.

CONCLUSIONS

This chapter describes physical commodity transportation and transmission instruments that have one thing in common: they depend on the spread of locational or cross-commodity prices and thus can be modelled by a family of spread options. As this dependency suggest, the value of the instrument in any future moment depends not only on the level of prices but also on their future distribution function. The inclusion of spikes changes this distribution and hence produces a different value for these assets. The correct calibration and inclusion of these sudden and big deviations from the normal level of prices is very important for correct valuations and mark-to-market procedures. A reader has a number of methods that may be utilised to solve for the value of assets once the price process is established. They range from approximate solutions under some simplifying assumptions that are quite easy to compute, to the exact solution that is reducible to single integrals. Numerical methods include Monte Carlo and binomial methods and have a different degree of robustness. Monte Carlo always converges and requires a simple algorithm, but may be slow. The binomial approach requires more analytics and programming but pays off in the speed of the computation.

BIBLIOGRAPHY

Acton, F. S., 1970, *Numerical Methods that Work* (New York: Harper and Row).

Barenblatt, G. I., 1996, *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge University Press).

Baron, M. I., 2002, "Bayes and asymptotically pointwise optimal stopping rules for the detection of influenza epidemics", in C. Gatsonis, R. E. Kass, A. Carriquiry, A. Gelman, D. Higdon, D. K. Pauler and I. Verdinelli, (eds), *Case Studies of Bayesian Statistics*, Volume 6 (New York: Springer-Verlag).

Baron, M., M. Rosenberg and N. Sidorenko, 2001, "Modelling and Prediction with Automatic Spike Detection", *Energy & Power Risk Management*, October, pp. 36–9.

Baron, M., M. Rosenberg and N. Sidorenko, 2002, "Divide and Conquer: Forecasting Power via Automatic Price Regime Separation", *Energy & Power Risk Management*, March, pp. 70–3.

Basseville, M., and I. V. Nikiforov, 1993, *Detection of Abrupt Changes: Theory and Application* (PTR Prentice-Hall, Inc).

Bridges, C. B., A. G. Winquist, K. Fukuda, N. J. Cox, J. A. Singleton, R. A. Strikas, 2000, "Prevention and Control of Influenza", *Morbidity and Mortality Weekly Reports*, 49, pp. 1–38 (Centers for Disease Control and Prevention).

Bridgman, P. W., 1931, *Dimensional Analysis* (New Haven: Yale University Press).

Brockwell, P. J., and R. A. Davis, 1991, *Time Series: Theory and Methods* (New York: Springer-Verlag).

Buckingham, E., 1914, "On Physically Similar Systems; Illustration of the Use of Dimension Equations", *Physical Review*, 4, pp. 345–76.

Calvin, W. H., 2002, *A Brain for All Seasons: Human Evolution and Abrupt Climate Change* (University of Chicago Press).

Clewlow, L., and C. Strickland, 1999, *Energy Derivatives: Pricing and Risk Management* (Lacima Publications).

Cloth, P., and R. Backofen, 2000, *Computational Molecular Biology* (Chichester: Wiley).

Dahlquist, G., and A. Bjork, 1974, *Numerical Methods*, translated by N. Anderson, (Englewood Cliffs, NJ: Prentice-Hall).

Durbin, S., S. Eddy, A. Krogh, and G. Mitchison, 1998, *Biological sequence analysis: Probabilistic models of proteins and nucleic acids* (Cambridge MA: Cambridge University Press).

Granott, N., and J. Parziale, 2002, "Microdevelopment: A process-oriented perspective for studying development and learning", in N. Granott and J. Parziale (eds), *Microdevelopment: Transition processes in development and learning*, pp. 1–28 (Cambridge University Press).

Hull, J. C., 2002, *Options, Futures, and Other Derivatives*, Fifth Edition, (Upper Saddle River, NJ: Prentice-Hall).

Judd, K. L., 1998, *Numerical Methods in Economics*, (Cambridge, MA: MIT Press).

Kamrad, B., and P. Ritchken, 1991, "Multinomial Approximating Models for Options with *k* State Variables", *Management Science*, 37(12), pp. 1640–52.

Kloeden, P. E., and E. Platen, 1994, *Numerical Solution of SDEs Through Computer Experiments* (New York: Springer-Verlag).

Lai, T. L., 1995, "Sequential changepoint detection in quality control and dynamical systems", *Journal of Royal Statistical Society B*, 57, pp. 613–58.

Lai, T. L., 2001, "Sequential analysis: some classical problems and new challenges", *Statistica Sinica*, 11, pp. 303–408.

Margrabe, W., 1978, "The Value of an Option to Exchange One Asset for Another", *Journal of Finance*, 33, pp. 177–86.

Montgomery, D. C., 1997, *Introduction to Statistical Quality Control*, Third Edition, (New York: Wiley).

Piaget, J., 1970, "Piaget's theory", in P. H. Mussen, *Carmichael's manual of child psychology*, pp. 703–32 (New York: Wiley).

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, 1992, *Numerical Recipes in C*, Second Edition, (Cambridge University Press).

Rabiner, L. R., 1989, "A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition", *IEEE Proceedings*, 77, pp. 257–85.

Stoer, J., and R. Bulirsch, 1980, *Introduction to Numerical Analysis* (Berlin: Springer-Verlag).

Wilson, Jr., E. B., 1990, *An Introduction to Scientific Research* (New York: Dover Publications).