DISTRIBUTION OF THE NUMBER OF VISITS OF A RANDOM WALK

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ABSTRACT

The distribution of the number of visits to a given state within an excursion of a simple random walk is derived. This distribution is shown to be a zero-modified geometric law, which extends a result of Revesz.

Key words: Excursion, Random walk, Testing randomness, Zero-modified geometric distribution.

1 The distribution of the number of visits for a random walk

Consider a simple random walk $S_k = X_1 + \cdots + X_k$, where X_i are independent random variables taking values $+1$ or -1 with probabilities p and $q = 1 - p$ respectively. With $S_0 = 0$, let $\rho_1 < \rho_2 < \ldots$ be the times when this random walk returns to the origin, i.e. $\rho_1 = \min\{k, k > 0, S_k = 0\}, \rho_2 = \min\{k, k > \rho_1, S_k = 0\}, \ldots$. Associate with this random walk a sequence of excursions

$$(S_0, \ldots, S_{\rho_1}), (S_{\rho_1}, \ldots, S_{\rho_2}), \ldots$$

When $p = 1/2$, any ρ_j is finite almost surely. If $p \neq 1/2$, $\rho = \rho_1 = \infty$ with the probability $|q - p|$.

Indeed, let $p < q$ and $M = \sup\{S_k,\ k \geq 0\}$. Then M is finite a.s. and it has a geometric distribution with the success probability p/q. Thus

$$P(\rho = \infty) = P(S_1 = -1)P(M = 0) = q(1 - p/q) = q - p.$$

Let $\xi(x)$ be the number of visits to $x, x \neq 0$, during one fixed, say, the first excursion. The distribution of $\xi(x)$ is obtained below.

Theorem 1 *For* $p \neq 1/2$

$$P\left(\xi(x)=0\right) = 1 - \frac{|p-q|}{\left|1-\left(\frac{q}{p}\right)^x\right|} \tag{1}$$

and for $k = 1, 2, \ldots$

$$P\left(\xi(x)=k\right)$$
$$= \begin{cases} \frac{|p-q|^2}{\left|1-\left(\frac{q}{p}\right)^x\right|^2}\left[1-\frac{|p-q|}{1-\left(\frac{q}{p}\right)^x}\right]^{k-1}, & p>q, x>0 \ \ or\ p<q, x<0, \\ \frac{|p-q|^2}{\left|1-\left(\frac{q}{p}\right)^x\right|\left|1-\left(\frac{p}{q}\right)^x\right|}\left[1-\frac{|p-q|}{1-\left(\frac{p}{q}\right)^x}\right]^{k-1}, & p>q, x<0 \ \ or\ p<q, x>0. \end{cases} \tag{2}$$

When $p = 1/2$

$$P\left(\xi(x)=0\right) = 1 - \frac{1}{2|x|}, \tag{3}$$

and for $k \geq 1$

$$P\left(\xi(x)=k\right) = \frac{1}{4x^2}\left(1-\frac{1}{2|x|}\right)^{k-1}. \tag{4}$$

Proof Observe that in a finite excursion, $\xi(x) = k$ with $k \geq 1$, if and only if the random walk S_k reaches the level x, then $k-1$ times, before it finally returns to 0, visits the state x. Then the process $S_k - x$ does not visit the state $-x$ during its first $k-1$ excursions. Therefore, by independence of these excursions,

$$P\left(\xi(x)=k;\ \rho<\infty\right)$$
$$= P\left(\xi(x)>0\right)\left[P\left(\xi(-x)=0;\ \rho<\infty\right)\right]^{k-1} P\left(\xi(-x)>0\right). \tag{5}$$

Assume first that $x > 0$. Then

$$P\left(\xi(x)>0\right) = P\left(S_1=1\right) P\left(\ x-1 \text{ is reached before } -1\right).$$

The probability that the state $x-1$ is visited before -1 represents the probability of winning in the classical ruin problem when the initial capital is 1 and the total capital is x. It follows from [1] Section XIV.2, (2.5) that

$$P\left(\xi(x)>0\right) = p\frac{\frac{q}{p}-1}{\left(\frac{q}{p}\right)^x-1} = \frac{q-p}{\left(\frac{q}{p}\right)^x-1}.$$

Replacing p by q, one obtains

$$P\left(\xi(-x) > 0\right) = \frac{p-q}{\left(\frac{p}{q}\right)^x - 1},$$

so that

$$P\left(\xi(x) > 0\right) = \left|\frac{p-q}{\left(\frac{q}{p}\right)^x - 1}\right|$$

for all x.

Let $I = 0$ if $p > q$, $x < 0$ or $p < q$, $x > 0$; otherwise $I = 1$. Then

$$P(\xi(x) = 0;\ \rho = \infty) = (1-I)P(\rho = \infty) = (1-I)|p-q|.$$

Hence, from (1),

$$P(\xi(x) = 0;\ \rho < \infty) = 1 - \left|\frac{p-q}{(q/p)^x - 1}\right| - (1-I)|p-q|. \tag{6}$$

Using this in (5), one obtains

$$P\left(\xi(x) = k;\ \rho < \infty\right) = (pq)^x \left(\frac{p-q}{p^x - q^x}\right)^2 \left[1 - \left|\frac{p-q}{(p/q)^x - 1}\right| - I|p-q|\right]^{k-1}.$$

In an infinite excursion, $\xi(x) = k > 0$ if and only if the random walk S_k visits the state x, then $k-1$ times returns to it, but not to the origin, without visiting x and 0 afterwards. This is possible only if $x > 0$, $p > q$ or $x < 0$, $p < q$; otherwise the walk has to visit the origin again. Hence, $P\left(\xi(x) = k;\ \rho = \infty\right) = 0$ if $I = 0$. Then, making use of (6) with $-x$ in place of x, one obtains

$$\begin{aligned} P(\xi(x) = k;\ \rho = \infty) &= IP(\xi(x) > 0)\left(P(\xi(-x) = 0;\ \rho < \infty)\right)^{k-1} P(\rho = \infty) \\ &= \frac{I(p-q)^2}{1-(q/p)^x}\left(1 - \left|\frac{p-q}{(p/q)^x - 1}\right| - |p-q|\right)^{k-1}. \end{aligned}$$

By adding the last two probabilities, one derives (2).

The probabilities (3) and (4) can be derived by taking limit for $p \to 1/2$ in (1) and (2). $\square$

When $p = 1/2$, the probability distribution of the number of visits is known (see [7] Theorem 9.7, p 96). However, our proof is different.

According to this Proposition $\xi(x) = 0$ with probability p_0 in (1). Given that $\xi(x) > 0$, $\xi(x)$ has a geometric distribution with the parameter

$$\pi = \begin{cases} \frac{|p-q|}{1-\left(\frac{q}{p}\right)^x} & p > q, x > 0 \text{ or } p < q, x < 0 \\ \frac{|p-q|}{1-\left(\frac{p}{q}\right)^x} & \text{otherwise .} \end{cases}$$

Thus $\xi(x)$ has a zero-modified geometric distribution (see [3] p 312) with probabilities

$$P(\xi(x) = 0) = p_0 \tag{7}$$

and for $k \geq 1$

$$P(\xi(x) = k) = (1-p_0)\pi(1-\pi)^{k-1}. \tag{8}$$

When $I = 1, 1 - p_0 = \pi$. For $p_0 = 0$, one obtains the geometric distribution.

The moment generating function of $\xi(x)$ has the form

$$\phi(t) = Ee^{t\xi(x)} = p_0 + \frac{e^t(1-p_0)\pi}{1-(1-\pi)e^t}. \tag{9}$$

Identity (9) shows that

$$E\xi(x) = \frac{1-p_0}{\pi},$$

and

$$Var(\xi(x)) = \frac{(1-p_0)(1-\pi+p_0)}{\pi^2}.$$

When $p = 1/2$, these formulas take the form

$$E\xi(x) = 1,$$

and

$$Var(\xi(x)) = 4|x| - 2.$$

Observe that for $a = 0, 1, 2, \ldots$

$$p(\xi(x) > a) = (1-p_0)(1-\pi)^a = \frac{P(\xi(x) = a+1)}{\pi},$$

which for $p = 1/2$ leads to a useful formula

$$P(\xi(x) > a) = \frac{1}{2|x|}\left(1 - \frac{1}{2|x|}\right)^a = 2|x|P(\xi(x) = a+1), \qquad a = 0, 1, 2, \ldots.$$

The distribution of the number of visits to a certain state within an excursion derived in Proposition 1.1 finds applications in testing randomness of a sequence of binary bits. The corresponding test is based on the comparison between the observed frequencies and the theoretical ones from (7) and (8) by the χ^2-statistic. As a matter of fact, in addition to some existing statistical procedures [2,4-6], this test belongs now to a battery of randomness tests at the Computer Security Division of the National Institute of Standards and Technology.

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