Grigori Perelman and the Saga of Poincaré Conjecture

by

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Beginning of the XXI century saw a major breakthrough in Mathematics. After eight years of hard theoretical work, Russian mathematician Grigori Perelman proved the famous *Poincaré conjecture*, thus solving a 100-year old problem that occupied great mathematical minds of the time and was named one of the seven Millennium Prize problems. The problem was deemed so difficult and important that a $1,000,000 prize was designated by Clay Mathematics Institute for its correct solution.

... Grisha Perelman grew up in St. Petersburg, the "Russian North capital", a city with deep and strong mathematical traditions. Leonard Euler, Christian Goldbach, Daniel and Nicolas Bernoulli worked in St. Petersburg in the XVIII century, invited by the Russian Academy of Sciences, which was inspired and advised by Gottfried Wilhelm Leibniz. Pafnuty Chebyshev, Sofia Kovalevskaya, and even young Georg Cantor lived and worked in the city in the XIX century, followed by Andrey Markov, Alexander Lyapunov, Lev Landau, Leonid Kantorovich, and many others.

Great mathematicians are not born overnight. They grow.

What was the mathematics world of St. Petersburg at the time when Grisha Perelman made his first steps in it? Yuri Matiyasevich has just completed solution of the 10-th Hilbert problem[1] [Mat]. Nikolay Nikolskii and his group were carefully verifying and simplifying Louis de Branges' proof of the famous Bieberbach conjecture [Bra]. Renowned mathematicians were working in St. Petersburg State University, Steklov Mathematical Institute, and Herzen Mathematical institute (where Isidor Natanson and other Jewish mathematicians landed after being denied a position at the University).

Hundreds of St. Petersburg children were going crazy about mathematics, reading mathematical books and solving problems during their after-school free time, the time that could have otherwise be used for soccer, skiing,

[1] Matiyasevich actually proved that the famous Tenth Problem is unsolvable.

movies, and parties. In different parts of the big city there were mathematical schools, clubs, and circles, ready to accept talented, enthusiastic, and just interested girls and boys into the attractive World of Mathematics. Grigori was not yet ten when he successfully participated in a local mathematical competition. This caused his mother, Lubov, teacher of mathematics at a commerce school, to recommend Grigori to mathematical circles.

Misha B. first met Grisha Perelman at the mathematical circle that which they attended twice a week in the Anichkov Palace built originally for Russian Empress Elizabeth. It was the Fall of 1980. The first acquaintance was rather remarkable.

... We usually came to the mathematical circle to learn some theory or to solve tricky problems, but once in a while we had *mathematical battles*, says Misha. Two teams had to challenge each other on mathematical problems and vigorously debate their solutions. Once we were set for the next math fight, when we discovered that the number of participants was odd, so we could not form two equal teams. We looked around, and there was Grisha Perelman, quietly sitting in the corner immersed, as usually, in his difficult math problems. "Let us invite Grisha to one of the teams", somebody proposed. "How about Grisha *being one of the teams*", replied our teacher S. E. Rukshin at once.

Sergey Rukshin, our teacher and leader, devoted his whole career to the creation of math circles and mathematical Olympic movement. His charm and enthusiasm attracted all of us to the Anichkov Palace. Every year, he leads the St. Petersburg team to the National Mathematical Olympiad, where his students always win high awards.

Very soon we realized that no matter whether Grisha joins one of the teams or becomes one of the teams, the mathematical battle against Grisha would be doomed.

Grisha Perelman, three years senior to us, also taught us from time to time. His first group of students included future Russian Olympiad winners Fedor N. and Anna B.

A typical lesson with Grisha was rather peculiar. Grisha was sitting and solving *his* problems while we were hopelessly solving *our* problems generously given to us by Grisha. We called these problems "coffins", because only once in a while someone could have any advancement in any of them. We even tried to collaborate on them. Sooner or later, our entire group gave up on all the Grisha's coffins, but trying to squeeze new problems out of

Perelman was to no purpose. Certainly, Grisha knew many more problems to entertain us. However, as we realize now, he strongly believed that mathematicians should not give up on a problem and should try to find the solution by all means.

Mathematical passion continued in summer. Every year, most kids from our mathematical circles went to a math summer camp to dive into Mathematics for the whole month. There, some two hours from the city, in cute barracks surrounded by the pine forest, we had mathematics classes from breakfast until lunch and then again from lunch until dinner. Only for the last couple of hours before bedtime, we had a choice between table tennis, books, classical music... and more math. In fact, very few chose math in the evening. Most, like me, chose table tennis. Do you think Grisha went for math? No, apparently he was fond of classical music concerts that consisted of recordings proposed and brought by S. E. Rukshin. Later I found out that Grisha played violin, and played rather well.

At the camp, Grisha taught our group a course. It seemed rather interesting, and sometimes some of us even had an illusion of understanding something... it was abstract algebra... I guess. At the end, there was an oral exam, where I got a D. That was the average grade. Timur O., for example, got an F whereas Sasha T. overscored us with a proud C.

After the 8th grade, almost everyone in our group left their schools and landed together in High School #239. Three years before us, the group of Grisha Perelman did the same. This school was very special. Easily the best school of the city, it was officially specialized in mathematical subjects, however, all the main disciplines were taught at the very high level. Gifted children from all over the city tried to win the competition and enter the school for two years. For some of them, it meant traveling more than an hour each way every day, six days a week.

When we were students of HS 239, it was preparing to celebrate its 250-th jubilee! The school alumni included known scientists, politicians, musicians, and actors. Almost every year, 239-ers were among the winners of International Mathematical Olympiad. Their portraits are in the school's Hall of Fame, and one of them, of course, is Grisha Perelman.

In 1981 Perelman took a second prize in the All Soviet Mathematical Olympiad. He was not happy with the result and worked very hard not to be second the next year. In 1982 he won the all Soviet Olympiad and

earned a perfect score and the gold medal at the International Mathematical Olympiad in Budapest.

Grisha studied in HS 239 from 1980 until 1982. The 10th grade was the highest grade in Russia at that time, after which 239-ers went straight to the entrance examinations to the nation's best universities.

The renowned University of St. Petersburg was the first choice for many. It is famous for eight Nobel Prize winners among its alumni, the Periodic Table of Elements discovered by Dmitri Mendeleev, the first radio receiver built by Alexander Popov, and many other scientific achievements.

As the absolute winner of the International Mathematical Olympiad, Grisha Perelman was admitted without any exams. In fact, the Olympiad was held concurrently with the entrance exams, and its participants were always admitted without a contest. Until the beginning of Russian Perestroika in 1985, this was practically the only way a Jewish youngster like Grisha could get admission to the leading university.

In the late 1980s, Perelman went on to earn a Candidate of Science degree (the Soviet equivalent to the Ph.D.). His dissertation was titled "Saddle surfaces in Euclidean spaces."

After graduation, he worked at the Steklov Institute of Mathematics of the USSR Academy of Sciences, where his advisors were Aleksandr Aleksandrov and Yuri Burago. Burago recommended Perelman to Mikhail Gromov, and Grigori was invited to spend a few months in the institute IHES in suburbs of Paris, after defending his dissertation in 1990. In 1991, Gromov arranged an invitation for a Geometry festival at Duke University. After his talk at the festival, Grigori gave presentations at several American Universities, including the University of Pennsylvania.

In 1992, Perelman was invited to spend a semester each at the Courant Institute in New York University and the State University of New York at Stony Brook. From there, he accepted a two-year Miller Research Fellowship at the University of California, Berkeley, in 1993. He was offered jobs at several top universities in the US, including Princeton and Stanford, but he rejected them all and returned to the Steklov Institute in the summer of 1995. At Stony Brook, Perelman met young Chinese mathematician Gang Tian and they were traveling together to seminars at Princeton (IAS). There Grigori met American mathematician Richard Hamilton for the first time.

In 1990s, Perelman was best known for his work in comparison theorems

in Riemannian geometry. Among his notable achievements was a short and elegant proof of the 20-year-old Soul conjecture of Cheeger and Gromoll (due to this result, he was invited to give a talk at the International Congress of Mathematicians in Zurich in 1994.)

In 1996, when Grigori was already back in Leningrad, he was awarded the prestigious prize of the European Mathematical Society. He refused accepting the prize saying that the work he is honored for is not yet finished and the committee is anyway unable to evaluate his work. Possibly Grigori was already seriously working on Poincare and Geometrization conjectures and didn't want any distraction.

In the next 6 years there are two facts worth of mentioning: mathematical and personal. In February 2000, Mike Anderson, a mathematician from Stony Brook who new Perelman well and who worked on geometrization conjecture, got an e-mail from Perelman asking about one of his papers or rather a mistake in it. The short exchange of e-mails stopped when Grigori wrote that he could not open a file send by Anderson, that his sister Elena used to help him with this but now she was studying for Ph.D. in Mathematics in Israel, and that he would not trust his colleagues from Steklov to open files for him.

Around the same time, Perelman left Burago's group; the dispute was over Grigori paper with the younger colleague A. Petrunin. Grigori, always a perfectionist, was so unhappy with some references in their joint paper that, the story goes, he was against hiring Petrunin and asked Burago to do the same. When Burago refused, Perelman left his laboratory for that of senior mathematician Olga Ladyzhenskaya.

In November 2002, Perelman posted the first of a series of e-prints to the arXiv[2], in which he claimed to have outlined a proof of the geometrization conjecture, of which the Poincar conjecture is a particular case. Immediately Perelman's papers attracted great attention from the mathematical community. In April 2003, he accepted an invitation to visit Massachusetts Institute of Technology, Princeton University, State University of New York at Stony Brook, Columbia University, and New York University, where he gave a series of talks on his work.

[2]It is an an archive for electronic preprints of scientific papers in the fields of mathematics, physics, computer science, quantitative biology and statistics which can be accessed via the world wide web. In many fields of mathematics and physics (including topology), almost all scientific papers are placed on the arXiv.

In May 2006, a committee of nine mathematicians voted to award Perelman a Fields Medal for his work on the Poincar conjecture. The Fields Medal is the highest award in mathematics; two to four medals are awarded every four years.

On August 22, 2006, Perelman was publicly offered the medal at the International Congress of Mathematicians in Madrid, "for his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow". He did not attend the ceremony and declined to accept the medal, making him the first person to decline this prestigious prize.

On March 18, 2010, eight years after the proof of Poincare conjecture appeared on ArXiv, CMI[3] announced that Dr. Grigoriy Perelman of St. Petersburg, Russia, is the recipient of the Millennium Prize for resolution of the Poincar conjecture. Despite another media speculation, there was no official response from Dr. Perelman on the acceptance of this $1 mln. prize. In fact he announced that the Clay Institute will be the first to know his decision.

Perelman proved the Poincare Conjecture (1904) and its broad generalization, the Geometrization Conjecture of Thurston (1978). Perelman's proof has fundamentally altered two distinct branches of mathematics. First, it solved a problem that for more than a century was the indigestible seed at the core of topology, the mathematical study of abstract shape. The broader result, a proof of the geometrization conjecture, is essentially a "periodic table" that brings clarity to the study of three-dimensional manifolds, much as Mendeleev's table did for chemistry.

While bringing new results to topology, Perelman's work brought new techniques to geometry. It cemented the central role of geometric evolution equations, powerful machinery for transforming hard-to-work-with spaces into more-manageable ones. Earlier studies of such equations (e.g. by Richard Hamilton) always ran into "singularities" at which the equations break down. Perelman dynamited that roadblock.

Henri Poincarè, who posed his problem in 1904, is generally regarded

[3]The Poincarè conjecture is one of the seven Millennium Prize Problems established by The Clay Mathematics Institute (CMI) in 2000. CMI was founded in September, 1998 by Mr. Landon T. Clay, a Boston businessman, and his wife, Lavinia D. Clay. It is dedicated to increasing and disseminating mathematical knowledge.

as the founder of algebraic topology. W. Thurston got a Fields medal for his work in 1982 (official ceremony was delayed to 1983 because of martial law in Poland: several mathematicians were arrested, interned or in hiding, including Roman Duda who was working on Poincaré Conjecture).

Currently...

Grigori Perelman, one of the most famous mathematicians of the world, lives modestly in St. Petersburg, declining multiple interviews and invitations and trying to avoid the storm of intrusive journalists and photographers by all means.

Apparently, not everyone wants to be a celebrity. "Money, glory don't interest me. I do have everything I need. I am not a hero of mathematics, I am only a mathematician, and not even a successful one", says Perelman.

Very few people know where he works and what challenges occupy his mind now. Any statements about his private life in a multitude of publications are most likely to be random unsupported guesses and speculations.

School # 239 continues its beautiful scientific and educational traditions and remains a great source of talented students in Mathematics and other fields.

S. E. Rukshin, now in his early 50s, taught in # 239 for a while but have never left his main mission of attracting and training young mathematicians. He still organizes mathematical circles, summer camps, and Olympiads.

Aleksandr Aleksandrov, the most influential advisor of Perelman and once President of St. Petersburg University, passed away in 1999 at the age of 86. His other advisor Dmitri Burago is Professor of Geometry at St. Petersburg University. After Perelman, he taught Riemannian Geometry to all the abovementioned students. Richard Hamilton is Professor of mathematics at Columbia University.

Perelman's students... Timur Oikhberg is now Associate Professor of Mathematics at the University of California, Irvine. Sasha Teplyaev is Associate Professor of Mathematics at the University of Connecticut, Storrs. Anton Petrunin is Associate Professor of Mathematics at Pennsylvania State University. Anna Bogomolnaia is Associate Professor of Economics at Rice University. Fedor Nazarov, a Salem prize winner, is Professor of Mathematics at the University of Wisconsin, Madison. Misha B., or Michael Baron, is (proudly) Professor of Statistics at UT-Dallas.

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