On the first passage time for queueing processes

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During the last 40 years queueing processes were studied in different fields of Probability and Statistics. D.Lindley ([8]), D.Kendall ([3]) and A.Borovkov ([1]) considered them in the theory of queues, E.Page ([10]) and R.Khan ([4] and [5]) in the cumulative sum procedure, N.Prabhu ([11]) in storage theory. In a number of papers for various reasons a quantity of interest was an epoch when a process hits some given level N given beforehand for the first time.

In this paper an asymptotics of the expected value of the first passage time as $N \to \infty$ is derived. Although in the situation when time is discrete the results of the first section essentially follow from the work of V.Lotov (see [9]), they can be obtained alternatively, by correcting the proof of V.Labkovskii ([7]).

In the second part we define a queueing process with continuous time, generated by a stochastically continuous random process with independent increments. A similar result for the first passage time happens to be valid also in this case.

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§**1.** Let $X = \{X_t,\ t \in I\!N\}$ be a sequence of iid random variables with a mean $\mathbf{E}\,X_1$, where $-\infty \le \mathbf{E}\,X_1 < 0$. A sequence $\{W_t, t \in I\!N_0\}$ defined by the equalities

$$W_0 = 0; \qquad W_{t+1} = \max\{0;\ W_t + X_{t+1}\} \tag{1}$$

is called *a queueing process* generated by the sequence X_t.

For every positive N consider $\tau^X(N) = \min\{t : W_t \ge N\}$ - level N first passage time for the process W_t.

The following notations will be used:

$\phi^X(z) = \mathbf{E}\, z^{X_1}$; $\phi(+\infty) = +\infty$ - a common moment generating function of random variables X_t;

R^X - an upper bound of the interval where $\phi(z)$ is finite;

$$\gamma^X = \begin{cases} R^X, & \text{if} \quad \phi^X(R^X) \leq 1; \\ \theta^X, & \text{if} \quad 1 < \phi^X(R^X) \leq +\infty, \end{cases} \tag{2}$$

where θ^X is the only root of the equation $\phi(z) = 1$, satisfying the condition $1 < \theta^X < R^X$.

In future references, the index X will be omitted for quantities related to the initial sequence. When considering a family of sequences Y_t^j or Z_t^j, we shall write the index j as a subscript: R_j^Y, $\phi_j^Y(z)$.

The following result was formulated in [6] for integer X_t :

Theorem 1.

There exists a limit $\lim\limits_{N\to\infty} \sqrt[N]{\mathbf{E}\,\tau(N)} = \gamma$, *where* $\gamma = \gamma^X$ *is defined by (2).*

However, only the inequality $\lim\limits_{N\to\infty} \inf \sqrt[N]{\mathbf{E}\,\tau(N)} \geq \gamma$ was proved correctly. The rest of the proof was given in [7], but the following additional assumption was made there for the case when $\phi(R) \leq 1$: *among all increasing sequences* N_j, *satisfying the condition* $\lim\limits_{j\to\infty} \mathbf{P}^{1/N_j}\{X_1 = N_j\} = R^{-1}$, *there exists such a sequence that* $\lim\limits_{j\to\infty} N_j/N_{j-1} = 1$.

Actually, the theorem is valid in its initial form and not only for integer but any real-valued sequence X_t.

Proof of Theorem 1:

1). At first assume that X_t are integers. To complete the proofs given in [6] and [7] it is enough to show that $\lim\limits_{N\to\infty} \sup \sqrt[N]{\mathbf{E}\,\tau(N)} \leq \gamma$ when $\phi(R) \leq 1$.

Consider a family of sequences of random variables $\{Y_t^{(j)},\ t \in I\!N\}_{j\in I\!N}$:

$$Y_t^{(j)} = \begin{cases} X_t & \text{if} \quad X_t \leq j; \\ 0 & \text{if} \quad X_t > j. \end{cases}$$

For any j and t one has $Y_t^{(j)} \leq X_t$ almost surely, hence, $\tau_j^Y(N) \geq \tau(N)$. Also, $R_j^Y = +\infty$ for any j because

$$\phi_j^Y(z) = \sum_{n=-\infty}^{j} z^n \mathbf{P}\{X_1 = n\} + \mathbf{P}\{X_1 > j\}, \tag{3}$$

and $\phi_j^Y(R_j^Y) = +\infty$. Thus, the results of [7] are applicable to $Y_t^{(j)}$ and according to them $\lim_{N\to\infty} \sup \sqrt[N]{\mathbf{E}\,\tau_j^Y(N)} \leq \theta_j^Y$, which yields to

$$\lim_{N\to\infty} \sup \sqrt[N]{\mathbf{E}\,\tau(N)} \leq \theta_j \quad \text{ for any } j. \tag{4}$$

Then, since $Y_t^{(j)} \leq Y_t^{(j+1)}$ for any j with probability one, $\phi_j^Y(z) \leq \phi_{j+1}^Y(z)$ for any $z \geq 1$, which immediately implies $\theta_j^Y \geq \theta_{j+1}^Y$. But $\theta_j^Y \geq 1$ for all j, therefore there exists a limit $\lim_{j\to\infty} \theta_j^Y = \theta_\infty$.

If $\theta_\infty < R$, then $\theta_j^Y < R$ for sufficiently large j, and this is a contradiction: $1 = \phi_j^Y(\theta_j^Y) \leq \phi(\theta_j^Y) < 1$, because from $\phi(R) < 1$ one has $\phi(z) < 1$ for all $z \in (1; R]$.

The inequality $\theta_\infty > R$ is also impossible, because $\theta_\infty \leq \theta_j^Y$ and $\phi_j^Y(\theta_j^Y) = 1$ imply that $\phi_j^Y(\theta_\infty) \leq 1$ for all j. Taking limit in (3) as $j \to \infty$ with $z = \theta_\infty$, one obtains:

$$\sum_{n=-\infty}^{\infty} \theta_\infty^n \,\mathbf{P}\{X_1 = n\} \leq 1.$$

Hence, θ_∞ belongs to the interval of convergence of the series which defines $\phi(z)$. But it means that $\theta_\infty \leq R$, and one has a contradiction again. Therefore, $\theta_\infty = R$, and taking limit in (4) for $j \to \infty$ one has the required inequality.

2). Let X_t be dyadic rational numbers, that is, $2^n X_t$ assumes only integer values for some n. Then $\tau^X(N) = \tau^{2^n X}(2^n N)$ with probability one, and according to [6], [7] and the first part of the proof,

$$\sqrt[N]{\mathbf{E}\,\tau(N)} = \sqrt[N]{\mathbf{E}\,\tau^{2^n X}(2^n N)} = \left(\sqrt[2^n N]{\mathbf{E}\,\tau^{2^n X}(2^n N)} \right)^{2^n} \underset{n\to\infty}{\longrightarrow} \left(\gamma^{2^n X}\right)^{2^n}.$$

Clearly, $\phi^{2^n X}(z) = \mathbf{E}\, z^{2^n X_1} = \phi^X(z^{2^n})$ yields to $\gamma^{2^n X} = (\gamma^X)^{2^{-n}}$, from where $\sqrt[N]{\mathbf{E}\,\tau(N)} \underset{N\to\infty}{\longrightarrow} \gamma$.

3). Finally, let us consider the general case of arbitrary real-valued X_t. For all integer n define random variables

$$Y_t^{(n)} = \frac{[2^n X_t]}{2^n} \quad \text{and} \quad Z_t^{(n)} = \begin{cases} X_t, & \text{if} \quad 2^n X_t \in \mathbb{Z}; \\ \dfrac{[2^n X_t]+1}{2^n}, & \text{if} \quad 2^n X_t \notin \mathbb{Z}. \end{cases},$$

where $[x]$ denotes the integer part of x. Notice that always $\mathbf{E}\, Y_t^{(n)} < 0$, because $\mathbf{E}\, X_t < 0$, and for sufficiently large n $\mathbf{E}\, Z_t^{(n)} \le \mathbf{E}\, X_t + 1/2^n < 0$. Hence, according to the previous result for dyadic rationals,

$$\sqrt[N]{\mathbf{E}\,\tau_n^Y(N)} \longrightarrow \gamma_n^Y \quad \text{and} \quad \sqrt[N]{\mathbf{E}\,\tau_n^Z(N)} \longrightarrow \gamma_n^Z \quad \text{for} \quad N\to\infty$$

Also, with probability one $Y_t^{(n)} \le X_t \le Z_t^{(n)}$ for all t and n, which yields to

$$\tau_n^Z(N) \le \tau(N) \le \tau_n^Y(N)\,.$$

Let us show that

$$\gamma_n^Y \to \gamma \quad \text{and} \quad \gamma_n^Z \to \gamma \ \text{ as } n\to\infty\,, \tag{5}$$

then the existence of $\lim\limits_{N\to\infty} \sqrt[N]{\mathbf{E}\,\tau(N)} = \gamma$ will follow.

It is clear that:

i) $R_n^Y = R_n^Z = R$ for all n;

ii) $\phi_n^Y(z)$ is non-decreasing in n for any $z \in [1;R]$ and tends to $\phi(z)$ as $n\to\infty$;

iii) $\phi_n^Z(z)$ is non-increasing in n for any $z \in [1;R]$ and also tends to $\phi(z)$ as $n\to\infty$.

Let us examine three cases.

A). $\phi(R) < 1$. Then $\phi_n^Y(R) < 1$ and for large n $\phi_n^Z(R) < 1$. Hence, $\gamma_n^Y = R_n^Y = R = R_n^Z = \gamma_n^Z$, and (5) holds.

B). $\phi(R) > 1$. Then $\phi_n^Z(R) > 1$ and for large n $\phi_n^Y(R) > 1$, from where $\gamma_n^Y = \theta_n^Y$ and $\gamma_n^Z = \theta_n^Z$. We use (ii) and (iii) to conclude that θ_n^Y is non-increasing and θ_n^Z is non-decreasing in n. Also, $\theta_n^Z \leq \theta \leq \theta_n^Y$. Hence, there exist $\lim\limits_{n\to\infty} \theta_n^Y = \theta_\infty^Y \geq \theta$ and $\lim\limits_{n\to\infty} \theta_n^Z = \theta_\infty^Z \leq \theta$. By the definition of θ, $\phi_n^Y(\theta_n^Y) = \phi_n^Z(\theta_n^Z) = 1$, therefore for any n $\phi_n^Y(\theta_\infty^Y) \leq 1$ and $\phi_n^Z(\theta_\infty^Z) \geq 1$. Taking the limit as $n \to \infty$, one has: $\phi(\theta_\infty^Y) \leq 1$ and $\phi(\theta_\infty^Z) \geq 1$.

On the other hand, $\theta_\infty^Z \leq \theta \leq \theta_\infty^Y$ and $\phi(\theta) = 1$, therefore, $\phi(\theta_\infty^Y) \geq 1$ and $\phi(\theta_\infty^Z) \leq 1$. This is possible only if $\theta_\infty^Z = \theta_\infty^Y = \theta$.

C). $\phi(R) = 1$. Then $\gamma = R$, $\gamma_n^Y = R_n^Y = R$ and $\gamma_n^Z = \theta_n^Z$. Let us prove that $\theta_n^Z \to R$ as $n \to \infty$. Similarly to the previous case, there exists $\lim\limits_{n\to\infty} \theta_n^Z = \theta_\infty^Z \leq R$ and $\phi(\theta_\infty^Z) \geq 1$. The equality $\phi(R) = 1$ implies that $\phi(z) < 1$ for all $1 < z < R$. Hence, $\theta_\infty^Z = R$, and this completes the proof.

□

§**2.** Similar results can be obtained for the continuous time processes. Of course, in this situation a new definition for the queueing process is required.

Let $\{S_t,\ t \in I\!\!R_+\}$ be a homogeneous stochastically continuous and right-cotinuous random process with independent increments, starting at the origin. Assume that there exists an expected value $-\infty \leq \mathbf{E}\, S_1 < 0$. We call a process $\{W_t,\ t \in I\!\!R_+\}$ *a queueing process* if

$$W_t = S_t - \inf_{[0;t]} S_u.$$

It is easy to see that in the discrete time case this definition agrees with (1) if one sets $X_t = S_t - S_{t-1}$.

Define $\tau(N) = \inf\{t:\ W_t \geq N\}$ and $\phi(z) = \mathbf{E}\, z^{S_1}$ and let r and R be respectively the lower and the upper bounds of the interval, where $\phi(z)$ is finite. The quantities γ and θ are defined similarly to (2).

Theorem 2.

There exists a limit $\lim\limits_{N\to\infty} \sqrt[N]{\mathbf{E}\,\tau(N)} = \gamma.$

Proof:

Let us consider a random process with discrete time $\{\bar{S}_t,\ t \in I\!N_0\}$ defined by the equality $\bar{S}_t = S_t$ for all $t \in I\!N_0$. Then one has for the corresponding queueing process $\bar{W}_t$:

$$\bar{W}_t = \bar{S}_t - \inf_{u \le t} \bar{S}_u = S_t - \inf\{S_u:\ u \le t,\ u \in I\!N_0\} \le W_t$$

almost surely for all $t \in I\!N_0$. Thus, if $\bar{\tau}(N) = \inf\{t :\ \bar{W}_t \ge N\}$ then $\bar{\tau}(N) \ge \tau(N)$.

The process $\bar{X}_t = \bar{S}_t - \bar{S}_{t-1}$ satisfies to all the conditions of Theorem 1, hence,

$$\sqrt[N]{\mathbf{E}\,\bar{\tau}(N)} \underset{N\to\infty}{\longrightarrow} \gamma, \tag{6}$$

from where $\limsup\limits_{N\to\infty} \sqrt[N]{\mathbf{E}\,\tau(N)} \le \lim\limits_{N\to\infty} \sqrt[N]{\mathbf{E}\,\bar{\tau}(N)} = \gamma.$

To complete the proof we show that $\liminf\limits_{N\to\infty} \sqrt[N]{\mathbf{E}\,\tau(N)} \ge \gamma.$

1). At first consider the case $r = 0$ which implies $\phi(z) < \infty$ for any $z \in (0;1]$. According to this assumption, $\phi(1/z) = \mathbf{E}\,z^{-S_1} < \infty$ for all $z \in [1;+\infty)$ and $\mathbf{E}\,z^{[-S_1]} = \sum\limits_{N=-\infty}^{\infty} z^n \mathbf{P}\{[S_1] = -N\} < \infty$ i.e. the radius of convergence of the last series is $+\infty$. Then, by Cauchy formula,

$$\limsup_{N\to\infty} \sqrt[N]{\mathbf{P}\{[S_1] = -N\}} = 0.$$

It is easy to show that for any sequence of non-negative numbers $a_1,\ a_2,\ \ldots,$ if $\sum\limits_{n=1}^{\infty} a_n < \infty$, then $\limsup\limits_{N\to\infty} \left(\sum\limits_{n=N}^{\infty} a_n\right)^{1/N} = \limsup\limits_{N\to\infty} \sqrt[N]{a_n}$. Using this fact, one has:

$$\begin{aligned} 0 &= \limsup_{N\to\infty} \sqrt[N]{\mathbf{P}\{[S_1] = -N\}} \\ &= \limsup_{N\to\infty} \sqrt[N]{\mathbf{P}\{[S_1] \le -N\}} = \limsup_{N\to\infty} \sqrt[N]{\mathbf{P}\{S_1 \le -N\}}. \end{aligned}$$

Hence, for any $c > 0 \quad \mathbf{P}\{S_1 \le -N\} = o(c^N)$ as $N \to \infty$. Define $p_N = \mathbf{P}\{\inf\limits_{[0;1]} S_t \le -N\}$, then, by Kolmogorov inequality,

$$p_N \le 2\mathbf{P}\{S_1 \le -N\} = o(c^N), \ N \to \infty. \tag{7}$$

For any fixed $\varepsilon > 0$ consider $t_N = [\tau(N + N\varepsilon)] + 1$. Let us prove that the probability of the event $A_N = \{t_N < \bar{\tau}(N)\}$ tends to zero as $N \to \infty$.

If A_N occurs then $\bar{W}_{t_N} < N$. On the other hand, the inequality $W_{\tau(N+N\varepsilon)} \ge N + N\varepsilon$ always holds, therefore,

$$\begin{aligned} A_N \quad \subset \quad & \{W_{\tau(N+N\varepsilon)} - \bar{W}_{t_N} > N\varepsilon\} = \\ & = \{(S_{\tau(N+N\varepsilon)} - \inf_{t \le \tau(N+N\varepsilon)} S_t) \quad - \quad (S_{t_N} - \inf_{\substack{t \le t_N \\ t \in \mathbb{N}_0}} S_t) > N\varepsilon\} \subset \\ & \subset (B_N \cup C_N), \end{aligned}$$

where events B_N and C_N are defined as follows:

$$\begin{aligned} B_N &= \{(S_{\tau(N+N\varepsilon)} - S_{t_N} \ge N\varepsilon/2\}; \\ C_N &= \{\inf_{\substack{t \le t_N \\ t \in \mathbb{N}_0}} S_t - \inf_{t \le \tau(N+N\varepsilon)} \ge N\varepsilon/2\}. \end{aligned}$$

The moment $\tau(N + N\varepsilon)$ is a Markov stopping time, hence,

$$S_{t+\tau(N+N\varepsilon)} - S_{\tau(N+N\varepsilon)} \stackrel{d}{=} S_t.$$

Then, since $t_N \in [\tau(N + N\varepsilon);\ \tau(N + N\varepsilon) + 1\,]$, one obtains the following estimate for the probability of B_N:

$$\mathbf{P}(B_N) \le \mathbf{P}\{\inf_{[0;1]} S_t \le -N\varepsilon/2\} = p_{N\varepsilon/2}. \tag{8}$$

As to the events C_N, one has $C_N = \bigcup\limits_{j=0}^{t_N} C_{jN}$ for $C_{jN} = \{S_j - \inf\limits_{[j;j+1]} S_t \ge N\varepsilon/2\}$. Also, since $S_{t+j} - S_j \stackrel{d}{=} S_t$ for any j,

$$\mathbf{P}(C_{jN}) = \mathbf{P}\{\inf_{[0;1]} S_t \le -N\varepsilon/2\} = p_{N\varepsilon/2}.$$

Thus $\mathbf{P}(C_N) \le \sum_{k=1}^{\infty} \mathbf{P}\{t_N = k\}(k+1)\mathbf{P}(C_{0N}) =$
$= \mathbf{E}(t_N+1)p_{N\varepsilon/2} \le (\mathbf{E}\,\tau(N+N\varepsilon)+2)p_{N\varepsilon/2} \le$
$\le (\mathbf{E}\,\bar\tau(N+N\varepsilon)+2)p_{N\varepsilon/2}.$

Combining this result with (8), one has:

$$\mathbf{P}(A_N) \le \mathbf{P}(B_N) + \mathbf{P}(C_N) \le (\mathbf{E}\,\bar\tau(N+N\varepsilon)+3)p_{N\varepsilon/2}. \tag{9}$$

According to (6), $\mathbf{E}\,\bar\tau(N+N\varepsilon) = o(2\gamma)^{N+N\varepsilon}$, $N \to \infty$, and using (9) and (7) with $c = (2\gamma)^{-2(1+\varepsilon)/\varepsilon}$, one obtains:

$$\mathbf{P}(A_N) \underset{N\to\infty}{\longrightarrow} 0. \tag{10}$$

Then,

$$\begin{aligned}
&\mathbf{E}\,\tau(N+N\varepsilon) \ge \mathbf{E}\,t_N - 1 = \\
&= \mathbf{E}\{t_N \mid A_N\}\mathbf{P}(A_N) + \mathbf{E}\{t_N \mid \bar A_N\}\mathbf{P}(\bar A_N) - 1 \ge \\
&\ge \mathbf{E}\{t_N \mid A_N\}\mathbf{P}(A_N) + \mathbf{E}\{\bar\tau(N) \mid \bar A_N\}\mathbf{P}(\bar A_N) - 1 = \\
&= \mathbf{E}\,\bar\tau(N) - \mathbf{E}\{\bar\tau(N) - t_N \mid A_N\}\mathbf{P}(A_n) - 1.
\end{aligned} \tag{11}$$

Let $\sigma_N = \min\{t_N;\, \bar\tau(N)\}$. This is a Markov stopping time, and the event A_N is an element of the corresponding σ-algebra $\mathcal{F}_{\sigma_N}$. Therefore (see [12], §4.1) the process $\bar S_t^{(\sigma)} = \bar S_{t+\sigma_N} - \bar S_{\sigma_N}$ is independent of A_N and $\bar S_t^{(\sigma)} \overset{d}{=} \bar S_t$.

Define a random process $\{\bar V_t,\ t \in I\!N_0\}$:

$$\bar V_0 = \bar W_{t_N}; \qquad \bar V_{t+1} = \max\{0;\ \bar V_t + \bar S_{t+1}^{(\sigma)} - \bar S_t^{(\sigma)}\}.$$

Then $\bar\tau(N) - t_N$ is a level N first passage time for this process.

If $\bar W_t^{(\sigma)}$ is a queueing process, generated by the process $S_t^{(\sigma)}$, and $\bar\tau^{(\sigma)}(N) = \inf\{t:\ \bar W_t^{(\sigma)} \ge N\}$, then $\bar W_t^{(\sigma)}$ is defined exactly like $\bar V_t$, with the only difference that $\bar W_0^{(\sigma)} = 0$ while $\bar V_0 = \bar W_{t_N} \ge 0$. Hence, $\bar W_t^{(\sigma)} \le \bar V_t$ with probability one and $\bar\tau(N) - t_N \le \bar\tau^{(\sigma)}(N) \overset{d}{=} \bar\tau(N)$. Also, $\bar\tau^{(\sigma)}(N)$ does not depend on A_N. Therefore,

$$\mathbf{E}\{\bar\tau(N) - t_N \mid A_N\} \le \mathbf{E}\{\bar\tau^{(\sigma)}(N)\} \mid A_N\} = \mathbf{E}\,\bar\tau^{(\sigma)}(N) = \mathbf{E}\,\bar\tau(N).$$

Applying this estimation to (11), one has:

$$\mathbf{E}\,\tau(N+N\varepsilon)\geq \mathbf{E}\,\bar{\tau}(N)\,(1-\mathbf{P}\,(A_N))-1,$$

and using (6) and (10):

$$\lim_{N\to\infty}\inf\sqrt[N]{\mathbf{E}\,\tau(N)}=(\lim_{N\to\infty}\inf\sqrt[N]{\mathbf{E}\,\tau(N+N\varepsilon)})^{1/(1+\varepsilon)}\geq\gamma^{1/(1+\varepsilon)},$$

Since ε was chosen arbitrarily, $\lim\limits_{N\to\infty}\inf\sqrt[N]{\mathbf{E}\,\tau(N)}\geq\gamma$.

2). Let us prove the theorem in its general form. The random process S_t admits the following decomposition formula:

$$S_t=\xi_t+\int_{|x|>1}x\nu_t(dx)+\int_{|x|\leq 1}x\tilde{\nu}_t(dx),$$

where ν_t and $\tilde{\nu}_t$ are random Poisson measures. (for details see [2], chapter VI).

For every integer k consider

$$S_t^{(k)}=\xi_t+\int_{|x|\leq 1}x\tilde{\nu}_t(dx)+\int_{-k}^{-1}x\nu_t(dx)+\int_{1}^{\infty}x\nu_t(dx).$$

Then $\quad S_t=S_t^{(k)}+\displaystyle\int_{-\infty}^{-k}x\nu_t(dx);$

$$S_t^{(k)}\geq S_t^{(k+1)}\geq\ldots\geq S_t \tag{12}$$

and $S_t^{(k)}\underset{k\to\infty}{\longrightarrow}S_t$ almost surely. For sufficiently large k $\mathbf{E}\,S_1^{(k)}<0$ because $\mathbf{E}\,S_1<0$. Let us show that for those k the process $S_t^{(k)}$ satisfies to the conditions of the first part of the proof, i.e. $\phi_k(z)=\mathbf{E}\,z^{S_1^{(k)}}<\infty$ for all $z\in(0;1]$.

Let $f_k(z)$ be a characteristic function of a random variable $S_1^{(k)}$. Then (see [2], VI, §4)

$$f_k(z)=\exp\left\{iaz-\frac{b}{2}z^2+\int_{-k}^{-1}(e^{izx}-1)\Pi_1(dx)\right.$$
$$\left.+\int_{1}^{\infty}(e^{izx}-1)\Pi_1(dx)+\int_{0<|x|\leq 1}(e^{izx}-1-izx)\Pi_1(dx)\right\}, \tag{13}$$

where $\Pi_1(A) = \mathbf{E}\,\nu_1(A)$ for any measurable A; $a = \mathbf{E}\,\xi_1$; $b = \mathbf{D}\xi_1$. The last integral is improper, and it is known that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<|x|\le 1} x^2\,\Pi_1(dx) < \infty. \tag{14}$$

Using (13) to calculate the moment generating function $\phi_k(z)$ of $S_1^{(k)}$, one has:

$$\phi_k(z) = f_k(-i\log z) = \exp\left\{a\log z - \frac{b}{2}\log^2 z + \int_{-k}^{-1}(z^x-1)\Pi_1(dx) + \right.$$

$$\left. + \int_1^\infty (z^x-1)\Pi_1(dx) + \int_{0<|x|\le 1}(z^x - 1 - x\log z)\Pi_1(dx)\right\}.$$

For $z \in (0;\,1]$

$$\phi_k(z) \;\le\; \exp\left\{a\log z - \frac{b}{2}\log^2 z\right\}\;\cdot\;\exp\left\{z^{-k}\Pi_1([-k;\,-1])\right\}\;\cdot \tag{15}$$

$$\cdot\;\exp\left\{\int_{0<|x|\le 1}(z^x - 1 - x\log z)\Pi_1(dx)\right\}.$$

The first factor in (15) is finite for all $z > 0$ as well as the second one, because $\nu_1([-k;\,-1])$ is a Poisson random variable and $\Pi_1([-k;\,-1])$ is its expected value. Consider the third multiplier.

For any x, $|x| \le 1$,

$$z^x - 1 - x\log z = \sum_{k=2}^{\infty} x^k\frac{\log^k z}{k!} \le x^2(z - \log z),$$

from where, using (14),

$$\int_{0<|x|\le 1}(z^x - 1 - x\log z)\Pi_1(dx) \le (z - \log z)\int_{0<|x|\le 1} x^2\,\Pi_1(dx) < \infty.$$

Therefore, $\phi_k(z)$ is finite for any $0 < z \le 1$, and the results of the first part of the proof are applicable to processes $S_t^{(k)}$. According to them, there exists $\lim\limits_{N\to\infty}\sqrt[N]{\mathbf{E}\,\tau_k^S(N)} = \gamma_k^S$.

It follows from (12) that $\tau(N) \geq \tau_k^S(N)$ for any k with probability one. Hence,

$$\liminf_{N\to\infty} \sqrt[N]{\mathbf{E}\,\tau(N)} \geq \lim_{N\to\infty} \sqrt[N]{\mathbf{E}\,\tau_k^S(N)} = \gamma_k^S,$$

and we have to prove only that $\gamma_k^S \underset{k\to\infty}{\longrightarrow} \gamma$.

Since

$$\phi(z) = \phi_k(z) \exp\{\int_{-\infty}^{-k} (z^x - 1)\Pi_1(dx)\},$$

then $R_k^S = R$ for all k. Also, $\phi(z) \leq \cdots \leq \phi_{k+1}(z) \leq \phi_k(z) \leq +\infty$ for all $z \geq 1$ and $\phi_k(z) \longrightarrow \phi(z)$ as $k \to \infty$. This situation is similar to the case with functions $\phi_n^Z(z)$ which we considered in the proof of Theorem 1. Having repeated that proof, one obtains: $\gamma_k^S \underset{k\to\infty}{\longrightarrow} \gamma$.

□

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