Fall 2019 Jingchen (Monika) Hu

MATH 347 Homework 2 (Total 30 points)

Due: Tuesday 9/24, at the beginning of the class

Name: ______________________________

- **Print out this cover page and staple with your homework.**
- **Show all work. Incomplete solutions will be given no credit.**
- **You may prepare either hand-written or typed solutions, but make sure that they are legible. Answers that cannot be read will be given no credit.**
- **R graphical outputs must be printed instead of hand-drawn.**

1. (*6 points; 2 points each part*)
Recall the Tokyo Express dining preference example covered in class. Suppose the Tokyo Express owner in the college town gives another survey to a different group of students. This time, he gives to 30 students and receive 10 of them saying Friday is their preferred day to eat out. Use the owner's prior (restated below) and calculate the following posterior probabilities.

$$\begin{aligned} p &= \{0.3, 0.4, 0.5, 0.6, 0.7, 0.8\} \\ \pi_{owner}(p) &= (0.125, 0.125, 0.250, 0.250, 0.125, 0.125) \end{aligned}$$

   (a) The probability that 30% of the students prefer eating out on Friday.

   (b) The probability that more than half of the students prefer eating out on Friday.

   (c) The probability that between 20% and 40% of the students prefer eating out on Friday.

2. (*8 points; 2 points each part*)
Revisit the figure in lecture slides page 23, where nine different Beta curves are displayed. In the context of Tokyo Express customers' dining preference example where p is the proportion of students preferring Friday, interpret the following prior choices in terms of the opinion of p. For example, $\text{Beta}(0.5, 0.5)$ represents the prior belief the extreme values $p = 0$ and $p = 1$ are more probable and $p = 0.5$ is the least probable. In the customers' dining preference example, specifying a $\text{Beta}(0.5, 0.5)$ prior indicates the owner thinks the students' preference of dining out on Friday is either very strong or very weak.

   (a) Interpret the $\text{Beta}(1, 1)$ curve.

   (b) Interpret the $\text{Beta}(0.5, 1)$ curve.

   (c) Interpret the $\text{Beta}(4, 2)$ curve.

   (d) Compare the opinion about p expressed by the two Beta curves: $\text{Beta}(4, 1)$ and $\text{Beta}(4, 2)$.

3. (*8 points; 2 points each part*)
Use any of the functions from this list i) `dbeta()`, ii) `pbeta()`, iii) `qbeta()`, iv) `rbeta()`, v) `beta_area()`, and vi) `beta_quantile()` to answer the following questions about Beta probabilities.

   (a) The density of $\text{Beta}(0.5, 0.5)$ at $p = \{0.1, 0.5, 0.9, 1.5\}$.

   (b) The probability $P(0.2 \leq p \leq 0.6)$ if $p \sim \text{Beta}(6, 3)$.

   (c) The quantile of the $\text{Beta}(10, 10)$ distribution at the probability values in the set $\{0.1, 0.5, 0.9, 1.5\}$.

(d) A sample of 100 random values from Beta$(4, 2)$.

4. (*4 points*)
   If the proportion has a Beta(a, b) prior and one observes Y from a Binomial distribution with parameters n and p, then if one observes $Y = y$, then the posterior density of p is Beta$(a+y, b+n-y)$.

   Recall that the mean of a Beta(a, b) random variable following is $\frac{a}{a+b}$. Show that the posterior mean of $p \mid Y = y \sim$ Beta$(a + y, b + n - y)$ is a weighted average of the prior mean of $p \sim$ Beta(a, b) and the sample mean $\hat{p} = \frac{y}{n}$. Find the two weights and explain their implication for the posterior being a combination of prior and data. Comment how Bayesian inference allows collected data to sharpen one's belief from prior to posterior.

5. (*4 points*)
   Derivation exercise of the Beta posterior. If the proportion has a Beta(a, b) prior and one samples Y from a Binomial distribution with parameters n and p, then if one observes $Y = y$, then the posterior density of p is Beta$(a + y, b + n - y)$. You need to perform the complete derivation, i.e. keep the constants.