An Introduction to Stan and RStan

Houston R Users Group

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Introduction

Credits

I (MW) am not a developer of `Stan`, only a very happy user.

Credit for `Stan` goes to the Stan Development Team: Andrew Gelman, Bob Carpenter, Daniel Lee, Ben Goodrich, Michael Betancourt, Marcus Brubaker, Jiqiang Guo, Allen Riddell, Marco Inacio, Jeffrey Arnold, Rob J. Goedman, Brian Lau, Mitzi Morris, Rob Trangucci, Jonah Sol Gabry, Alp Kucukelbir, Robert. L. Grant, Dustin Tran, Krzysztof Sakrejda, Aki Vehtari, Rayleigh Lei, Sebastian Weber, Chalres Margossian, Thel Seraphim, Vincent Picaud, Matt Hoffman, Michael Malecki, Peter Li, Yuanjun Guo.

Much of the material in this presentation mirrors the excellent `Stan` manual. Any mistakes in exposition are solely the responsibility of MW.

Background

How familiar are you with the following concepts?

- Bayesian Statistics

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Bayesian Inference

> *Statistics is the science of learning from data, and of measuring, controlling, and communicating uncertainty. ([DL12])*

Bayesian Statistics emphasizes the use of *probability* as a language for describing uncertainty.

An All-Too-Brief Introduction to Bayesian Inference

Bayesian statistics uses the language of probability to quantify information. Anything which is (treated as) unknown has a probability distribution associated with it.

The tools of probability, in particular Bayes' Rule, give a mathematically coherent framework for updating beliefs in the presence of data. Classical works on this point-of-view are Ramsey, Savage, Jeffreys and Jaynes [Ram31, Sav54, Jef61, Jay03].[1][2]

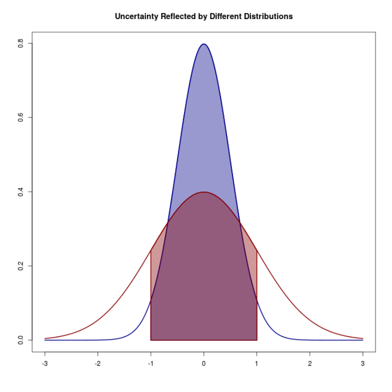
The probabilistic content of Bayes' Theorem is trivial. The statistical content is not. – Steve Huntsman (MathOverflow)

[1] While Bayesian inference is typically promoted on the basis of incorporating prior information and inferential flexibility, it can be shown to have good frequentist properties in a range of circumstances as well [Efr15].

[2] Two expositions of "subjective" probability of particular interest to finance are [Key21] and [SV01].

Two normal distributions with different standard deviations.

The blue distribution is more “precise” than the red one.[3]



[3] The connection between curvature and inferential precision is found in classical statistics as well: the Fisher information is a measure of curvature of the likelihood function. The field of *information geometry* uses differential geometry to develop this relationship in much more generality; see, *e.g.*, [ABNK+87] for more.

Basically all Bayesian inference problems[4] are of the canonical form:

- Given prior information about an unknown quantity θ, express that information as a probability distribution $p(\theta)$, conventionally known as the *prior*;
- Given data observed according to some random process depending on θ, construct an appropriate *likelihood* $p(X|\theta)$;
- Using Bayes' rule, calculate a new distribution of θ, conditioned on the data X; this distribution is conventionally known as the *posterior*:

$$\pi(\theta|X) = \frac{p(X|\theta)p(\theta)}{p(X)} = \frac{p(X|\theta)p(\theta)}{\int p(X|\theta)p(\theta)\,\mathrm{d}\theta}$$

(Almost) All inference reduces to taking expectations under $\pi(\cdot)$.

[4] For an introduction to Bayesian methods see [McE15], [GH06], or [Hof09]; [GCS+14] is the Bayesian "Bible" for applied statistics. [Rob07] is an excellent text on Bayesian foundations.

The choice of prior has historically been one of the more controversial aspects of Bayesian analysis. Roughly, three classes of priors:

- Informative: Provide significant information and help guide analysis.
- Weakly Informative: Avoid pathologies, but let the data dominate. Similar to weak regularization in non-Bayesian analysis.
- Non-Informative. Attempt to provide no information: hard to achieve in practice.[5]

Technical Warning: If you don't provide a prior, `Stan` will implicitly use a uniform (flat) prior. For unbounded parameters, this gives an improper distribution and strange things can occur (*e.g.*, [HC96]). *Caveat emptor*

[5] See, *e.g.*, [KW96, BBS09].

Having defined our posterior distribution

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we want to actually perform statistical inference (make claims about the unobserved parameters).

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In classical statistics, these often require three different sets of tools. In Bayesian statistics, all three can be accomplished using the posterior.

Bayesian Inference

Typically, the posterior is actually a pretty difficult distribution to work with directly, so we actually work with samples from the distribution. Historically, obtaining these samples has been the hardest part of Bayesian inference, but new tools like `Stan` have made it significantly easier.

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Slides in the Computation with Stan section explain a bit more about how this is done. For now, we'll take it as a given.

Generalized Linear Mixed Effect Models

In "standard" linear regression, the error terms are considered to be *independent and identically distributed*.

$$y_i = \mathbf{x}_i^T \beta + \epsilon_i \qquad \epsilon_i \overset{\text{IID}}{\sim} \mathcal{N}(0, \sigma^2)$$

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Mixed-Effects Models let us deal with all of these phenomena. Bayesians use these all of the time, but typically prefer the name "multilevel-" or "hierarchical-" models.

For mixed-effects models, data is typically arranged in a "hierarchy" of observational units: *e.g.*

Exams *within* students *within* classes *within* schools *within* districts *within* states

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We model each grouping as an linear/additive effect:

$$\text{Score}_{ijklm} = \overbrace{\mu}^{\text{National Baseline}} + \overbrace{\nu_i}^{\text{State Effect}} + \overbrace{\kappa_{ij}}^{\text{District Effect}} + \cdots + \overbrace{\epsilon_{ijklm}}^{\text{Exam Specific Randomness}}$$

Notation can be a bit overwhelming, but key idea is that we are partitioning observed variance to different layers of the hierarchy

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Fitting Mixed-Effects Models

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Let’s try this out an an example of tumor growth data from [HHW$^+$04] (example courtesy of Peter Hoff, [Hof09, Section 11.4]):

www.stat.washington.edu/people/pdhoff/Book/Data/XY.tumor

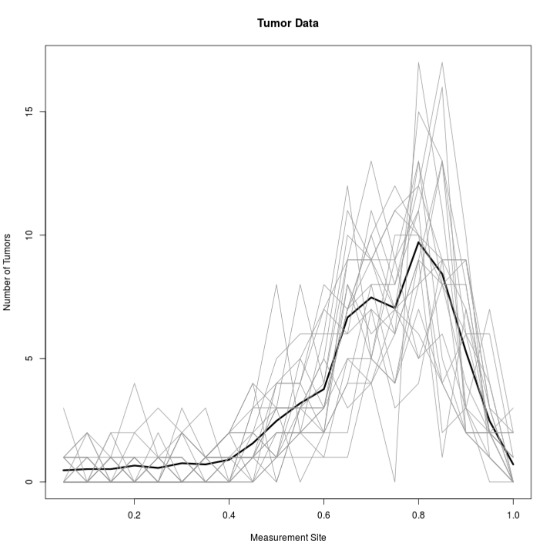
This data records the number of tumors found in different sections of the intestines of 21 different mice. The data follows a natural hierarchy (counts within mice) and suggests a mixed-effect model should be used to capture the "mouse effect".

If we look at the data, we see that certain mice have higher overall cancer incidence than others, but that there is still a consistent rise near the end of the intestine:

```
URL = "http://www.stat.washington.edu/people/pdhoff/Book/Data/data/XY.tumor"
TUMOR_DATA <- dget(URL)
plot(seq(0.05, 1, by=0.05), colMeans(TUMOR_DATA$Y), col="black", lwd=3,
     ylim=range(TUMOR_DATA$Y), type="l", main="Tumor Data",
     xlab="Measurement Site", ylab="Number of Tumors")

for(i in 1:21){
    lines(seq(0.05, 1, by=0.05), TUMOR_DATA$Y[i, ], col="grey80")
}
```

Tumor Data



Tumor Data
Number of Tumors
0
5
10
15
0.2
0.4
0.6
0.8
1.0
Measurement Site

Poisson GLMM

Since this is count data, we'll use Poisson regression.

The data is a bit messily formatted, but after some cleaning, we can fit the model with a mouse-effect and a 4th degree polynomial for the location effect using `rstanarm` as follows:

```
library(reshape2); library(rstanarm);
options(mc.cores=parallel::detectCores())
DATA <- cbind(expand.grid(mouse_id=1:21,
                          location=seq(0.05, 1, by=0.05)),
              count=melt(TUMOR_DATA$Y)$value)

M <- stan_glmer(count ~ poly(location, 4) + (1|mouse_id),
                family=poisson, data=DATA)
```

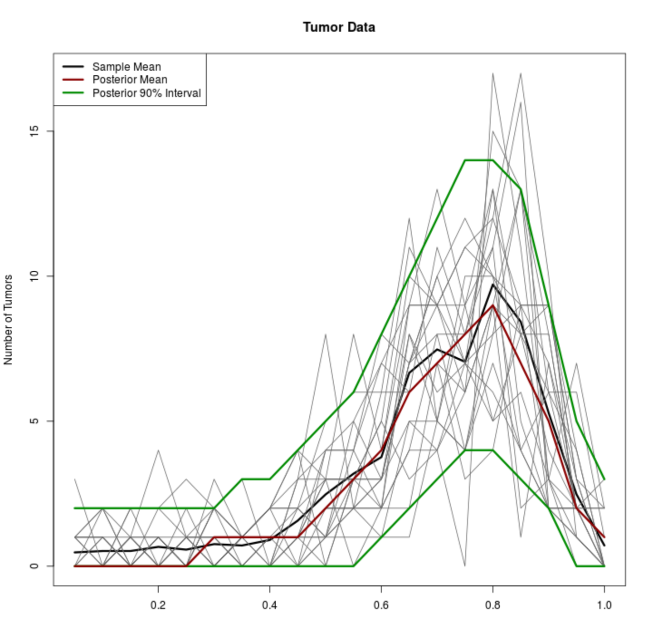
Takes about 15 seconds to get 4000 samples on this data set, which is more than enough for posterior inference.

Say we were interested in the prediction for the “typical” mouse. We make posterior predictions with the mouse-effect set to zero and examine the 5%, 50%, and 95% quantiles of the posterior (giving a 90% prediction interval):

```
X <- seq(0.05, 1, by=0.05)
PP <- posterior_predict(M, re.form=~0, newdata=data.frame(location=X))
apply(PP, 2, quantile, c(0.05, 0.50, 0.95))
```

The model appears to fit the data well:

Poisson GLMM



MCMC Convergence and Model Assessment

Checking Model Fit

In our Poisson GLMM example, the model fit the data well and sampling converged rapidly. We are not always so lucky. In general, we should always check convergence diagnostics before proceeding to inference.

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The `shinystan` and `bayesplot` packages provide tools for visualizing and checking the results of MCMC inference.

Demo Time!

```
launch_shinystan(M)
```

Computation with Stan

Stan

Stan [Sta15c, CGH+, GLG15] is a probabilistic programming language superficially like `BUGS` or `JAGS` [LJB+03, Plu03]. Unlike BUGS and JAGS, not restricted to Gibbs sampling or conjugate (exponential family graphical) models.

Provides:

- Full Bayesian Inference (via Hamiltonian Monte Carlo)
- Variational Bayesian Inference (via ADVI [KRGB15, KTR+16, BKM16])
- Penalized MLE (Bayesian MAP)[6]

Best thought of as a DSL for specifying a distribution and sampling from it.

Named after *Stanislaw Ulam*, co-author, with N. Metropolis, of *The Monte Carlo Method* (JASA-1949, [MU49])

[6] Useful for quickly checking results against non-Bayesian software. MAP with uniform priors should recover the MLE (modulo optimization issues for non-convex problems).

Language- and platform-agnostic back-end ([Sta15c, Sta15d]) with a wide range of front-ends:

- Shell (“`CmdStan`”)
- `R` (“`RStan`”, [Sta15b]) ⋆
- `Python` (“`PyStan`”, [Sta15a]) ⋆
- MATLAB (“`MatlabStan`”, [Lau15])
- `Stata` (“`StataStan`”, [GS15])
- `Julia` (“`JuliaStan`”, [Goe15])

⋆: In process interface. The others “simply” wrap `CmdStan`.

`Stan` uses a two step compilation process: the user writes a model in pure `Stan` code[7] which is then translated to `C++` by the `stanc` compiler. The translated program is then compiled like any other `C++` program into a stand alone binary.

Once compiled, the model can be re-run with different inputs / parameters.

Requires a working `C++` compiler unlike the (interpreted) BUGS/JAGS to compile new models. Once compiled, the binary can be moved between machines (*modulo* standard linking and library constraints).

Higher level interfaces (*e.g.* `RStan`) can run the compilation in the background.

[7] It is possible to embed `Stan` directly within a `C++` program, but more advanced.

`Stan` provides a wide range of built-in data types:

- Data primitives: `real`, `int`
- Mathematical structures: `vector`, `matrix` – can hold `real` and `int`
- Programming structures: `array` – can hold *any* other `Stan` type
- Constrained structures: `ordered`, `positive_ordered`, `simplex`, `unit_vector`
- Matrix types: `cov_matrix`, `corr_matrix`, `cholesky_factor_cov`, `cholesky_factor_corr`

Constraints on data types are used to transform into an *unconstrained space* where `Stan` performs inference.

```
real<lower=0> sigma;
real<lower=0,upper=1> p;
```

`sigma` is log-transformed to be supported on $\mathbb{R}$; similarly `p` is logit^{-1}-transformed.[8] Since there is an change of variables in these transforms, `Stan` automatically adds a Jacobian to the target density. When you perform similar "left-hand-side" transformations, `Stan` will warn that a manual Jacobian adjustment may be necessary [Sta15d, Chapter 20].

Warning: Because `Stan` works on a (transformed) $\mathbb{R}$, *discrete parameters* are not directly supported. (Discrete data is fine.)

[8] `Stan` has a range of transformations into unconstrained space:

- Positivity constraints use a $\log(\cdot)$-transform
- Boundedness constraints use a (scaled) $\text{logit}(\cdot)$-transform
- Simplex constraints use a stick-breaking transform ($\mathbb{R}^K \to \mathbb{R}^{K-1}$)
- Matrix constraints (PD) use Cholesky-based transforms (see [PB96])

A `Stan` program is divided into *blocks*. The key blocks are:

- `data`: Defines the external data which `Stan` will read at the beginning of execution
- `parameters`: Defines the variables which will be inferred
- `model`: Defines the probability model relating the data and parameters. Both the prior and the likelihood are coded in this block

Additional blocks, *e.g.*, `transformed data`, `generated quantities` are useful for performing additional transformations within `Stan`. Less useful when using `Stan` through the interfaces.

Toy example (Beta-Bernoulli):

```
data{
    int<lower=0> N; // Number of observations
    int<lower=0,upper=1> y[N]; // observed 0/1 variables
}
parameters{
    real<lower=0,upper=1> p; // unknown p
}
model{
    p ~ beta(1, 1); // weak prior
    y ~ bernoulli(p); // vectorized across elements of y
}
```

Stan Language Model

The “sampling statements” in the model block are syntactic sugar for direct manipulation of the log-posterior.

Equivalent:

```
data{
    int<lower=0> N; // Number of observations
    int<lower=0,upper=1> y[N]; // observed 0/1 variables
}
parameters{
    real<lower=0,upper=1> p; // unknown p
}
model{
    target += beta_log(p, 1, 1); // weak prior
    for(i in 1:N){ // likelihood
        target += bernoulli_log(p, y[i]);
    }
}
```

$$\pi(\theta|X) = \frac{p(X|\theta)p(\theta)}{\int p(X|\theta)p(\theta)\,\mathrm{d}\theta}$$

The denominator of this quantity (the “partition function”) is often analytically intractable so we are left with

$$\pi(\theta|X) \propto p(X|\theta)p(\theta)$$

How can we calculate $\mathbb{E}[F(\theta)]$ for arbitrary (measurable) $F(\cdot)$ when we can only calculate π up to a normalizing constant?

In theory, we *sample* from π and invoke the Law of Large Numbers:

$$\frac{1}{N}\sum_{i=1}^{N} F(\theta_i) \overset{n\to\infty}{\longrightarrow} \mathbb{E}[F(\theta)]$$

In practice, we cannot sample directly from π either.

Not entirely true. We can (sort of) sample from π, but not IID.

We construct an (ergodic) Markov chain with transition kernel Π chosen to have the same stationary distribution as π (see, *e.g.*, [Tie94] for details). Then, samples from this Markov chain are samples from π if either:

- We initialize the chain with a draw from π;
- We run the chain long enough (infinitely long!) so that it converges to π.

The first is, again, impossible. Let's look more closely at the second.

A Markov chain is a map from the space of probability distributions onto itself.

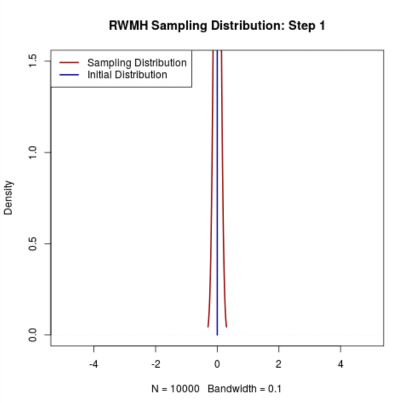
Given an initialization distribution (which may be a point mass) P_0, application of the transition kernel Π gives a new distribution P_1.

Repeated application gives $\{P_n\}$ which tend to π as $n \to \infty$. If P_0 is "close" to π, the convergence is rapid.

π is the stationary point of Π so $\Pi\pi = \pi$.

Example:

- $P_0 = \delta_0$;
- Π is a Random Walk Metropolis Hastings update (proposal distribution: $X_t \sim \mathcal{N}(X_{t-1}, 1)$);
- π is $\mathcal{N}(2, 5^2)$.



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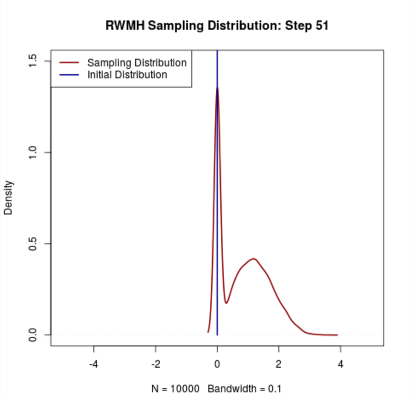

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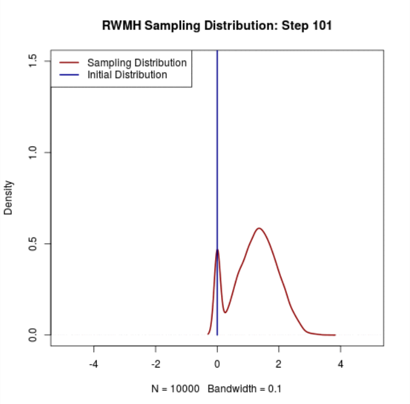

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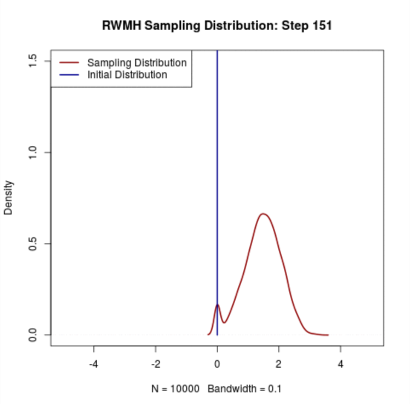

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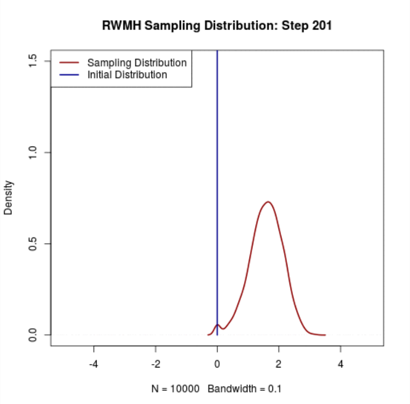

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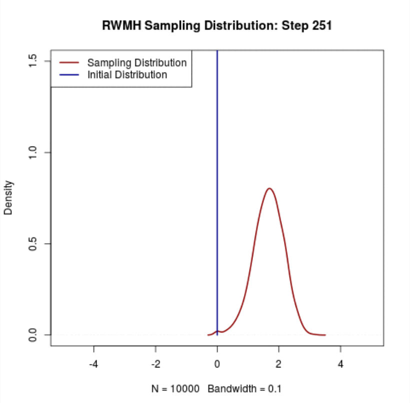

Convergence of MCMC

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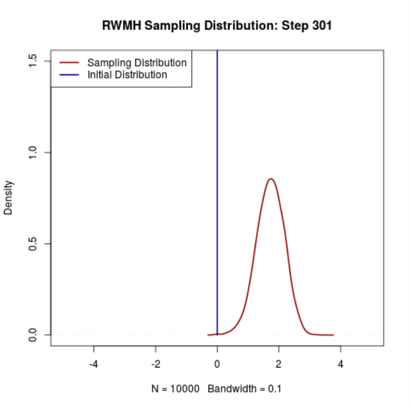

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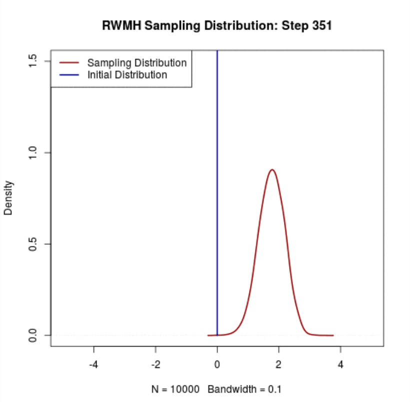

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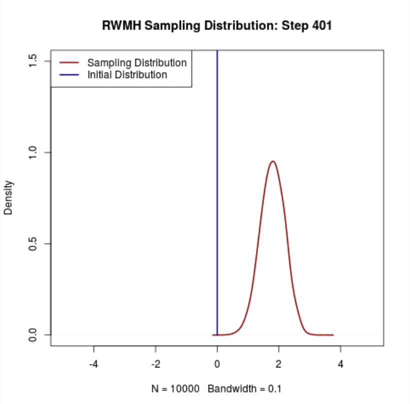

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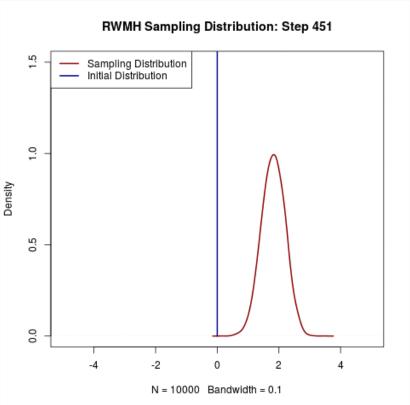

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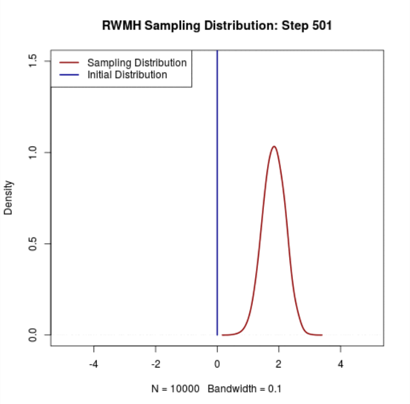


How many samples from π do we need?

Depends what we want to do: let's take calculating the posterior mean as a typical task.[9]

Two possible sources of variance:

- The inherent variance of the posterior;
- Additional variance from having a finite number of draws ("Monte Carlo error")

The first is unavoidable (and a key advantage of the Bayesian approach); the second can be reduced with more samples.

[9] See [Jon04] for a discussion of the *Markov Chain Central Limit Theorem*; see [RR04] for a discussion of the general conditions required for MCMC convergence.

Accuracy of MCMC: Effective Sample Size

If we have n IID samples from π, our Monte Carlo standard error (ratio of total variance to inherent variance) is approximately $\sqrt{1+n^{-1}}$ [GCS+14, Section 10.5].

With autocorrelation, we instead look at the *effective sample size*:[10]

$$\text{ESS} = \frac{n}{1 + \sum_{t=1}^{\infty} \rho_t}$$

See [GCS+14, Section 11.5] or [KCGN98, Section 2.9] for details.

Technically, there is a different ACF for each $\mathbb{E}[f(\cdot)]$ we estimate, but this isn't usually a big deal in practice.[11]

---

[10] The exact formula has $n^2/(n + \sum_{t=1}^{n}(n-t)\rho_t)$ but for large n this is approximately equal (and faster to calculate).

[11] There is a disconnect between practice and theory here. Theory establishes conditions for accurate inference for *all* possible f (see, *e.g.*, [LPW08]), but we usually only care about a few f. Some (very) recent work attempts to establish convergence rates for restricted classes of f [RRJW16].

Choice of Markov Kernel

Since ESS controls the quality of our inference, ESS/second is the appropriate metric for comparing samplers.

Different choices of the Markov transition kernel Π can give radically different ESS/second.

Standard Metropolis-Hastings or Gibbs Samplers struggle for complex (hierarchical) and high-dimensional (many parameters) models.

Hamiltonian Monte Carlo ([Nea11]) performs much more efficiently for a wide range of problems.[12]

[12] The Markov Chains constructed by HMC can be shown to be *geometrically ergodic* (quick mixing) under relatively weak conditions [LBBG16].

In its default mode, `Stan` uses a technique known as *Hamiltonian Monte Carlo* to sample from the posterior with minimal autocorrelation. These draws are typically more expensive than from other methods, *e.g.* Gibbs samplers, but the reduced correlation leads to a (much) higher ESS/second.

Very roughly: Metropolis-Hastings methods ([MRR+53, Has70]) move around the probability space randomly (without knowledge of the underlying geometry) and use a accept-reject step to adjust probabilities accordingly.

Hamiltonian Monte Carlo gives a particle a random "kick" and samples based on the path of the particle: uses *Hamiltonian mechanics* to simulate the path of the particle in an energy field induced by the target density π.

Hamiltonian dynamics (*a.k.a.* Hamiltonian mechanics) is an equivalent formulation of Newtonian mechanics. Let $\boldsymbol{p}$ be the momenta of all particles in the system and $\boldsymbol{q}$ be the position of the particles.

The evolution of the system over time is uniquely defined by:

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}$$
$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}$$

where $\mathcal{H}$ is the *Hamiltonian* of the system, a function which measures the total energy of the system.

Hamiltonian mechanics is easily extended to relativistic systems.

Once the Hamiltonian $\mathcal{H}$ and the initial conditions are specified, the time evolution of the system is known deterministically. In practice, the PDEs cannot be solved explicitly and a numerical integrator must be used.

A common choice of integrator is the *leapfrog integrator* which has the nice properties of being:

- time-reversibility
- symplectic (energy conserving)

See [LR05] for details. With step size ϵ (requiring $L\epsilon$ steps to integrate over a time interval of length L), the leapfrog integrator has ϵ^2 error.

Hamiltonian Monte Carlo (originally *Hybrid* Monte Carlo) [Nea11] builds a Hamiltonian system to sample from a target density π.

We let $\boldsymbol{q}$ (the "position") be the parameters of the target density and add an auxiliary momentum variable $\boldsymbol{p}$. The joint distribution of $\boldsymbol{p}, \boldsymbol{q}$ can be used to define a Hamiltonian $\mathcal{H}$:

$$\begin{aligned} H(\boldsymbol{p}, \boldsymbol{q}) &= -\log p(\boldsymbol{p}, \boldsymbol{q}) \\ &= -\log p(\boldsymbol{q}|\boldsymbol{p}) - \log p(\boldsymbol{p}) \\ &= \underbrace{T(\boldsymbol{q}|\boldsymbol{p})}_{\text{Kinetic energy}} + \underbrace{V(\boldsymbol{p})}_{\text{Potential energy}} \end{aligned}$$

Solving the Hamiltonian equations for this system, we find

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} &= \frac{\partial T}{\partial \boldsymbol{p}} \\ \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= -\frac{\partial V}{\partial \boldsymbol{q}} \end{aligned}$$

We can (approximately) solve these equations using a leapfrog integrator. To introduce randomness, $\boldsymbol{p}_0$ is initialized from a $\mathcal{N}(0, \Sigma)$ matrix where Σ is independent of prior samples and of $\boldsymbol{q}_0$.

Leapfrog integration is then simply:

$$\rho \leftarrow \rho - \frac{\epsilon}{2} \frac{\partial V}{\partial \boldsymbol{q}}$$

$$\theta \leftarrow \theta + \epsilon \Sigma \boldsymbol{p}$$

$$\rho \leftarrow \rho - \frac{\epsilon}{2} \frac{\partial V}{\partial \boldsymbol{q}}$$

repeated L times.

If leapfrog integration were exact, we could directly accept the result of the leapfrog step. In reality, we have to use a Metropolis acceptance step [MRR+53] to account for the error. Thus, HMC *as implemented* is strictly a form of Metropolis MCMC, but with a *highly efficient* transition kernel Π which moves along the geometric contours of the target distribution π rather than randomly.

The form of Σ controls the implicit geometry of the the Hamiltonian dynamics [BS11, BBLG14]. In particular, Σ^{-1} is the mass matrix of the particle undergoing Hamiltonian evolution.

Fixed Σ (either diagonal or full) corresponds to a Euclidean metric on the parameter space and gives *Euclidean Hamiltonian Monte Carlo*.

Current research considers changing Σ for each sample: this corresponds to sampling on a Riemannian manifold and gives rise to *Riemannian Hamiltonian Monte Carlo* [GC11, Bet13]. By varying Σ, RHMC can adapt to the “funnels” found in complex hierarchical variables more efficiently [BG13].

The No-U-Turn Sampler (NUTS)

For large L, running HMC to termination is wasteful when the particle begins to retrace its steps. Early termination would save computation time but biases our sampling.

To avoid this, the *No-U-Turn Sampler* ("NUTS") enhances HMC by allowing time to intermittently run backwards: see [HG14] for details. The time-reversability of the leapfrog integrator is key for allowing NUTS to work properly.

NUTS is the default sampler used in `Stan` though "pure" HMC is also available.

Automatic Differentiation (“AutoDiff”) is a technique for automatically calculating the numerical gradient of a function at a fixed point.

AutoDiff expresses computations in terms of language primitives (addition, multiplication, and function calls) and uses the chain rule to calculate the gradient as part of regular function evaluation.

`Stan` uses autodiff to efficiently calculate the gradients necessary for HMC.

AutoDiff vs Other Gradient Calculation Techniques

AutoDiff is not

- Symbolic Differentiation
- Numerical Differentiation (finite difference approximations)

Unlike symbolic differentiation, AutoDiff has no knowledge about the function being evaluated: only the arithmetic primitives.[13] Unlike numerical differentiation, AutoDiff provides exact gradient calculations with a single function call.

[13] Consequently, AutoDiff provides an exact derivative for an *approximation* of the function of interest rather than an approximation to the exact function of interest.

`Stan` provides a fully AutoDiff-equipped math library ([CHB+15]) built on Boost and Eigen [Sch11, GJ+10].

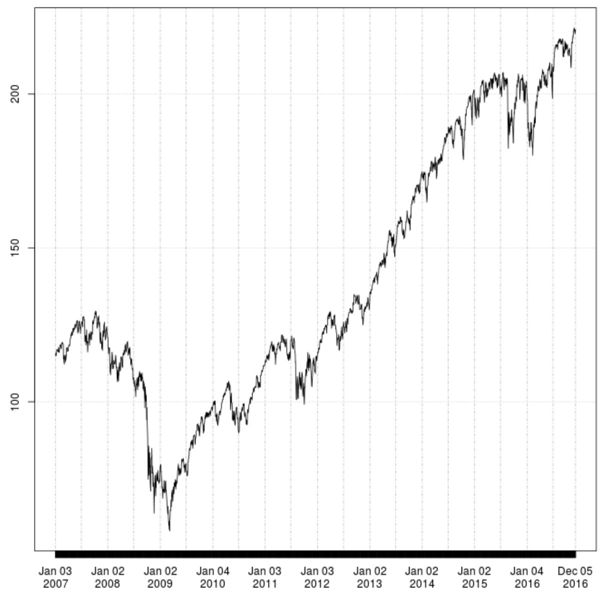
Currently `Stan` only uses *first-order* AutoDiff but *second-order* AutoDiff will be released soon.

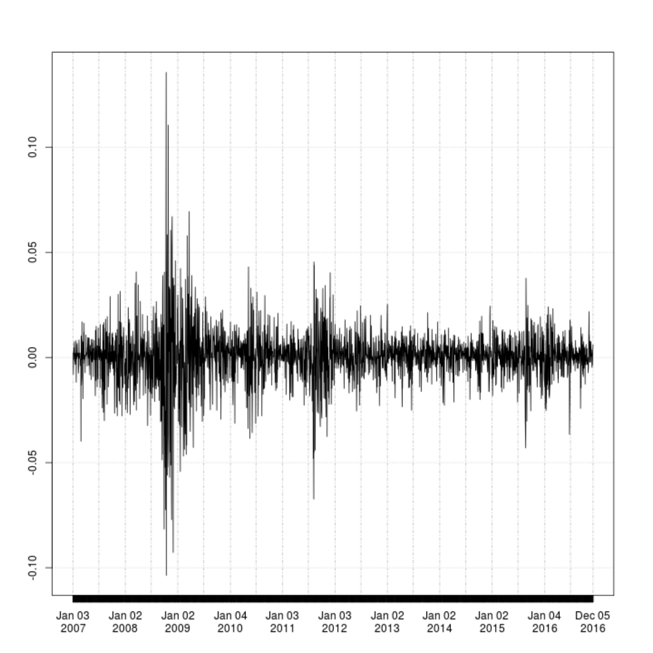
`Stan`'s AutoDiff is *reverse-mode* which means that it works "down" the function call chain:

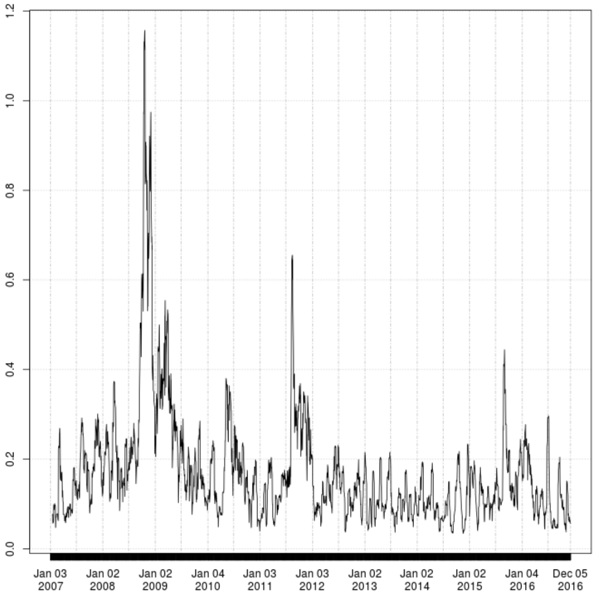
$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial w_1}\frac{\partial w_1}{\partial x} = \frac{\partial y}{\partial w_2}\frac{\partial w_2}{\partial w_1}\frac{\partial w_1}{\partial x} = \dots$$

When computing derivatives of functions $f : \mathbb{R}^n \to \mathbb{R}^m$, this is more efficient for $m \ll n$; for `Stan`, $m = 1$.

Financial Time Series: Stochastic Volatility Models


200
150
100
Jan 03 2007
Jan 02 2008
Jan 02 2009
Jan 04 2010
Jan 03 2011
Jan 03 2012
Jan 02 2013
Jan 02 2014
Jan 02 2015
Jan 04 2016
Dec 05 2016


0.10
0.05
0.00
-0.05
-0.10
Jan 03 2007
Jan 02 2008
Jan 02 2009
Jan 04 2010
Jan 03 2011
Jan 03 2012
Jan 02 2013
Jan 02 2014
Jan 02 2015
Jan 04 2016
Dec 05 2016



1.2
1.0
0.8
0.6
0.4
0.2
0.0
Jan 03 2007
Jan 02 2008
Jan 02 2009
Jan 04 2010
Jan 03 2011
Jan 03 2012
Jan 02 2013
Jan 02 2014
Jan 02 2015
Jan 04 2016
Dec 05 2016

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$$r_t \overset{\text{IID}}{\sim} \mathcal{N}(0, \sigma_t^2)$$

where σ_t^2 is the "instantaneous volatility."

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Volatility models add additional structure to the time-dynamics of σ_t^2 so that we can estimate it from observed data.

The Stochastic Volatility model of [KSC98] models volatility as a latent mean-reverting AR(1) process.

$$h_{t+1} \sim \mathcal{N}\left(\mu + \phi(h_t - \mu), \sigma^2\right)$$
$$r_t \sim \mathcal{N}\left(0, \exp\{h_t\}\right)$$

Implemented in `stochvol` package for `R` [Kas16].

If we want the volatility at each time t, this is a *high-dimensional* model: quantities–$\{h_t\}_{t=1}^{T}, \mu, \phi, \sigma$–than we have observations.

Normally, this is impossible without further constraints or regularization, but the prior fulfills that role in the Bayesian context.

The Stan manual [Sta15d, Section 9.5] describes how to code this model efficiently:

```
data {
    int<lower=0> T;
    vector[T] y;
}
parameters {
    real mu;
    real<lower=-1, upper=1> phi; // Stationary volatility
    real<lower=0> sigma;
    vector[T] h_std;
}
```

(continued)

```
transformed parameters {
    vector[T] h;
    h = h_std * sigma;
    h[1] = h[1] / sqrt(1 - phi * phi);
    h = h + mu;
    for(t in 2:T){
        h[t] = h[t] + phi * (h[t-1] - mu);
    }
}
```

```
model {
    // Priors
    phi ~ uniform(-1, 1);
    sigma ~ cauchy(0, 5);
    mu ~ cauchy(0, 10);
    // Scaled Innovations in h process are IID N(0,1)
    h_std ~ normal(0, 1);
    // Observation likelihood.
    // Note exp(h/2) since Stan uses normal(mean, SD)
    y ~ normal(0, exp(h/2));
}
```

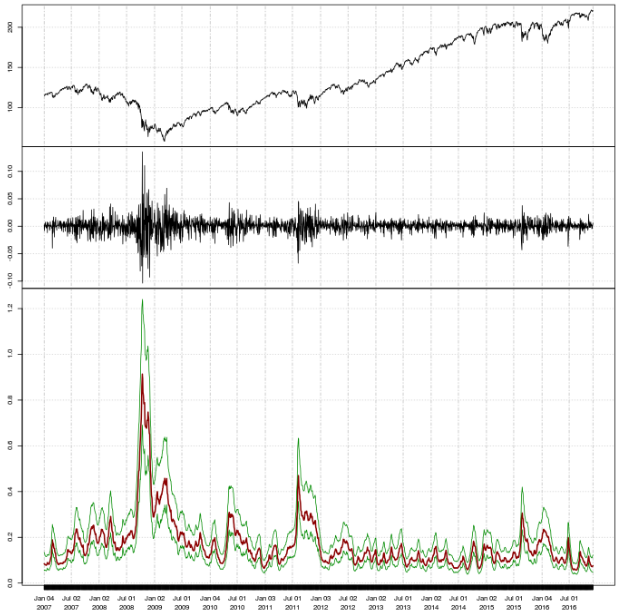
Running this model, we can plot the estimated volatility with its confidence interval over time:

```
library(quantmod)

SPY <- getSymbols("SPY", auto.assign=FALSE)
R <- na.omit(ROC(Ad(SPY)))

SAMPLES <- stan("sv.stan", data=list(y=as.vector(R), T=length(R)))
PP <- apply(exp(extract(SAMPLES, "h")[[1]]/2), 2,
            quantile, c(0.05, 0.50, 0.95))
```

SPY & Inferred Volatility



Once coded-up, adapting the model to use a heavy-tailed or skewed error process is straightforward:

t-errors (inferring the degrees of freedom)

```
...
    real<lower=0> nu;
...
    nu ~ cauchy(0, 5);
    y ~ student_t(nu, 0, exp(h/2));
...
```

Skew-normal errors (inferring the skewness parameter):

```
...
    real alpha;
...
    alpha ~ cauchy(0, 5);
    y ~ skew_normal(0, exp(h/2), alpha);
...
```

Thank you

If you're interested in learning more, start with Michael Betancourt's talks to the Tokyo Stan Users' Group: (Modeling (link) and HMC (link)).

Bob Carpenter's MLSS-2015 talk (link) is a bit more "hands-on" with the Stan language. (Michael's talks go into more MCMC and HMC theory)

The `Stan` manual (link) is remarkably readable.

The `stan-users` mailing list (link) is a good place to ask for help with more detailed issues.

Pierre-Antoine Kremp built a fully-open source political forecasting model using Stan. Check it out!

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