WRITTEN ASSIGNMENT 1

(Final due date: 5-11-25)

`Problems worth 10 points each.`

Problems 1-6: Solve the online quiz.

Problem 7

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & p & p^2 & p^3 \\ 1 & q & q^2 & q^3 \\ 1 & p+q & (p+q)^2 & (p+q)^3 \end{pmatrix},$$

where p, q are integer parameters. Using `sagemath`:
(a) Define the matrix A.
(b) Compute the determinant of A and factorize it.
(c) Find the constraints on p and q for which the matrix A is non-invertible.
In the case that p and q are the last two digits of your student ID:
(d) Bring A to its reduced row echelon form.
(e) Compute the rank of A.

Problem 8

Let

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 4 & 1 & 0 \end{pmatrix}.$$

Define

$$C = A^T (BB^T)^{-1} A$$

and consider the linear system

$$Cx = y, \text{ where } \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \quad y = \begin{pmatrix} 25 \\ 15 \end{pmatrix}.$$

Using `sagemath`
(a) Define the matrices A, B and the vectors x, y.
(b) Compute BB^T and check if it is singular.
(c) Compute C.
(d) Solve the linear system $Cx = y$ in terms of $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ using (i) matrix inversion and (ii) the `sagemath` function `solve`.

Problem 9

Using by hand calculation, find the values of k for which the system

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y - 5z &= 2 \\ 4x + y + (k^2 - 9)z &= k + 5 \end{aligned}$$

has:
(a) No solution,
(b) Infinitely many solutions,
(c) Exactly one solution. Find this solution for k=2.

Problem 10

Consider matrix $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 2 & 3 & 1 & 0 \end{pmatrix}$ and the linear mapping $\Phi : \mathbb{R}^4 \to \mathbb{R}^3, x \to Ax$.
(a) Find the rank of matrix A.
(b) Find the reduced row echelon form of A.
(c) Find the image of mapping Φ.
(d) Find the kernel of mapping Φ.