Problem 9

Using hand calculation, find the values of k for which the system

$$\begin{cases} x + 2y - 3z = 4, \\ 3x - y - 5z = 2, \\ 4x + y + (k^2 - 9)z = k + 5 \end{cases}$$

has:

(a) No solution,

(b) Infinitely many solutions,

(c) Exactly one solution (and find this solution for $k = 2$).

Step 1: Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & -5 & 2 \\ 4 & 1 & k^2 - 9 & k + 5 \end{array}\right]$$

Step 2: Eliminate x from the 2nd and 3rd rows

Perform:

$$R_2 \leftarrow R_2 - 3R_1, \quad R_3 \leftarrow R_3 - 4R_1$$

Compute:

$$R_2 = [3, -1, -5, 2] - 3[1, 2, -3, 4] = [0, -7, 4, -10],$$

$$R_3 = [4, 1, k^2 - 9, k + 5] - 4[1, 2, -3, 4] = [0, -7, k^2 + 3, k - 11].$$

So the new matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 4 & -10 \\ 0 & -7 & k^2 + 3 & k - 11 \end{array}\right]$$

Step 3: Eliminate y from the 3rd row

$$R_3 \leftarrow R_3 - R_2$$

Compute:

$$[0, -7, k^2 + 3, k - 11] - [0, -7, 4, -10] = [0, 0, k^2 - 1, k - 1].$$

Thus the reduced matrix is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 4 & -10 \\ 0 & 0 & k^2 - 1 & k - 1 \end{array}\right]$$

Step 4: Analyze the third equation

From the last row we get:

$$(k^2-1)z = k-1$$

- If $k^2-1 \neq 0 \Rightarrow k \neq \pm 1$, the system has a **unique solution**.
- If $k=1$, the equation becomes $0=0$, giving **infinitely many solutions**.
- If $k=-1$, the equation becomes $0=-2$, giving **no solution**.

Step 5: Summary

$$\begin{aligned}
&(a) \text{ No solution: } k=-1,\\
&(b) \text{ Infinitely many solutions: } k=1,\\
&(c) \text{ Exactly one solution: } k \neq \pm 1.
\end{aligned}$$

Step 6: Find the unique solution for $k=2$

For $k=2$:

$$(4-1)z = 2-1 \implies 3z=1 \implies z=\frac{1}{3}.$$

From the second equation:

$$-7y+4z=-10 \implies -7y+\frac{4}{3}=-10 \implies -7y=-\frac{34}{3} \implies y=\frac{34}{21}.$$

From the first equation:

$$x+2y-3z=4 \implies x+2\left(\frac{34}{21}\right)-3\left(\frac{1}{3}\right)=4,$$

$$x+\frac{68}{21}-1=4 \implies x=5-\frac{68}{21}=\frac{37}{21}.$$

Hence:

$$\boxed{x=\frac{37}{21}, \quad y=\frac{34}{21}, \quad z=\frac{1}{3}}$$

Final Answers

$$\begin{aligned}
&(a)\ k=-1 \Rightarrow \text{No solution},\\
&(b)\ k=1 \Rightarrow \text{Infinitely many solutions},\\
&(c)\ k\neq \pm 1 \Rightarrow \text{Exactly one solution. For } k=2:\\
&\quad x=\tfrac{37}{21},\ y=\tfrac{34}{21},\ z=\tfrac{1}{3}.
\end{aligned}$$