Problem 10

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

and the linear mapping

$$\Phi : \mathbb{R}^4 \to \mathbb{R}^3, \quad \Phi(x) = Ax.$$

We are asked to:

(a) Find the rank of A,

(b) Find the reduced row echelon form (RREF) of A,

(c) Find the image (range) of mapping Φ,

(d) Find the kernel (null space) of mapping Φ.

(a) Rank of A

Start with the augmented matrix representation for A:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

We perform Gaussian elimination to find the row echelon form.

Step 1: Eliminate the first column below the pivot

$$R_3 \to R_3 - 2R_1$$

$$R_3 = [2, 3, 1, 0] - 2[1, 0, 2, 1] = [0, 3, -3, -2].$$

Matrix becomes:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & -3 & -2 \end{bmatrix}$$

Step 2: Eliminate the entry below the second pivot

$$R_3 \to R_3 - 3R_2$$

$$R_3 = [0, 3, -3, -2] - 3[0, 1, 3, 4] = [0, 0, -12, -14].$$

Matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -12 & -14 \end{bmatrix}$$

Step 3: Simplify third row

Divide by -2:

$$R_3 \to \frac{1}{-2} R_3 = [0, 0, 6, 7]$$

But more simply, divide by -12:

$$R_3 \to \frac{1}{-12} R_3 = [0, 0, 1, \tfrac{7}{6}]$$

Now we can eliminate the third column entries above.

Step 4: Eliminate above the third pivot

$$R_1 \to R_1 - 2R_3, \qquad R_2 \to R_2 - 3R_3.$$

Compute:

$$\begin{aligned} R_1 &= [1, 0, 2, 1] - 2[0, 0, 1, \tfrac{7}{6}] = [1, 0, 0, 1 - \tfrac{7}{3}] = [1, 0, 0, -\tfrac{4}{3}], \\ R_2 &= [0, 1, 3, 4] - 3[0, 0, 1, \tfrac{7}{6}] = [0, 1, 0, 4 - \tfrac{7}{2}] = [0, 1, 0, \tfrac{1}{2}], \\ R_3 &= [0, 0, 1, \tfrac{7}{6}]. \end{aligned}$$

Thus, the **reduced row echelon form (RREF)** is:

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{7}{6} \end{bmatrix}}$$

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(b) Rank of A

There are 3 pivot positions (columns 1, 2, and 3). Hence,

$$\boxed{\text{rank}(A) = 3.}$$

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(c) Image of mapping Φ

The image (or column space) of A is the span of its pivot columns (the first three columns of A):

$$\operatorname{Im}(\Phi) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Since the rank is 3, these three vectors are linearly independent in $\mathbb{R}^3$, so:

$$\boxed{\operatorname{Im}(\Phi) = \mathbb{R}^3.}$$

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(d) Kernel of mapping Φ

Let $x = (x_1, x_2, x_3, x_4)^T$. From the RREF we have:

$$\begin{cases} x_1 = \frac{4}{3}x_4, \\ x_2 = -\frac{1}{2}x_4, \\ x_3 = -\frac{7}{6}x_4, \\ x_4 \text{ free.} \end{cases}$$

Hence the kernel is the set of all vectors of the form:

$$x = x_4 \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{2} \\ -\frac{7}{6} \\ 1 \end{pmatrix}$$

$$\boxed{\ker(\Phi) = \operatorname{span}\left\{ \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{2} \\ -\frac{7}{6} \\ 1 \end{pmatrix} \right\}}$$

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Final Summary

$$(a)\ \ \mathrm{rank}(A) = 3,$$

$$(b)\ \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{7}{6} \end{bmatrix},$$

$$(c)\ \mathrm{Im}(\Phi) = \mathbb{R}^3,$$

$$(d)\ \ \ker(\Phi) = \mathrm{span}\left\{ \begin{pmatrix} \frac{4}{3} \\ -\frac{1}{2} \\ -\frac{7}{6} \\ 1 \end{pmatrix} \right\}.$$