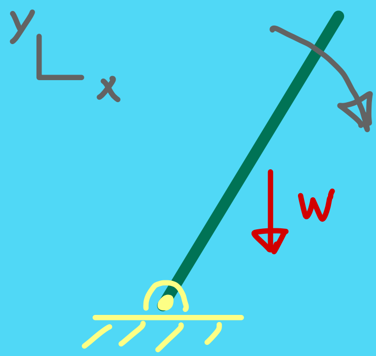
The Inverted Pendulum

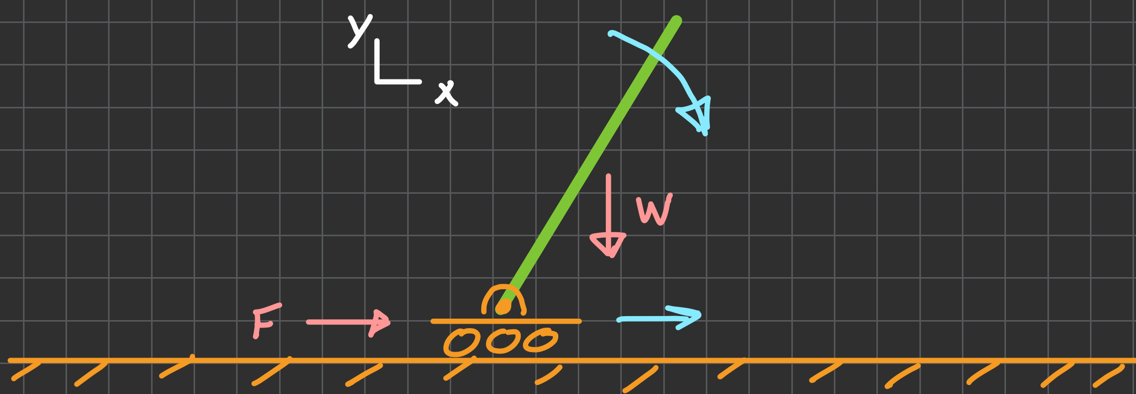


Calculus skills in this example:

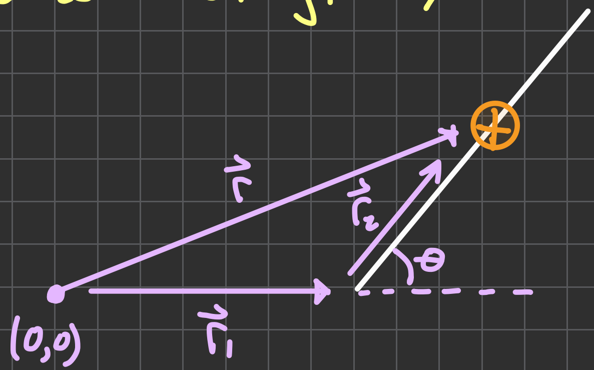
- Derivatives
- Integrals
- Fundamental theorem
- Taylor Series
- Tangent Line
- Linear Approximation
- Numerical Methods
- Polar Coordinates
- Vector Calculus

Part 2:

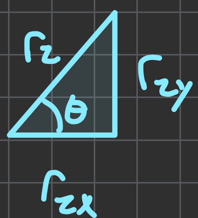
What happens if the base of the pendulum is able to translate in the horizontal direction?



Lets use a vector approach to define position of the center of gravity.



Note: $|\vec{r}_2| = \frac{L}{2}$ (constant)



$\sin\theta = r_{2y}/r_2 \rightarrow r_{2y} = r_2 \sin\theta$

$\cos\theta = r_{2x}/r_2 \rightarrow r_{2x} = r_2 \cos\theta$

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$
$$= r_1 \hat{i} + r_{2x}\hat{i} + r_{2y}\hat{j}$$
$$= r_1 \hat{i} + r_2 \cos\theta\, \hat{i} + r_2 \sin\theta\, \hat{j}$$

$\theta = \theta(t)$

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r_1 \hat{i} + r_2 \cos\theta\, \hat{i} + r_2 \sin\theta\, \hat{j} \right)$$

$$= \frac{dr_1}{dt}\hat{i} + \underbrace{\cancel{\frac{dr_2}{dt}}}_{0}\cos\theta\, \hat{i} - r_2 \sin\theta \frac{d\theta}{dt}\hat{i}$$

$$+ \underbrace{\cancel{\frac{dr_2}{dt}}}_{0}\sin\theta\, \hat{j} + r_2 \cos\theta \frac{d\theta}{dt}\hat{j}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \underbrace{\frac{dr_1}{dt}}_{u}\hat{i} - \underbrace{r_2}_{L/2}\sin\theta\,\underbrace{\frac{d\theta}{dt}}_{\omega}\hat{i} + \underbrace{r_2}_{L/2}\cos\theta\,\underbrace{\frac{d\theta}{dt}}_{\omega}\hat{j}$$

x-direction: $V_x = u - \frac{L}{2}\omega\sin\theta$

y-direction: $V_y = \frac{L}{2}\omega\cos\theta$

Velocity of center of gravity, θ

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\frac{dr_1}{dt}\hat{i} - r_2\sin\theta\frac{d\theta}{dt}\hat{i} + r_2\cos\theta\frac{d\theta}{dt}\hat{j}\right)$$

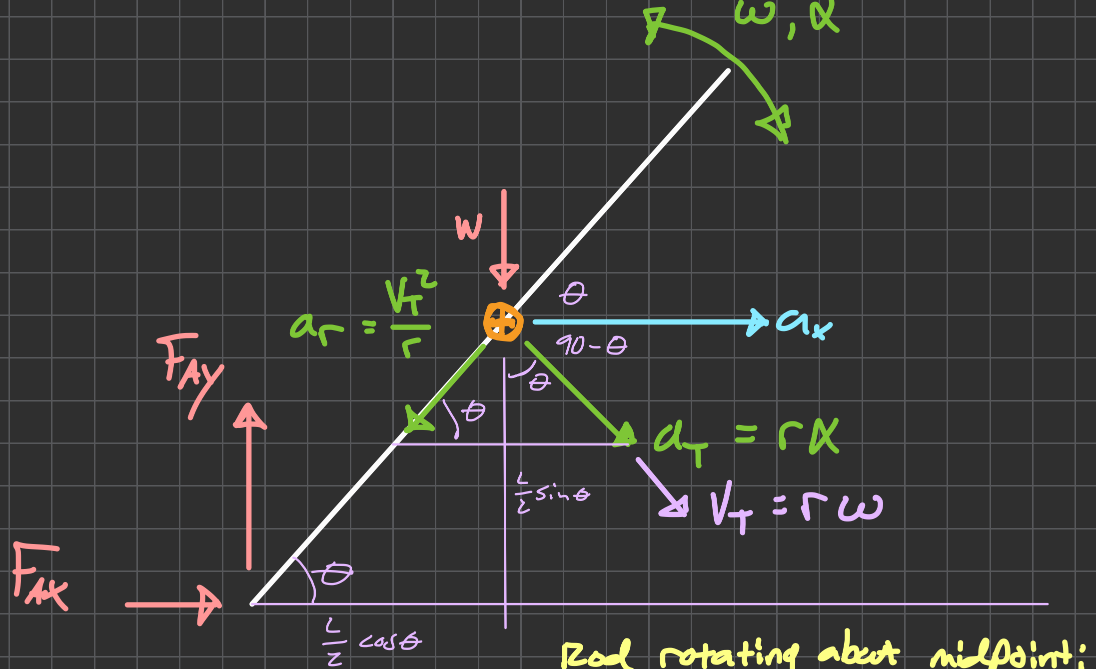
$$= \underbrace{\frac{d^2 r_1}{dt^2}}_{a_x}\hat{i} - \underbrace{r_2}_{L/2}\cos\theta\,\underbrace{\frac{d\theta}{dt}}_{\omega}\,\underbrace{\frac{d\theta}{dt}}_{\omega}\hat{i} - \underbrace{r_2}_{L/2}\sin\theta\,\underbrace{\frac{d^2\theta}{dt^2}}_{\alpha}\hat{i}$$

$$- \underbrace{r_2}_{L/2}\sin\theta\,\underbrace{\frac{d\theta}{dt}}_{\omega}\,\underbrace{\frac{d\theta}{dt}}_{\omega}\hat{j} + \underbrace{r_2}_{L/2}\cos\theta\,\underbrace{\frac{d^2\theta}{dt^2}}_{\alpha}\hat{j}$$

x-direction: $A_x = a_x - \frac{L}{2}\omega^2\cos\theta - \frac{L}{2}\alpha\sin\theta$

y-direction: $A_y = -\frac{L}{2}\omega^2\sin\theta + \frac{L}{2}\alpha\cos\theta$

acceleration of center of gravity



Rod rotating about midpoint:

$$I = \frac{1}{12}mL^2$$

$$\Sigma T_G = I\alpha$$

$$\boxed{-F_{Ay}\frac{L}{2}\cos\theta + F_{Ax}\frac{L}{2}\sin\theta = \frac{1}{12}mL^2\alpha}$$

$$\Sigma F_x = mA_x$$

$$F_{Ax} = mA_x$$

$$\Sigma F_y = mA_y$$

$$F_{Ay} - W = mA_y$$

$$F_{Ay} - mg = mA_y$$

$$\boxed{\begin{aligned} F_{Ax} &= mA_x \\ F_{Ay} - mg &= mA_y \end{aligned}}$$

Forces

Governing Equations - Kinematic

(1) x-direction: $V_x = u - \frac{L}{2}\omega \sin\theta$

(2) y-direction: $V_y = \frac{L}{2}\omega \cos\theta$

Velocity of center of gravity, $\oplus$

(3) x-direction: $A_x = a_x - \frac{L}{2}\omega^2 \cos\theta - \frac{L}{2}\alpha \sin\theta$

(4) y-direction: $A_y = -\frac{L}{2}\omega^2 \sin\theta + \frac{L}{2}\alpha \cos\theta$

acceleration of center of gravity

Governing Equations - Dynamic

(5) x-direction: $F_{Ax} = mA_x \rightarrow A_x = F_{Ax}/m$

(6) y-direction: $F_{Ay} - mg = mA_y$

(7) θ-direction: $-F_{Ay}\frac{L}{2}\cos\theta + F_{Ax}\frac{L}{2}\sin\theta = \frac{1}{12}mL^2\alpha$

Integration Plan

Choose: F_{Ax} @ all t

m, g, L

$\theta_0 \quad \omega_0 \quad \alpha_0$

(7)

$\omega_2 = \omega_1 + \alpha_1 \Delta t$ ✓

$\theta_2 = \theta_1 + \omega_1 \Delta t$ ✓

(5)

$V_{x_2} = V_{x_1} + A_{x_1}\Delta t$

$x_2 = x_1 + V_{x_1}\Delta t$ ✓

$$-F_{Ay}\frac{L}{2}\cos\theta + F_{Ax}\frac{L}{2}\sin\theta = \frac{1}{12}mL^2\alpha \quad (7)$$

$$\rightarrow \alpha = \frac{\cancel{12}\,6}{mL^{\cancel{2}}}\left(-F_{Ay}\frac{\cancel{L}}{\cancel{2}}\cos\theta + F_{Ax}\frac{\cancel{L}}{\cancel{2}}\sin\theta\right)$$

$$\alpha = \frac{6}{mL}\left(-F_{Ay}\cos\theta + F_{Ax}\sin\theta\right)$$

$$F_{Ay} - mg = mA_y \quad (6)$$

$$\rightarrow F_{Ay} = mA_y + mg$$

$$F_{Ay} = m(A_y + g)$$

$$-\frac{L}{2}\omega^2\sin\theta + \frac{L}{2}\alpha\cos\theta = A_y \quad (4)$$

$$F_{Ay} = m\left(-\frac{L}{2}\omega^2\sin\theta + \frac{L}{2}\alpha\cos\theta + g\right)$$

$$= -\frac{mL}{2}\omega^2\sin\theta + \frac{mL}{2}\alpha\cos\theta + mg$$

$$\rightarrow \alpha = \frac{6}{mL}\left[-\left(-\frac{mL}{2}\omega^2\sin\theta + \frac{mL}{2}\alpha\cos\theta + mg\right)\cos\theta + F_{Ax}\sin\theta\right]$$

$$\downarrow$$

$$\alpha = \frac{\overset{3}{\cancel{6}}}{m\cancel{L}}\left[-\left(-\frac{m\cancel{L}}{\cancel{2}}\omega^2\sin\theta + \frac{\cancel{mL}}{\cancel{2}}\alpha\cos\theta + 2\frac{\cancel{m}g}{L}\right)\cos\theta\right] + \frac{6}{mL}\left[F_{Ax}\sin\theta\right]$$

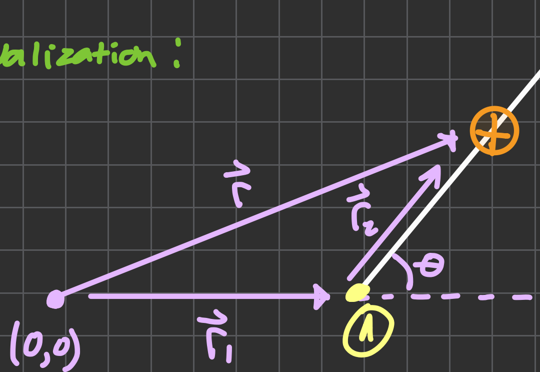
$$\alpha = 3\omega^2\sin\theta\cos\theta - 3\alpha\cos^2\theta - \frac{2g}{L}\cos\theta + \frac{6}{mL}\left(F_{Ax}\sin\theta\right)$$

$$\alpha + 3\alpha\cos^2\theta = 3\omega^2\sin\theta\cos\theta - \frac{2g}{L}\cos\theta + \frac{6}{mL}\left(F_{Ax}\sin\theta\right)$$

$$\alpha\left(1 + 3\cos^2\theta\right) = 3\omega^2\sin\theta\cos\theta - \frac{2g}{L}\cos\theta + \frac{6}{mL}\left(F_{Ax}\sin\theta\right)$$

$$\boxed{\alpha = \frac{3\omega^2\sin\theta\cos\theta - \frac{2g}{L}\cos\theta + \frac{6}{mL}\left(F_{Ax}\sin\theta\right)}{\left(1 + 3\cos^2\theta\right)}} \tag{8}$$

Visualization:



Note: $|\vec{r}_2| = \frac{L}{2}$

(constant)

7

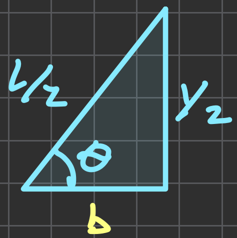
$\omega_2 = \omega_1 + \alpha_1 \Delta t$

$\theta_2 = \theta_1 + \omega_1 \Delta t$

5

$V_{x_2} = V_{x_1} + A_{x_1} \Delta t$

$x_2 = x_1 + V_{x_1} \Delta t$



CG coordinates: (x_2, y_2)

$$\boxed{x_2 = x_1 + V_{x_1} \Delta t}$$

$\sin\theta = y_2 / L/2 \rightarrow \boxed{y_2 = \frac{L}{2}\sin\theta}$

A: (x_A, y_A)

$x_A = x_2 - b$ $\qquad \cos\theta = b / L/2 \rightarrow b = \frac{L}{2}\cos\theta$

$\boxed{x_A = x_2 - \frac{L}{2}\cos\theta}$ $\qquad \boxed{y_A = 0}$ ← cart on flat surface

B (x_B, y_B)

$x_B = x_2 + b = \boxed{x_2 + \frac{L}{2}\cos\theta = x_B}$

$y_B = y_2 + y_2 \rightarrow \boxed{y_B = L\sin\theta}$