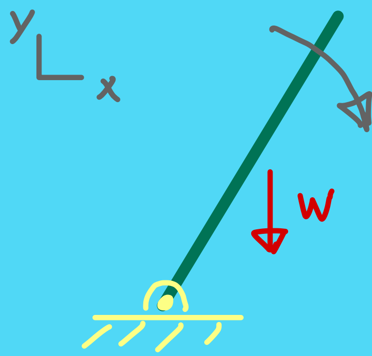
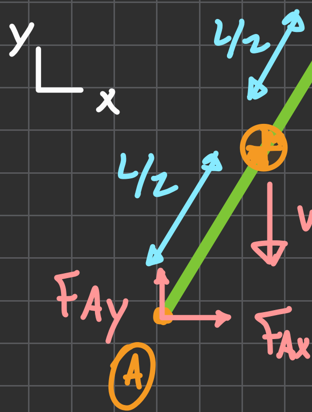
The Inverted Pendulum



Calculus skills in this example:

- Derivatives
- Integrals
- Fundamental theorem
- Taylor Series
- Tangent Line
- Linear Approximation
- Numerical Methods
- Polar Coordinates

The Inverted Pendulum



Conservation of linear momentum

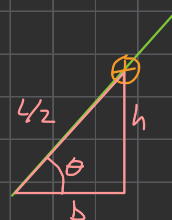
$$\sum F_x = ma_x \qquad \sum F_y = ma_y$$

$$F_{Ax} = ma_x \qquad F_{Ay} - W = ma_y$$

If A is fixed, $a_x = a_y = 0$

$$\boxed{\begin{aligned} F_{Ax} &= 0 \\ F_{Ay} &= W \end{aligned}}$$

Conservation of Angular Momentum



$$\sum T_A = I\alpha$$

$$-Wb = I\alpha$$

$$-W\frac{L}{2}\cos\theta = I\frac{d^2\theta}{dt^2}$$

$$\cos\theta = \frac{b}{L/2}$$

$$\rightarrow b = \frac{L}{2}\cos\theta$$

$$\rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{W}{I}\frac{L}{2}\cos\theta}$$

Strategy:

The Following relationship is the same For both translation and rotation!

$$\begin{tikzcd} \text{Position} \arrow[d, bend left, "d/dt"] \\ \text{Velocity} \arrow[u, bend left, "\int (\,) dt"] \arrow[d, bend left, "d/dt"] \\ \text{Acceleration} \arrow[u, bend left, "\int (\,) dt"] \end{tikzcd}$$

$$\theta = \theta(t)$$

$$\omega = d\theta/dt$$

$$\alpha = d^2\theta/dt^2 = -\frac{W}{I}\frac{L}{2}\cos[\theta(t)]$$

We know this From Force/torque analysis on previous page. Integrate twice to obtain position as a Function of time.

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{W}{I}\frac{L}{2}\cos\theta}$$

For a solid rod, with axis of rotation at the end,

$$I = \frac{1}{3}mL^2$$

Weight: $W = mg$

$$\rightarrow \frac{d^2\theta}{dt^2} = \frac{-\cancel{m}g}{\frac{1}{3}\cancel{m}L^{\cancel{2}}}\frac{\cancel{L}}{2}\cos\theta = -\frac{g}{L}\frac{3}{1}\frac{1}{2}\cos\theta$$

$$\rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\cos\theta}$$

note: the length matters, but not the weight. This is surprising

can we solve this?

note: $\theta = \theta(t)$

An unknown function, trapped inside cos

u-sub would work if we knew $\theta(t)$

$$\int \frac{d^2\theta}{dt^2}\,dt = \int -\frac{3}{2}\frac{g}{L}\cos\theta\,dt$$

$$\frac{d\theta}{dt} = \boxed{\omega(t) = -\frac{3}{2}\frac{g}{L}\int \sin\theta\,dt}$$

$$\int \frac{d\theta}{dt}\,dt = \int\left[-\frac{3}{2}\frac{g}{L}\int \sin\theta\,dt\right]dt$$

$$\boxed{\theta(t) = \int\left[-\frac{3}{2}\frac{g}{L}\int \sin\theta\,dt\right]dt}$$

Without differential equations, we're stuck!

Could we integrate numerically?

$$\frac{d^2\theta}{dt^2} = \alpha(t) = -\frac{W}{I}\frac{L}{2}\cos[\theta(t)]$$

unknown, except for starting point $\theta(t=0) = \theta_0$ (initial condition)

$$\omega(t) = \int \alpha(t)\,dt$$

Consider a Taylor Series:

$$f(x) \cong \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

remember: the more terms, the wider range of t values this is able to represent

$$\omega(t) \cong \frac{\omega^{(0)}(a)}{0!}(t-a)^0 + \frac{\omega^{(1)}(a)}{1!}(t-a)^1 + \frac{\omega^{(2)}(a)}{2!}(t-a)^2 + \ldots$$

$$= \frac{\omega(a)}{1}(1) + \left.\frac{d\omega}{dt}\right|_a \frac{(t-a)}{1} + \left.\frac{d^2\omega}{dt^2}\right|_a \frac{(t-a)^2}{2} + \ldots$$

here, "a" represents a value for t at which we know ω already. Lets call that t_1

$$\omega(t) \cong \omega(t_1) + \left.\frac{d\omega}{dt}\right|_{t_1}(t-t_1) + \left.\frac{d^2\omega}{dt^2}\right|_{t_1}\frac{(t-t_1)^2}{2} + \ldots$$

For t close to t_1, this $\rightarrow 0$

$$\omega(t) \cong \omega(t_1) + \left.\frac{d\omega}{dt}\right|_{t_1}(t-t_1) + \cancel{\left.\frac{d^2\omega}{dt^2}\right|_{t_1}\underbrace{\frac{(t-t_1)}{2}}} + \ldots$$

0

neglect higher order terms, keep $(t-t_1)$ close to zero!

For t close to t_1, this $\Rightarrow 0$

$$\rightarrow \quad \omega(t) \cong \omega(t_1) + \left.\frac{d\omega}{dt}\right|_{t_1}(t-t_1)$$

look familiar? this is just the tangent line, aka linearization

Here's the idea: if we know initial conditions, we can find what's happening a short time into the future. Lets use t_1 to represent the start of the interval, and t_2 to represent the end of the interval.

$$\omega(t_2) \cong \omega(t_1) + \left.\frac{d\omega}{dt}\right|_{t_1}(t_2-t_1)$$

$$\frac{d\omega}{dt} = \alpha = -\frac{W}{I}\frac{L}{2}\cos[\theta(t_1)]$$

$$\rightarrow \quad \omega(t_2) \cong \omega(t_1) + \alpha(t_1)\underbrace{(t_2-t_1)}_{\Delta t}$$

$$\boxed{\omega(t_2) \cong \omega(t_1) + \alpha(t_1)\,\Delta t}$$

So Far:

$\int (\,)\,dt$ ↗ $\theta(t)$ our goal

$\omega(t) = \frac{d\theta}{dt}$

$\int (\,)\,dt$ ↗

$\alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = -\frac{g}{L}\frac{3}{1}\frac{1}{2}\cos\theta(t)$

$$\boxed{\omega(t_2) \cong \omega(t_1) + \alpha(t_1)\,\Delta t} \quad (1)$$

by similar argument, we can integrate ω to find θ

$$\boxed{\theta(t_2) \cong \theta(t_1) + \omega(t_1)\,\Delta t} \quad (2)$$

Initial Value Problem: We know starting condition, so we can use (1) + (2) to find the condition a <u>short</u> time later. Repeat to find overall behavior!

$\omega(t_2) \cong \omega(t_1) + \alpha(t_1)\,\Delta t$

$\theta(t_2) \cong \theta(t_1) + \omega(t_1)\,\Delta t$

$\omega(t_3) \cong \omega(t_2) + \alpha(t_2)\,\Delta t$

Loop!

$\theta(t_3) \cong \theta(t_2) + \omega(t_2)\,\Delta t$

$\omega(t_4) \cong \omega(t_3) + \alpha(t_3)\,\Delta t$

$\theta(t_4) \cong \theta(t_3) + \omega(t_3)\,\Delta t$

⋮

Head over to my GitHub to play with the code for yourself!

https://github.com/Dr-F-Research