Structural VARs

Barcelona GSE. Empirical Time Series Methods for Macroeconomic Analysis.

Instructor: Luca Gambetti.
TA: Nicolò Maffei Faccioli.

1 Introduction:

In the following notes, we will review short-run and long-run restrictions, the variance decomposition analysis, and see how to implement these in MATLAB, using data on TFP growth (henceforth, $grTFP$) and on real Standard & Poor 500 Index growth (henceforth, $grSP500$). In what follows, growth rates are computed as the $diff(log(.))$ of the series in levels.

2 Motivation - Orthogonalisation of Shocks:

Before kicking off with the introduction of the Cholesky decomposition, short-run and long-run restrictions, I think it's worth spending some comments on the orthogonalisation shocks. Impulse response functions, as we saw in the previous notes and in class, are interpreted under the assumption that, given our shock of interest, all the other shocks are held constant. However, in the Wold representation, the shocks are **not** orthogonal. Consider the example of a $VAR(4)$ with two variables, $grTFP$ and $grSP500$. If the two shocks ϵ_{1t} ($grTFP$) and ϵ_{2t} ($grSP500$) are correlated, it doesn't make much sense to ask "what if ϵ_{2t} has a unit impulse?" with no change in ϵ_{1t} (i.e. keeping constant the other shock), since the two usually come at the same time. More precisely, we would like to start thinking about the impulse responses in causal terms: in our case, the "effect" of news shocks (stock prices shocks) on real TFP growth. If the shock in $grSP500$ is correlated with the $grTFP$ shock, you don't know if the response you're seeing is the response of $grTFP$ to stock prices growth, or if it is affected by other distubances that we can't quantify. The basic idea is that, if the variance-covariance matrix Ω is not diagonal, then ϵ_{1t} and ϵ_{2t} are contemporaneously correlated. Thus, if at time t we have a unit increase in a shock – say $grSP500$ – which is correlated with another shock – say $grTFP$ – then it is hard to disentangle the effects of each of them separately at future horizons $t+j$, as at time t both disturbances moved together. An easy way to bypass this problem is to **orthogonalise** the shocks in order to make sure that they're not correlated. This is exactly what we do when we impose short-run and long-run restrictions.

3 Cholesky Decomposition - Short-Run Restrictions:

Consider the variance-covariance matrix Ω. The Cholesky factor of the matrix Ω is defined as the unique lower triangular matrix S such that $SS' = \Omega$. This implies that we can rewrite a $VAR(p)$ process in terms of orthogonal shocks $\eta_t = S^{-1}\epsilon_t$ with identity variance covariance matrix:

$$A(L)y_t = S\eta_t \tag{1}$$

where $var(\eta_t) = S^{-1}\Omega(S^{-1})' = S^{-1}(SS')(S^{-1})' = S^{-1}SS'(S^{-1})' = I$. Then, the impulse responses to orthogonalised shocks η_t are computed from the $VMA(\infty)$ representation:

$$y_t = A(L)^{-1}S\eta_t = C(L)S\eta_t = \sum_{j=0}^{\infty} C_j S\eta_{t-j} \tag{2}$$

where C_jS has the follwing interpretation:

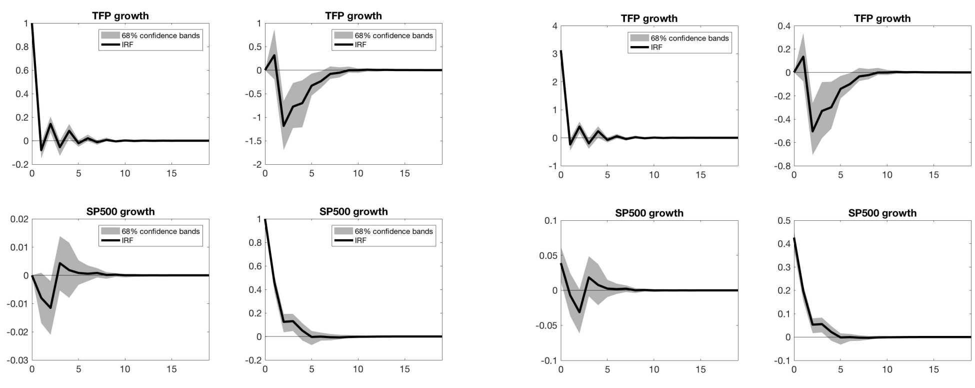
$$\frac{\delta Y_{t+j}}{\delta \eta_t} = C_jS$$

Hence, the row i, column k element of C_jS identifies the consequences of a unit increase in η_k at date t for the value of the ith variable at time $t+j$ holding all other η_{-k} constant. How can we compute the impulse responses of the orthogonalised shock using MATLAB? First of all, we have to compute the Cholesky factor S. We can do so by estimating Ω with our OLS residuals and then we can use the function *chol* of MATLAB: $S = chol(omega,' lower')$ uses only the diagonal and the lower triangle of Ω to produce a lower triangular S so that $SS' = \Omega$. If Ω is not positive definite, an error message is printed. When Ω is sparse, this syntax of *chol* is typically faster. In the following script, I compute the impulse response functions using the lower triangular Cholesky factor:

```
% Cholesky:

T=length(finaldata)-n*p-n; % -p lags for n variables and -n constants
omega=(err'*err)./T; %estimate of omega
S=chol(omega,'lower'); %cholesky factorization, lower triangular matrix

D=zeros(n,n,20);
for i=1:20
    D(:,:,i)=C(:,:,i)*S; %cholesky wold respesentation
end

D_wold=(reshape(permute(D,[3 2 1]),20,n*n,[]));

[HighD,LowD]=bootstrapchol(finaldata,hor,c,iter,conf,p,n); % function that ...
    performs bootstrap for cholesky

% Plot:

figure3=figure(3);

for k=1:n
    for j=1:n
    subplot(n,n,j+n*k-n)
    plot(D_wold(:,j+n*k-n),'LineWidth',1.2,'Color','b'); hold on;
    plot(LowD(:,j+n*k-n),'LineWidth',0.8,'Color','r','LineStyle','-.');
    plot(HighD(:,j+n*k-n),'LineWidth',0.8,'Color','r','LineStyle','-.'); ...
        hold off
    end
end
```

What are we actually doing when we implement Cholesky? We are imposing, in this case where $n = 2$, the restriction that the shock of the second variable, in this case stock prices growth, has **no contemporaneous effect** on TFP growth. Why is so? $C_0S = IS = S =$

$\begin{bmatrix} s_{11} & 0 \\ s_{21} & s_{22} \end{bmatrix}$. Clearly, as $s_{12} = 0$, the shock in the second variable has no contemporaneous effect on the first one. This is different from the case in which we had no restriction, where the matrix of contemporaneous effects was the identity matrix, meaning that, not only the second variable had no impact contemporaneously on the first one, but also that the first one had no impact contemporaneously on the second one. This is clear by comparing the results in Figure 1.



(a) Wold

(b) Cholesky

Figure 1: Impulse Response Functions - Wold & Cholesky

4 Blanchard & Quah's Long-Run Restriction:

To identify a technology shock using the restriction that the shock is the only one having effect in the long run on the TFP, we can use the identification scheme proposed by Blanchard and Quah in 1989. The restriction can be implemented, again, using a Cholesky factorization. Let $S = chol(C(1)\Omega C(1)')$ and $K = C(1)^{-1}S$. We can rewrite (2) as follows:

$$y_t = C(L)Kw_t = H(L)w_t \tag{3}$$

where $w_t = K^{-1}\epsilon_t$. By doing so, the long run restriction is implemented as:

$$H(1) = C(1)K = C(1)C(1)^{-1}S = S$$

As S is lower triangular, this means that $H(1)_{12} = 0$, hence the shock of stock prices growth doesn't have effect on the long run of the level of TFP. As in the Cholesky decomposition, the errors are orthogonal as:

$$var(w_t) = var(K^{-1}\epsilon_t) = (K^{-1})\Omega(K^{-1})' = S^{-1}C(1)\Omega C(1)'(S^{-1})' = S^{-1}(SS')(S^{-1})' = I$$

The IRFs are computed in MATLAB as follows::

```
%% Blanchard & Quah - Long-run restriction:

A1=sum(C,3);
S=chol(A1*omega*A1')';
K=A1\S;
F=zeros(2,2,20);

for i=1:20
    F(:,:,i)=C(:,:,i)*K;
end

F_wold=reshape(permute(F, [3 2 1]),[],n*n);

[HighF,LowF]=bootstrapBQ(finaldata,hor,1,iter,conf,p,n);

figure5=figure(5);

for k=1:n
    for j=1:n
    subplot(n,n,j+n*k-n)
    plot(F_wold(:,j+n*k-n),'LineWidth',1,'Color','b'); hold on;
    plot(LowF(:,j+n*k-n),'LineWidth',0.5,'Color','r','LineStyle','-.');
    plot(HighF(:,j+n*k-n),'LineWidth',0.5,'Color','r','LineStyle','-.'); ...
        hold off
    end
end
```

The results are presented in Figure 2.

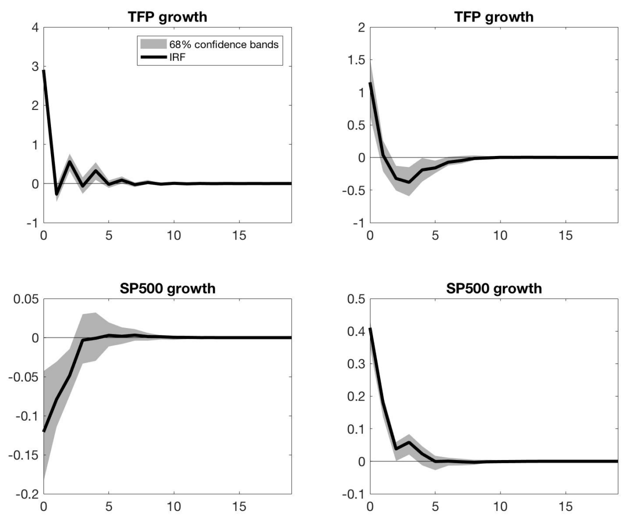


Figure 2: Impulse Response Functions - Blanchard & Quah

5 Cumulative Impulse Responses:

In this set of notes, we considered a set of variables in $diff(log(.))$. What if we were interested in the responses of log levels, rather than the ones in differences? Suppose that X_t is a vector of trending variables, namely $X_t = (logTFP, logSP500)$, so that we considered, above, $\Delta X_t = Y_t = C(L)\epsilon_t$, to reach stationarity. Notice that, as $\Delta X_t = X_t - X_{t-1} = C(L)\epsilon_t$, we can transform this model as follows:

$$X_t = X_{t-1} + C(L)\epsilon_t \tag{4}$$

Substituting forward one period ahead:

$$\begin{aligned} X_{t+1} &= X_{t-1} + C(L)\epsilon_t + C(L)\epsilon_{t+1} \\ &= X_{t-1} + C_0\epsilon_{t+1} + (C_0 + C_1)\epsilon_t + ... \end{aligned} \tag{5}$$

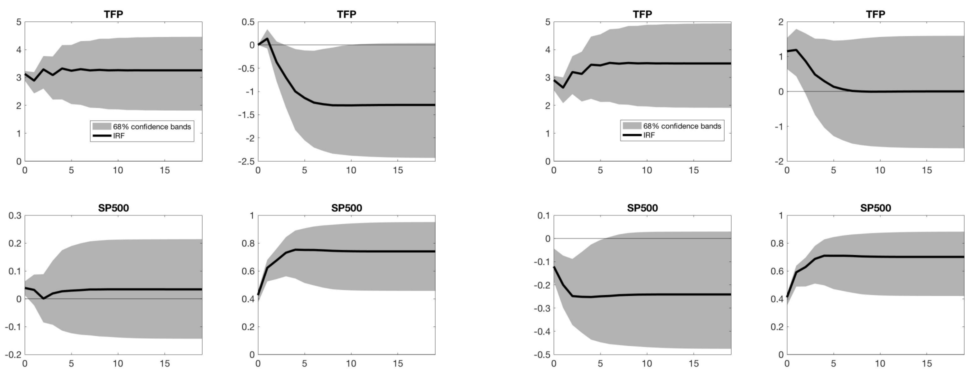
Two periods ahead:

$$\begin{aligned} X_{t+2} &= X_{t-1} + C(L)\epsilon_t + C(L)\epsilon_{t+1} + C(L)\epsilon_{t+2} \\ &= X_{t-1} + C_0\epsilon_{t+2} + (C_0 + C_1)\epsilon_{t+1} + (C_0 + C_1 + C_2)\epsilon_t + ... \end{aligned} \tag{6}$$

Therefore, the effect of ϵ_t on X_{t+1} is given by $(C_0 + C_1)$ and the effect on X_{t+2} by $(C_0 + C_1 + C_2)$. For a general horizon j, the cumulative impulse responses are then given by $(C_0 + C_1 + C_2 + ... + C_j)$. How can we perform this in MATLAB? There is a nice function, named *cumsum*, that does the job.

```
%% Cumulative Impulse Responses:

% Cholesky:

D_wold_cum=cumsum(D_wold);
HighD_cum=cumsum(HighD);
LowD_cum=cumsum(LowD);

figure7=figure(7);

for k=1:n
    for j=1:n
    subplot(n,n,j+n*k-n)
    plot(D_wold_cum(:,j+n*k-n),'LineWidth',1,'Color','b'); hold on;
    plot(LowD_cum(:,j+n*k-n),'LineWidth',0.5,'Color','r','LineStyle','-.');
    plot(HighD_cum(:,j+n*k-n),'LineWidth',0.5,'Color','r','LineStyle','-.'); ...
        hold off
    end
end

% Blanchard and Quah:

F_wold_cum=cumsum(F_wold);
HighF_cum=cumsum(HighF);
LowF_cum=cumsum(LowF);

```

```
figure8=figure(8);

for k=1:n
    for j=1:n
    subplot(n,n,j+n*k-n)
    plot(F_wold_cum(:,j+n*k-n),'LineWidth',1,'Color','b'); hold on;
    plot(LowF_cum(:,j+n*k-n),'LineWidth',0.5,'Color','r','LineStyle','-.');
    plot(HighF_cum(:,j+n*k-n),'LineWidth',0.5,'Color','r','LineStyle','-.'); ...
        hold off
    end
end
```



(a) Cholesky

(b) Blanchard & Quah

Figure 3: Cumulative Impulse Response Functions - Cholesky and Blanchard & Quah

6 Variance Decomposition:

The second type of analysis that is usually conducted, in the field of structural VARs, is the variance decomposition analysis. The idea is to decompose the total variance of a time series into the percentages attributable to each structural shock. For example, you might be interested to which extent the variance of log TFP is attributable to shocks in stock prices. In general, the variance decomposition analysis might useful in order to address questions like "What are the sources of the business cycle?" or "Is the shock important for economic fluctuations?".

Consider, for example, the VMA representation of an identified SVAR with the Cholesky decomposition as in (3), hence:

$$y_t = D(L)\eta_t = \sum_{j=0}^{\infty} D_j \eta_{t-j}$$

The variance of y_{it} is given by:

$$var(y_{it}) = \sum_{k=1}^{n} \sum_{j=0}^{\infty} D_{jik}^2 var(\eta_t) = \sum_{k=1}^{n} \sum_{j=0}^{\infty} D_{jik}^2$$

since $var(\eta_t) = I$ by construction. But what is D^2_{jik}? It is simply the matrix $D_j = C_j S$, where C_j will be the $n \times n$ upper left matrix of C_j, as we saw in the previous set of notes. Notice that $\sum_{j=0}^{\infty} D^2_{jik}$ is the variance of y_{it} generated by the kth variable. Therefore, the percentage of variance of y_{it} explained by the kth shock is given by:

$$\frac{\sum_{j=0}^{\infty} D^2_{jik}}{\sum_{k=1}^{n} \sum_{j=0}^{\infty} D^2_{jik}} \tag{7}$$

How can we compute (4) using MATLAB?

```
%% Variance Decomposition Analysis:

% Cholesky:

F_TFP=sum((D_wold(:,1)).*(D_wold(:,1)))+sum((D_wold(:,2)).*(D_wold(:,2)));
F_SP=sum((D_wold(:,3)).*(D_wold(:,3)))+sum((D_wold(:,4)).*(D_wold(:,4)));
decomposion_TFP1=sum((D_wold(:,1)).*(D_wold(:,1)))/F_TFP;
decomposion_TFP2=sum((D_wold(:,2)).*(D_wold(:,2)))/F_TFP;
decomposion_SP1=sum((D_wold(:,3)).*(D_wold(:,3)))/F_SP;
decomposion_SP2=sum((D_wold(:,4)).*(D_wold(:,4)))/F_SP;

table(decomposion_TFP1,decomposion_TFP2)
table(decomposion_SP1, decomposion_SP2)

% Blanchard & Quah:

F_TFPBQ=sum((F_wold(:,1)).*(F_wold(:,1)))+sum((F_wold(:,2)).*(F_wold(:,2)));
F_SPBQ=sum((F_wold(:,3)).*(F_wold(:,3)))+sum((F_wold(:,4)).*(F_wold(:,4)));
decomposion_TFP1BQ=sum((F_wold(:,1)).*(F_wold(:,1)))/F_TFPBQ;
decomposion_TFP2BQ=sum((F_wold(:,2)).*(F_wold(:,2)))/F_TFPBQ;
decomposion_SP1BQ=sum((F_wold(:,3)).*(F_wold(:,3)))/F_SPBQ;
decomposion_SP2BQ=sum((F_wold(:,4)).*(F_wold(:,4)))/F_SPBQ;

table(decomposion_TFP1BQ,decomposion_TFP2BQ)
table(decomposion_SP1BQ, decomposion_SP2BQ)
```

What are the percentages of the variance explained by the different shocks?

Cholesky	$grTFP$ shock	$grSP$ shocks
$grTFP$	0.95252	0.047482
$grSP$	0.012892	0.98711

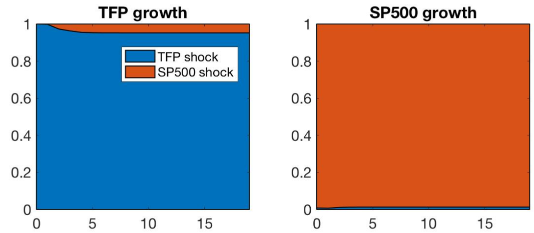
Blanchard & Quah	$grTFP$ shock	$grSP$ shocks
$grTFP$	0.84449	0.15551
$grSP$	0.10124	0.89876

It's interesting to point out that news shock are more important as drivers of the business cycle in the case in which we impose the long run restriction. Why is so? By comparing Figure 2 and Figure 1, it is clear that the restriction that news shock have no impact contemporaneously on TFP growth plays a big role.

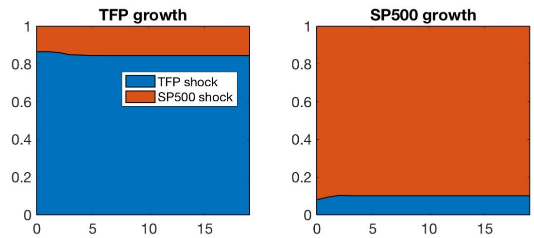
We can also present the results above in terms of the evolution of the variance decomposition over the horizon considered and not only at the last horizon. The percentage of variance of y_{it} explained by the kth shock at horizon J is given by:

$$\frac{\sum_{j=0}^{J} D_{jik}^2}{\sum_{k=1}^{n} \sum_{j=0}^{J} D_{jik}^2} \tag{8}$$

```
%% Variance Decomposition:

%MiddleDsquare=D_wold.^2; % For Cholesky, growth rates
MiddleDsquare=F_wold.^2; % For B&Q, growth rates

%MiddleDsquare=D_wold_cum.^2; % For Cholesky, levels
%MiddleDsquare=F_wold_cum.^2; % For B&Q, levels

denom=zeros(hor,n);

for k=[1:n;1:n:n*n]

    denom(:,k(1))=cumsum(sum(MiddleDsquare(:,k(2):k(2)+n-1),2));

end

denomtot=zeros(hor,n*n);

for k=[1:n;1:n:n*n]

    denomtot(:,k(2):k(2)+n-1)=denom(:,k(1)).*ones(hor,n);

end

vardec=zeros(hor,n);

for j=1:n*n

    vardec(:,j)=cumsum(MiddleDsquare(:,j))./denomtot(:,j);

end

figure2=figure(2);

% VARnames={'TFP growth'; 'SP500 growth'}; % growth rates
VARnames={'TFP'; 'SP500'}; % levels



for k=[1:n;1:n:n*n]
    subplot(1,n,k(1))
    area(0:hor-1,vardec(:,k(2):k(2)+n-1),'LineWidth',1); hold on;
    legend({'TFP shock','SP500 shock'},'FontSize',13)
    set(gca,'FontSize',15)
    title(VARnames{k(1)})
    xlim([0 hor-1]);
    ylim([0 1])
end
```

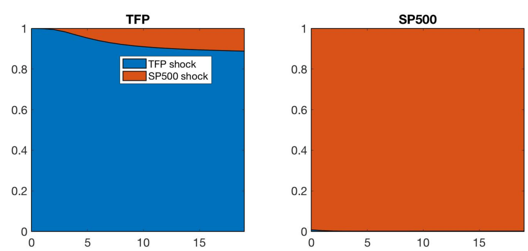


(a) Cholesky

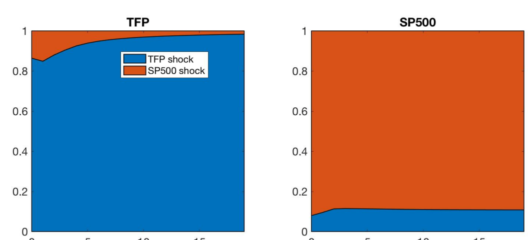


(b) B&Q

Figure 4: Variance Decomposition - Cholesky & B&Q - Growth Rates



(a) Cholesky



(b) B&Q

Figure 5: Variance Decomposition - Cholesky & B&Q - Levels