DISTANCE GEOMETRY FOR KISSING SPHERES

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Abstract. A kissing sphere is a sphere that is tangent to a fixed reference ball. We develop in this paper a distance geometry for kissing spheres, which turns out to be a generalization of the classical Euclidean distance geometry.

1. Introduction

Distance geometry studies the geometry based only on knowledge of distances. A basic problem of distance geometry is the embeddability problem. A typical embeddability problem in Euclidean distance geometry is like this: Are there three points A, B, C in a plane such that the distances are $d_E(AB) = 3$, $d_E(BC) = 4$ and $d_E(CA) = 5$, where d_E is Euclidean distance?

In classical Euclidean distance geometry, previous works show that information about embeddability is encoded in the distance matrix and the Cayley–Menger matrix. This will be reviewed in detail in Section 2.

A kissing sphere is a sphere that is tangent to a fixed reference ball. The name comes from the billards term. The classical kissing number problem [9] asks for the maximal number of unit balls with pairwisely disjoint interiors that can touch a reference unit ball. However, here we *do not* require the kissing spheres to have a fixed radius.

Following the approach of Euclidean distance geometry, we develop in this paper a distance geometry for kissing spheres based on a Möbius invariant distance function, defined in Section 4. A key observation is that the distance matrix for kissing spheres also plays the role of a Cayley–Menger matrix. It is then possible to adapt proof techniques from Euclidean distance geometry for our use.

The embeddability problem for kissing spheres asks whether there is a set of kissing spheres that realizes given distances. Our first main result, Theorem 5.1, says that the answer lies in the distance matrix in a similar way as in Euclidean distance geometry.

If for some pairs of points, distances are not given, one would like to find values for these unknown distances that can be realized by a set of points. This is the distance completion problem, another basic problem in distance geometry. In general it is very difficult to tell if the completion is possible. In Euclidean distance geometry, the completion problem is much easier if the known distances form a chordal graph. By embedding the kissing spheres into Minkowski space (Section 6), we come to a same conclusion for the distance completion problem for kissing spheres, which is our second main result, Theorem 7.2.

Apart from obvious similarities between the theorems above, we also notice that the distance geometry for kissing spheres does degenerate to Euclidean distance

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geometry in several ways. In this sense, the distance geometry developed in this paper generalizes Euclidean distance geometry.

We would like to explain the initial motivation for this study. The ball packing problem asks for the possibility to realize a given graph by the tangency relations of a collection of balls with disjoint interiors. Disk packings in dimension two are well understood thanks to Koebe–Andreev–Thurston's disk packing theorem, but a generalization to higher dimensions turns out to be very difficult. Some attempts can be found in [3, 10, 20, 25].

Kissing spheres are balls restricted to the surface of the reference ball. A packing of balls touching an n-dimensional reference ball can be viewed as a ball packing of "dimension $n - 1/2$": It's a special case of ball packing, where one ball touches every others.

Our hope is that the study of kissing spheres may help understanding ball packings. In this paper, kissing spheres are allowed to intersect each other, so we are not restricted to the packing configuration. However, the distance function in Section 4 is intentionally designed to indicate the relations between kissing spheres (intersection, tangency or disjointness).

Our research provides a geometric picture for the set of kissing spheres, but we do not expect a direct application to the packing problem. In fact, since the distance geometry for kissing spheres degenerates to Euclidean distance geometry, the packing problem for kissing spheres contains as a special case the packing problem for unit balls, which is an NP-hard problem [18]. This will be discussed in detail at the end of the paper.

2. Preliminaries

In this section, we introduce distance geometry in a very general framework, and present some results from Euclidean distance geometry.

Let I be a set. A non-negative function $d : I \times I \to \mathbb{R}_{\geq 0}$ is a *distance function* on I if $d(i, j) = d(j, i)$ and $d(i, i) = 0$ for all $i, j \in I$. The pair (I, d) is called a *distance space*. So the Euclidean distance space $(\mathbb{E}^n, d_E)$ is the Euclidean space $\mathbb{E}^n$ equipped with the Euclidean distance d_E.

Note that the distance function is not necessarily a metric, as $d(i, j) = 0$ is allowed for $i \neq j$, and the triangular inequality is not required. Our distance function d_K for kissing spheres, defined in Section 4, is an example of non-metric distance function.

Definition 2.1 (Isometrically embeddable)**.** Let (I, d) and (I', d') be two distance spaces. We say that (I, d) is *isometrically embeddable* into (I', d') if there exists a map $\sigma : I \to I'$ such that $d'(\sigma(i), \sigma(j)) = d(i, j)$ for all $i, j \in I$. In this case, we call σ an *isometric embedding*.

For the embeddability problems studied in distance geometry, the set I is often finite. In this case, we can assume that $I = \{0, \dots, k\}$, and write σ_i instead of $\sigma(i)$ for $i \in I$.

For a subset $J \subset I$, we denote by (J, d) the distance space consisting of J and the distance function d restricted to J.

So in Euclidean distance geometry, the *embeddability problem* asks: Given a finite distance space (I, d), is it isometrically embeddable into the Euclidean distance space $(\mathbb{E}^n, d_E)$?

For a finite distance space (I, d), where $I = \{0, \dots, k\}$, there are two powerful tools for solving this problem. One is the *distance matrix* $D(I, d)$, defined as the $(k+1) \times (k+1)$ matrix whose i, j entry is the squared distance $d(i, j)^2$ for $i, j \in I$. The other is the *Cayley–Menger matrix* $M(I, d)$, defined as the $(k + 2) \times (k + 2)$

matrix

$$M(I,d) = \begin{pmatrix} D(I,d) & e \\ e^T & 0 \end{pmatrix},$$

where e denotes the all-ones column vector of length $k+1$.

The following theorem combines some important results of Euclidean distance geometry (see [14, 15, 16, 24]).

Theorem 2.2. *For a finite distance space (I,d), where d is non-zero on I, consider the following statements:*

(i) *(I,d) is isometrically embeddable into $(\mathbb{E}^n, d_E)$.*
(ii) *The rank of the Cayley–Menger matrix $M(I,d)$ is at most $n+2$ and*

$$(-1)^{|J|} \det M(J,d) \geq 0$$

for all $J \subseteq I$.
(iii) *The Cayley–Menger matrix $M(I,d)$ has exactly one positive eigenvalue and at most $n+1$ negative eigenvalues.*
(iv) *The distance matrix $D(I,d)$ has exactly one positive eigenvalue and at most $n+1$ negative eigenvalues.*

Then, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

If we are not given a complete information about the distances, a natural question to ask is the distance completion problem. In the language of graph theory:

Definition 2.3 ((Distance) completable)**.** Let an undirected graph $G = (V, \mathcal{E})$ be given with a *length function* $\ell : \mathcal{E} \to \mathbb{R}_{\geq 0}$. We say that (G, ℓ) is *completable* in a distance space (I,d) if there is a distance function d_V on the vertex set V such that (V, d_V) is isometrically embeddable into (I,d) and $d(u,v) = \ell(u,v)$ for all $(u,v) \in \mathcal{E}$.

For a subgraph $H \subset G$, we denote by (H, ℓ) the graph H equipped with the length function ℓ restricted to H.

So in Euclidean distance geometry, the *distance completion problem* asks: Given a graph $G = (V, \mathcal{E})$ and a length function ℓ on G, is (G, ℓ) completable in $(\mathbb{E}^n, d_E)$?

We define two sets of length functions as follows:

$$\mathcal{C}^n_E(G) = \{\ell : \mathcal{E} \to \mathbb{R}_{\geq 0} \mid (G, \ell) \text{ is completable in } (\mathbb{E}^n, d_E)\},$$
$$\mathcal{K}^n_E(G) = \{\ell : \mathcal{E} \to \mathbb{R}_{\geq 0} \mid \text{for all cliques } K \subset G, (K, \ell) \text{ is completable in } (\mathbb{E}^n, d_E)\}.$$

In general these two sets are not equal, but we have the following theorem:

Theorem 2.4 (Laurent [21])**.** *$\mathcal{C}^n_E(G) = \mathcal{K}^n_E(G)$ if and only if G is chordal.*

Here, a *chordal graph* is a graph without chordless cycles with more than three edges, or equivalently, every such cycle has an edge joining two vertices not adjacent in the cycle. This is a very important class of graphs [13].

Therefore, for a chordal graph, in order to tell for a length function whether it is completable in Euclidean distance space, we only need to check all the maximal cliques for the embeddability, which makes the distance completion problem much easier.

3. Kissing spheres

We now formally define kissing spheres, the main object of study of this paper. For this we work in the extended n-dimensional Euclidean space $\hat{\mathbb{E}}^n = \mathbb{E}^n \cup \{\infty\}$, with Cartesian coordinate system $(x_0, \ldots, x_{n-1})$. The sphere centered at $\mathbf{o} \in \mathbb{E}^n$ with diameter $\phi > 0$ is the set $\{\mathbf{x} \in \mathbb{E}^n \mid d_E(\mathbf{x}, \mathbf{o}) = \phi/2\}$. We regard $(n-1)$-hyperplanes (with the adjoined infinity point) as spheres of infinite diameter, whose center $\mathbf{o}$ is at infinity.

Let κ be a real number. A *ball* of curvature κ may be one of the following:

- a set $\{\mathbf{x} \mid d_E(\mathbf{x}, \mathbf{o}) \leq 1/\kappa\}$ if $\kappa > 0$;
- a set $\{\mathbf{x} \mid d_E(\mathbf{x}, \mathbf{o}) \geq -1/\kappa\}$ if $\kappa < 0$;
- a closed half-space if $\kappa = 0$.

In the first two cases, $\mathbf{o} \in \mathbb{E}^n$ is the center of the ball.

A sphere is said to be *tangent* to a ball at a point $\mathbf{t} \in \hat{\mathbb{E}}^n$ if $\mathbf{t}$ is the only element in their intersection. We call $\mathbf{t}$ the *tangency point*, which can be at infinity if it involves a ball of curvature 0 and a sphere of infinit diameter.

Definition 3.1 (Kissing sphere)**.** Fix a ball in $\hat{\mathbb{E}}^n$ as the reference ball. A *kissing sphere* is a sphere tangent to the reference ball.

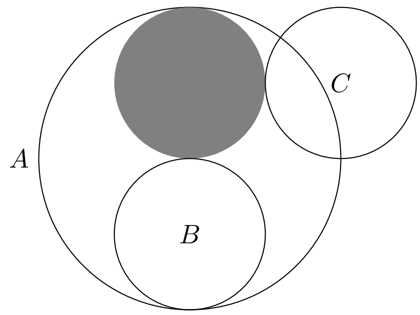

FIGURE 1. Three kissing spheres to the gray reference ball in $\hat{\mathbb{E}}^2$, as defined in Definition 3.1.

Our main concern is the combinatorics, which means relations like tangency, intersection and disjointness between the kissing spheres. These are defined as follows: Two kissing spheres are *tangent* to each other if their intersection consists of a single point which is not on the boundary of the reference ball. Two kissing spheres *intersect* if their intersection consists of more than one point, or if they are tangent to the reference ball at a same tangency point. Two kissing spheres are *disjoint* if their intersection is empty. For example, in Figure 1, A is tangent to B, B is disjoint from C, and C intersects A.

In the following, we assume the reference ball to be the half-space $x_0 \leq 0$. If it is not the case, we can always find a Möbius transformation that sends the reference ball to the half-space without any change to the combinatorics of kissing spheres.

Therefore we define a kissing sphere alternatively as follows.

Definition 3.1′ (Kissing sphere, alternative)**.** A *kissing sphere* in $\hat{\mathbb{E}}^n$ is a sphere tangent to the half-space $x_0 \leq 0$.

Such a kissing sphere must lie in the half-space $x_0 \geq 0$. Note that the tangency point can be at infinity. In this case, the kissing sphere is a hyperplane $x_0 = h > 0$ (with the adjoined infinity point), and we say that it is a hyperplane at level h.

We denote by $\mathbb{K}^n$ the set of kissing spheres in $\hat{\mathbb{E}}^n$ as defined in Definition 3.1′. For a kissing sphere $p \in \mathbb{K}^n$, as shown in Figure 2 for $n = 2$, we denote by $\mathbf{t}(p) \in \hat{\mathbb{E}}^{n-1}$ the tangency point on the $(n-1)$-dimensional hyperplane $\hat{\mathbb{E}}^{n-1}$, and by $\phi(p) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ the diameter of p. Therefore if $\phi(p) < \infty$, the pair $(\phi(p), \mathbf{t}(p))$ is the "north pole" of p, situated in the half-space $x_0 > 0$.

4. A DISTANCE FUNCTION

Möbius transformations are conformal diffeomorphisms from S^n to S^n. Viewing $\hat{\mathbb{E}}^n$ as S^n, Möbius transformations map spheres to spheres. They form a group

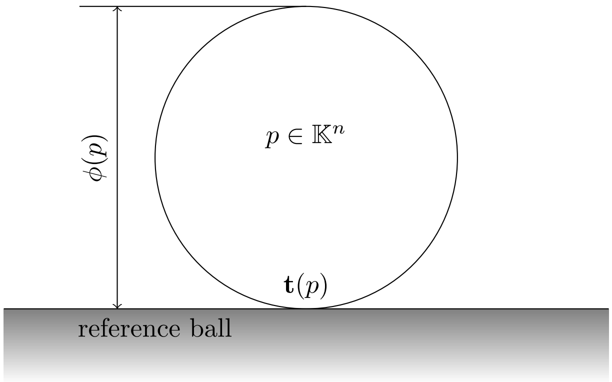

FIGURE 2. A kissing sphere as defined in Definition 3.1′.

called the *Möbius group*, denoted by Möb(n). Please note that, as in [17], we do not require Möbius transformations to preserve the orientation. Therefore reflections and inversions are also Möbius transformations.

Let $T \in$ Möb(n) be a Möbius transformation that preserves the half-space $x_0 \leq 0$. Then T maps kissing spheres to kissing spheres. For a sphere centered on the hyperplane $x_0 = 0$, its image under T is also centered on $x_0 = 0$. Therefore the restriction of T on the hyperplane $x_0 = 0$ is a Möbius transformation on $\hat{\mathbb{E}}^{n-1}$. Conversely, by *Poincaré extension* [2, Section 3.3], any Möbius transformation on $\hat{\mathbb{E}}^{n-1}$ can be naturally extended to a Möbius transformation on $\hat{\mathbb{E}}^n$ that preserves the half-space $x_0 \leq 0$. We thus define the action of a Möbius transformation $T \in$ Möb($n-1$) on $\mathbb{K}^n$ to be the action of its Poincaré extension.

We define a distance function d_K on $\mathbb{K}^n$ as follows.

Definition 4.1 (Distance function for kissing spheres)**.** Let $p, q \in \mathbb{K}^n$ be two kissing spheres. If there is a Möbius transformation T that preserves the half-space $x_0 \leq 0$ such that

$$\phi(Tp) = \phi(Tq) = 1,$$

then the distance between p and q is $d_K(p,q) := d_E(\mathbf{t}(Tp), \mathbf{t}(Tq))$. If such a T does not exist, $d_K(p,q) := 0$.

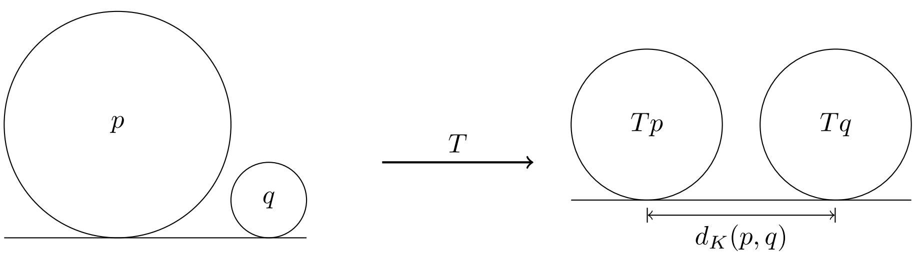

FIGURE 3. Definition of the distance, when T exists.

The required transformation T does not exist if and only if $\mathbf{t}(p) = \mathbf{t}(q)$ but $p \neq q$.

Theorem 4.2. *The distance function d_K of Definition 4.1 is well defined. That is, d_K is independent of the choice of T.*

In other words, d_K defined on $\mathbb{K}^n$ is invariant under the action of Möb($n-1$).

As a warm-up before the proof, we shall look at the effect that an inversion preserving the half-space $x_0 \leq 0$ has on a kissing sphere. Let s be a sphere with radius r centered at a point $\mathbf{o}$ on the hyperplane $x_0 = 0$. For a kissing sphere $p \in \mathbb{K}^n$, we denote by p^s the image of p under the inversion transform with respect to s. We have

$$\phi(p^s) = \frac{r^2\phi(p)}{d_E(\mathbf{o}, \mathbf{t}(p))^2} \tag{1}$$

and

$$d_E(\mathbf{o}, \mathbf{t}(p^s)) = \frac{r^2}{d_E(\mathbf{o}, \mathbf{t}(p))}, \tag{2}$$

where $d_E(\mathbf{x}, \mathbf{y})$ is the Euclidean distance on the hyperplane $x_0 = 0$.

The effect of such an inversion on p is then the same as the effect of a dilation of scale factor $r^2/d_E(\mathbf{o}, \mathbf{t}(p))^2$. The scale factor does not depend on the diameter of p. This will be a useful fact.

Proof of Theorem 4.2. An explicit calculation is not necessary, but may help to understand the situation.

Let p, q be two kissing spheres such that $\phi(p), \phi(q) < \infty$ and $d_E(\mathbf{t}(p), \mathbf{t}(q)) > 0$. The infinite case and the degenerate case will be discussed later.

Choose a point $\mathbf{o}$ on the line segment $\mathbf{t}(p)\mathbf{t}(q)$, such that $d_E(\mathbf{o}, \mathbf{t}(p))/d_E(\mathbf{o}, \mathbf{t}(q)) = \sqrt{\phi(p)/\phi(q)}$. That is,

$$d_E(\mathbf{o}, \mathbf{t}(p)) = \frac{d_E(\mathbf{t}(p), \mathbf{t}(q))\sqrt{\phi(p)}}{\sqrt{\phi(p)} + \sqrt{\phi(q)}},$$

$$d_E(\mathbf{o}, \mathbf{t}(q)) = \frac{d_E(\mathbf{t}(p), \mathbf{t}(q))\sqrt{\phi(q)}}{\sqrt{\phi(p)} + \sqrt{\phi(q)}}.$$

Let s be a sphere centered at $\mathbf{o}$ with radius

$$r = \frac{d_E(\mathbf{t}(p), \mathbf{t}(q))}{\sqrt{\phi(p)} + \sqrt{\phi(q)}}.$$

Then, by (1) and (2), we have $\phi(p^s) = \phi(q^s) = 1$, and

$$d_K(p, q) = d_E(\mathbf{t}(p^s), \mathbf{t}(q^s)) = d_E(\mathbf{o}, \mathbf{t}(p^s)) + d_E(\mathbf{o}, \mathbf{t}(q^s)) = \frac{d_E(\mathbf{t}(p), \mathbf{t}(q))}{\sqrt{\phi(p)\phi(q)}}. \tag{3}$$

Any Möbius transformation is generated by reflections and inversions. For details, see [2, Definition 3.1.1], where inversions are interpreted as reflections with respect to a sphere, or [7, Theorem 3.8], where reflections are viewed as inversions with respect to a plane. Theorem 4.2 is obviously true for reflections. We shall study the inversions in detail.

An inversion preserving the half-space $x_0 \leq 0$ must have its inversion sphere centered at a point $\mathbf{o}$ on the hyperplane $x_0 = 0$. Let s be such an inversion sphere of radius r. By (2), we have $d_E(\mathbf{o}, \mathbf{t}(p^s)) = r^2/d_E(\mathbf{o}, \mathbf{t}(p))$ and $d_E(\mathbf{o}, \mathbf{t}(q^s)) = r^2/d_E(\mathbf{o}, \mathbf{t}(q))$. Thanks to the independence of the scale factor of the diameter, the triangle $\mathbf{ot}(p)\mathbf{t}(q)$ and the triangle $\mathbf{ot}(q^s)\mathbf{t}(p^s)$ are similar, and

$$d_E(\mathbf{t}(p^s), \mathbf{t}(q^s)) = \frac{r^2 d_E(\mathbf{t}(p), \mathbf{t}(q))}{d_E(\mathbf{o}, \mathbf{t}(p))d_E(\mathbf{o}, \mathbf{t}(q))}.$$

We then have

$$d_K(p^s, q^s) = \frac{d_E(\mathbf{t}(p^s), \mathbf{t}(q^s))}{\sqrt{\phi(p^s)\phi(q^s)}} = \frac{d_E(\mathbf{t}(p), \mathbf{t}(q))}{\sqrt{\phi(p)\phi(q)}} = d_K(p, q),$$

which proves the theorem.

We now extend the calculation to the infinite case. Let p be a hyperplane at level h. Consider again a sphere s centered at a point $\mathbf{o}$ on $x_0 = 0$ with radius r. Then $\phi(q^s) = r^2\phi(q)/d_E(\mathbf{o}, \mathbf{t}(q))^2$, $\phi(p^s) = r^2/h$, and $d_E(\mathbf{t}(p^s), \mathbf{t}(q^s)) = r^2/d_E(\mathbf{o}, \mathbf{t}(q))$. Since the inversion preserves the distance d_K, by (3), we have

$$d_K(p,q) = \frac{d_E(\mathbf{t}(p^s), \mathbf{t}(q^s))}{\sqrt{\phi(p^s)\phi(q^s)}} = \sqrt{\frac{h}{\phi(q)}}. \tag{4}$$

Finally we study the degenerate case. That is, $\mathbf{t}(p) = \mathbf{t}(q)$ but $p \neq q$.

Since a Möbius transformation is bijective, it is impossible to transform p and q into spheres of same diameter. According to the definition, $d_K(p,q) = 0$. This is reasonable, since it's the limit of (3) as $d_E(\mathbf{t}(p), \mathbf{t}(q))$ tends to 0, or the limit of (4) as $\phi(q)$ tends to infinity. This is Möbius invariant since $\mathbf{t}(p) = \mathbf{t}(q)$ holds under any Möbius transformation. □

Remark. As mentioned in the preliminaries, d_K is not a metric. More specifically, the triangle inequality is not satisfied in general, and there are $p, q \in \mathbb{K}^n$ such that $p \neq q$ but $d_K(p,q) = 0$.

Theorem 4.3. *The distance function d_K reflects the combinatorics as follows:*

$$d_K(p,q) \begin{cases} > 1 & \text{if } p \text{ is disjoint from } q; \\ = 1 & \text{if } p \text{ is tangent to } q; \\ < 1 & \text{if } p \text{ intersects } q; \\ = 0 & \text{if } \mathbf{t}(p) = \mathbf{t}(q). \end{cases}$$

We notice from (3) that the distance space for a set of kissing spheres is conformally Euclidean, as defined in the following (compare conformal equivalence defined by Bobenko et al. [4] and Luo [22]):

Definition 4.4 (Conformal embedding)**.** Let (I, d) and (I', d') be two distance spaces. We say that (I, d) is *conformally embeddable* into (I', d') if there exists a map $\xi : I \to I'$ and a real valued function $f : I \to \mathbb{R}_{\geq 0}$, such that $d'(\xi_i, \xi_j) = f(i)f(j)d(i,j)$ for all $i, j \in I$. We say that ξ is a *conformal embedding* with the *conformal factor* f.

(I, d) and (I', d') are *conformally equivalent* if the conformal embedding is bijective.

(I, d) is *conformally Euclidean* if it is conformally embeddable into the Euclidean distance space.

In fact, if a finite distance space (I, d) is embeddable into $(\mathbb{K}^n, d_K)$, we can always choose an isometric embedding σ such that $\phi(\sigma_i) < \infty$ for all $i \in I$. We then recognise in (3) that $\mathbf{t} \circ \sigma$ conformally maps (I, d) to $(\mathbb{E}^{n-1}, d_E)$, with conformal factor $\sqrt{\phi \circ \sigma}$.

Degeneration, Case 1. If there exists an embedding σ such that $\phi(\sigma_i) = 1$ for all $i \in I$, then d_K degenerates to Euclidean distance.

5. Embeddability problem

We now prove our first main theorem.

Theorem 5.1. *Given a finite distance space (I, d), where d is non-zero on I, the following statements are equivalent:*

(i) *(I, d) is isometrically embeddable into $(\mathbb{K}^n, d_K)$.*

(ii) *The rank of the distance matrix $D(I,d)$ is at most $n+1$ and*

$$(-1)^{|J|}\det D(J,d) \le 0$$

for all $J \subseteq I$.

(iii) *The distance matrix $D(I,d)$ has exactly one positive eigenvalue and at most n negative eigenvalues.*

Our proof is inspired by the proofs in [11, Sect. 6.2]. It will use the notion of *Schur complement*: Consider a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A and D are square matrices, and D is invertible. After a block Gaussian elimination, it becomes $\left(\begin{smallmatrix} P & 0 \\ 0 & D \end{smallmatrix}\right)$, where $P = A - BD^{-1}C$ is called the Schur complement of D.

Proof of Theorem 5.1. Assume that $I = \{0, \cdots, k\}$ and that for all $i,j \in I$, $d(i,j) = 0$ if and only if $i = j$. So the off-diagonal entries are all positive. The degenerate case, where some off-diagonal entries vanish, will be discussed later.

If (I,d) is isometrically embeddable into $(\mathbb{K}^n, d_K)$, we can choose an embedding $\sigma : I \to \mathbb{K}^n$ such that σ_k is the hyperplane at level 1. This is always possible, because d_K is invariant under the action of $\mathrm{Möb}(n-1)$. We then write the distance matrix $D(I,d)$ explicitly. It will be in the form

$$D(I,d) = \begin{pmatrix} D(I \setminus \{k\}, d) & \Phi \\ \Phi^t & 0 \end{pmatrix}, \tag{5}$$

where Φ denotes a $k \times 1$ column matrix whose i-th entry is $1/\phi(\sigma_i)$ for $0 \le i < k$.

Consider the sub-distance-matrix $D(\{0,k\},d)$, one can easily verify that its Schur complement, denoted by $P(I,d)$, is in the form

$$\begin{aligned} P(I,d)_{ij} &= \frac{d_E(\mathbf{t}(\sigma_i), \mathbf{t}(\sigma_j))^2 - d_E(\mathbf{t}(\sigma_i), \mathbf{t}(\sigma_0))^2 - d_E(\mathbf{t}(\sigma_0), \mathbf{t}(\sigma_j))^2}{\phi(\sigma_i)\phi(\sigma_j)} \\ &= -2\left\langle \frac{\mathbf{t}(\sigma_i) - \mathbf{t}(\sigma_0)}{\phi(\sigma_i)}, \frac{\mathbf{t}(\sigma_j) - \mathbf{t}(\sigma_0)}{\phi(\sigma_j)} \right\rangle \end{aligned}$$

for $i,j \in I \setminus \{0,k\}$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. Since $I \setminus \{k\}$ is embedded by σ into $\mathbb{K}^n$, $\mathbf{x}_i = \mathbf{t}(\sigma_i) - \mathbf{t}(\sigma_0)$ are vectors in the $(n-1)$-dimensional hyperplane $x_0 = 0$. Therefore, $P(I,d)$ is negative semi-definite, whose rank is at most $n-1$.

Inversely, let $P(I,d)$ be the Schur complement of the submatrix $D(\{0,k\},d)$. If it is negative semi-definite with rank at most $n-1$, it can be written in the form $P(I,d)_{ij} = -2\langle \mathbf{x}_i, \mathbf{x}_j \rangle$ for $0 < i,j < k$, where $\mathbf{x}_i$ are vectors in an $(n-1)$-dimensional hyperplane, which can be, without loss of generality, assumed to be $x_0 = 0$. Then an embedding can be constructed by setting σ_k as the hyperplane at level 1, and

$$\begin{aligned} \phi(\sigma_i) &= 1/D(I,d)_{ik} \\ \mathbf{t}(\sigma_i) &= \begin{cases} \mathbf{x}_i / D(I,d)_{ik} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases} \end{aligned}$$

for $0 \le i < k$.

We have proved that (I,d) is isometrically embeddable into $(\mathbb{K}^n, d_K)$ if and only if $P(I,d)$ is negative semi-definite of rank at most $n-1$. We also have the following relations for the Schur complement

$$\det D(I,d) = -\frac{\det P(I,d)}{\phi(\sigma_0)^2} \tag{6a}$$

$$\mathrm{In}\, D(I,d) = (1,1,0) + \mathrm{In}\, P(I,d) \tag{6b}$$

$$\mathrm{rk}\, D(I,d) = \mathrm{rk}\, P(I,d) + 2 \tag{6c}$$

where $\operatorname{rk} M$ is the rank of M, and $\operatorname{In} M$ is the inertia of matrix M, which is a triple indicating (in order) the number of positive, negative and zero eigenvalues of M. Here we use the convention that the determinant of an empty matrix is 1.

These relations allow us to express the negative semi-definiteness and the rank of $P(I,d)$ by the distance matrices.

A principal submatrix of $P(I,d)$ is of the form $P(J,d)$ for some subset J of I such that $J \supseteq \{0,k\}$. $P(I,d)$ is negative semi-definite if and only if for any principal submatrix $P(J,d)$, $(-1)^{|J|} \det P(J,d) \geq 0$. Notice that the choice of $\{0,k\}$ is arbitrary, since we can always apply a permutation on I, to bring any index to 0 or k. So we have $(-1)^{|J|} \det P(J,d) \geq 0$ for all $J \subseteq I$ (the case $|J| \leq 1$ is trivial). Then Relations (6a) and (6c) prove (i) $\Leftrightarrow$ (ii).

$P(I,d)$ is negative semi-definite if and only if all its eigenvalues are nonpositive. Then Relations (6b) and (6c) prove (i) $\Leftrightarrow$ (iii).

For the degenerate case, suppose that the $d(0,k) = 0$ and keep other distances positive. Now the sub-distance-matrix $D(\{0,k\},d)$ is a zero matrix, so we can not argue by Schur complement.

However, since d is not zero on I, we can always relabel the elements of I so that $d(0,k) > 0$. So the argument for the rank and the inertia is still valid. For a subset $J \subseteq I$, if 0 or k is not in J, the non-degenerate results applies. Otherwise, if $\{0,k\} \subseteq J$, the 0-th row(and column) of $D(J,d)$ is a multiple of the k-th row(and column), so $\det D(J,d) = 0$.

By induction, the theorem remains valid as long as the off-diagonal entries of $D(I,d)$ are not all zero, as assumed in the theorem. Anyway, a zero distance function is not interesting. $\square$

Equation (5) shows that the distance matrix for kissing spheres is playing dual roles: It combines the power of the distance matrix and the Cayley–Menger matrix. Alternatively, this can be seen by comparing Theorem 2.2 and Theorem 5.1,

Degeneration, Case 2. We notice from (5) that if $\phi(\sigma_i) = 1$ for all $i \in I \setminus \{k\}$, then $\Phi = e$ and $D(I,d)$ degenerates to an Euclidean Cayley–Menger matrix.

6. Embedding into the lightcone

We now show that the space of kissing spheres in $\hat{\mathbb{E}}^n$ can be embedded into the Minkowski space $\mathbb{R}^{n,1}$, which will be useful later for the study of distance completion problem.

The *Minkowski space* $\mathbb{R}^{n,1}$ is an $(n+1)$-dimensional real vector space with an indefinite inner product of signature $(n,1)$. Explicitly, with the coordinate system $\mathbf{x} = (x_0, \ldots, x_{n-1}, t)$, the *Minkowskian inner product* $\langle \cdot, \cdot \rangle_{n,1}$ on $\mathbb{R}^{n,1}$ is defined as

$$\langle \mathbf{x}, \mathbf{x}' \rangle_{n,1} = x_0 x_0' + \ldots + x_{n-1} x_{n-1}' - tt'$$

for two vectors $\mathbf{x}$ and $\mathbf{x}'$ in $\mathbb{R}^{n,1}$. A vector $\mathbf{x}$ is *space-like* (resp. *null, time-like*) iff $\langle \mathbf{x}, \mathbf{x} \rangle_{n,1}$ is positive (resp. zero, negative). The *lightcone*, denoted by $\mathbb{L}^n$, is the set of null vectors, $\mathbb{L}^n = \{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{x} \rangle_{n,1} = 0\}$. A vector $\mathbf{x}$ is *future-directed* (resp. *past-directed*) if its last coordinate t is positive (resp. negative). We denote by $\mathbb{L}^n_+$ the future-directed lightcone, that is, the set of future directed null vectors (the origin excluded).

Let (I,d) be a distance space isometrically embeddable into $(\mathbb{K}^n, d_K)$. Since $D(I,d)$ has exactly one positive eigenvalue and at most n negative eigenvalues, we can factor it into $D(I,d) = Q^t \Lambda Q$, where Λ is an $(n+1) \times (n+1)$ diagonal matrix, with 1 as its last entry, and all other entries being -1. The columns of Q can be viewed as vectors in the Minkowski space $\mathbb{R}^{n,1}$, indexed by the elements of I. We can therefore write $D(I,d)_{ij} = -\langle \mathbf{x}_i, \mathbf{x}_j \rangle_{n,1}$ for a system of vectors $\{\mathbf{x}_i\}_{i \in I}$ in $\mathbb{R}^{n,1}$.

Since $\langle \mathbf{x}_i, \mathbf{x}_i \rangle_{n,1} = -D(I,d)_{ii} = 0$, all the vectors $\mathbf{x}_i$ are on the lightcone $\mathbb{L}^n$. Since

$$-\langle \mathbf{x}_i, \mathbf{x}_j \rangle_{n,1} = \frac{1}{2}\langle \mathbf{x}_i - \mathbf{x}_j, \mathbf{x}_i - \mathbf{x}_j \rangle_{n,1} = D(I,d)_{ij} \geq 0$$

for all $i \neq j$, the difference between any two vectors can not be time-like. Therefore, $\{\mathbf{x}_i\}_{i \in I}$ have to be either all future-directed, or all past-directed. The Minkowskian inner product induces a distance function

$$d_M(\mathbf{x}, \mathbf{x}')^2 := -\langle \mathbf{x}, \mathbf{x}' \rangle_{n,1} = \frac{1}{2}\langle \mathbf{x} - \mathbf{x}', \mathbf{x} - \mathbf{x}' \rangle_{n,1} =\geq 0 \tag{7}$$

on the future-directed lightcone $\mathbb{L}^n_+$. We can therefore view $D(I,d)$ as the Minkowskian distance matrix for a set of future-directed null vectors.

In fact, we just proved the following theorem:

Theorem 6.1. *$(\mathbb{K}^n, d_K)$ and $(\mathbb{L}^n_+, d_M)$ are isometric. That is, there is a bijective isometric embedding between $(\mathbb{K}^n, d_K)$ and $(\mathbb{L}^n_+, d_M)$.*

For example, let $\mathbb{K}^n_*$ be the set of kissing spheres of finite diameter, the following map Ψ_* from $\mathbb{K}^n_*$ to $\mathbb{L}^n_+$ is an isometric embedding:

$$\Psi_*(p) = \frac{\sqrt{2}}{2\phi(p)}\left(1 - \|\mathbf{t}(p)\|_2^2, 2\mathbf{t}(p), 1 + \|\mathbf{t}(p)\|_2^2\right). \tag{8}$$

It is easy to verify that $\langle \Psi(p), \Psi(p) \rangle_{n,1} = 0$ and $d_K(p,q) = d_M(\Psi(p), \Psi(q))$. We can extend Ψ_* to a map Ψ defined on $\mathbb{K}^n$ by setting

$$\Psi(p) = \frac{\sqrt{2}}{2}(-h, 0, \dots, 0, h)$$

if p is a hyperplane at level h. One can check that Ψ is bijective.

Remark (Continuous analogy). A continuous version of Theorem 6.1 can be found in [5], which states that a Riemann space is conformally Euclidean if and only if it can be embedded into the lightcone. A generalisation for conformally flat Riemann manifolds can be found in [1].

Remark (Reference ball of finite radius). If the reference ball has a non-zero curvature κ, we can either make it zero by a Möbus transformation before applying the isometric embedding in (8), or directly apply the following embedding to a kissing sphere p:

$$\Psi(p) = \Big(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{\kappa\phi(p)}\Big)(\hat{\mathbf{t}}(p), 1), \tag{9}$$

where $\phi(p)$ is now the signed diameter, negative if the sphere surrounds the reference ball, and $\hat{\mathbf{t}}(p)$ refers to the unit direction vector for the tangency point, taking the center of the reference ball as the origin. The Minkowski distance $d_M(\Psi(p), \Psi(q))$ can be used as a distance function for kissing spheres to a reference ball of non-zero curvature.

Remark (Lorentz invariance). The invariance of $D(I,d)$ under the action of Möb$(n-1)$ is reflected in $\mathbb{R}^{n,1}$ as the invariance under Lorentz transformations that preserves the direction of time. Indeed, Möb$(n-1)$ is isomorphic to the *orthochronous* Lorentz group $O_+(n,1)$ [7, Corollary 3.3].

Remark (Möbius geometry). In the projective model of Möbius geometry, points are mapped to null directions [7, Equation 2.3]. In fact, our isometric embedding maps the tangency points of kissing spheres to null directions in a same way, and use the vector lengths to distinguish different kissing spheres with a same tangency point.

Furthermore, there is a conventional way [7,17,23] to represent spheres in $\hat{\mathbb{E}}^n$ by vectors on the one-sheet hyperboloid

$$\mathbb{H}^{n+1,1} = \{\mathbf{x} \in \mathbb{R}^{n+1,1} \mid \langle \mathbf{x}, \mathbf{x} \rangle_{n+1,1} = 1\}. \tag{10}$$

If we take the Minkowski "distance" $d_M(\mathbf{x}, \mathbf{x}')^2 := -\langle \mathbf{x}, \mathbf{x}' \rangle_{n,1}$ (which is not a distance on $\mathbb{H}^{n+1,1}$), it reflects the relation between the correponding spheres (tangency, disjointness, intersection etc.) in a similar way as the isometric embedding in (8).

From the point of view of Klein's Erlangen Program, we can regard the geometry of kissing spheres as a "subgeometry" of Möbius geometry (geometry of spheres), which is in turn a "subgeometry" of Lie sphere geometry (geometry of oriented spheres) (see [7]).

We observe two more situations where distance geometry for kissing spheres degenerates to Euclidean distance geometry.

A non-zero vector $\mathbf{y}$ and a real number c determine a hyperplane $H = \{\mathbf{x} \mid \langle \mathbf{x}, \mathbf{y} \rangle_{n,1} = c\}$. We call $\mathbf{y}$ a *normal vector* of H. A hyperplane is said to be *space-like* (*resp. null, time-like*) if its normal vector is time-like (reps. null, space-like).

Degeneration, Case 3. Let I be a set of kissing spheres. Suppose that for all $p \in I$, $\Psi(p)$ lies on a same null hyperplane H. That is, H can be written in the form $H = \{\mathbf{x} \mid -\langle \mathbf{x}, \mathbf{y} \rangle_{n,1} = 1\}$, where $\mathbf{y}$ is a null vector. Then $\Psi^{-1}(\mathbf{y})$ is tangent to all the elements of I. There is a Lorentz transformation $\mathcal{L}$ that sends $\mathbf{y}$ to $(-\frac{\sqrt{2}}{2}, 0, \cdots, 0, \frac{\sqrt{2}}{2})$. For all $p \in I$, $\Psi^{-1}\mathcal{L}\Psi(p)$ is a unit kissing sphere (kissing spheres of unit diameter) since they are tangent to the hyperplane at level 1. Therefore $D(I, d)$ degenerates to Euclidean distance matrix, as discussed in Case 1.

Degeneration, Case 4. If for all $p \in I$, $\Psi(p)$ lies on a same space-like hyperplane H, then there is a Lorentz transformation $\mathcal{L}$ that sends H to a time-constant hyperplane. The intersection of $\mathcal{L}H$ with the lightcone is an $(n-1)$-sphere, and d_M degenerates to Euclidean distance d_E. One way to view this is to consider unit kissing spheres kissing a reference ball of *finite* radius. It turns out that d_K, induced by (9), equals the Euclidean distance between their centers.

Seidel also observed in [26, Theorem 4.5.3] that the distance matrix for points in Euclidean space can be written as the Gram matrix of null vectors on a time-constant hyperplane.

7. Distance completion problem

We now study the distance completion problem in $(\mathbb{K}^n, d_K)$. The following are consequences of our results for the embeddability problem.

Theorem 7.1. *Let a graph $G = (V, \mathcal{E})$ be given with a length function $\ell : \mathcal{E} \to \mathbb{R}_{\geq 0}$. Then (G, ℓ) is completable in $(\mathbb{K}^n, d_K)$ if and only if there is a non-negative symmetric matrix D satisfying*

- (**C1**) *$D_{uv} = 0$ if $u = v$.*
- (**C2**) *$D_{uv} = \ell(u, v)^2$ if and only if $(u, v) \in \mathcal{E}$.*
- (**C3**) *the rank of D is at most $n + 1$.*
- (**C4**) *D has exactly one positive eigenvalue.*

D is in fact the distance matrix corresponding to a distance function realising the given edge lengths. We call D a *target matrix*.

This theorem transforms a distance completion problem to a matrix completion problem: Some entries of the matrix being given (**C1** and **C2**), find the values for the other entries, so that the rank of matrix is low (**C3**). We refer to [6, 19] for more about matrix completion.

Compared to the classical matrix completion problem, (**C4**) is new. The target matrix is usually positive semi-definite, but here we need it to be indefinite. The result in the previous section can help on this point as follows:

If (G,ℓ) is completable in $(\mathbb{K}^n, d_K)$, then for every clique K of G, (K,ℓ) is completable in $(\mathbb{K}^n, d_K)$. The inverse is in general not true. Define two sets as in Euclidean case,

$$\mathcal{C}_K^n(G) = \{\ell : \mathcal{E} \to \mathbb{R}_{\geq 0} \mid (G,\ell) \text{ is completable in } (\mathbb{K}^n, d_K)\},$$
$$\mathcal{K}_K^n(G) = \{\ell : \mathcal{E} \to \mathbb{R}_{\geq 0} \mid \text{for all cliques } K \subset G, (K,\ell) \text{ is completable in } (\mathbb{K}^n, d_K)\}.$$

We now prove our second main theorem

Theorem 7.2. *$\mathcal{C}_K^n(G) = \mathcal{K}_K^n(G)$ if and only if G is chordal.*

This is almost the same as Theorem 2.4. In fact, we will employ the proof techniques in [21], but with some necessary adaptions.

Proof of the "only if" part of Theorem 7.2. If G is not chordal, consider a chordless cycle C of length at least 4, and pick an edge $e_0 \in C$. Construct a length function ℓ by setting $\ell(e) = 1$ if e has exactly one end in C or if $e = e_0$, otherwise $\ell(e) = 0$. Then $\ell \in \mathcal{K}_K^n(G)$ but $\ell \notin \mathcal{C}_K^n(G)$. □

The "if" part will be derived from the following.

If two graphs each has a clique of a same size, the *clique-sum* glues them together by identifying that clique. In the language of mathematics, consider a graph $G = (V,\mathcal{E})$ and two of its subgraphs $G_1 = (V_1,\mathcal{E}_1)$ and $G_2 = (V_2,\mathcal{E}_2)$. G is the clique-sum of G_1 and G_2 if $V = V_1 \cup V_2$ and $W = V_1 \cap V_2$ induces a clique in G, and there is no edge joining a vertex in $V_1 \setminus W$ and a vertex in $V_2 \setminus W$.

We use Theorem 6.1 to prove the following lemma.

Lemma 7.3. *Let G be a clique-sum of $G_1(V_1,\mathcal{E}_1)$ and $G_2(V_2,\mathcal{E}_2)$. If $\mathcal{C}_K^n(G_i) = \mathcal{K}_K^n(G_i)$ for $i = 1,2$, then $\mathcal{C}_K^n(G) = \mathcal{K}_K^n(G)$.*

Proof. Let ℓ be an element in $\mathcal{K}_K^n(G)$. Obviously, $\ell \in \mathcal{K}_K^n(G_i) = \mathcal{C}_K^n(G_i)$, for $i = 1,2$. Since (G_1,ℓ) and (G_2,ℓ) are n-completable, let D_1 and D_2 be the corresponding target matrices. We can find a system of future-directed null vectors $\mathbf{x}_u \in \mathbb{L}_+^n$ for $u \in V_1$ such that $(D_1)_{uv} = d_M(\mathbf{x}_u,\mathbf{x}_v)^2$ for $u,v \in V_1$, and $\mathbf{y}_u \in \mathbb{L}_+^n$ for $u \in V_2$ such that $(D_2)_{uv} = d_M(\mathbf{y}_u,\mathbf{y}_v)^2$ for $u,v \in V_2$. On the common clique, for all $u,v \in V_1 \cap V_2$, we have $d_M(\mathbf{x}_u,\mathbf{x}_v) = d_M(\mathbf{y}_u,\mathbf{y}_v) = \ell(u,v)$. Therefore, there is a Lorentz transformation $\mathcal{L}$, such that $\mathcal{L}\mathbf{x}_u = \mathbf{y}_u$ for $u \in V_1 \cap V_2$. Now we construct a system of vectors by setting $\mathbf{z}_u = \mathcal{L}\mathbf{x}_u$ for $u \in V_1$, and $\mathbf{z}_u = \mathbf{y}_u$ for $u \in V_2 \setminus V_1$. The matrix $D_{uv} = d_M(\mathbf{z}_u,\mathbf{z}_v)$ is a target matrix for (G,ℓ), therefore $\ell \in \mathcal{C}_K^n(G)$. □

Proof of the "if" part of Theorem 7.2. It follows from Lemma 7.3 since any chordal graph can be built up by clique-sums. □

Therefore, for a chordal graph, in order to tell for a length function whether it is completable in $(\mathbb{K}^n, d_K)$, we only need to check all the maximum cliques for embeddability, which makes the distance completion problem much easier.

Remark (Packing problem). It is a natural idea to apply distance geometry to the packing problem, which is also our motivation for studying kissing spheres, as mentioned in the introduction. We would like to point out that this is difficult.

Formally, a *packing of kissing spheres* is a set of kissing spheres, no two of which intersect. A famous example would be the *Ford circles* [12]. The *tangency graph* of such a packing takes the kissing spheres as vertices and the tangency relations as edges. The packing problem asks: Given a graph G, is there a packing of kissing spheres whose tangency graph is isomorphic to G? If such a packing exists, we say that G is *kissing-sphere packable*.

In the language of distance completion problem, consider a length function $\ell = 1$ on G. If G is kissing-sphere packable, then (G, ℓ) is completable in $(\mathbb{K}^n, d_K)$. However, the inverse is in general not true. Since no intersection is allowed in the packing, one is only allowed to complete the distances with values ≥ 1, which makes the situation much more complicated. In fact, a special case of kissing sphere packing is the unit ball packing, the recognition of whose tangency graph is proved to be an NP-hard problem [18].

Refering to the spherical code [9], a packing of kissing spheres, embedded into the lightcone, can be viewed as a "quasi-spherical code" on the lightcone with minimum Minkowski distance 1. Similarly, following the remarks in Section 6, a packing of spheres can be viewed as a "quasi-spherical code" on the one-sheet hyperboloid (cf. (10)) with minimum Minkowski "distance" 1. With this point of view, we would expect more investigations on discrete geometry in Minkowski space.

Inspired by the fact that chordal graphs are easier for distance completion, we managed to characerise the stacked polytopes whose graph (chordal) is the tangency graph of a sphere packing. A paper [8] on this result is in preparation.

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