ON THE BALL-PACKABILITY OF GRAPH JOINS

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Abstract. A graph is ball packable if it can be realized as the tangency relations of a ball packing in a given dimension. We study in this paper the ball-packability of some small graphs that can be written in the form of graph joins. This is then applied to characterise the ball-packability of the 1-skeletons of stacked 4-polytopes.

1. Introduction and Preliminaries

The combinatorics of disk packing is well understood thanks to Koebe–Andreev–Thurston's disk packing theorem, but we know few about the combinatorics of ball packings in higher dimensions.

For a better understanding of ball packings, we focus in this paper on some small graphs that can be written in form of graph joins. The ball-packability becomes much easier to investigate because of the graph join structure. For graphs that are packable, we will explicitely construct their ball packings.

As an application, one graph investigated in Section 2 that is not packable is proved to be the forbidden indued subgraph characterising the ball-packability of 1-skeletons of stacked 4-polytopes. This result is based on previous works on Apollonian ball packings.

The paper is organised in three sections: Graph joins will be studied in Section 2, stacked polytopes will be studied in Section 3, while the present section is dedicated to definitions, notations, useful tools and other necessary preliminaries, as well as reviews of previous works on ball packings.

1.1. **Ball packings.** We work in the d-dimensional extended Euclidean space $\hat{\mathbb{R}}^d = \mathbb{R}^n \cup \{\infty\}$. A *$d$-ball* of curvature κ means one of the followings:

$$\begin{cases} \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{c}\| \leq 1/\kappa\} & \text{if } \kappa > 0 \\ \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{c}\| \geq -1/\kappa\} & \text{if } \kappa < 0 \\ \{\mathbf{x} \mid \langle \mathbf{x}, \hat{\mathbf{n}} \rangle \geq b\} & \text{if } \kappa = 0 \end{cases}$$

where $\|\cdot\|$ is the Euclidean norm, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. In the first two cases, $\mathbf{c} \in \mathbb{R}^d$ is called the *center* of the ball. In the last case, the unit vector $\hat{\mathbf{n}}$ is called the *normal vector* of a half-space.

Two balls are tangent at a point $\mathbf{t} \in \hat{\mathbb{R}}^d$ if $\mathbf{t}$ is the only element of their intersection. We call $\mathbf{t}$ the *tangency point*, which can be the infinity point if it involves two disjoint half-spaces.

For a ball $S \subset \hat{\mathbb{R}}^d$, the *curvature-center coordinates* is introduced in [19]

$$\mathbf{m}(S) = \begin{cases} (\kappa, \kappa\mathbf{c}) & \text{if } \kappa \neq 0 \\ (0, \hat{\mathbf{n}}) & \text{if } \kappa = 0 \end{cases}$$

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Here, the term "coordinate" is an abuse of language, since the curvature-center coordinates do not uniquely determine a ball when $\kappa = 0$. A real global coordinate system would be the *augmented curvature-center coordinates* [19]. But in this paper, the curvature-center coordinates are good enough for our use.

Definition 1.1. A *d-ball packing* is a collection of d-balls with disjoint interiors.

For a ball packing $\mathcal{S}$, its *tangency graph $G(\mathcal{S})$* takes the balls as vertices and the tangency relations as the edges. It is invariant under Möbius transformations and reflections.

Definition 1.2. A graph G is said to be *d-ball packable* if there is a d-ball packing $\mathcal{S}$ whose tangency graph is isomorphic to G. In this case, we say that $\mathcal{S}$ is a d-ball packing of G.

Disk packings, or 2-ball packings, are well understood thanks to the following famous theorem:

Theorem 1.3 (Koebe–Andreev–Thurston theorem)**.** *Every connected simple planar graph is disk packable. If the graph is a finite triangulated planar graph, then it has a unique disk packing up to Möbius transformations and reflections.*

We know few about combinatorics of ball packings in higher dimensions. Some attempts of generalizing the disk packing theorem to higher dimensions include [3, 8, 18, 21].

An induced subgraph of a ball packable graph is also ball packable. In other words, the class of ball packable graphs is closed under the induced subgraph operation. This gives the motivation for investigating small graphs.

Notation and convention. Throughout this paper, a ball packing is always in dimension d. Correspondingly, a polytope is in dimension $d+1$, d or $d-1$ depending on the context.

We denote by G_n any graph on n vertices.

We use the following notations for some special graphs.

- P_n**:** the path on n vertices (therefore of length $n-1$);
- C_n**:** the cycle on n vertices;
- K_n**:** the complete graph on n vertices;
- $\bar{K}_n$**:** the empty graph on n vertices;
- $\Diamond_d$**:** the 1-skeleton of the d-dimensional orthoplex;

The *join* of two graphs G and H, denoted by $G \star H$, is the graph obtained by connecting every vertex of G to every vertex of H. Most of the graphs in this paper will be expressed in term of graph joins. Notably, $\Diamond_d = \underbrace{\bar{K}_2 \star \cdots \star \bar{K}_2}_{d}$.

1.2. **Descartes configurations.** A *Descartes configuration* in dimension d is a d-ball packing consisting of $d+2$ pairwisely tangent balls. The tangency graph of a Descartes configuration is K_{d+2}. This is the basic element for the construction of many ball packings in this paper.

The following relation was first established for dimension 2 by René Descartes in a letter [10] to Princess Elizabeth of Bohemia, then generalized to dimension 3 by Soddy in the form of a poem [26], and finally generalized to any dimension by Gossett [12].

Theorem 1.4 (Descartes–Soddy–Gossett Theorem)**.** *In dimension d, if $d+2$ balls $S_0, \cdots, S_{d+1}$ form a Descartes configuration, let κ_i be the curvature of S_i $(0 \leq i \leq$*

$d+1$), *then*

$$\sum_{i=0}^{d+1} \kappa_i^2 = \frac{1}{d}\Big(\sum_{i=0}^{d+1} \kappa_i\Big)^2 \tag{1}$$

Equivalently, $\mathbf{K}^\intercal \mathbf{Q}_d \mathbf{K} = 0$, where $\mathbf{K} = (\kappa_0, \cdots, \kappa_{d+1})^\intercal$ is the vector of curvatures, and $\mathbf{Q}_d := \mathbf{I}_{d+2} - \frac{1}{d}\mathbf{e}_{d+2}\mathbf{e}_{d+2}^\intercal$ where $\mathbf{e}_{d+2}$ is the all-one column vector, and $\mathbf{I}_{d+2}$ is the identity matrix.

A more generalized relation relating the curvature-center coordinates of the balls was proved in [19]:

Theorem 1.5 (Generalized Descartes–Soddy–Gossett Theorem)**.** *In dimension d, if $d+2$ balls $S_0, \cdots, S_{d+1}$ form a Descartes configuration, then*

$$\mathbf{M}^\intercal \mathbf{Q}_d \mathbf{M} = \begin{pmatrix} 0 & 0 \\ 0 & 2\mathbf{I}_d \end{pmatrix} \tag{2}$$

where $\mathbf{M}$ *is the* curvature-center matrix *whose i-th row is $m(S_i)$.*

Given a Descartes configuration $S_0, \cdots, S_{d+1}$, we can construct another Descartes configuration by replacing S_0 with an S_{d+2}, such that the curvatures κ_0 and κ_{d+2} are the two roots of (1) treating κ_0 as unknown. So we have the relation

$$\kappa_0 + \kappa_{d+2} = \frac{2}{d-1}\sum_{i=1}^{d+1} \kappa_i \tag{3}$$

We see from (2) that the same relation holds for all the entries in the curvature-center coordinates, i.e.

$$\mathbf{m}(S_0) + \mathbf{m}(S_{d+2}) = \frac{2}{d-1}\sum_{i=1}^{d+1} \mathbf{m}(S_i) \tag{4}$$

These equations are essential for the calculations in the paper.

By recursively replacing S_i with a new ball S_{i+d+2} in this way, we obtain an infinit sequence of balls $S_0, S_1, S_2, \cdots$, in which any $d+2$ consecutive balls form a Descartes configuration. This is Coxeter's loxodromic sequences of tangent balls [9].

1.3. **Apollonian cluster of balls.** A collection of d-balls is said to be *Apollonian* if it can be built from a Descartes configuration by repeatedly introducing, for $d+1$ pairwisely tangent balls, a new ball that is tangent to all of them. The Coxeter's sequence is an example of Apollonian balls.

A stacked polytope is constructed by repeatedly gluing a simplex onto a facet. There is a one-to-one correspondance between stacked polytopes in dimension 3 and Apollonian disk packings. More detailed and formal introduction to stacked polytopes can be found in Section 3.1. Unfortunatly, this correspondance has no analogue in higer dimensions, because some induced subgraphs are not ball packable according to Section 2.

However, with a group-theoretical view of Apollonian ball packings, we managed to prove that (Theorem 3.10) the forbidden induced subgraph $K_3 \star G_6$ suffices to fully characterise the ball packable stacked 4-polytopal graphs. In the following, we introduce some necessary preliminaries for the proof.

We reformulate the replacing operation in the previous part by inversions: Given a Descartes configuration $\mathcal{S} = \{S_0, \cdots, S_{d+1}\}$, let R_i be the inversion in the sphere that orthogonally intersects the boundary of S_j for all $0 \leq j \neq i \leq d+1$, then $R_i\mathcal{S}$ forms a new Descartes configuration, which keeps all the balls except that S_i is replaced by R_iS_i.

With this point of view, a Coxeter's sequence can be obtained from an initial Descartes configuration $\mathcal{S}_0$ by recursively constructing a sequence of Descartes configurations by $\mathcal{S}_{n+1} = R_j\mathcal{S}_n$ where $j = n \pmod{d+2}$, and taking the union.

More generally, the group W generated by the R_i's is called the *Apollonian group*. The union of the orbits $\cup_{S\in\mathcal{S}_0} WS$ is called the *apollonian cluster* (of balls) [14].

The Apollonian cluster is an infinite ball packing in dimensions 2 [13] and 3 [4]. This means that the interiors of any two balls in the cluster are either identical or disjoint.

This is unfortunately not true for higher dimensions. However, if we do not require $\mathcal{S}_0$ to be a Descartes configuration, i.e. $S_0, \cdots, S_{d+1}$ may not be pairwisely tangent, then a similar generating method yields a non-Apollonian infinite ball packing under certain conditions [4]. Maxwell [20] related this fact to hyperbolic reflection groups, and showed that this generating method works only up to dimension 9.

Define

$$\mathbf{R}_i := \mathbf{I} + \frac{2}{d-1}\mathbf{e}_i\mathbf{e}^\intercal - \frac{2d}{d-1}\mathbf{e}_i\mathbf{e}_i^\intercal$$

where $\mathbf{e}_i$ is a $(d+2)$-vector whose entries are 0 except the i-th entry being 1. So $\mathbf{R}_i$ coincide with the identity matrix at all rows except the i-th row, where the diagonal entry is -1 and the off-diagonal entries is $2/(d-1)$.

One can then verify that $\mathbf{R}_i$ induces a representation of the Apollonian group. In fact, if $\mathbf{M}$ is the curvature-center matrix of a Descartes configuration $\mathcal{S}$, then $\mathbf{R}_i\mathbf{M}$ is the curvature-center matrix of $R_i\mathcal{S}$.

2. Ball-packability of Graph Joins

2.1. **Graphs in form of $K_d \star P_m$.** The following theorem is a reformulation of a result first obtained by Wilker [27]. A proof was sketched in [4]. Here we present a very elementary proof, but more suitable for our further generalization.

Theorem 2.1. *Let $d \geq 2$ and $m \geq 0$.*

(i) *$K_2 \star P_m$ is 2-ball packable for any m;*
(ii) *$K_d \star P_m$ is d-ball packable if $m \leq 4$;*
(iii) *$K_d \star P_m$ is* not *d-ball packable if $m \geq 6$;*
(iv) *$K_d \star P_5$ is d-ball packable if and only if $2 \leq d \leq 4$;*

Proof. (i) is from Koebe–Andreev–Thurston's disk packing theorem (Theorem 1.3) since $K_2 \star P_m$ is planar for any $m \geq 0$.

For dimensions $d > 2$, we construct a ball packing for the $(d+1)$-simplex $K_{d+2} = K_d \star P_2$ as follows: The two vertices of P_2 are represented by two disjoint half-spaces A and B distance 2 apart, and the d vertices of K_d are represented by d pairwisely tangent unit balls touching both A and B. Figure 1 shows the situation for $d = 3$, where red balls represent vertices of K_3.

The idea of the proof is the following: We construct the ball packing of $K_d \star P_m$ from the ball packing above of $K_d \star P_2$, by appending new balls to the chain of balls representing the path. Every new ball is forced to touch all the d unit balls of K_d, therefore must center on a straight line perpendicular to the hyperplanes defining A and B. The construction fails if the sum of the diameters exceeds 2.

We now construct $K_d \star P_3$ by adding a ball C tangent to A. By (3), the diameter of C is $2/\kappa_C = (d-1)/d < 1$. So the construction of $K_d \star P_3$ succeeded since C is disjoint from B.

Then we add the ball D tangent to B. It has the same diameter as C, and they sum up to $2(d-1)/d < 2$. So the construction of $K_d \star P_4$ succeeded, which proves the statement (ii).

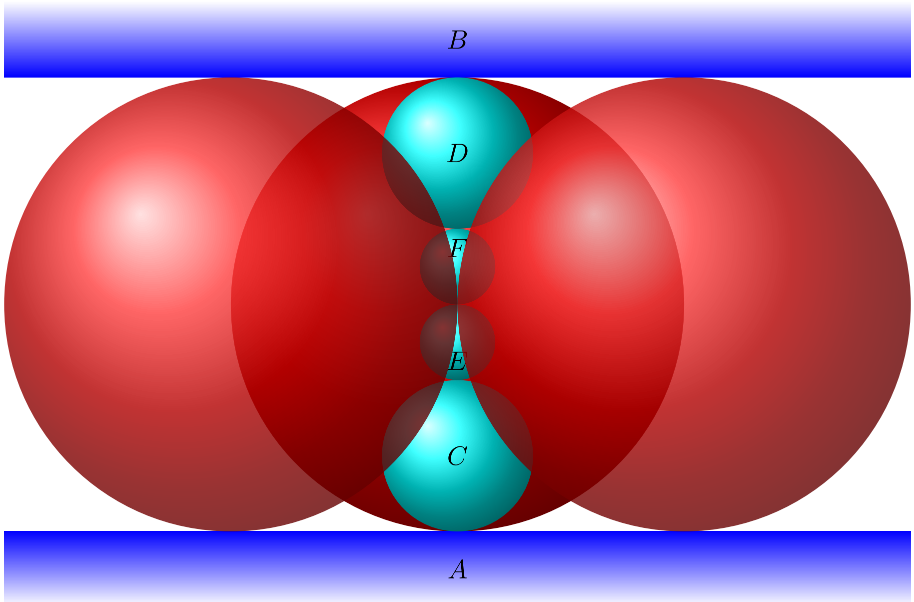

Figure 1. An attemp of constructing the ball packing of $K_3 \star P_6$ results in $K_3 \star C_6$.

We now add the ball E tangent to C. Still by (3), the diameter of E is

$$\frac{2}{\kappa_E} = \frac{(d-1)^2}{d(d+1)}$$

If we sum up the diameters of C, D and E, we get

$$2\frac{d-1}{d} + \frac{(d-1)^2}{d(d+1)} = \frac{3d^2-2d-1}{d(d+1)} \tag{5}$$

which is smaller then 2 if and only if $d \leq 4$. Therefore the construction can succeed only if $2 \leq d \leq 4$, which proves (iv).

Now for $2 \leq d \leq 4$, we continue to add the ball F tangent to D. It has the same diameter as E. If we sum up the diameters of C, D, E and F, we get

$$2\left(\frac{d-1}{d} + \frac{(d-1)^2}{d(d+1)}\right) = 4\frac{d-1}{d+1} \tag{6}$$

which is smaller then 2 if and only if $d < 3$. Therefore (iii) is proved. $\square$

Remark. Figure 1 shows the attempt of constructing the ball packing of $K_3 \star P_6$ but yields the ball packing of $K_3 \star C_6$. This packing is called Soddy's hexlet [25] (published again as a poem). It's an interesting configuration since the sum of diameters of C, D, E and F is exactly 2.

Remark. Let's point out the main differences between dimension 2 and higher dimensions: If $d = 2$, a Descartes configuration divides the space into 4 disjoint regions, and the radius of a circle tangent to the two unit circles of K_2 can tends to 0. However, if $d > 2$, the complement of a Descartes configuration is always connected, and the radius of a ball tangent to all the d balls of K_d can not be too small. Using the Descartes–Soddy–Gossett theorem, one can verify that the radius is at least $\frac{d-2}{d+\sqrt{2d^2-2d}}$, which tends to $\frac{1}{1+\sqrt{2}}$ as d tends to infinity.

2.2. **Graphs in form of $K_n \star G_m$.** The following is a corollary of Theorem 2.1.

Corollary 2.2. *For $d = 3$ or 4, $K_d \star G_6$ are not d-ball packable, with the exception of $K_3 \star C_6$. For $d \geq 5$, $K_d \star G_5$ are not d-ball packable.*

Proof. For construction of $K_d \star G_m$, we just repeat the construction in the proof of Theorem 2.1.

Since the centers of the balls of G_m are situated on a straight line, G_m can only be a path, a cycle C_m or a disjoint union of paths. The first possibility is ruled out by Theorem 2.1. The cycle is only possible when $d = 3$ and $m = 6$, in which case the ball packing of $K_3 \star C_6$ is Soddy's hexlet.

If G_m is a disjoint union of paths, we have to leave gaps between balls, which makes it more difficult to avoid self-intersection. For the graphs in the theorem, a ball packing is impossible even without any gap. So the construction must fail. □

We study in the following some other graphs in form of $K_n \star G_m$, using kissing configuration and spherical codes.

A *d-kissing configuration* is a packing of unit d-balls all touching another unit ball. The *d-kissing number* $k(d, 1)$ (we use this notation for the convenience of later generalizations) is the maximum number of balls in a d-kissing configuration.

For lower dimensions, the kissing number are known to be 2 for dimension 1, 6 for dimension 2, 12 for dimension 3 [7], 24 for dimension 4 [22], 240 for dimension 8 and 196560 for dimension 24 [23].

We have immediately the following theorem.

Theorem 2.3. *$K_3 \star G$ is d-ball packable if and only if G is the tangency graph of a $(d-1)$-kissing configuration.*

For the proof, just represent K_3 by one unit ball and two disjoint half-spaces distance 2 apart, then the other balls must form a $(d-1)$-kissing configuration.

For example, $K_3 \star G_{13}$ is not 4-ball packable, $K_3 \star G_{25}$ is not 5-ball packable, and in general, $K_3 \star G_m$ is not d-ball packable if $m > k(d-1, 1)$.

We can generalize this idea as follows: A *(d, α)-kissing configuration* is a packing of unit balls touching α pairwisely tangent unit balls. The *(d, α)-kissing number* $k(d, \alpha)$ is the maximum number of balls in a (d, α)-kissing configuration.

So the d-kissing configuration discussed before is actually the $(d, 1)$-kissing configuration. It is easy to see that if G is the tangency graph of a (d, α)-kissing configuration, $G \star K_1$ must be the graph of a $(d, \alpha-1)$-kissing configuration, and $G \star K_{\alpha-1}$ must be the graph of a d-kissing configuration.

With a similar argument as before, we have

Theorem 2.4. *$K_{2+\alpha} \star G$ is d-ball packable if and only if G is the tangency graph of a $(d-1, \alpha)$-kissing configuration.*

For the proof, just represent $K_{2+\alpha}$ by two half-spaces distance 2 apart and α pairwisely tangent unit balls, then the other balls must form a $(d-1, \alpha)$-kissing configuration.

For example, $K_{2+\alpha} \star G_m$ is not d-ball packable if $m > k(d-1, \alpha)$. The following corollary is from the fact that $k(d, d) = 2$ for all $d > 0$

Corollary 2.5. *$K_{d+1} \star G_3$ is not d-ball packable.*

A *$(d, \cos\theta)$-spherical code* [7] of minimal angle θ is a set of points on the unit $(d-1)$-sphere such that the spherical distance between any two points in the set is at least θ. We denote by $A(d, \theta)$ the maximal number of points in such a spherical code. This is in fact a generalization of kissing configurations: the minimal angle corresponds to the tangency relations, and $A(d, \cos\theta) = k(1, d)$ if $\theta = \pi/3$. Corresponding to the tangency graph, the *minimal-angle graph* of a spherical code takes the points as vertices and connects two vertices if the corresponding points achieve the minimal spherical distance.

As noticed by Bannai and Sloane [2, Theorem 1], the centers of unit balls in a (d,α)-kissing configuration correspond to a $(d-\alpha+1,\frac{1}{\alpha+1})$-spherical code after rescaling. Therefore:

Corollary 2.6. *$K_{2+\alpha}\star G$ is $(d+\alpha)$-ball packable if and only if G is the minimal-angle graph of a $(d,\frac{1}{\alpha+1})$-spherical code.*

We give in Table 1 an incomplete list of $(d,\frac{1}{\alpha+1})$-spherical codes for integer values of α. They are therefore $(d+\alpha-1,\alpha)$-kissing configurations for the α and d given in the table.

The first column is the name of the polytope whose verices form the spherical code. Some of them are from Klitzing's list of segmentochora [16], which can be viewed as a special type of spherical codes. Some others are inspired from Sloane's collection of optimal spherical codes [24]. For those polytopes with no conventional name, we keep Klitzing's notation, or give a name following Klitzing's method.

The second column is the corresponding minimal-angle graph, if it is possible to write out. Here are some notations used in the table: For a graph G, its *line graph* $L(G)$ takes the edges of G as vertices, and two vertices are adjacent whenver the corresponding edges share a vertex in G. The *Johnson graph* $J_{n,k}$ takes the k-element subsets of an n-element set as vertices, and two vertices are adjacent whenever their intersection contains $k-1$ elements. Especially, $J_{n,2}=L(K_n)$. For two graph G and H, $G\square H$ denotes the *Cartesian product*.

TABLE 1. Some known $(d,\frac{1}{\alpha+1})$-spherical codes for integer α

spherical code	minimal distance graph	α	d
k-orthoplicial prism	$\Diamond_k\square K_2$	2	$k+1$
k-orthoplicial-pyramidal prism	$(\Diamond_k\star K_1)\square K_2$	2	$k+2$
rectified k-orthoplex	$L(\Diamond_k)$	1	k
augmented k-simplicial prism		k	$k+1$
2-simplical prism(-1_{21}) [16, 3.4.1]	$K_3\square K_2$	6	3
3-simplical prism(-1_{31}) [16, 4.9.2]	$K_4\square K_2$	4	4
5-simplical prism	$K_6\square K_2$	3	6
triangle-triangle duoprism(-1_{22})[16, 4.10]	$K_3\square K_3$	3	4
tetrahedron-tetrahedron duoprism	$K_4\square K_4$	2	6
triangle-hexahedron duoprism	$K_3\square K_6$	2	7
rectified 4-simplex(0_{21}) [1]	$J_{5,2}$	5	4
rectified 5-simplex(0_{31})	$J_{6,2}$	3	5
rectified 7-simplex	$J_{8,2}$	2	7
birectified 5-simplex(0_{22})	$J_{6,3}$	2	5
birectified 8-simplex	$J_{9,3}$	1	8
trirectified 7-simplex	$J_{8,4}$	1	7
5-demicube(1_{21}) [24, pack.5.16]		4	5
6-demicube(1_{31})		2	6
8-demicube		1	8
1_{22}		1	6
2_{31}		1	7
2_{21} [6, Appendix A]		3	6
3_{21} [2]		2	7
4_{21} [2]		1	8
3p‖refl ortho 3p [16, 4.13]		2	4
3g‖gyro 3p [16, 4.6.2]		5	4
3g‖ortho 4g [16, 4.7.3]		5	4
3p‖ortho line [16, 4.8.2]		5	4
oct‖hex [24, pack.5.14]		4	5

We would like to point out that for $1 \leq \alpha \leq 6$, vertices of the uniform $(5-\alpha)_{21}$ polytope form an $(8,\alpha)$-kissing configuration. These codes are derived from the E_8 root lattice [2, Example 2]. They are optimal and unique except for the trigonal prism($(-1)_{21}$ polytope) [1; 6, Appendix A]. There are also spherical codes similarly derived from the *Leech lattice* [2, Example 3; 5].

As another example, since

$$k(d,\alpha) = A\Big(d-\alpha+1, \frac{1}{\alpha+1}\Big).$$

the following fact provides another way for proving Corollary 2.2:

$$k(d,d-1) = A(2,1/d) = \begin{cases} 4 & \text{if } d \geq 4 \\ 5 & \text{if } d = 3 \\ 6 & \text{if } d = 2 \text{(optimal)} \end{cases}$$

To end this part, the following theorem is trivial but more general.

Theorem 2.7. *$K_2 \star G$ is d-ball packable if and only if G is $(d-1)$-unit-ball packable.*

For the proof, just use disjoint half-spaces to represent K_2, then G has to be representable by a packing of unit balls.

2.3. **Graphs in form of $\Diamond_d \star G_m$.**

Theorem 2.8. *$\Diamond_{d-1} \star P_4$ is not d-ball packable, but $\Diamond_{d+1} = \Diamond_{d-1} \star C_4$ is.*

Proof. $\Diamond_{d-1}$ is the graph of the $(d-1)$-dimensional orthoplex. The vertices of a regular orthoplex of edge length $\sqrt{2}$ forms an optimal spherical code of minimal angle $\pi/2$.

As in the proof of Theorem 2.1, we first construct the ball packing of $\Diamond_{d-1} \star P_2$, where P_2 is represented by two disjoint half-spaces, and $\Diamond_{d-1}$ is represented by $2(d-1)$ unit balls, whose centers are the vertices of a regular $(d-1)$-dimensional orthoplex of edge length 2. Therefore the unit balls are centered on a $(d-2)$-sphere of radius $\sqrt{2}$.

We now construct $\Diamond_{d-1} \star P_3$ by adding the unique ball that is tangent to all the unit balls and also to one half-space. After an elementary calculation, the radius of this ball is $1/2$. By symmetry, a ball touching the other half-space has the same radius. These two balls must be tangent since their diameters sum up to 2.

Therefore, an attempt for constructing a ball packing of $\Diamond_{d-1} \star P_4$ results in a ball packing of $\Diamond_{d+1} = \Diamond_{d-1} \star C_4$. $\square$

For example, $C_4 \star C_4$ is 3-ball packable.

By the same argument as in the proof of Corollary 2.2, we have

Corollary 2.9. *$\Diamond_{d-1} \star G_4$ is not d-ball packable, with the exception of $\Diamond_{d+1} = \Diamond_{d-1} \star C_4$.*

2.4. **Graphs in form of $G_n \star G_m$.** The following is a corollary of Corollary 2.2.

Corollary 2.10. *$G_6 \star G_3$ is not 3-ball packable, with the exception of $C_6 \star C_3$.*

Proof. As in the proof of Theorem 2.1, we first construct a 3-ball packing of $P_2 \star G_3$, where P_2 is represented by two disjoint half-spaces distance 2 apart, and G_3 by three unit balls touching both hyperplanes, whose tangency graph is G_d.

If the centers of these unit balls are colinear, further construction is not possible. Otherwise, they must be on a circle in a 2-dimensional hyperplane parallel to the half-spaces. A ball touching all the three unit balls must center on the straight line passing through the center of this circle and penpendicular to this hyperplane.

We then continue the construction as in the proof of Theorem 2.1. However, since the three unit balls are not pairwisely tangent, a ball must have a larger radius in order to touch all of them. Again, this makes it more difficult to avoid self-intersection, which is already a mission impossible when the unit balls are pairwisely tangent.

And as in the proof of Corollary 2.2, gaps on the path make the situation even worse. So the construction must fail. □

Therefore, if a graph is 3-ball packable, any induced subgraph in form of $G_6 \star G_3$ must be in form of $C_6 \star K_3$.

By the same argument, we derive the following corollary from the fact that $C_4 \star P_4$ is not 3-ball packable

Corollary 2.11. *$G_4 \star G_4$ is not 3-ball packable, with the exception of $C_4 \star C_4$.*

Therefore, if a graph is 3-ball packable, any induced subgraph in form of $G_4 \star G_4$ must be in form of $C_4 \star C_4$.

From the fact that $\Diamond_3 \star P_4$ is not 4-ball packable, we derive the following corollary, but the argument is slightly different:

Corollary 2.12. *$G_4 \star G_6$ is not 4-ball packable, with the exception of $C_4 \star \Diamond_3$.*

Proof. The proof is basically the same as Corollary 2.10.

Two vertices of G_4 are represented by half-spaces, and G_6 are represented by unit balls. If the centers of these unit balls are collinear, further construction is not possible. If the centers are on a 2-sphere, its diameter reaches its minimum *only* when $G_6 = \Diamond_3$. If G_6 is in any other form, a ball touching all the unit balls must have a larger radius, and the construction must fail.

We should be careful that it is possible to have the six unit balls centered on a circle. In this case, the radius of a ball touching all of them is at least 1 (luckily), which rules out the possibility of further construction. □

Remark. The argument in the proof of Corollaries 2.10 and 2.12 should be used with caution. As mentioned in the proof of Corollary 2.12, one must check carefully the non-generic cases, and make sure that nothing goes wrong.

For example, Corollary 2.12 can *not* be derived from the fact that $K_4 \star P_6$ is not 4-ball packable. If we use the same argument, unit balls representing G_4 must be centered on a 2-sphere, whose radius is minimum when $G_4 = K_4$. However, it is possible to have the centers on a circle, for example when $G_4 = C_4$. In this case, a ball touching all the four unit balls can have a radius as small as $1/2$, and its center is not restricted on a line. Indeed, we have the counterexample $\Diamond_3 \star C_4$.

3. Ball Packable Stacked-polytopal Graphs

3.1. **Stacked polytopes.** For a simplicial polytope, a *stacking operation* glues a new simplex onto a facet.

Definition 3.1. A simplicial d-polytope is *stacked* if it can be iteratively constructed from a d-simplex by a sequence of *stacking operations*.

We call the 1-skeleton of a polytope $\mathcal{P}$ the *graph* of $\mathcal{P}$, denoted by $G(\mathcal{P})$. For example, the graph of a d-simplex is K_{d+1}. The graph of a stacked d-polytope is a *d-tree*, that is, a chordal graph whose maximal cliques are of a same size $d+1$. Inversely, a d-tree is the graph of a stacked d-polytope if and only if there is no three $(d+1)$-cliques sharing d vertices [15]. Therefore, by Corollary 2.5, a $(d+1)$-tree is d-ball packable only if it is the graph of a stacked $(d+1)$-polytope.

A simplicial d-polytope $\mathcal{P}$ is stacked if and only if it admits a triangulation $\mathcal{T}$ with only interior faces of dimension $(d-1)$. For $d \geq 3$, this triangulation is unique, whose simplices correspond to the maximal cliques of $G(\mathcal{P})$. This implies that the combinatorial type of a stacked polytope is uniquely determined by its graph. The *dual tree* [11] of $\mathcal{P}$ takes the simplices of $\mathcal{T}$ as vertices, and connect two vertices if the corresponding simplices share a $(d-1)$-face.

There is a 1-to-1 correspondence between Apollonian 2-ball packings and stacked 3-polytopes: Every Apollonian disk packing is a 2-ball packing of a stacked 3-polytopal graph, and the graph of every stacked 3-polytope has a disk packing that is Apollonian and unique up to Möbius transformations and reflections. The relation between 3-tree, stacked 3-polytope and Apollonian 2-ball packing can be illustrated as follows:

$$\text{3-tree} \xrightarrow[\text{sharing a 3-clique}]{\text{no three 4-cliques}} \text{stacked 3-polytope} \longleftrightarrow \text{Apollonian 2-ball packing}$$

However, in higher dimensions, this correspondence does not hold any more. Counterexamples are given in Theorem 2.1. In the following, we would like to characterise the stacked 4-polytopes whose graphs are 3-ball packable.

A weak relation remains between Apollonian ball packings and stacked polytopes in higher dimensions.

Theorem 3.2. *If the graph of a stacked $(d+1)$-polytope is d-ball packable, its ball packing is Apollonian and unique up to Möbius transformations and reflections.*

Proof. The Apollonianity can be easily seen by comparing the construction process.

The uniqueness can be proved by an induction on the construction process. While a stacked polytope is built from a simplex, we construct its ball packing from a Descarte configuration, which is unique up to Möbius transformations and reflections. For every stacking operation, a new ball was added into the packing to form a new Descartes configuration. We have an unique choice for every newly added ball, so the uniqueness is preserved at every step of construction. □

Given a ball packing $\mathcal{S} = \{S_0, \cdots, S_n\}$, let $\mathbf{c}_i$ be the center of S_i, a *stress* of $\mathcal{S}$ is a real function T on the edge set of $G(\mathcal{S})$ such that for all $S_i \in \mathcal{S}$

$$\sum_{S_iS_j \text{ edge of } G(\mathcal{S})} T(S_iS_j)(\mathbf{c}_j - \mathbf{c}_i) = 0$$

We can view stress as forces between tangent balls when all the balls are in equilibrium. We say that $\mathcal{S}$ is *stress-free* if it has no non-zero stress.

Theorem 3.3 (Stress free)**.** *If the graph of a stacked $(d+1)$-polytope is d-ball packable, its ball packing is stress-free.*

Proof. We construct the ball packing as we did in the proof of Theorem 3.2, and assume a non-zero stress.

The last ball S that is added into the packing has d "neighbor" balls tangent to it. If the stress is not zero on all the d edges incident to S, they can not be of the same sign, so there must be a hyperplane separating positive edges and negative edges. This contradicts the assumption that S is in equilibrium. So the stress must vanish on the edges incident to S. We then remove S and repeat the same argument on the second last ball, and so on, and finally conclude that the stress has to be zero on all the edges of $G(\mathcal{S})$. □

The theorem and the proof above was informally discussed in Section 8 in Kotlov, Lovász and Vempala's famous paper on Colin de Verdière number [17]. In that paper, they defined another graph invariant $\nu(G)$ using the notion of stress-freeness (for vector labellings, slightly different from above), which turns out to be strongly related to Colin de Verdière number. Applying their results on stress-freeness, they concluded that (in our formulation) if the graph G of a stacked $(d+1)$-polytope with n vertices is d-ball packable, then $\nu(G) \le d+2$, and the upper bound is achieved if $n \ge d+4$. However, they didn't pay much attention to the existence of the ball packing that they described. Theorem 2.1 shows that this kind of ball packing is not always constructible.

The following observation will be useful:

Lemma 3.4. *Let $\mathcal{P}$ be a stacked d-polytope. For be a k-face F of $\mathcal{P}$, let $H(F)$ be the subgraph induced by the common neighbors of the vertices of F, then $H(F)$ is the graph of a stacked $(d-k+1)$-polytope.*

3.2. **Weighted mass of a word.** The following theorem was proved in [14]

Theorem 3.5. *The 3-dimensional Apollonian group is a hyperbolic Coxeter group generated by the relations $\mathbf{R}_i\mathbf{R}_i = \mathbf{I}$ and $(\mathbf{R}_i\mathbf{R}_j)^3 = \mathbf{I}$ for $0 \le i \ne j \le 4$.*

As a sketch, their proof was through the study of reduced words.

Definition 3.6. A word $\mathbf{U} = \mathbf{U}_1\mathbf{U}_2\cdots\mathbf{U}_n$ over the generator of the 3-dimensional Apollonian group (i.e. $\mathbf{U}_i \in \{\mathbf{R}_0, \cdots, \mathbf{R}_4\}$) is *reduced* if it does not contain

- subword of form $\mathbf{R}_i\mathbf{R}_i$ for $0 \le i \le 4$; or
- subword of form $\mathbf{V}_1\mathbf{V}_2\cdots\mathbf{V}_{2m}$ in which $\mathbf{V}_1 = \mathbf{V}_3$, $\mathbf{V}_{2m-2} = \mathbf{V}_{2m}$ and $\mathbf{V}_{2j} = \mathbf{V}_{2j+3}$ for $1 \le j \le 2m-2$.

Notice that $m = 2$ excludes the subwords of form $(\mathbf{R}_i\mathbf{R}_j)^2$. One can verify that non-reduced words can be simplified to reduced words using the generating relations. Then it suffices to prove that no nonempty reduced word, treated as a matrix, is identity.

For proving this, they studied the sum of entries in the i-th row of $\mathbf{U}$, i.e. $\sigma_i(\mathbf{U}) := \mathbf{e}_i^\intercal\mathbf{U}\mathbf{e}$, and the sum of all the entries in $\mathbf{U}$, i.e. $\Sigma(\mathbf{U}) := \mathbf{e}^\intercal\mathbf{U}\mathbf{e}$. The latter is called the *mass*. The quantities $\Sigma(\mathbf{U}), \Sigma(\mathbf{R}_j\mathbf{U}), \sigma_i(\mathbf{U}), \sigma_i(\mathbf{R}_j\mathbf{U})$ satisfy a series of linear equations, which was used to inductively prove that $\Sigma(\mathbf{U}) > \Sigma(\mathbf{U}')$ for a reduced word $\mathbf{U} = \mathbf{R}_i\mathbf{U}'$. Therefore $\mathbf{U}$ is not an identity since $\Sigma(\mathbf{U}) \ge \Sigma(\mathbf{R}_i) = 7 > \Sigma(\mathbf{I}) = 5$.

We propose the following adaption: Given a weight vector $\mathbf{w}$, we define $\sigma_i^w(\mathbf{U}) = \mathbf{e}_i^\intercal\mathbf{U}\mathbf{w}$ the weighted sum of entries in the i-th row of $\mathbf{U}$, and $\Sigma^w(\mathbf{U}) = \mathbf{e}^\intercal\mathbf{U}\mathbf{w}$ the *weighted mass* of $\mathbf{U}$. We find that the following lemma can be proved with an argument similar as above:

Lemma 3.7. *For dimension 3, if $\Sigma^w(\mathbf{R}_i) \ge \Sigma^w(\mathbf{I})$ for any $0 \le i \le 4$, then for a reduced word $\mathbf{U} = \mathbf{R}_i\mathbf{U}'$, we have $\Sigma^w(\mathbf{U}) \ge \Sigma^w(\mathbf{U}')$.*

Sketch of proof. It suffices to replace "sum" by "weighted sum", "mass" by "weighted mass", and ">" by "≥" in the proof of [14, Theorem 5.1].

It turns out that the following relations hold for $0 \le i, j \le d+1$.

$$\sigma_i^w(\mathbf{R}_j\mathbf{U}) = \begin{cases} \sigma_i^w(\mathbf{U}) & \text{if } i \ne j \\ \Sigma^w(\mathbf{U}) - 2\sigma_i^w(\mathbf{U}) & \text{if } i = j \end{cases} \tag{7}$$
$$\Sigma^w(\mathbf{R}_i\mathbf{U}) = 2\Sigma^w(\mathbf{U}) - 3\sigma_i^w(\mathbf{U})$$

Then, if we define $\delta_i^w(\mathbf{U}) := \Sigma^w(\mathbf{R}_i\mathbf{U}) - \Sigma^w(\mathbf{U})$, the following relations hold:

$$\delta_i^w(\mathbf{R}_j\mathbf{U}) = \begin{cases} \delta_i^w(\mathbf{U}) + \delta_j^w(\mathbf{U}) & \text{if } i \ne j \\ -\delta_i^w(\mathbf{U}) & \text{if } i = j \end{cases}$$
$$\delta_i^w(\mathbf{R}_j\mathbf{U}) = \delta_j^w(\mathbf{R}_i\mathbf{U}) \text{ if } i \ne j$$
$$\delta_i^w(\mathbf{R}_j\mathbf{R}_i\mathbf{U}) = \delta_j^w(\mathbf{U})$$

These are all the relations that are useful for the induction. The base case is already assumed in the condition of the theorem, which reads $\delta_i^w(\mathbf{I}) \ge 0$ for $0 \le i \le 4$.

So the rest of the proof is exactly the same as in the proof of [14, Theorem 5.1]. For details of the induction, please refer to the original proof.

The conclusion is $\delta_i^w(\mathbf{U}') \geq 0$, i.e. $\Sigma^w(\mathbf{U}) \geq \Sigma^w(\mathbf{U}')$. □

3.3. **A generalization of Coxeter's sequence.** Let $\mathbf{U} = \mathbf{U}_n \cdots \mathbf{U}_2\mathbf{U}_1$ be a word over the generators of the 3-dimensional Apollonian group (we have a good reason for inversing the order of the index). Let $\mathbf{M}_0$ be the curvature-center matrix of an initial Descartes configuration, consisting of the first five balls in the sequence $S_0, \cdots, S_4$.

The curvature-center matrices recursively defined by $\mathbf{M}_i = \mathbf{U}_i\mathbf{M}_{i-1} (1 \leq i \leq n)$, define a sequence of Descartes configurations. We take S_{4+i} to be the single ball that is in the configuration at step i but not in the configuration at step $i-1$. This generates a sequence of $4+n$ balls.

Coxeter's loxodromic sequence in dimension 3 is therefore generated by an infinite word of period 5, e.g. $\mathbf{U} = \cdots \mathbf{R}_1\mathbf{R}_0\mathbf{R}_4\mathbf{R}_3\mathbf{R}_2\mathbf{R}_1\mathbf{R}_0$, which can be viewed as a special case of our sequence.

The following is a corollary of Lemma 3.7:

Corollary 3.8. *If $\mathbf{U}$ is reduced and $\mathbf{U}_1 = \mathbf{R}_0$, then in the sequence constructed as above, S_0 is disjoint from all balls except the first five.*

Proof. We take the initial configuration to be the configuration used in the proof of Theorem 2.1. Assume S_0 to be the lower half-space $x_1 \leq 0$, then the initial curvature-center matrix is

$$\mathbf{M}_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & \sqrt{1/3} \\ 1 & 1 & -1 & \sqrt{1/3} \\ 1 & 1 & 0 & -2\sqrt{1/3} \end{pmatrix}$$

Every row corresponds to the curvature-center coordinates $\mathbf{m}$ of a ball. The first coordinate m_1 is the curvature κ. If the curvature is not zero, the second coordinate m_2 is the "height" of the center times the curvature, i.e. $x_1\kappa$.

Now take the second column of $\mathbf{M}_0$ to be the weight vector $\mathbf{w}$. We have $\Sigma^w(\mathbf{R}_0) = 9 > \Sigma^w(\mathbf{I}) = 3$ and $\Sigma^w(\mathbf{R}_j) = 3 = \Sigma^w(\mathbf{I})$ for $j > 0$. Applying Lemma 3.7 we have

$$\Sigma^w(\mathbf{U}_k\mathbf{U}_{k-1} \cdots \mathbf{U}_2\mathbf{R}_0) \geq \Sigma^w(\mathbf{U}_{k-1} \cdots \mathbf{U}_2\mathbf{R}_0)$$

By (7), this means that

$$\sigma_j^w(\mathbf{U}_k \cdots \mathbf{U}_2\mathbf{R}_0) \geq \sigma_j^w(\mathbf{U}_{k-1} \cdots \mathbf{U}_2\mathbf{R}_0)$$

if $\mathbf{U}_k = \mathbf{R}_j$, or equality if $\mathbf{U}_k \neq \mathbf{R}_j$.

The key observation is that $\sigma_j^w(\mathbf{U}_k \cdots \mathbf{R}_0)$ is exactly the second curvature-center coordinate m_2 of j-th ball in the k-th Descartes configuration. So at every step, a ball is replaced by another ball with a larger or same value for m_2. Especially, since $\sigma_j^w(\mathbf{R}_0) \geq 1$ for $0 \leq j \leq 4$, we conclude that $m_2 \geq 1$ for every ball.

Four balls in the initial configuration have $m_2 = 1$. Once they are replaced, the new ball must have a strictly larger value of m_2. This can be seen from (4) and notice that the r.h.s. of (4) is at least 4 since the very first step of the construction. We then conclude that $m_2 > 1$ for all balls except the first five. This exclude the possibility of curvature zero, so $x_1\kappa > 1$ for all balls except the first five.

For dimension 3, Equation (4) is integral. Therefore the curvature-center coordinates of all balls are integral (see [14] for more on integrality of Apollonian packings). Since the sequence is a packing (by the result of [4]), no ball in the sequence has a negative curvature. By the definition of the curvature-center coordinates, the fact that $m_2 > 1$ exclude the possibility of curvature 0. Therefore all balls have a positive curvature $\kappa \geq 1$ except the first two.

For conclusion, $x_1\kappa > 1$ and $\kappa \geq 1$ implies that $x_1 > 1/\kappa$, therefore disjoint from the half-space $x_1 \leq 0$. □

3.4. **Characterising 3-ball packable stacked 4-polytopal graphs.**

Lemma 3.9. *Let G be the graph of a stacked 4-polytope. If G has an induced subgraph of form $G_3 \star G_6$, G must have an induced subgraph of form $K_3 \star P_6$.*

Note that $C_6 \star K_3$ is not an induced subgraph of any stacked polytopal graph.

Proof. Let H be an induced subgraph of G of form $G_3 \star G_6$. Let $v \in V(H)$ be the last vertex in H that is added into the polytope during the construction. We have $\deg_H v = d + 1$, and the neighbors of v induce a complete graph.

Since $\ell(d) = 6 > d + 1 = 4$, v must be a vertex in the part $G_{\ell(d)} = G_6$. so that the other part, being an induced subgraph of K_4, is the complete graph K_3. Therefore H is of the form $K_3 \star G_6$.

By Lemma 3.4, the commun neighbors of the vertices of K_3 induce a path P_n where $n \geq 6$. Therefore G must have an induced subgraph of form $P_6 \star K_3$. □

Theorem 3.10 (Characterising 3-ball packable stacked 4-polytopal graphs)**.** *The graph of a stacked 4-polytope is 3-ball packable if and only if it does not contain six 4-cliques sharing a 3-clique.*

proof of Theorem 3.10. The "only if" is by Theorem 2.1 and Lemma 3.9. We prove "if" by induction on number of vertices.

The complete graph on 5 vertices is of course 3-ball packable. Assume that every stacked 4-polytope with less than n vertices satisfies this theorem. We now study a stacked 4-polytope $\mathcal{P}$ of $n + 1$ vertices that do not have six 4-cliques in its graph with 3 vertices in commun, and assume that $G(\mathcal{P})$ is not ball packable.

Let u, v be two vertices of $G(\mathcal{P})$ of degree 4. Deleting v from $\mathcal{P}$ leaves a stacked polytope $\mathcal{P}'$ of n vertices that satisfies the condition of the theorem, so $G(\mathcal{P}')$ is ball packable by the assumption of induction. In the ball packing of $\mathcal{P}'$, the four balls corresponding to the neighbors of v are pairwisely tangent. We then construct the ball packing of $\mathcal{P}$ by adding a ball S_v that is tangent to these four balls. We have only one choice (the other choice will coincide with another ball), but since $G(\mathcal{P})$ is not ball packable, S_v must intersect some other balls.

However, deleting u also leaves a stacked polytope whose graph is ball packable. Therefore S_v must intersect S_u and only S_u. Now if there is another vertex w of degree 4 different from u and v, deleting w leaves a stacked polytope whose graph is ball packable, which produces a contradiction. Therefore u and v are the only vertices of degree 4.

Let $\mathcal{T}$ be the dual tree of $\mathcal{P}$, its leaves correspond to vertices of degree 4. So $\mathcal{T}$ must be a path, whose two ends correspond to u and v.

We can therefore construct the ball packing of $\mathcal{P}$ as a generalised Coxeter's sequence that we just studied. The first ball is S_u. The construction word does not contain any subword of form $(\mathbf{R}_i\mathbf{R}_j)^2$ (which produces $C_6 \star K_3$ and violates the condition) or $\mathbf{R}_i\mathbf{R}_i$, one can therefore always simplify the word into a *non-empty* reduced word. This does not change the corresponding matrix, so the last ball in the sequence, S_v, remains the same.

Then Corollary 3.8 implies that S_u and S_v are disjoint which contradicts our previous discussion. Therefore $G(\mathcal{P})$ is ball packable. □

Therefore, the relation between 4-trees, stacked 4-polytopes and Apollonian 3-ball packings can be illustrated as follows:

$$\text{4-tree} \xrightarrow[\text{sharing a 4-clique}]{\text{no three 5-cliques}} \text{stacked 4-polytope} \xrightarrow[\text{sharing a 3-clique}]{\text{no six 4-cliques}} \text{Apollonian 3-ball packing}$$

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