

# THE CHIRAL GYRATING $H'$ - $T$ SURFACE FAMILY: CONSTRUCTION FROM THE DUAL QTZ–QZD NETS AND EXISTENCE PROOF USING A TOROIDAL WEIERSTRASS METHOD

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**ABSTRACT.** This paper provides a construction and existence proof for a 1-parameter family of chiral unbalanced triply-periodic minimal surfaces of genus 4. We name these *gyrating  $H'$ - $T$*  surfaces, because they are related to Schoen's  $H'$ - $T$  surfaces in a similar way as the Gyroid is to the Primitive surface. Their chirality is manifest in a screw symmetry of order six. The two labyrinthine domains on either side of the surface are not congruent, rather one representing the quartz net ( $\text{qtz}$ ) and the other one the dual of the quartz net ( $\text{qzd}$ ). The family tends to the Scherk saddle tower in one limit and to the doubly periodic Scherk surface in the other. The motivation for the construction was to construct a chiral tunable unbalanced surface family, originally as a template for photonic materials. The numeric construction is based on reverse-engineering of the tubular surface of two suitably chosen dual nets, using the *Surface Evolver* to minimize area or curvature variations. The existence is proved using Weierstrass parametrizations defined on the branched torus.

Triply-periodic minimal surfaces (TPMSs) are periodic symmetric-saddle surfaces that divide space into two network-like domains. They are epitomized by Alan Schoen's *Gyroid* surface [Sch70, Sch12]. The interdisciplinary interest in triply-periodic bicontinuous structures is driven by the occurrence of related structures in chemistry and biology [HAL<sup>+</sup>97, HOP08] and as a useful material design such as the now common 'gyroid infill' in 3d printing applications. The cubic (G)yroid and its two close relatives, namely Schwarz' (D)iamond and (P)rimitive surfaces are, among the many triply-periodic minimal surfaces, those that are most widely found in nano- or micro-structured materials and tissues.

The study of TPMS has advanced through contributions from both the mathematical and natural sciences. The productivity of this relationship, although perhaps not always fully recognized [HST24], is epitomized by Schoen's discovery of his Gyroid. Schoen, a trained physicist employed by NASA, constructed the Gyroid using an 'infinite polyhedral model' and devised a Weierstrass parametrization for the Gyroid [Sch70, Sch12]), with mathematicians Große-Brauckmann and Wohlgemuth later proving its existence and that it is a minimal embedding [GBW96]. Große-Brauckmann went on to establish the existence (and partial uniqueness) of constant-mean-curvature companion surfaces to the gyroid [GB97] as well as numerical constructions of these that have since been used in physical applications [SKM<sup>+</sup>07, STdCE<sup>+</sup>13].

A similar interdisciplinary interplay established our understanding of families of TPMSs that contain the Gyroid, Diamond, or Primitive surfaces as particular members, and therefore provide transition pathways between these surfaces within the space of TPMSs. Meeks mathematically described a five-parameter family containing the cubic P and D surfaces, but not the Gyroid [WHM75, Mee90]. Hyde and Fogden enumerated Weierstrass parametrisations of an extended set of these surfaces [FH92a, FH92b, Fog93] including surface families rG and tGL containing the Gyroid [FH99]; these were later examined mathematically by Weyhaupt [Wey08] and their existence eventually proved by Chen [Che21a]. Numerical analysis of their packing and curvature properties was carried out by Fogden & Hyde [FH99] and others [STFH06, STVdC<sup>+</sup>11, MSTM12]. Recent work in mathematics and materials science has identified further deformation families, particularly an alternative deformation family of the Diamond surface [CW21a] that is different from the classical tD family, an orthorhombic deformation family of Schwarz' H surfaces [CW21b],

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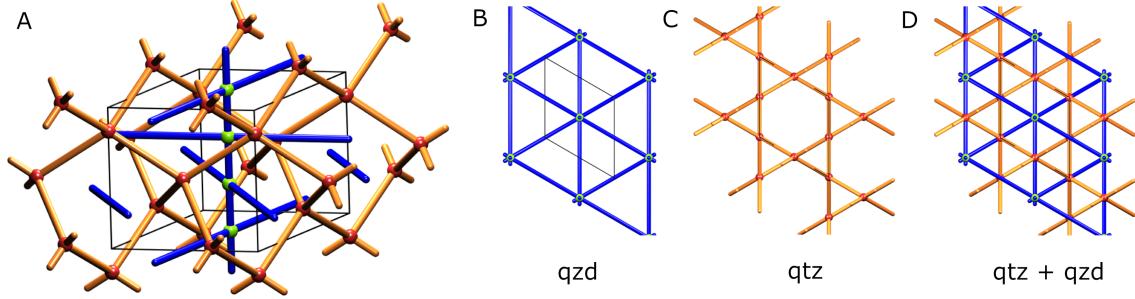


FIGURE 1. The pair of interpenetrating dual nets that lead to the **qtz-qzd** surface family, the quartz (**qtz**) net (orange) and its proper dual **qzd** network (blue). (A) Perspective view; (B-D) top views along the  $c$  axis of the **qzd** (A), **qtz** (B) **qzd** and **qtz** nets. The translational unit cell ( $P6_222$ ) is the gray frame.

and a deformation of a (shifted) Gyroid [WCZ<sup>+</sup>24]; this deformation may relate to the recent finding that the cubic Diamond is merely a saddle-point (rather than an unstable minimum) of the free-energy functional for a copolymeric model [DGMG25].

Over the years, research in both natural sciences and mathematics has proposed numerous ways to describe, enumerate or parametrize new TPMSs. In the natural sciences, there are approaches based on space group or crystallographic symmetries by Fischer & Koch ([KF88] and other papers in that series) and Lord & Mackay [LM03], on tilings of hyperbolic plane by Sadoc & Charvolin [SC89], on Weierstrass formula derived from Schwarz triangular tilings of the two sphere by Fogden & Hyde [FH92a, FH92b], and other explicit parametrizations [GK00]. In mathematics, there are approaches based on a Schwarz–Christoffel formula for periodic polygons in the plane by Fujimori & Weber [FW09], on the concept of discrete minimal surfaces [PP93, KP96], on gluing methods [Tra08, CF22, CT24]. New TPMSs were recently constructed by deliberately breaking symmetries [CW21a, CW21b, Che21a].

In addition to exact descriptions, methods to approximate or simulate TPMSs have been instrumental in their exploration: Nodal surface formulae that approximate minimal or CMC surfaces [vSN91] have become popular for their ease of use. Furthermore, a tool that has been widely used in mathematics and natural sciences for the exploration of TPMSs is Kenneth Brakke’s *Surface Evolver* [Bra92]. This software package uses a conjugate gradient solver to minimize energy functionals (including the area functional, Willmore functional, etc.) on a surface, and integrates functionality for crystallographic symmetries, boundary conditions, volume constraints, etc. Brakke’s extensive TPMS library is testimony to its usefulness [Bra25] as are many other applications in the context of area-minimizing surfaces [GB97, HdCO09] and its use in this work.

Some properties of a minimal surface relate to intrinsic properties, unrelated to the embedding in  $\mathbb{E}^3$ . In particular, the Gauss curvature  $K$ , the integral of which provides the Euler characteristic and the variations of which are a homogeneity measure relevant for self-assembly [Hyd90], is an intrinsic property. There are also many extrinsic properties. In particular, the relationship between the minimal surface interface and the two skeletal (medial) graphs or surfaces often used to illustrate the connectivity and geometry of the two labyrinthine domains on either side of the surface.

This paper is the result of another interplay between mathematics and natural sciences on the topic of TPMSs: We here describe the numerical and analytic constructions of a TPMS without in-surface symmetries (that is, without mirror planes and without two-fold rotation axes embedded in the surface). The absence of such symmetries means that a surface patch bounded by straight lines or a plane line of curvature does not exist; therefore a bottom-up approach starting from a Plateau patch is not possible.

Our numerical construction of the gyrating H'-T is a top-down approach. We start with two specific interpenetrating nets (namely the chiral **qtz-qzd** pair), with specific topology and symmetry, with the goal of determining a minimal surface interface that separates those nets. We

determine a mesh approximation using a numerical construction in Surface Evolver (based on a procedure suggested by de Campo, Hyde et al [HdCO09]), which is good enough to determine the location and nature of the flat points (those with zero Gauss curvature).

Based on that information, we determine a Weierstrass parametrization of the surface. It turns out that this can be done in one of two ways. One way, explored in an earlier unpublished version of this manuscript by a subset of the current authors (available as an arXiv preprint [MSSTM18]), is using the knowledge of the flat points to determine a Weierstrass function in the spirit of work by Fogden & Hyde [FH92a, FH92b], particularly in relation to the so-called *irregular* class [Fog93]. This is based on the “traditional” spherical Weierstrass parametrization, which is the inverse of the stereographically projected Gauss map.

Before joining the project, Chen had been working on the existence proof of the tG and rGL families [Che21a] and a construction of minimal surfaces by gluing Scherk saddle towers [CT24]. Upon reading [MSSTM18], he noticed that the Scherk tower limit of the family can be produced by the gluing method [CT24]. Moreover, he recognized that the gyrating H'-T surfaces are to the H'-T surfaces what the tG surfaces are to the tP surfaces. Therefore, the existence can be proved using a Weierstrass parametrization defined on a torus, see section 2, in a similar manner as was done for the deformation families of the Gyroid [Che21a].

## 1. NUMERICAL CONSTRUCTION FROM THE DUAL INTERTHREADED QTZ–QZD NETS

We construct a unique triply-periodic minimal surface from a dual pair of interpenetrating nets (specifically the quartz net and its dual). A tubular representation of one of the nets is ‘evolved’ numerically to a TPMS. One of its two disjoint domains is homeomorphic to a tubular neighbourhood of the **qtz** net while the other is homeomorphic to the **qzd** net.

**1.1. Dual pair of **qtz** and **qzd** nets.** We consider the pair of dual interpenetrating 3D networks known as the quartz (**qtz**) net and the **qzd** net [DFOY03], both documented in reticular chemistry structure resource (RCSR, [OPRY08]), see Figure 1.

The **qtz** net has symmetry group  $P6_222$ , with identical four-coordinated vertices, all at symmetry site 3c (in Wyckoff notation), and with a single type of edge along the two-fold axes with Wyckoff symbol 6j. The dual net of **qtz** is the **qzd** (quartz dual) net, with the same space group. The **qzd** net has a single type of vertex, at Wyckoff position 3a with symmetry 222, on the six-fold  $6_2$  screw axis in vertical (c) direction. The **qzd** net has two edge types: a vertical edge along the  $6_2$  screw axis and a horizontal edge along a two-fold rotational symmetry axis. The nodes of the **qzd** net are planar and four-coordinated, with edges at  $90^\circ$ . The **qtz**-**qzd** structure has one free parameter (the value of the crystallographic  $c/a$ ), in addition to the lattice parameter  $a$  which only represents an affine scaling.

The **qtz**-**qzd** network pair was chosen, following advice by Michael O’Keeffe, as it represents two dual nets that are not identical, that have chiral screw symmetries and a free parameter. This

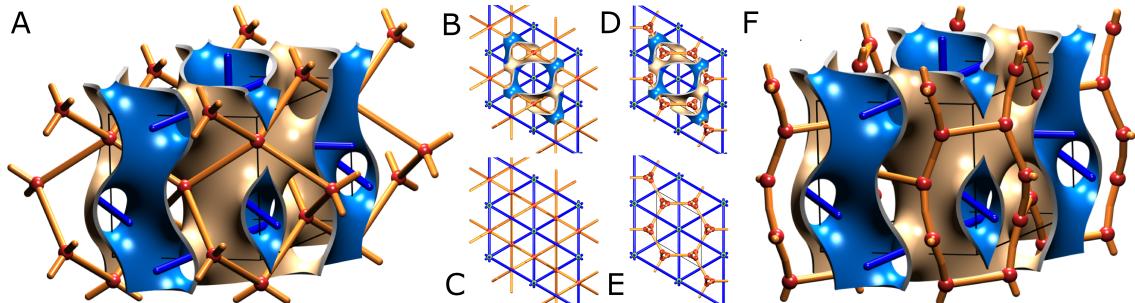


FIGURE 2. The numerically evolved minimal surface that separates the **qtz** and the **qzd** net. (A-C) the networks shown are the **qtz** and the **qzd** net. (D-F) the networks shown are the medial axes as calculated from the minimal surface, as the curves tracing the center lines that are most distance from the minimal surface.

was motivated by our optical work in which we sought chiral sheet-like prototype geometries for photonic crystals with a tunable chiral pitch.

Following the philosophy set out by O'Keeffe, Hyde and others, the quartz (or **qtz**) net and the **qzd** net, shown in Figure 1(a) and (b), are an adequate choice for the generation of a well-defined dividing surface, as they are duals (in the notion of Appendix II of [Sch70]) of one another and of the same symmetry (called proper duals [BDFOP07]).

**1.2. Evolution of tubular net to a mesh approximation of the minimal surface.** As the next stage of the construction, a discrete representation (triangulation) of an interface between the **qtz** and the **qzd** nets can be generated, which is neither smooth nor area-minimizing. This can be the tubular surface, or a discretisation thereof, around one of the two networks; or a culled Voronoi diagram of a set of points on the two graphs (with all Voronoi faces removed that separate points on the same graph [dCDFHO13, HPK<sup>+</sup>25]). The resulting triangulated surface can be evolved towards a minimal surface, by using a conjugate gradient scheme with a function  $f[S] = \int_S (H - H_0)^2 dS$  with  $H_0 = 0$  and  $H$  the point-wise mean curvature. Leaving mathematical details aside, minimisation of this functional to  $f[S] = 0$  yields the same minimal surface as does area-minimisation with a fixed (but here unknown) volume constraint [GB97]. For a general surface, it is however not guaranteed that a minimum with constant  $H = 0$  can be reached by a topology-and/or symmetry-preserving evolution, however for the **qtz**-**qzd** surface it works (We have used this method for  $0.4 \leq c/a \leq 2.9$ ).

We have used Brakke's conjugate gradient solver **Surface Evolver** [Bra92] to determine a surface minimising  $f[S]$ , starting from a tubular mesh around the **qtz** net. To a high degree of numerical accuracy, the resulting surface has  $f[S] = 0$ , with discrete point-wise mean curvatures verified to vanish. By computer-visual inspection, all symmetries of the original symmetry group are preserved.

The mesh representation we obtain is of sufficiently high quality to determine the flat points (the points with Gauss curvature  $K = 0$ ) and their nature. In an earlier pre-print version of this article [MSSTM18] (see in particular section 4.2), we identified these flat points.

When one constructs the medial surface of the **qtz** domain and reduces it to a one-dimensional net, with nodes and edges tracing lines of maximal distance to the surface, it turns out that that 'skeletal graph' is not the four-coordinated **qtz** net, but rather a net, known as **eta** [RKE<sup>+</sup>05], with the same symmetry group with three-coordinated vertices. See Fig. 2 (F).

## 2. TORIC WEIERSTRASS PARAMETERIZATION

There is another way to approach the previously described surface. It relates to Allan Schoen's model of the H'-T surface [Sch70] and gives rise to our name for this surface as the *gyrating H'-T* surface family.

Schoen described a surface that he called H'-T [Sch70]. Its translational unit consists of three "catenoids", one bounded by two regular hexagons and the other two bounded by two equiangular triangles, with catenoidal axis all in vertical direction (i.e., crystallographic  $c$  axis), see Fig. 3. Apart from periodic translations, the H'-T surface also admits a six-fold rotation symmetry around the vertical axis through the center of the hexagons, and two-fold rotational symmetries around the horizontal axes that interchange the two hexagons.

The skeletal graphs of the H'-T surface are, as per the reticular chemistry structure resource, the graphs **bnn** and **hex** (which, individually and as a pair, have symmetry group 191,  $P6/mmm$ ), see Fig. 3 (C and D). The space group of the H'-T surface is, therefore, space group 191 ( $P6/mmm$ ).

This means that the surface has six-fold rotational symmetry around a vertical axes (the vertical edges of the **hex** net) and horizontal two-fold rotational symmetry axes (the horizontal edges of the **hex** net).

The construction of the *gyrating H'-T* surface (separating the **qtz**-**qzd** nets) is based on the following idea: The six-fold rotational axis of the H'-T surface changes into a six-fold screw axis ( $6_2$ ) in the *gyrating H'-T* surface. The two-fold rotations, while preserved, no longer intersect in the same point on the screw axis, but rather become a staggered 'ladder' of two-fold axes shown in Fig. 1.

It was observed in [GBW96, Lemma 4] that, as one travels along the associate family, the rotational symmetries listed above are preserved, except for the rotational symmetry around the vertical axis, which is reduced to a screw symmetry.

We define  $\mathcal{H}$  to be the set of embedded TPMSs of genus 4 that admit order-6 screw symmetries around vertical axes and order-2 rotational symmetries around horizontal axes.

Note that pure rotations are screw transforms with 0 translation. So Schoen's H'-T surfaces belong to  $\mathcal{H}$ .

Our key result is the statement that there exists another 1-parameter family in  $\mathcal{H}$ , different to Schoen's H'-T surface. We derive an explicit Weierstrass formula for that surface and prove the following theorem:

**Theorem 2.1.** *Apart from Schoen's H'-T surfaces, there is another 1-parameter family of TPMS of genus 4 with the following properties:*

- (1) *Each TPMS in the family admits a screw symmetry of order 6 around a vertical axis, and rotational symmetries of order 2 around horizontal axes.*
- (2) *The family tends to Scherk saddle towers in one limit and to doubly periodic Scherk surfaces in the other limit.*

In [CT24], this family was constructed implicitly and proved to be unique near the saddle tower limit. This guarantees that the two constructions we provide (that is, above in terms of the dual qtz-qzd net and below in terms of the gyrated H'-T surface) give the same surface family.

**2.1. Weierstrass parametrization.** Let  $M$  be a TPMS of genus  $g$  invariant under the lattice translations  $\Lambda$  (i.e.,  $M/\Lambda$  is the translational unit cell). Meeks [Mee90] proved that the Gauss map  $G$  represents  $M/\Lambda$  as a  $(g - 1)$ -sheeted conformal branched cover of the sphere  $\mathbb{S}^2$ , and Riemann-Hurwitz formula implies  $4(g - 1)$  branch points (counted with multiplicity) corresponding to the zeros of the Gauss curvature (counted with multiplicity).

We use the following form of Weierstrass parametrization that maps a point  $p$  on the Riemann surface to a point in  $\mathbb{R}^3$  and that traces back to Osserman [Oss64],

$$(1) \quad p \mapsto \operatorname{Re} \int^p (\omega_1, \omega_2, \omega_3) = \operatorname{Re} \int^p \left( \frac{G^{-1} - G}{2}, \frac{i(G^{-1} + G)}{2}, 1 \right) dh .$$

On the Riemann surface  $\Sigma$  the functions  $\omega_1, \omega_2, \omega_3$  must all be holomorphic. In particular, the holomorphic differential  $\omega_3 = dh$  is called the *height differential*.  $G$  denotes (the stereographic projection of) the Gauss map. The triple  $(\Sigma, G, dh)$  is called Weierstrass data.

The purpose of this section is to determine the Weierstrass data for surfaces in  $\mathcal{H}$  from their symmetries. In particular,  $\Sigma$  will be a branched torus, whose branch points are determined in Lemma 2.6. The height differential is determined in Lemma 2.4, and the Gauss map is explicitly given by (2).

**Remark 2.2.** Most textbooks only mention the Weierstrass parameterization where  $\Sigma$  is taken as a simply connected domain in the complex plane. Now that we allow  $\Sigma$  to be any Riemann surface, the parameterization becomes more flexible and powerful. Different choices of  $\Sigma$  should lead to the same result, but particular choices often bring convenience. For instance, to parametrise a TPMS, one could use a branched cover of the sphere as  $\Sigma$  (as was done by Schoen [Sch70], Meeks[WHM75, Mee90], Fogden & Hyde [FH92a, FH92b, Fog93], and many others, and as we also did in an earlier arXiv version of this manuscript [MSSTM18]). In this paper, as screw symmetries are concerned, we will instead use branched cover of tori as  $\Sigma$ , which allows us to employ elliptic functions.

**2.2. Weierstrass data on tori.** Surfaces in  $\mathcal{H}$  all admit screw symmetries. The following Lemma justifies our choice of branched tori for  $\Sigma$ . Here,  $M$  is the infinite periodic surface,  $\Lambda$  are the lattice translations and therefore  $M/\Lambda$  the (primitive) translational unit cell.  $S$  is the screw symmetry,

and therefore  $(M/\Lambda)/S$  the surface patch from which the translational unit is obtained by applying the screw symmetry  $S$ .

**Lemma 2.3.** *If a TPMS  $M$  of genus 4 admits a screw symmetry  $S$ , then  $(M/\Lambda)/S$  is of genus one.*

The simple proof (given in section C) uses the Riemann–Hurwitz formula

$$g = n(g' - 1) + 1 + B/2$$

that relates the genus  $g$  of the translational unit cell  $M/\Lambda$  (in our case  $g = 4$ ) to the genus  $g'$  of the translational unit cell modulo  $S$ , i.e.  $(M/\Lambda)/S$ .  $n$  is the degree of the quotient map, and  $B$  is the total branching number. The proof simply considers each of the small number of discrete possibilities.

The height differential  $dh$ , being a holomorphic 1-form on the torus, must be of the form  $re^{-i\theta}dz$ . Varying the modulus  $r$  only results in a scaling. Varying the argument  $\theta$  gives the associate family, so we call  $\theta$  the *associate angle*:

**Lemma 2.4.** *If a TPMS  $M/\Lambda$  with a screw symmetry  $S$  is represented on the branched cover of the torus  $(M/\Lambda)/S$ , then up to the scaling, the height differential  $dh$  must be the lift of  $e^{-i\theta}dz$  (note the sign!).*

**2.3. Locating branch points.** If the order of  $S$  is prime, the following formula from [FK92] allows us to calculate the number of fixed points:

$$|\text{fix}(S)| = 2 + \frac{2g - 2g' \text{order}(S)}{\text{order}(S) - 1}.$$

In particular, a screw symmetry of order 2 fixes exactly six points. They correspond to zeroes and poles of  $G^2$ . The following lemma follows from the same argument as in [Wey06, Lemmas 3.9, 3.13].

**Lemma 2.5.** *If a TPMS of genus 4  $M/\Lambda$  admits a screw symmetry  $S$  of order 2, then  $G^2$  descends to an elliptic function on the torus  $(M/\Lambda)/S$  with three simple zeros and three simple poles.*

We now try to locate the branch points of the covering map for surfaces in  $\mathcal{H}$ . Since the ramification points on  $M/\Lambda$  (i.e. points where the covering map fails to be a local homeomorphism) are all poles and zeros of the Gauss map, our main tool is naturally Abel’s Theorem, which states that the difference between the sum of poles and the sum of zeros (counting multiplicity) is a lattice point.

A surface in  $\mathcal{H}$  admits a screw symmetry  $S$  of order 6,  $S^3$  is then a screw symmetry of order 2.

**Lemma 2.6** (Compare [Wey06, Lemma 3.10]). *Let  $M$  be a TPMS of genus 4 admitting a screw symmetry  $S$  of order 6, hence parametrized on a branched double cover of the torus  $(M/\Lambda)/S^3$ . If one branch point is placed at 0, then the other branch points must be placed at 3-division points.*

Here, a point  $p$  in the torus is a *3-division point* if  $3p = 0$ . Or, equivalently, if we see the torus as the quotient of the complex plane over a lattice, then a point is a 3-division point if  $3p$  is a lattice point. See Appendix D for the proof of this lemma.

Let the quotient torus be spanned by 1 and  $\tau \in \mathbb{C}$ . Without loss of generality, we may then assume that the zeros of  $G^2$  are at 0,  $1/3$ , and  $2/3$ , while the poles of  $G^2$  are at  $\tau/3$ ,  $\tau/3 + 1/3$ , and  $\tau/3 + 2/3$ . See Figure 4.

Then  $G^2$  has the explicit form

$$G^2(z) = \rho \frac{\theta(z; \tau)\theta(z - \frac{1}{3}; \tau)\theta(z + \frac{1}{3}; \tau)}{\theta(z + \frac{2\tau}{3}; \tau)\theta(z - \frac{\tau}{3} - \frac{1}{3}; \tau)\theta(z - \frac{\tau}{3} + \frac{1}{3}; \tau)},$$

where  $\theta(z; \tau)$  is the Jacobi Theta function.

The factor  $\rho$  is called the López–Ros factor [LR91]. Varying its argument only results in a rotation of the surface in the space, hence only the norm  $|\rho|$  concerns us. We choose  $\rho$  so that  $G^2 = 1$  at the rotation center  $\tau/6$ .

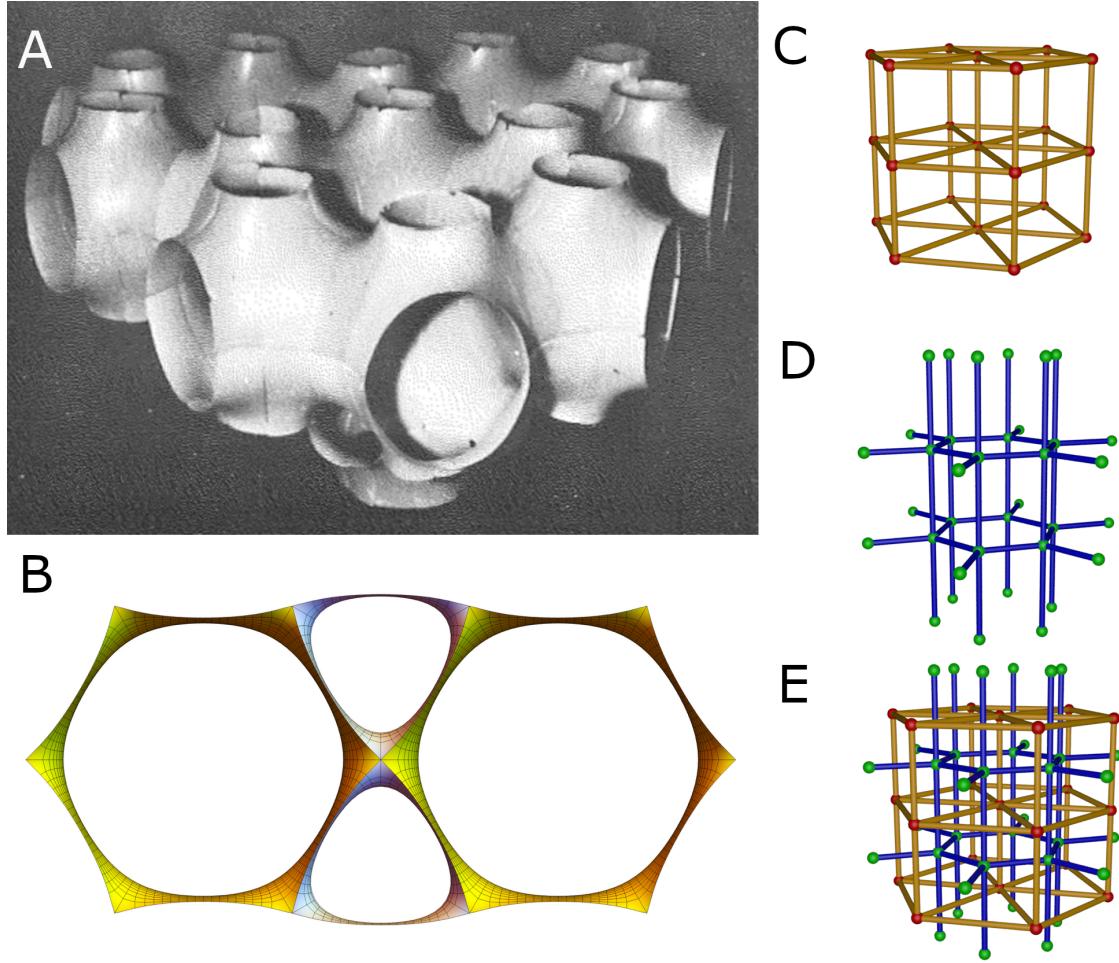


FIGURE 3. Schoen's H'-T surface: (A) Photograph of a plasticine model created by Alan Schoen. (B) The H'-T surface can be constructed by copy-translations from a hexagonal catenoidal patch and two adjacent triangular catenoidal necks, all aligned in the vertical axis (that is, the crystallographic  $c$  axis). (C) The hexagonal lattice (**hex**) representing the skeletal graphs on one side of the surface (D) The **bnn** net representing the other skeletal graph. (E) The dual pair of **hex** and **bnn** nets. (Image (A) reproduced from Figure 13 of [Sch70]).

Equivalently, by [Law89, (9.6.6)], we can write

$$(2) \quad G^2(z) = \rho' e^{2i\pi z} \frac{\theta(3z; 3\tau)}{\theta(3z - \tau; 3\tau)},$$

where  $\rho' = -e^{i\pi\tau/3}$  is again chosen so that  $G^2(\tau/6) = 1$ .

**2.4. Twisted catenoids.** We choose three branch cuts along the straight segments from  $\tau/3 + k/3$  to  $\tau + k/3$ ,  $k = 0, 1, 2$ . Assume  $dh = e^{-i\pi/2} = -idz$  for the moment. We now study the image of the Weierstrass parametrization

From the quasi-periodicity of  $\theta$  function

$$\theta(z + (m + n\tau); \tau) = (-1)^{m+n} e^{-in^2\pi\tau} e^{-2in\pi z} \theta(z; \tau),$$

we note that

$$G^2(z + 1/3) = -e^{2i\pi/3} G^2(z).$$

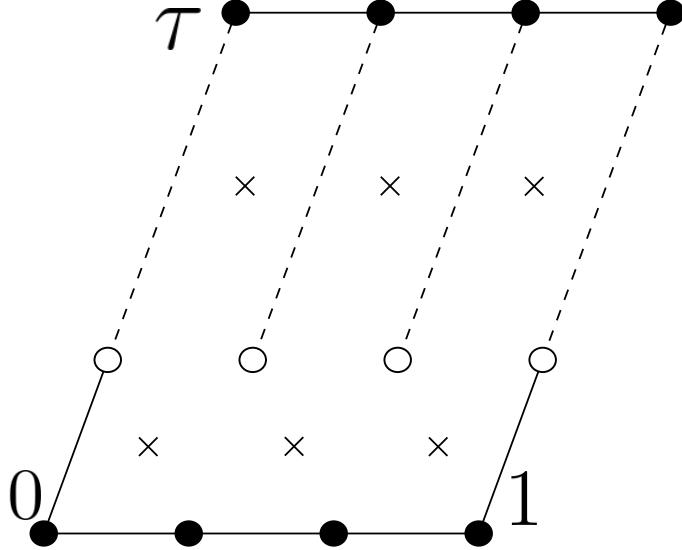


FIGURE 4. Branched torus on which our surface is parametrized. Dashed segments are branched cuts. Solid circles are the zeros of  $G^2$ , empty circles are the poles, and  $\times$  represent symmetry centers, corresponding to fixed points on the surface under order-2 rotations with horizontal axis.

By the same argument as in [Che21b], we know the following

- (1) For each branch of the branched torus, the part  $0 < \text{Im } z < \text{Im } \tau/3$  is mapped by the Weierstrass representation to a twisted catenoids bounded by curved triangles. These triangles are congruent, right-angled, and lie in horizontal planes at height 0 and  $\text{Im } \tau/3$ , respectively. Moreover, they share a rotational symmetry of order 3 around vertical axis.
- (2) The part  $\text{Im } \tau/3 < \text{Im } z < \text{Im } \tau$  of the branched torus is mapped to a twisted catenoid bounded by curved hexagons. These hexagons are congruent, right-angled, and lie in horizontal planes at height  $\text{Im } \tau/3$  and  $\text{Im } \tau$ , respectively. Moreover, they share a rotational symmetry of order 6 around a vertical axis.
- (3) Because of our choice of the López–Ros factor  $\rho$ , all these twisted catenoids admit rotational symmetries of order 2 around horizontal axes that exchange its boundaries.

See Figure 5 for a typical example.

**2.5. Period problem.** As one travels from the twisted catenoids along the associate family, the twisted catenoid opens up into a ribbon bounded by triangular or hexagonal helices. For adjacent ribbons to “fit exactly into each other” [GBW96], we need that the poles and zeros of the Gauss map are (i) aligned along vertical lines over the vertices of a hexagon-triangle tiling, and (ii) alternatingly arranged and the vertical width of the hexagonal ribbon doubles that of the triangular ribbon.

The ratio of the vertical width is actually a consequence of the positions of the branch points. The requirements listed above form a sufficient condition for the immersion.

To be more precise, we define the pitch of a helix to be the increase of height after the helix makes a full turn. We need that the pitch of each triangular helix is three times the minimum vertical distance between the poles and the zeros. As a consequence, the pitch of each hexagonal helix, which doubles the pitch of triangular helices, must be six times the minimum vertical distance between the poles and the zeros.

This means that the integral of the height differential  $dh$  from 0 to 1 triples the integral from 0 to  $\tau/3$ . Or equivalently, the integral from 0 to  $1/3$  equals the integral from 0 to  $\tau/3$ . This can be easily achieved by adjusting the associate angle  $\theta$  to (compare [Wey08, Definition 4.2])

$$\theta = \theta_v(\tau) = \arg(\tau - 1) - \pi/2.$$

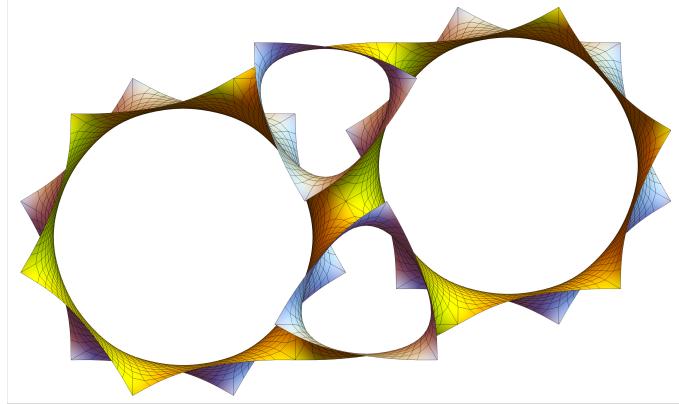


FIGURE 5. Image of Weierstrass representation assuming bonnet angle  $\theta = \pi/2$ , seen from below, showing two twisted triangular catenoids (corresponding to the lower third of the branched torus), and two twisted hexagonal catenoids (one of these corresponds to the upper two thirds of the branched torus, the other one is a lattice translation included for clarity).

We now calculate the associate angle in another way, using the fact that the images of 0 and  $\tau$  are vertically aligned, i.e. have the same horizontal coordinates.

First note that, because  $\tau/6$  is a center of symmetry, we have

$$\int_0^{\tau/3} dz/G = \int_0^{\tau/3} dz \cdot G =: \psi(\tau)$$

We may place the image of 0 at the origin. First look at the surface with  $\theta = 0$ , hence  $dh = dz$ . Then the horizontal coordinates of the image of  $\tau/3$  are

$$\operatorname{Re} \int_0^{\tau/3} \left( \frac{1}{2} \left( \frac{1}{G} - G \right), \frac{i}{2} \left( \frac{1}{G} + G \right) \right) dz = (0, -\operatorname{Im} \psi(\tau)).$$

Then we look at the surface with  $\theta = \pi/2$ , hence  $dh = e^{-i\pi/2} dz = -idz$  (the conjugate surface). Then the coordinates are

$$\operatorname{Re} \int_0^{\tau/3} \left( -\frac{i}{2} \left( \frac{1}{G} - G \right), \frac{1}{2} \left( \frac{1}{G} + G \right) \right) dz = (0, \operatorname{Re} \psi(\tau)).$$

So for the surface with associate angle  $\theta$ , the first coordinate is always 0, while the second coordinate

$$-\cos \theta \operatorname{Im} \psi(\tau) + \sin \theta \operatorname{Re} \psi(\tau)$$

vanishes when

$$\theta = \theta_h(\tau) := \arg \psi(\tau).$$

We have shown that

**Lemma 2.7.** *The Weierstrass data given by Lemmas 2.4 and (2) define an immersion if and only if*

$$\theta_h(\tau) = \theta_v(\tau),$$

*or more explicitly,*

$$(3) \quad \arg \psi(\tau) = \arg(\tau - 1) - \pi/2$$

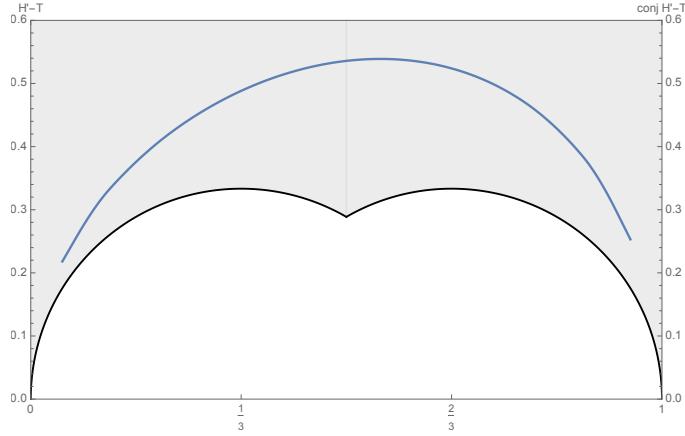


FIGURE 6. Solution to the period problem (3) with  $0 < \operatorname{Re} \tau < 1$ .

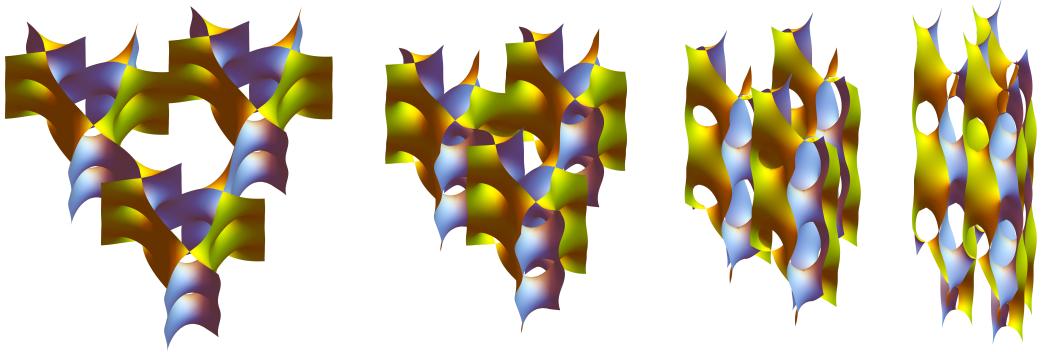


FIGURE 7. Gyrating H'-T surfaces with  $\operatorname{Re} \tau = 0.2, 0.4, 0.6, 0.8$ , from left to right.

### 3. EXISTENCE PROOF

In Figure 6, we show the numerical solutions to (3) with  $0 < \operatorname{Re} \tau < 1$ , accompanied by two circular arcs. Our task is to prove the existence of the continuous 1-parameter solution curve that we see in the picture. Let the shaded domain in the figure be denoted by  $\Omega_t := \{\tau \mid \operatorname{Im} \tau > 0, 0 < \operatorname{Re} \tau < 1, |\tau - 1/3| > 1/3, |\tau - 2/3| > 1/3\}$ .

**Proposition 3.1.** *There exists a continuous 1-dimensional curve of  $\tau$  in  $\Omega_t$  that solves (3). This curve tends to 0 at one end and to 1 at the other end. Moreover, the TPMSSs of genus 4 represented by points on the curve are all embedded.*

The proof for this proposition is found in Appendix E. In its essence, it constructs a solution to the period problem in terms of the parameter  $\tau$ . Surfaces with the same symmetry near the saddle tower limit have been constructed in [CT24] by a gluing constructions. Those surfaces have been proven to be embedded and unique, and hence must belong to the curve above near the limit 0. Their embeddedness then ensures the embeddedness of all TPMSSs on the curve. This follows from [Wey06, Proposition 5.6], which was essentially proved in [Mee90].

In Figure 7, we show four gyrating H'-T surfaces.

**3.1. Acknowledgement.** We are grateful to Michael O’Keeffe for his suggestion of the **qtz-qzd** as a chiral pair of networks with a tunable pitch. We are grateful to Robert Kusner and Karsten Große-Brauckmann for early discussions of the period problem.

## REFERENCES

- [BDFOP07] Vladislav A. Blatov, Olaf Delgado-Friedrichs, Michael O’Keeffe, and Davide M. Proserpio. Three-periodic nets and tilings: natural tilings for nets. *Acta Crystallographica Section A Foundations of Crystallography*, 63(5):418–425, August 2007.
- [Bra92] Kenneth A. Brakke. The surface evolver. *Experimental Mathematics*, 1(2):141–165, January 1992.
- [Bra25] Ken Brakke. Triply-periodic minimal surfaces. Technical report, <https://kenbrakke.com/evolver/examples/periodic/periodic.html>, 2025.
- [CF22] Hao Chen and Daniel Freese. Helicoids and vortices. *Proc. A.*, 478(2267):Paper No. 20220431, 18, 2022.
- [Che21a] Hao Chen. Existence of the tetragonal and rhombohedral deformation families of the gyroid. *Indiana University Mathematics Journal*, 70(4):1543–1576, 2021.
- [Che21b] Hao Chen. Existence of the tetragonal and rhombohedral deformation families of the gyroid. *Indiana Univ. Math. J.*, 70(4):1543–1576, 2021.
- [CT24] Hao Chen and Martin Traizet. Gluing Karcher-Scherk saddle towers I: triply periodic minimal surfaces. *J. Reine Angew. Math.*, 808:1–47, 2024.
- [CW21a] Hao Chen and Matthias Weber. A new deformation family of schwarz’ d surface. *Transactions of the American Mathematical Society*, 374(4):2785–2803, January 2021.
- [CW21b] Hao Chen and Matthias Weber. An orthorhombic deformation family of Schwarz’ H surfaces. *Trans. Amer. Math. Soc.*, 374(3):2057–2078, 2021.
- [dCDFHO13] Liliana de Campo, Olaf Delgado-Friedrichs, Stephen T. Hyde, and Michael O’Keeffe. Minimal nets and minimal minimal surfaces. *Acta Crystallographica Section A Foundations of Crystallography*, 69(5):483–489, July 2013.
- [DFOY03] Olaf Delgado Friedrichs, Michael O’Keeffe, and Omar M. Yaghi. The cdso4, rutile, cooperite and quartz dual nets: interpenetration and catenation. *Solid State Sciences*, 5(1):73 – 78, 2003. Dedicated to Sten Andersson for his scientific contribution to Solid State and Structural Chemistry.
- [DGMG25] Michael S. Dimitriev, Benjamin R. Greenvall, Rejoy Mathew, and Gregory M. Grason. Not even metastable: Cubic double-diamond in diblock copolymer melts. *ACS Macro Letters*, 14(9):1291–1298, 2025. PMID: 40864671.
- [DLMF] *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.0.19 of 2018-06-22. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.
- [FH92a] A. Fogden and S. T. Hyde. Parametrization of triply periodic minimal surfaces. i. mathematical basis of the construction algorithm for the regular class. *Acta Crystallographica Section A Foundations of Crystallography*, 48(4):442–451, July 1992.
- [FH92b] A. Fogden and S. T. Hyde. Parametrization of triply periodic minimal surfaces. ii. regular class solutions. *Acta Crystallographica Section A Foundations of Crystallography*, 48(4):575–591, July 1992.
- [FH99] Andrew Fogden and Stephan T. Hyde. Continuous transformations of cubic minimal surfaces. *The European Physical Journal B-Condensed Matter and Complex Systems*, 7(1):91–104, 1999.
- [FK92] H. M. Farkas and I. Kra. *Riemann surfaces*, volume 71 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992.
- [Fog93] A. Fogden. Parametrization of triply periodic minimal surfaces. iii. general algorithm and specific examples for the irregular class. *Acta Crystallographica Section A Foundations of Crystallography*, 49(3):409–421, May 1993.
- [FW09] Shoichi Fujimori and Matthias Weber. Triply periodic minimal surfaces bounded by vertical symmetry planes. *manuscripta mathematica*, 129(1):29–53, May 2009.
- [GB97] Karsten Große-Brauckmann. Gyroids of constant mean curvature. *Experimental Mathematics*, 6(1):33–50, January 1997.
- [GBW96] Karsten Große-Brauckmann and Meinhard Wohlgemuth. The gyroid is embedded and has constant mean curvature companions. *Calc. Var. Partial Differential Equations*, 4(6):499–523, 1996.
- [GK00] Paul J.F Gandy and Jacek Klinowski. Exact computation of the triply periodic g (‘gyroid’) minimal surface. *Chemical Physics Letters*, 321(5):363 – 371, 2000.
- [HAL<sup>+</sup>97] S. T Hyde, S. Andersson, K. Larsson, Z. Blum, T. Landh, S. Lidin, and B.W. Ninham. *The language of shape: the role of curvature in condensed matter: physics, chemistry and biology*. Elsevier, Amsterdam, 1997.
- [HdCO09] Stephen T. Hyde, Liliana de Campo, and Christophe Oguey. Tricontinuous mesophases of balanced three-arm “star polyphiles”. *Soft Matter*, 5(14):2782, 2009.
- [HOP08] Stephen T. Hyde, Michael O’Keeffe, and Davide M. Proserpio. A short history of an elusive yet ubiquitous structure in chemistry, materials, and mathematics. *Angewandte Chemie International Edition*, 47(42):7996–8000, 2008.
- [HPK<sup>+</sup>25] M Himmelmann, MC Pedersen, MA Klatt, PWA Schönhöfer, ME Evans, and GE Schröder-Turk. Amorphous bicontinuous minimal surface models and the superior gaussian curvature uniformity of diamond, primitive and gyroid surfaces. *Proc. R. Soc. A*, 2025.

- [HST24] Stephen T. Hyde and Gerd E Schroeder-Turk. Alan hugh schoen. *Physics Today*, 2024(01):ykfu, January 2024.
- [Hyd90] S. T. Hyde. Curvature and the global structure of interfaces in surfactant-water systems. *Le Journal de Physique Colloques*, 51(C7):C7–209–C7–228, December 1990.
- [KF88] Elke Koch and Werner Fischer. On 3-periodic minimal surfaces with non-cubic symmetry. *Zeitschrift für Kristallographie - Crystalline Materials*, 183(1-4), January 1988.
- [KP96] H. Karcher and K. Polthier. Construction of triply periodic minimal surfaces. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 354(1715):2077–2104, September 1996.
- [Law89] Derek F. Lawden. *Elliptic functions and applications*, volume 80 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.
- [LM03] Eric A. Lord and Alan L. Mackay. Periodic minimal surfaces of cubic symmetry. *Current Science*, 85(3):346 – 362, 2003.
- [LR91] Francisco J. López and Antonio Ros. On embedded complete minimal surfaces of genus zero. *J. Differential Geom.*, 33(1):293–300, 1991.
- [Mee90] William H. Meeks, III. The theory of triply periodic minimal surfaces. *Indiana Univ. Math. J.*, 39(3):877–936, 1990.
- [MSSTM18] Shashank Ganesh Markande, Matthias Saba, Gerd Schroeder-Turk, and Elisabetta A. Matsumoto. A chiral family of triply-periodic minimal surfaces derived from the quartz network, 2018.
- [MSTM12] Walter Mickel, Gerd E. Schröder-Turk, and Klaus Mecke. Tensorial minkowski functionals of triply periodic minimal surfaces. *Interface Focus*, 2(5):623–633, June 2012.
- [OPRY08] Michael O’Keeffe, Maxim A. Peskov, Stuart J. Ramsden, and Omar M. Yaghi. The reticular chemistry structure resource (rcsr) database of, and symbols for, crystal nets. *Accounts of Chemical Research*, 41(12):1782–1789, 2008. PMID: 18834152.
- [Oss64] Robert Osserman. Global properties of minimal surfaces in  $E^3$  and  $E^n$ . *Ann. of Math. (2)*, 80:340–364, 1964.
- [PP93] Ulrich Pinkall and Konrad Polthier. Computing discrete minimal surfaces and their conjugates. *Experimental Mathematics*, 2(1):15–36, January 1993.
- [RKE<sup>+</sup>05] Nathaniel L. Rosi, Jaheon Kim, Mohamed Eddaoudi, Banglin Chen, Michael O’Keeffe, and Omar M. Yaghi. Rod packings and metal-organic frameworks constructed from rod-shaped secondary building units. *Journal of the American Chemical Society*, 127(5):1504–1518, 2005. PMID: 15686384.
- [SC89] J F. Sadoc and J. Charvolin. Infinite periodic minimal surfaces and their crystallography in the hyperbolic plane. *Acta Crystallographica Section A Foundations of Crystallography*, 45(1):10–20, January 1989.
- [Sch70] A. H. Schoen. Infinite periodic minimal surfaces without self-intersections. *NASA Technical Note*, 1970.
- [Sch12] Alan H. Schoen. Reflections concerning triply-periodic minimal surfaces. *Interface Focus*, 2(5):658 – 668, 2012.
- [SKM<sup>+</sup>07] Gemma C. Shearman, Bee J. Khoo, Mary-Lynn Motherwell, Kenneth A. Brakke, Oscar Ces, Charlotte E. Conn, John M. Seddon, and Richard H. Templer. Calculations of and evidence for chain packing stress in inverse lyotropic bicontinuous cubic phases. *Langmuir*, 23(13):7276–7285, 2007. PMID: 17503862.
- [STdCE<sup>+</sup>13] Gerd E. Schröder-Turk, Liliana de Campo, Myfanwy E. Evans, Matthias Saba, Sebastian C. Kapfer, Trond Varslot, Karsten Grosse-Brauckmann, Stuart Ramsden, and Stephen T. Hyde. Polycontinuous geometries for inverse lipid phases with more than two aqueous network domains. *Faraday Discuss.*, 161:215–247, 2013.
- [STFH06] Gerd E Schröder-Turk, Andrew Fogden, and Stephen T Hyde. Bicontinuous geometries and molecular self-assembly: comparison of local curvature and global packing variations in genus-three cubic, tetragonal and rhombohedral surfaces. *The European Physical Journal B-Condensed Matter and Complex Systems*, 54(4):509–524, 2006.
- [STVdC<sup>+</sup>11] Gerd E. Schröder-Turk, Trond Varslot, Liliana de Campo, Sebastian C. Kapfer, and Walter Mickel. A bicontinuous mesophase geometry with hexagonal symmetry. *Langmuir*, 27(17):10475–10483, August 2011.
- [Tra08] Martin Traizet. On the genus of triply periodic minimal surfaces. *J. Differential Geom.*, 79(2):243–275, 2008.
- [vSN91] H. G. von Schnering and R. Nesper. Nodal surfaces of fourier series: Fundamental invariants of structured matter. *Zeitschrift für Physik B Condensed Matter*, 83(3):407–412, October 1991.
- [WCZ<sup>+</sup>24] Shuqi Wang, Hao Chen, Tianyu Zhong, Quanzheng Deng, Shaobo Yang, Yuanyuan Cao, Yongsheng Li, and Lu Han. Tetragonal gyroid structure from symmetry manipulation: A brand-new member of the gyroid surface family. *Chem*, 10(5):1406–1424, May 2024.
- [Wey06] Adam G. Weyhaupt. *New families of embedded triply periodic minimal surfaces of genus three in euclidean space*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–Indiana University.
- [Wey08] Adam G. Weyhaupt. Deformations of the gyroid and Lidinoid minimal surfaces. *Pacific J. Math.*, 235(1):137–171, 2008.

[WHM75] III William H. Meeks. The Geometry and the Conformal Structure of Triply Periodic Minimal Surfaces in R<sup>3</sup>. PhD thesis, University of California Berkeley, 1975.

#### APPENDIX A. USE OF SURFACE EVOLVER TO OBTAIN APPROXIMATE MINIMAL SURFACE

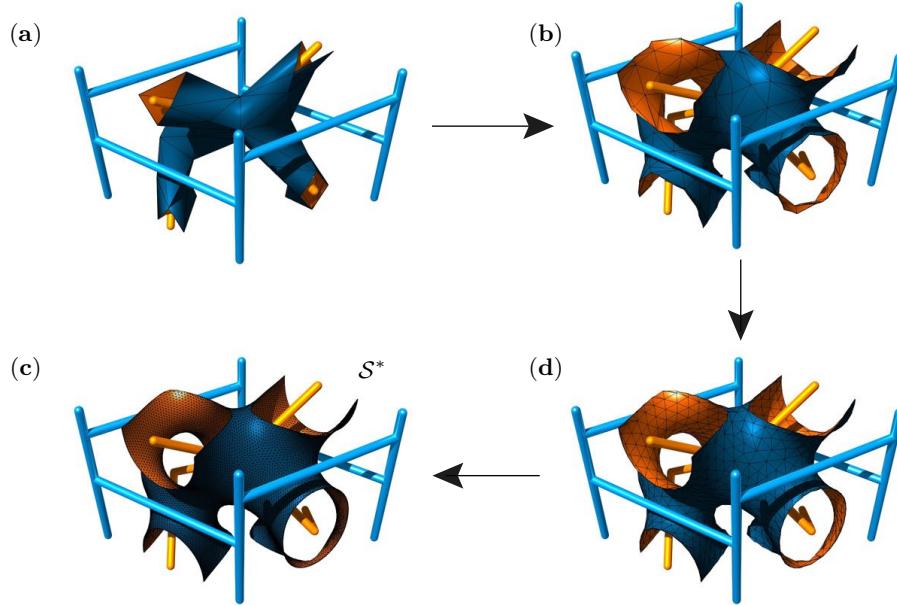


FIGURE 8. Four different stages illustrating the minimisation of the area functional using conjugate gradient descent method in Surface Evolver [Bra92]. See Appendix A2.2. The boundary conditions respect the translational periodicities imposed by the  $P6_222$  space group symmetry with lattice parameters  $c = a = 1$ . The blue coloured network is the  $\text{qzd}$  network and the orange coloured network is the  $\text{qtz}$  (quartz) network. Figure (a) shows a triangulated tubular neighbourhood of the quartz network and figure (c) shows the final surface,  $\mathcal{S}^*$  after convergence.

## APPENDIX B. ASYMPTOTIC BEHAVIOR

We now prove that

**Lemma B.1.**

$$\lim_{\operatorname{Im} \tau \rightarrow +\infty} \arg \psi(\tau) = \lim_{\operatorname{Im} \tau \rightarrow +\infty} \arg \int_0^{\tau/3} G(z) dz = \frac{1}{2} - \frac{\operatorname{Re} \tau}{3}.$$

*Proof.* In the limit  $\operatorname{Im} \tau \rightarrow +\infty$ , by the expansion [DLMF, (20.2.1)]

$$\theta(z; \tau) = 2q^{1/4} \sin(z) + o(q^{1/4}),$$

where  $q = \exp(i\pi\tau)$ , we have

$$\begin{aligned} G(z) &\sim \sqrt{-\frac{\sin(\pi 3z)}{\sin(\pi(3z - \tau))}} e^{i\pi(2z - \tau/3)} \\ &= \sqrt{-\frac{e^{i\pi 3z} - e^{-i\pi 3z}}{e^{i\pi(3z - \tau)} - e^{-i\pi(3z - \tau)}}} e^{i\pi(z - \tau/6)}, \end{aligned}$$

as  $\operatorname{Im} \tau \rightarrow +\infty$ .

Now we divide  $\psi(\tau)$  into two integrals

$$\psi(\tau) = \psi_1(\tau) + \psi_2(\tau) = \int_0^{\tau/6} G(z) dz + \int_{\tau/6}^{\tau/3} G(z) dz$$

In the integral  $\psi_1(\tau)$ ,  $\tau - 3z > \tau/2$ . So as  $\operatorname{Im} \tau \rightarrow +\infty$ , we

$$\begin{aligned} \psi_1(\tau) &\sim \int_0^{\tau/6} \sqrt{\frac{e^{i\pi 3z} - e^{-i\pi 3z}}{e^{-i\pi(3z - \tau)}}} e^{i\pi(z - \tau/6)} dz \\ &= e^{i\pi\tau/3} \int_0^{\tau/6} \sqrt{e^{i\pi 6z} - 1} e^{i\pi z} dz \\ (4) \quad &= \frac{i}{\pi} e^{i\pi\tau/3} \int_1^{e^{-i\pi\tau/6}} \frac{\sqrt{w^6 - 1}}{w^2} dw \quad (w = e^{-i\pi z}). \end{aligned}$$

Note that

$$\left| \frac{\sqrt{w^6 - 1}}{w^2} \right| < |w|,$$

we have

$$\begin{aligned} |(4)| &\leq \frac{1}{\pi} e^{-\pi \operatorname{Im} \tau/3} \int_1^{e^{\pi \operatorname{Im} \tau/6}} x dx \\ &= \frac{1}{2\pi} (1 - e^{-\pi \operatorname{Im} \tau/3}) \rightarrow \frac{1}{2\pi} \end{aligned}$$

as  $\operatorname{Im} \tau \rightarrow +\infty$ . So  $\psi_1(\tau)$  is bounded.

In the integral  $\psi_2(\tau)$ ,  $3z > \tau/2$ . So as  $\operatorname{Im} \tau \rightarrow +\infty$ , we

$$\begin{aligned} \psi_2(\tau) &\sim \int_{\tau/6}^{\tau/3} \sqrt{\frac{e^{-i\pi 3z}}{e^{i\pi(3z - \tau)} - e^{-i\pi(3z - \tau)}}} e^{i\pi(z - \tau/6)} dz \\ &= \int_0^{\tau/6} \sqrt{\frac{e^{-i\pi(\tau - 3z)}}{e^{-i\pi 3z} - e^{i\pi 3z}}} e^{i\pi(\tau/6 - z)} dz \quad (3\zeta = \tau - 3z) \\ &= e^{-i\pi\tau/3} \int_0^{\tau/6} \frac{e^{-i\pi\zeta}}{\sqrt{e^{-i\pi 6\zeta} - 1}} d\zeta \\ &= \frac{i}{\pi} e^{-i\pi\tau/3} \int_1^{e^{-i\pi\tau/6}} \frac{du}{\sqrt{u^6 - 1}} \quad (u = e^{-i\pi\zeta}) \\ &= \frac{i}{\pi} e^{-i\pi\tau/3} \int_1^{e^{i\pi\tau/6}} \frac{-vdv}{\sqrt{1 - v^6}} \quad (v = 1/w) \end{aligned}$$

Note that the integral in the last line converge to the bounded real integral

$$I = \int_0^1 \frac{x dx}{\sqrt{1-x^6}} \in \mathbb{R}$$

as  $\text{Im } \tau \rightarrow +\infty$ , so

$$\psi_2(\tau) \sim \frac{i}{\pi} I e^{-i\pi\tau/3}.$$

As  $\text{Im } \tau \rightarrow +\infty$ ,  $\psi_2$  is dominant, so

$$\lim_{\text{Im } \tau \rightarrow +\infty} \arg \psi(\tau) = \lim_{\text{Im } \tau \rightarrow +\infty} \arg \psi_2(\tau) = \frac{1}{2} - \frac{\tau}{3}.$$

□

### APPENDIX C. PROOF OF LEMMA 2.3

*Proof.* The height differential  $dh$  is invariant under  $S$ , hence descends holomorphically to the quotient  $(M/\Lambda)/S$ . Since there is no holomorphic differential on the sphere, the genus of  $(M/\Lambda)/S$  cannot be 0.

Recall the Riemann–Hurwitz formula

$$g = n(g' - 1) + 1 + B/2.$$

In our case,  $g = 4$  is the genus of  $M/\Lambda$ ,  $g'$  is the genus of  $(M/\Lambda)/S$ ,  $n$  is the degree of the quotient map, and  $B$  is the total branching number. Since the  $S$  is of order at least 2, we conclude immediately that  $g' < 3$ . It remains to examine the case  $g' = 2$ . Without loss of generality, we may assume the screw axis to be vertical. When  $g' = 2$ , we have  $3 = n + B/2$ . Recall that  $n > 1$  and that the Gauss map  $G$  represents  $M/\Lambda$  as a 3-sheeted branched cover of  $\mathbb{S}^2$  with branching number 12. More specifically, we have 12 branch points of branching number 1, none fixed by the screw symmetry.

One solution is given by  $n = 3$  and  $B = 0$ . Then  $S$  is a screw symmetry of order 3 and  $G^3$  represents  $(M/\Lambda)/S$  as a 3-sheeted branched cover of  $\mathbb{S}^2$  with branching number 4. By Riemann–Hurwitz formula,  $(M/\Lambda)/S$  has genus

$$g' = 3 \times (0 - 1) + 1 + 4/2 = 0 \neq 2,$$

which is impossible.

Another solution is given by  $n = 2$  and  $B = 2$ . Then  $S$  is a screw symmetry of order 2 and  $G^2$  represents  $(M/\Lambda)/S$  as a 3-sheeted branched cover of  $\mathbb{S}^2$  with branching number 6. Again By Riemann–Hurwitz formula,  $(M/\Lambda)$  has genus

$$g' = 3 \times (0 - 1) + 1 + 6/2 = 1 \neq 2,$$

again impossible.

So  $g' = 1$  is the only possibility. □

### APPENDIX D. PROOF OF LEMMA 2.6

*Proof.* The screw symmetry of order 6 descends to the quotient torus as a translation that permutes the zeroes and, respectively, the poles. Let  $t \in \mathbb{C}$  be the translation vector. Then  $t$  must be a 3-division point. That is,  $3t$  must be a lattice point.

We may assume that the branch point at 0 corresponds to a zero of  $G^2$ . Then  $t$  and  $-t$  are also zeroes of  $G^2$ . Let the poles of  $G^2$  be  $p$ ,  $p+t$  and  $p-t$ . Then Abel's Theorem requires that  $3p$  is a lattice point. In other words,  $p$  is a 3-division point.

We have then proved that all branch points are at 3-division points. □

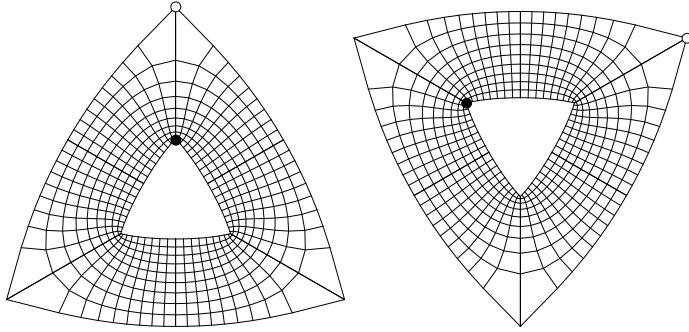


FIGURE 9. Image of  $\int G \cdot dz$  when  $\operatorname{Re} \tau = 0$  (left) and when  $\operatorname{Re} \tau = 1$ . The solid circle is the image of 0, the empty circle is the image of  $\tau/3$ .

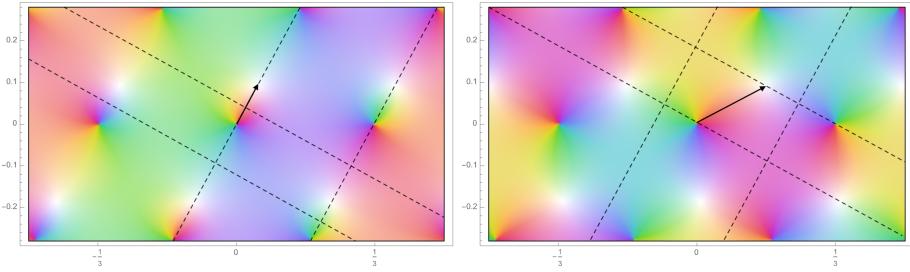


FIGURE 10. Plot of  $G^2$  for with  $|\tau - 1/3| = 1/3$  (left) and  $|\tau - 2/3| = 1/3$ . The arrow points from 0 to  $\tau/3$ . The dashed lines highlight the symmetries.

#### APPENDIX E. PROOF OF PROPOSITION 3.1 (EXISTENCE)

*Proof.* We examine the angles  $\theta_v$  and  $\theta_h$  on the boundaries of  $\Omega_t$ .

On the vertical line  $\operatorname{Re} \tau = 0$ , we see immediately that  $0 < \theta_v < \pi/2$ . With  $\theta = \pi/2$ , this line corresponds to the H'-T family, we know very well that the image of  $\tau$  is directly above the image of 0 when  $dh = e^{-i\pi/2}dz$ . Hence  $\theta_h = \pi/2 > \theta_v$ . In particular, the image of  $\int G \cdot dz$  with  $\theta = \pi/2$  is shown in Figure 9 (left).

On the vertical line  $\operatorname{Re} \tau = 1$ , we see immediately that  $\theta_v = 0$ . With  $\theta = 0$ , this line corresponds to the conjugate of H'-T surfaces. In particular, the image of  $\int G \cdot dz$  with  $\theta = 0$  is shown in Figure 9 (right), from which we can read that  $\theta_h > 0 = \theta_v$ .

As  $\operatorname{Im} \tau \rightarrow \infty$ , we see immediately that  $\theta_v \rightarrow 0^+$ . In Appendix B, we have compute that  $\theta_h \rightarrow (1/2 - \operatorname{Re} \tau/3)\pi$ . Hence for any value of  $\operatorname{Re} \tau < 1$ , we have  $\theta_h > \theta_v$  as long as  $\operatorname{Im} \tau$  is sufficiently large.

When  $\tau$  is on the circular arc  $|\tau - 1/3| = 1/3$ , the Gauss map exhibits many symmetries similar to CLP surfaces; see Figure 10 (left). In particular,  $G$  is real positive on the straight segment from 0 to  $\tau/3$ , which gives immediately that  $\theta_h = \arg \psi(\tau) = \arg \tau$ . But it follows from elementary geometry that  $\theta_v > \arg \tau = \theta_h$ .

When  $\tau$  is on the circular arc  $|\tau - 2/3| = 1/3$ , the Gauss map again exhibits many symmetries, similar to the previous case; see Figure 10 (right). In particular

- on the straight segment from 0 to  $(1 - \tau)/3$ ,  $\arg(G) = -\pi/6$  and  $|G| < 1$ ;
- on the straight segment from  $(1 - \tau)/3$  to  $(1 + \tau)/6$ ,  $|G| = 1$  and  $\arg(G)$  decrease from  $-\pi/6$  to  $-\pi/3$ ;
- on the straight segment from  $(1 + \tau)/6$  to  $\tau/3$ ,  $\arg(G) = -\pi/3$  and  $|G| > 1$ ;
- on the straight segment from  $\tau/3$  to  $(2\tau - 1)/3$ ,  $\arg(G) = \pi/6$  and  $|G| > 1$ ;
- on the straight segment from  $(2\tau - 1)/3$  to  $-(1 - \tau)/6$ ,  $|G| = 1$  and  $\arg(G)$  increase from  $\pi/6$  to  $\pi/3$ ;
- on the straight segment from  $-(1 - \tau)/6$  to 0,  $\arg(G) = \pi/3$  and  $|G| < 1$ .

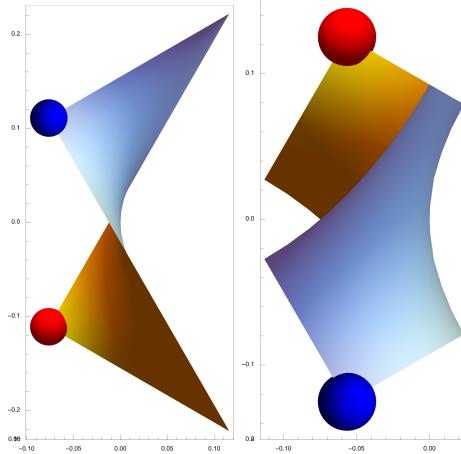


FIGURE 11. Image of Weierstrass parametrization assuming  $\theta = \theta_v - \pi/2$  (left) and  $\theta = \theta_v$  (right). Both seen from above. The red point is the image of 0. The blue point is the image of  $\tau/3$ .

The region bounded by these segments is exactly the region bounded by the dashed lines in Figure 10 (right).

This allows us to explicitly sketch the image of the Weierstrass parametrization with  $\theta = \arg(1 - \tau) = \theta_v - \pi/2$ . Up to translation and scaling, it is bounded by six straight segments between the points

$$(0, 2, h), \quad (-\sqrt{3}, 1, h), \quad (0, -2, h), \quad (0, -2, 0), \quad (-\sqrt{3}, -1, 0), \quad (0, 2, 0),$$

in this cyclic order. See Figure 11 (left). Then by conjugation, we can sketch the image with  $\theta = \arg(\tau/2 - 1/6) = \theta_v$ . It is a minimal surface with free boundary condition on the faces of a prism over a kite; see Figure 11 (right). We see that the  $y$ -coordinates of the images of 0 and  $\tau/3$  switched order, implying that  $\theta_h \in (\theta_v - \pi/2, \theta_v)$ . In particular,  $\theta_h < \theta_v$ .

Note that  $\theta_h$  and  $\theta_v$  are both real analytic functions in the real and imaginary part of  $\tau$ , hence the solution set of the period condition (3) is an analytic set. By the continuity, we conclude that the solution set contains a connected component that separates the circular arcs from the vertical lines and the infinity. Because of the analyticity, we may extract a continuous curve from the connected component. In particular, this curve must tend to 0 in one limit (correspond to Karcher–Scherk saddle towers with six wings), and tend to 1 in the other limit (correspond to doubly periodic Scherk surfaces).

Surfaces with the same symmetry near the saddle tower limit have been constructed in [CT24] by gluing constructions. They are proved to be embedded and unique, hence must belong to the curve above near the limit 0. Their embeddedness then ensures the embeddedness of all TPMSSs on the curve. This follows from [Wey06, Proposition 5.6], which was essentially proved in [Mee90].  $\square$

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