Supplementary material to "Goodness of fit tests in spatial autoregressive stochastic frontier models"

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November 30, 2024

This material provides

1. **the comparison of finite sample performance of our proposed trigonometry tests and the classic Kolmogorov-Smirnov (KS) and Cramér-von Mises (CvM) counterparts;**
2. **the corresponding analysis.**

Experimental comparison

The experimental design is entirely the same as that in "**Section 5 Monte Carlo simulations**" of the original paper. Here, Table 1 and Table 2 report, respectively, the comparison of empirical sizes and powers for "$H_0^1 : u_{ni} \sim H^+(0, \sigma_{uo}^2)$". Similarly, Table 3 and Table 4 are for "$H_0^2 : u_{ni} \sim Exp(\gamma_{uo})$".

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Table 1: Comparison of empirical sizes for $H_0^1 : u_{ni} \sim N^+(0, \sigma_{uo}^2)$.

nominal size	$\frac{\mathrm{Var}[u_{ni}]}{\mathrm{Var}[v_{ni}]}$	n	10^2	20^2	30^2	40^2	50^2	60^2
1%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.014	0.014	0.011	0.012	0.006	0.006
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.003	0.012	0.005	0.011	0.009	0.013
		KS	0.000	0.000	0.000	0.000	0.000	0.000
		CvM	0.000	0.000	0.000	0.000	0.000	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.016	0.012	0.014	0.007	0.011	0.008
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.009	0.011	0.010	0.006	0.012	0.014
		KS	0.000	0.000	0.000	0.000	0.000	0.000
		CvM	0.000	0.000	0.000	0.000	0.000	0.000
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.017	0.008	0.014	0.012	0.006	0.008
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.007	0.016	0.007	0.014	0.015	0.013
		KS	0.000	0.000	0.000	0.000	0.000	0.000
		CvM	0.000	0.000	0.000	0.000	0.000	0.000
5%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.053	0.050	0.040	0.047	0.038	0.033
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.053	0.050	0.040	0.047	0.038	0.033
		KS	0.000	0.000	0.000	0.000	0.000	0.000
		CvM	0.000	0.000	0.000	0.000	0.000	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.067	0.060	0.043	0.047	0.053	0.031
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.067	0.060	0.043	0.047	0.053	0.031
		KS	0.001	0.000	0.000	0.000	0.000	0.000
		CvM	0.000	0.000	0.000	0.000	0.000	0.000
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.079	0.047	0.060	0.046	0.043	0.054
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.079	0.047	0.060	0.046	0.043	0.054
		KS	0.003	0.001	0.000	0.000	0.000	0.000
		CvM	0.000	0.000	0.000	0.000	0.000	0.000
10%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.096	0.110	0.080	0.093	0.085	0.078
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.072	0.094	0.089	0.098	0.095	0.114
		KS	0.000	0.000	0.000	0.001	0.000	0.000
		CvM	0.001	0.000	0.000	0.000	0.000	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.111	0.102	0.094	0.092	0.114	0.062
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.072	0.101	0.115	0.100	0.116	0.111
		KS	0.001	0.000	0.000	0.000	0.000	0.000
		CvM	0.002	0.000	0.000	0.000	0.000	0.001
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.130	0.084	0.110	0.107	0.087	0.104
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.125	0.116	0.113	0.100	0.088	0.117
		KS	0.002	0.002	0.001	0.002	0.001	0.000
		CvM	0.000	0.001	0.000	0.000	0.000	0.001

Table 2: Comparison of empirical powers for $H_0^1 : u_{ni} \sim N^+(0, \sigma_{uo}^2)$.

nominal size	$\frac{\mathrm{Var}[u_{ni}]}{\mathrm{Var}[v_{ni}]}$	n	10^2	20^2	30^2	40^2	50^2	60^2
1%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.016	0.009	0.041	0.133	0.262	0.513
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.010	0.007	0.015	0.060	0.108	0.197
		KS	0.000	0.002	0.001	0.004	0.009	0.014
		CvM	0.000	0.001	0.000	0.001	0.002	0.009
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.049	0.099	0.289	0.631	0.902	0.965
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.012	0.018	0.104	0.290	0.595	0.810
		KS	0.000	0.014	0.090	0.276	0.480	0.664
		CvM	0.000	0.003	0.071	0.200	0.434	0.598
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.080	0.259	0.545	0.891	0.991	1.000
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.009	0.028	0.140	0.377	0.746	0.920
		KS	0.000	0.013	0.189	0.478	0.667	0.784
		CvM	0.000	0.003	0.140	0.406	0.639	0.780
5%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.063	0.069	0.164	0.366	0.578	0.773
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.043	0.058	0.120	0.252	0.376	0.580
		KS	0.000	0.014	0.021	0.026	0.040	0.075
		CvM	0.000	0.008	0.014	0.017	0.026	0.052
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.133	0.267	0.565	0.846	0.978	1.000
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.052	0.132	0.343	0.654	0.860	0.985
		KS	0.000	0.061	0.245	0.443	0.700	0.805
		CvM	0.000	0.052	0.203	0.427	0.683	0.806
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.160	0.474	0.816	0.975	1.000	1.000
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.060	0.122	0.395	0.712	0.940	0.987
		KS	0.001	0.076	0.384	0.607	0.707	0.876
		CvM	0.000	0.045	0.343	0.590	0.713	0.871
10%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.120	0.141	0.287	0.490	0.710	0.853
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.100	0.138	0.229	0.405	0.526	0.723
		KS	0.002	0.018	0.026	0.051	0.090	0.142
		CvM	0.001	0.012	0.028	0.044	0.068	0.133
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.190	0.392	0.711	0.927	0.985	1.000
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.102	0.239	0.508	0.806	0.930	0.980
		KS	0.003	0.087	0.314	0.559	0.791	0.931
		CvM	0.001	0.075	0.306	0.542	0.801	0.940
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.244	0.586	0.897	0.989	1.000	1.000
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.107	0.229	0.566	0.848	0.977	1.000
		KS	0.003	0.126	0.452	0.639	0.773	0.776
		CvM	0.002	0.104	0.428	0.647	0.767	0.776

Table 3: Comparison of empirical sizes for $H_0^2 : u_{ni} \sim Exp(\gamma_{uo})$.

nominal size	$\frac{\mathrm{Var}[u_{ni}]}{\mathrm{Var}[v_{ni}]}$	n	10^2	20^2	30^2	40^2	50^2	60^2
1%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.018	0.010	0.006	0.011	0.006	0.002
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.006	0.003	0.012	0.017	0.008	0.010
		KS	0.004	0.002	0.002	0.000	0.000	0.000
		CvM	0.002	0.002	0.001	0.000	0.000	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.013	0.016	0.012	0.007	0.011	0.007
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.015	0.015	0.012	0.014	0.012	0.009
		KS	0.000	0.000	0.001	0.002	0.002	0.000
		CvM	0.000	0.000	0.001	0.002	0.000	0.000
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.015	0.011	0.010	0.006	0.005	0.012
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.014	0.018	0.020	0.018	0.012	0.015
		KS	0.000	0.004	0.003	0.007	0.002	0.000
		CvM	0.000	0.003	0.002	0.004	0.001	0.000
5%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.049	0.054	0.044	0.036	0.045	0.036
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.043	0.057	0.041	0.053	0.046	0.054
		KS	0.005	0.001	0.000	0.001	0.000	0.001
		CvM	0.004	0.001	0.000	0.001	0.000	0.001
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.052	0.044	0.044	0.038	0.049	0.044
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.056	0.064	0.061	0.058	0.063	0.050
		KS	0.004	0.003	0.003	0.008	0.003	0.003
		CvM	0.004	0.002	0.002	0.005	0.002	0.001
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.061	0.045	0.036	0.057	0.053	0.048
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.060	0.064	0.058	0.073	0.059	0.065
		KS	0.001	0.007	0.014	0.010	0.003	0.009
		CvM	0.000	0.006	0.012	0.008	0.003	0.005
10%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.132	0.104	0.092	0.112	0.090	0.098
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.102	0.104	0.116	0.111	0.088	0.087
		KS	0.006	0.002	0.000	0.000	0.004	0.000
		CvM	0.006	0.002	0.000	0.000	0.002	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.116	0.116	0.096	0.094	0.084	0.115
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.137	0.131	0.091	0.118	0.113	0.098
		KS	0.006	0.010	0.006	0.009	0.010	0.014
		CvM	0.006	0.006	0.004	0.005	0.010	0.013
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.110	0.089	0.093	0.074	0.107	0.097
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.110	0.098	0.132	0.123	0.103	0.101
		KS	0.009	0.019	0.018	0.023	0.016	0.014
		CvM	0.004	0.014	0.017	0.018	0.015	0.006

Table 4: Comparison of empirical powers for $H_0^2 : u_{ni} \sim Exp(\gamma_{uo})$.

nominal size	$\frac{\mathrm{Var}[u_{ni}]}{\mathrm{Var}[v_{ni}]}$	n	10^2	20^2	30^2	40^2	50^2	60^2
1%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.029	0.030	0.045	0.069	0.090	0.153
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.012	0.016	0.023	0.037	0.070	0.090
		KS	0.007	0.005	0.006	0.000	0.000	0.000
		CvM	0.006	0.005	0.006	0.000	0.000	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.037	0.075	0.151	0.275	0.417	0.575
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.013	0.039	0.084	0.189	0.308	0.480
		KS	0.006	0.000	0.000	0.000	0.000	0.000
		CvM	0.006	0.000	0.000	0.000	0.000	0.000
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.033	0.132	0.218	0.434	0.652	0.890
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.026	0.058	0.136	0.289	0.465	0.685
		KS	0.000	0.000	0.000	0.000	0.000	0.006
		CvM	0.000	0.000	0.000	0.000	0.000	0.000
5%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.108	0.118	0.133	0.195	0.240	0.283
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.050	0.064	0.092	0.137	0.186	0.263
		KS	0.009	0.008	0.000	0.000	0.000	0.000
		CvM	0.008	0.006	0.000	0.000	0.000	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.140	0.195	0.333	0.494	0.627	0.800
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.071	0.122	0.235	0.377	0.545	0.675
		KS	0.002	0.008	0.000	0.001	0.000	0.014
		CvM	0.001	0.000	0.000	0.000	0.001	0.001
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.122	0.263	0.427	0.672	0.835	0.970
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.077	0.155	0.347	0.509	0.705	0.860
		KS	0.002	0.000	0.004	0.007	0.037	0.128
		CvM	0.002	0.000	0.000	0.001	0.018	0.086
10%	0.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.164	0.193	0.202	0.268	0.350	0.397
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.098	0.120	0.173	0.245	0.298	0.390
		KS	0.006	0.002	0.000	0.001	0.002	0.003
		CvM	0.006	0.001	0.000	0.000	0.000	0.000
	1.0	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.214	0.294	0.444	0.620	0.740	0.880
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.136	0.205	0.334	0.497	0.647	0.735
		KS	0.005	0.002	0.002	0.003	0.020	0.040
		CvM	0.004	0.000	0.001	0.001	0.008	0.023
	1.5	$T_2^{\cos}(\omega_{\mathrm{opt}})$	0.198	0.371	0.546	0.780	0.920	0.980
		$T_2^{\sin}(\omega_{\mathrm{opt}})$	0.127	0.250	0.464	0.640	0.797	0.890
		KS	0.003	0.002	0.012	0.036	0.118	0.264
		CvM	0.003	0.000	0.004	0.017	0.091	0.255

Comparison results

H_0^1 **(Half normal)** In terms of empirical sizes, Table 1 shows that the best cosine and sine tests (with the optimal parameter ω) respect the nominal size to a satisfactory degree, but the KS and CvM tests do not.

In terms of empirical powers reported in Table 2, although the empirical powers of the KS and CvM tests gradually approach to 1 as n increases, the powers of both nonparametric are obviously smaller than those of our tests.

H_0^2 **(exponential)** In terms of the empirical sizes in Table 3, the pattern is similar to that in Table 1 that (i) both nonparametric tests have some size distortions, and (ii) our tests are also robust.

In terms of the empirical powers, we can see from Table 4 that in all scenarios, the trigonometry tests also have the largest powers. Additionally, the powers of the KS and CvM tests can increase only when the nominal size is 10% and $\frac{\mathrm{Var}[u_{ni}]}{\mathrm{Var}[v_{ni}]} = 1.5$. Even so, the related speed is slow.

In summary, we conclude that

i) our proposed trigonometry tests outperform the classic KS and CvM tests;

ii) no matter how well the statistic performs in the goodness of fit test for (non-spatial) SF model, it still needs to verify the mathematical/statistical theory when such statistic is employed in spatial SF models, such as the SARSF model.