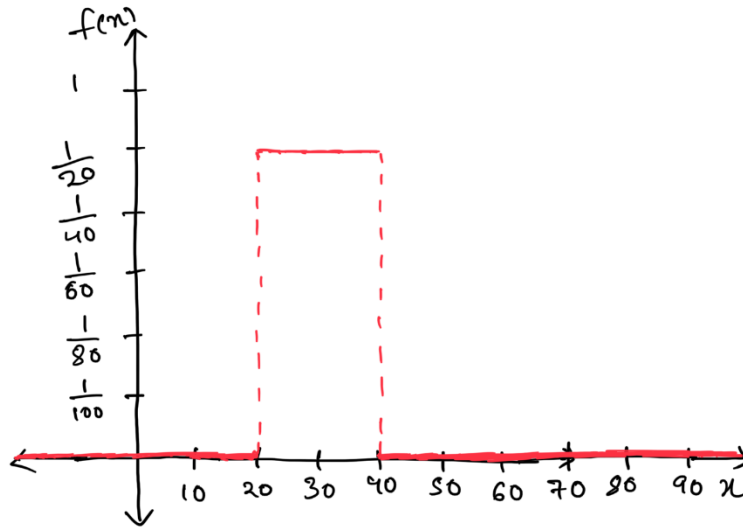# STAT-S520: Homework #3

1)

a) Following is the graph of f(x).

b) f(x) can be considered as the probability density function of a continuous random variable X if it satisfies the following conditions:
   a. f(x) is non-negative for all real values of x.
      i. From the graph of f(x) we can clearly observe that f(x) is positive valued for all real values.
   b. Area under the curve of f(x) is 1, i.e., $\int_{-\infty}^{-\infty} f(x)dx = 1$
      i. We can calculate the area under the curve of f(x) using its graph. We can observe that the value of f(x) is 0 when x<20 or x>40. So, the area of the graph in these regions is 0. Now, when x is between [20,40], f(x)=1/20. So, the area of f(x) between 20 and 40= 1/20 *(40-20) =1/20*20 =1.

   Since, f(x) satisfies the above conditions it <mark>can be considered as a probability density function.</mark>

c) P(0<X<35) = P(0<X≤20) + P(20≤X≤35) = $\int_{0}^{20} f(x)dx + \int_{20}^{35} f(x)dx$ =0+ 1/20 *(35-20) = 3/4= <mark>0.75.</mark>

d) The CDF of X can be written as F(y)=$\int_{-\infty}^{y} f(x)dx$. So, CDF of X for all y, can be written as follows:

For y<20, F(y)=0 as the value of f(x) (pdf) is 0 in this range. Similarly, for $20 \leq y \leq 40$, f(x)=1/20, so, in this range, $F(y)=\int_{20}^{y} f(x)dx=\frac{1}{20} * (y - 20)$. Finally, for y>40, F(y)=1.

$$F(y) = \begin{cases} 0, & y < 20 \\ \frac{1}{20} * (y - 20), & 20 \leq y \leq 40 \\ 1, & y > 40 \end{cases}$$

2) $f(x) = \begin{cases} 0, & if\ x < 1 \\ 2(x - 1), & if\ 1 \leq x \leq 2 \\ 0, if\ x > 2 \end{cases}$

a) Following is the graph of f(x).
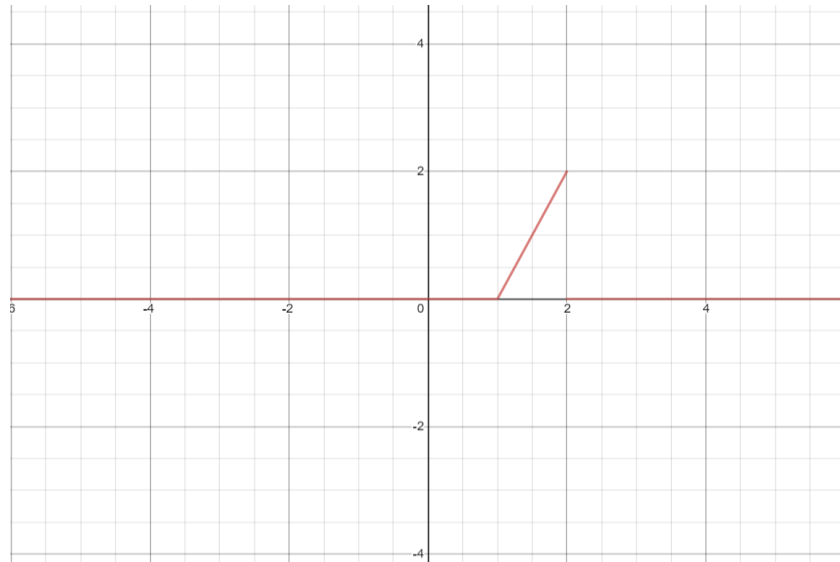
b)   f(x) can be considered as the probability density function of continuous random variable X if it satisfies the following conditions:

   i.   f(x) is non-negative for all real values of x.
        1.   From the graph of f(x) we can clearly observe that f(x) is non-negative for all real values of x.

   ii. Area under the curve of f(x) is 1, i.e., $\int_{-\infty}^{-\infty} f(x)dx = 1$
        1.   We can calculate the area under the curve of f(x) by using its graph. We can observe that the value of f(x) is zero ∀ x<1 or x>2. So, the area of f(x) in these regions is 0. But, for x in range [1,2], f(x)=2(x-1), i.e., f(x) takes the shape of a right-angled triangle when x is between [1,2]. The height and base of this triangle can be calculated as follows:
             a.   Height (h)= f(x=2) = 2(2-1) =2.
             b.   Base (b)= 2-1=1

        So, the area of the triangle is ½*2*1=<mark>1</mark>.

   Since, f(x) satisfies the above conditions it <mark>can be considered as a probability density function.</mark>

c) $P(1.5 < X < 1.75) = \int_{1.5}^{1.75} f(x)\, dx = \int_{1.5}^{1.75} 2(x-1)\, dx = 2 * \frac{(1.75^2 - 1.5^2)}{2} - 2 * (1.75\text{-}1.5)$

$= (3.0625\text{-}2.25)\text{-} 0.5 = $ <mark>0.3125</mark>.

3) $f(x) = \begin{cases} \frac{1}{30}, & 0 \le x < 20 \\ \frac{1}{60}, & 20 \le x \le 40 \\ 0, & \text{otherwise} \end{cases}$

We know that the expected value of a continuous random variable can be written as,

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx, \text{where f(x) is the probability distribution function of X.}$$

From the probability distribution function of X, we know that the random variable is X is defined between [0,40]. So, the above integral becomes-

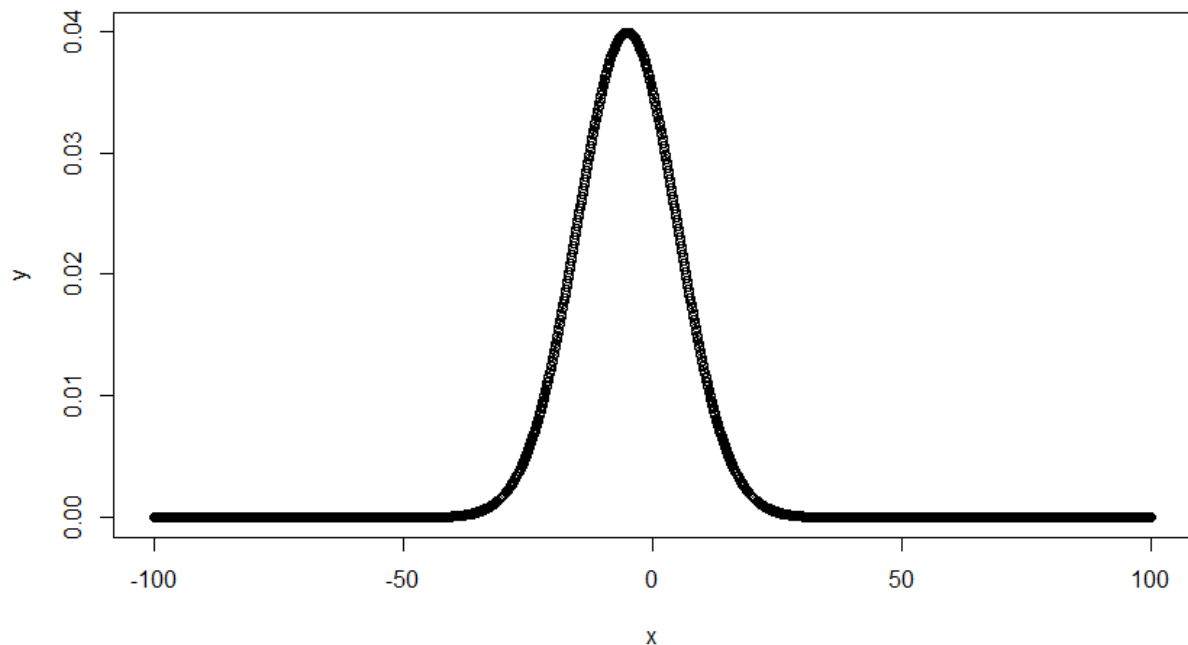$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{20} x f(x)\, dx + \int_{20}^{40} x f(x)\, dx.$ Therefore,

$$E(X) = \frac{1}{30} * \frac{1}{2} * (20^2 - 0^2) + \frac{1}{60} * \frac{1}{2} * (40^2 - 20^2) = \frac{400}{60} + 10 = 6.67 + 10 = 16.67.$$

4) The probability density function of the normal random variable X with mean(μ)=-5 and standard deviation(σ)=10 can be written as-

$$f(x) = \frac{1}{(2\Pi)^{0.5}\sigma} * exp\left[-\frac{1}{2} * \left(\frac{(x-\mu)}{\sigma}\right)^2\right] = \frac{1}{(2\Pi)^{0.5}10} * exp\left[-\frac{1}{2} * \left(\frac{(x+5)}{10}\right)^2\right]$$

The PDF of the above normal random variable is a bell-shaped curve centered at mean(μ)=-5.

Plot-

a) $P(X<0) = \int_{-\infty}^{0} f(x)dx$, where f(x) is the pdf of the random variable X; The value of this integral can be calculated using R with the following code-

Code: pnorm (0, mean=-5, sd=10) = 0.6914625.

b) $P(X>5)=1-P(X\leq5)= 1-\int_{-\infty}^{5} f(x)dx$, where f(x) is the pdf of the random variable X; The value of this integral can be calculated using R with the following code-

Code: 1-pnorm (5, mean=-5, sd=10) = 0.1586553.

c) $P(-3<X<7) = P(X<7)-P(X<-3) = \int_{-\infty}^{7} f(x)dx - \int_{-\infty}^{-3} f(x)dx$, where f(x) is the pdf of the random variable X; The value of this integral can be calculated using R with the following code-

Code: pnorm (7, mean=-5, sd=10) – pnorm (-3, mean=-5, sd=10) = 0.3056706.

d) $P(|X+5|<10) = P(-10<X+5<10) = P(-15<X<5)= P(-15<X\leq5)= P(X\leq5)-P(X\leq-15) = \int_{-\infty}^{5} f(x)dx - \int_{-\infty}^{-15} f(x)dx$; The value of this integral can be calculated using R with the following code-

Code: pnorm (5, mean=-5, sd=10) – pnorm (-15, mean=-5, sd=10) = 0.6826895.

e) $P(|X-3|>2) = P(X-3>2) + P(X-3<-2) = P(X>5) + P(X<1) = 1-P(X\leq5) + P(X\leq1)= 1-\int_{-\infty}^{5} f(x)dx + \int_{-\infty}^{1} f(x)dx$; The value of this integral can be calculated using R with the following code-

Code: 1-pnorm (5, mean=-5, sd=10) + pnorm (1, mean=-5, sd=10) = 0.8844021.

5) Given, X1 ~ Normal (1; 9) and X2 ~ Normal (3,16), which means that X1 and X2 are two normal random variables distributed such that-
   a) Mean (X1) = μ_1= 1, Variance (X1) = σ_1^2= 9.
   b) Mean (X2) = μ_2=3, Variance (X2) = σ_2^2= 16.

For the given linear transformations on X1 and X2, the mean and variance can be written as follows-

a) X1+X2

Mean(X1+X2) = E(X1+X2) = E(X1) + E(X2) = Mean(X1) + Mean(X2) = 1+3 =4.

Var(X1+X2) = Var(X1) + Var(X2) = 9+16 = 25.

b) -X2
Mean (-X2) = E(-X2) =-E(X2) =-Mean(X2) =-3.
Var(-X2) = $(-1)^2$ * Var(X2) = Var(X2) = 16.

c) X_1-X_2

Mean (X1-X2) = E(X1-X2) = E(X1)-E(X2) = Mean(X1)-Mean(X2) = 1-3= -2.

Var (X1- X2) = Var(X1) + Var(X2) = Var(X1) + Var(X2) = 9+16= 25.

d) 2X1

Mean (2X1) = E(2X1) =2* E(X1) = 2* Mean(X1) = 2*1=2.

Variance (2X1) = 2^2 * Var(X1) = 4*9= 36.

e) 2X1-2X2

Mean (2X1-2X2) = 2*Mean(X1)-2* Mean(X2) = 2*1-2*3= -4.

Var (2X1-2X2) = 2^2 * Var(X1) + $(-2)^2$ * Var(X2) = 4*9+ 4*16= 4*25= 100.

6) The demand (D) during lead time follows a normal distribution with Mean (μ)=930 gallons, and Standard Deviation (σ)= 140 gallons. So, the probability distribution function of demand (D) can be written as:

$$f(x) = \frac{1}{(2\Pi)^{0.5}\sigma} * exp\left[-\frac{1}{2} * \left(\frac{(x-\mu)}{\sigma}\right)^2\right] = \frac{1}{(2\Pi)^{0.5}140} * exp\left[-\frac{1}{2} * \left(\frac{(x-930)}{140}\right)^2\right]$$

The station would run out of gas if the demand during lead times rises above the inventory, i.e. 1200 gallons. So, the probability that the station will run out of gas can be written as:

$$P(D>1200) = 1-P(D\leq1200) = 1- \int_{-\infty}^{1200} f(x)dx$$

Again, the value of the above integral can be calculated using R.

Code: 1-pnorm (1200, mean=930, sd=140)

Output: 0.02689204

So, there is 0.026 or 2.6% chance that the gas station would run out of inventory or gas during the lead time.

7) Given that X is a standard normal variable, i.e., the mean(X)=0, and standard deviation (σ)=1.

The probability distribution function (pdf) of X can be written as follows:

$$f(x) = \frac{1}{(2\Pi)^{0.5}\sigma} * exp\left[-\frac{1}{2} * \left(\frac{(x-\mu)}{\sigma}\right)^2\right] = \frac{1}{(2\Pi)^{0.5}} * exp\left[-\frac{1}{2} * x^2\right]$$

a) P(-1.5<X<2.5) = P(X<2.5)- P(X<-1.5) = P(X≤2.5)- P(X≤-1.5) = $\int_{-\infty}^{2.5} f(x)dx$- $\int_{-\infty}^{-1.5} f(x)dx$. Using the following code in R, we can calculate the value of the integral-

Code- pnorm (2.5)-pnorm (-1.5)

Output: 0.9269831

b) P(X^2>1) = P(X>1) + P(X<-1) = 1- P(X≤1) + P(X<-1) = 1- $\int_{-\infty}^{1} f(x)dx$+ $\int_{-\infty}^{-1} f(x)dx$. Using the following code in R, we can calculate the value of the integral-

Code- pnorm (-1) + (1-pnorm (1))

Output: 0.3173105