Norm-an conquest

The absolute value $|x|$ represents the distance on the number line between the point x and the origin. We can do the same for a vector of n numbers, plotting an arrows whose tail is at the origin and applying the Pythagorean theorem to conclude that $\boldsymbol{x}$ represents a point that is a distance $(x_1^2 + \cdots + x_n^2)^{1/2}$ from the origin. This is what we call the 2-norm, $\|\boldsymbol{x}\|_2$. It turns out that if we abstract the notion of "distance" a little to its essential properties, we can get other vector norms as well. Each norm has its own set of unit vectors (those having norm equal to 1).

For matrices, a definition based on the action of the matrix as a linear transformation is quite handy for analyzing linear algebra algorithms. The definition starts with all the unit vectors in whatever norm we like. Each of these vectors $\boldsymbol{v}$ is mapped to the vector $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}$. The maximum of $\|\boldsymbol{u}\|$ for all such $\boldsymbol{v}$ is the value of $\|\boldsymbol{A}\|$. This definition can be visualized easily only when both $\boldsymbol{v}$ and $\boldsymbol{u}$ are low-dimensional.

Goals

You will use experiments based on the definitions of norms to approximate the calculation of norms of a small matrix.

Preparation

Read section 2.7. Recall that the equations $x = \cos(\theta)$, $y = \sin(\theta)$, $0 \le \theta \le 2\pi$ parameterize the unit circle (set of all unit vectors) defined by the 2-norm in $\mathbf{R}^2$.

Procedure

Download the script template and complete it to perform the following steps.

1. Define a vector `theta` of 200 equally spaced values from 0 to 2π. Use it to define vectors `x` and `y` via $x_j = \cos(\theta_j)$, $y_j = \sin(\theta_j)$ (do this without any loops). Plot these points, which lie on the unit circle.

2. Let `A=magic(2)`. For each j, define the vector $\boldsymbol{v} = [x_j \; y_j]^T$ and let $\boldsymbol{u} = \boldsymbol{A}\boldsymbol{v}$. On a new graph, plot all the $\boldsymbol{u}$ points, and make a vector storing $\|\boldsymbol{u}\|_2$ for all j.

3. Plot $\|\boldsymbol{u}\|_2$ as a function of θ.

4. Using `format long` and `max`, calculate and show the maximum value of $\|\boldsymbol{u}\|_2$ over all j. This estimates $\|\boldsymbol{A}\|_2$. Also compute the actual value `norm(A,2)`.

5. The script creates new vectors `x` and `y` that are unit vectors in the ∞-norm. Plot these points, giving a title and axis labels.

6. Repeat step 2, this time collecting $\|\boldsymbol{u}\|_\infty$ as a function of the point index j.

7. Make a plot as in step 3. This time, the x-axis will the index number j.

8. Estimate the norm $\|\boldsymbol{A}\|_\infty$ and compute its exact value, as in step 4.