Let's be rational about this

Our go-to form of interpolating function is the polynomial,

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}. \tag{1}$$

When the monomials are evaluated at m nodes, we get the Vandermonde matrix

$$\boldsymbol{V} = \begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^{m-1} \\ 1 & t_1 & t_1^2 & \cdots & t_1^{m-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_{m-1} & t_{m-1}^2 & \cdots & t_{m-1}^{m-1} \end{bmatrix}. \tag{2}$$

Given a function f, setting $y_i = f(t_i)$ and solving $\boldsymbol{V}\boldsymbol{c} = \boldsymbol{y}$ produces the coefficients of the interpolating polynomial.

Polynomials can, in principle, converge pointwise to any continuous function. However, they are not equally efficient at doing so for all functions. One way to obtain faster convergence in some cases is to turn to a different category of functions. An interesting choice are the **rational functions**,

$$r(x) = \frac{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}}{b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + b_n x^n}. \tag{3}$$

Because they can have zeros in the denominator, rational functions are often superior for approximating functions that blow up or have steep gradients.

Observe that in (3), multiplying all of the coefficients by the same constant leaves r unchanged. Thus we make a normalization, $b_n = 1$. This leaves a total of $n + n = 2n$ coefficients to be determined in r, so we require $2n$ interpolation nodes. From (3), setting $r(t_i) = f(t_i) = y_i$, for $i = 0, \ldots, 2n-1$ and clearing the denominator, we eventually obtain

$$\boldsymbol{W}\boldsymbol{a} - \boldsymbol{Y}\boldsymbol{W}\boldsymbol{b} = \boldsymbol{Y} \begin{bmatrix} t_0^n \\ t_1^n \\ \vdots \\ t_{2n-1}^n \end{bmatrix}, \tag{4}$$

where $\boldsymbol{W}$ is a $2n \times n$ Vandermonde-style matrix, $\boldsymbol{Y} = \operatorname{diag}(y_0, \ldots, y_{2n-1})$, and $\boldsymbol{a}$ and $\boldsymbol{b}$ collect the polynomial coefficients in the numerator and denominator respectively. This equation is actually a square linear system,

$$\begin{bmatrix} \boldsymbol{W} & -\boldsymbol{Y}\boldsymbol{W} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix} = \boldsymbol{Y} \begin{bmatrix} t_0^n \\ t_1^n \\ \vdots \\ t_{2n-1}^n \end{bmatrix}, \tag{5}$$

easily solved for the vector $\boldsymbol{c} = \begin{bmatrix} \boldsymbol{a} ; \boldsymbol{b} \end{bmatrix}$.

Goals

You will compute a rational interpolant and compare it to a polynomial interpolant for the same points.

Preparation

Read section 9.1 and answer the following questions.

1. Derive (4).
2. Write out the linear system (5) for the rational interpolant to the four points $(-1,0)$, $(0,-1)$, $(1,1)$, $(2,1)$.

Procedure

Download templates for the script and for the function `ratinterp.m`.

1. Complete the function `ratinterp` that computes a rational interpolant to given data using the algorithm outlined above.
2. Define the function $g(x) = \tanh(10x) + 2x^2$ and plot it over the interval $[-1,1]$.
3. To your plot add the polynomial interpolant using $m = 18$ equally spaced nodes in $[1,1]$. (It will not be a good result.)
4. Find the rational interpolant with $n = 9$ and the same nodes. Start a new plot and plot the error $g(x) - r(x)$ over $[-1,1]$. It should be fairly small over the whole interval.

Discussion

Returning to the $n = 9$ rational interpolant of g, find the poles of r (i.e., the roots of the denominator). Several of them are conjugate pairs that lie essentially on the imaginary axis of the complex plane. Referring to a property of g, explain the two poles closest to the origin. You may want to prove and then use the identity $\cosh(x) = \cos(ix)$.