You've got potential

One reason the Laplace equation is considered so important is its relationship to *potential functions*. Broadly speaking, a potential function usually replaces a vector quantity with a scalar one. The standard relationship is via a gradient, e.g., $\boldsymbol{F} = \nabla u$. Familiar forces from physics are subject to this relationship, as are certain fluid flows and other phenomena.

In the absence of sources or sinks, a conservative force is divergence-free, which implies that $\nabla^2 u = 0$. So Laplace's equation can be solved for the potential, which can be considerably easier than solving the equations for the orignal force. Level curves of $u(x, y)$ are called *equipotential curves*. Thanks to the properties of the gradient, the force is everywhere orthogonal to the equipotential curves.

Goals

You will use numerical solutions of Laplace's equation to find the potential and force for a two different boundary conditions on a square.

Preparation

Read section 13.3. All of the following questions are about the potential function $u(x, y) = (\cos\theta)x + (\sin\theta)y$, for an arbitrary fixed real θ.

1. Show that u is a solution of Laplace's equation.

2. Find ∇u.

3. Suppose $\theta = \pi/3$. In the square $[0,1]\times[0,1]$, sketch 5 equipotential curves. Draw arrows representing the gradient at 3 points along each equipotential curve.

Procedure

Download the template script and complete it to perform the following tasks.

1. Let the boundary values be given by $g(x, y) = \tanh\bigl(12w(x, y) - 7\bigr)$, where $w(x, y) = (\cos\theta)x + (\sin\theta)y$ and $\theta = \pi/5$. Solve Laplace's equation on $[0,1]\times[0,1]$ with $u = g$ on the boundary and $m = n = 144$. Make a surface plot of the solution, labelling the axes.

2. Use differentiation matrices to compute $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ on the grid. Use a 1×2 subplot grid to make surface plots of them.

3. In a new graph, use `contour` to make a contour plot of u with 32 level curves. To this plot you should add a `quiver` plot that draws arrows representing the gradient of u. In order to avoid having too many arrows, use only every 6th row and column in all of the matrix arguments to `quiver`. Use `axis equal` after the plot.

4. Repeat the previous steps for $\theta = 3\pi/5$.

Extra

E1. In two dimensions, solutions of Laplace's equation have a close connection with complex analysis. For instance, let $z = x + iy$, let $\zeta = h(z)$ for a differentiable function h whose derivative never vanishes in the square, and let $\xi = \mathrm{Re}(\zeta)$, $\eta = \mathrm{Im}(\zeta)$. If $u_{xx} + u_{yy} = 0$, and if we define

$$\tilde{u}(\xi, \eta) = u(x, y),$$

then $\tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} = 0$ and $\tilde{u}$ is a potential function in the geometry that results from applying h to every point in the (x, y) domain.

Let $h(z) = \exp(\pi z/2)$. Calculate u as in step 1 above. Then make a `pcolor` plot of $\tilde{u}$ over its domain (which should look like a quarter of a donut). Don't forget `axis equal`!