What goes around, comes around

A small satellite in the Earth–moon system can be modeled as a *restricted three-body problem*, in which one object has so little relative mass that the motions essentially take place in a plane. In this formulation, Newton's laws of motion are simplified by a transformation so that the Earth has mass μ_* and remains at $(0,0)$, while the moon has mass μ and is at $(1,0)$, where $\mu+\mu_*=1$. The position of the satellite is $(x(t), y(t))$ and satisfies

$$\begin{aligned} x'' &= x + 2y' - \mu_* \frac{x+\mu}{r} - \mu \frac{x-\mu_*}{r_*} \\ y'' &= y - 2x' - \mu_* \frac{y}{r} - \mu \frac{y}{r_*}, \end{aligned} \tag{1}$$

where

$$\begin{aligned} r &= \left[(x+\mu)^2 + y^2\right]^{3/2} \\ r_* &= \left[(x-\mu_*)^2 + y^2\right]^{3/2}. \end{aligned}$$

For the Earth/moon system, $\mu = 0.012277471$.

The key to computing solutions using standard software is to transform the original ODE system into a first-order one. Since there are two dependent variables x and y, and both of these appear to second order, there will be 4 variables and 4 equations in the first-order version. Define $u_1 = x$, $u_2 = y$, $u_3 = x'$, $u_4 = y'$. Two trivial equations in the new system are $u_1' = u_3$ and $u_2' = u_4$. The other two equations, for u_3' and u_4', come from substitution into the original system (1). For example,

$$u_3' = u_1 + 2u_4 - \mu_* \frac{u_1+\mu}{r} - \mu \frac{u_1-\mu_*}{r_*},$$

and so on.

Goals

You will explore unlikely-looking satellite orbits that have fascinated mathematicians since Poincaré. They are of more than academic interest, because despite their strangeness, they are energy-efficient.

Preparation

Read section 6.1. Fully write out the first-order system of ODEs $\boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u})$ that is equivalent to (1).

Procedure

Download the script template and complete it to execute the following steps.

1. Set

```
opt = odeset('reltol',1e-13,'abstol',1e-13);
```

This will be used below to require strict error tolerances in the ODE solutions. The solutions below will all use `ode113` with `opt` as a fourth input argument.

2. Using the expression of equation (1) as a first-order system, $\boldsymbol{u}' = \boldsymbol{f}(t, \boldsymbol{u})$, write the function

   ```
   function dudt = r3body(t,u)
   ```

   in which `u` is a vector containing numerical values for the dependent variables, and `dudt` returns the components of $\boldsymbol{u}'$.

3. Using the initial conditions

   $$\begin{aligned} x(0) &= 1.2, & x'(0) &= 0, \\ y(0) &= 0, & y'(0) &= -1.049357510, \end{aligned}$$

   solve the problem on the interval $[0, 6.192169331]$, using `ode113` and passing `opt` as a fourth input argument. Plot the trajectory in the (x, y) phase plane. It will have two major loops, one to each side of the Earth. Add the moon and Earth positions as points in the plot.

4. Repeat step 3 for

   $$\begin{aligned} x(0) &= 0.994 & x'(0) &= 0 \\ y(0) &= 0 & y'(0) &= -2.03173262955734 \end{aligned}$$

   and solve over $[0, 11.124340337266]$. This orbit has three loops.

5. Repeat step 3 for

   $$\begin{aligned} x(0) &= 0.994 & x'(0) &= 0 \\ y(0) &= 0 & y'(0) &= -2.00158510637908 \end{aligned}$$

   and solve over $[0, 17.06521656015796]$. This one has four loops.

6. These orbits are very sensitive to perturbations, and while our convergence theorems apply to a fixed time interval and $h \to 0$, the case of $t \to \infty$ is quite different. Re-solve step 5 for $0 \le t \le 100$. Your plot will show the path of the satellite departing from the apparently periodic orbit.

Discussion

In step 3 above, what is the maximum speed of the satellite over the whole orbit?