On Additive Combinatorics of Permutations of $\mathbb{Z}_n$.

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Abstract

Let $\mathbb{Z}_n$ denote the ring of integers modulo n. In this paper we consider two extremal problems on permutations of $\mathbb{Z}_n$, namely, the maximum size of a collection of permutations such that the sum of any two distinct permutations in the collection is again a permutation, and the maximum size of a collection of permutations such that the sum of any two distinct permutations in the collection is not a permutation. Let the sizes be denoted by $s(n)$ and $t(n)$ respectively. The case when n is even is trivial in both the cases, with $s(n) = 1$ and $t(n) = n!$. For n odd, we prove $s(n) \geq (n\phi(n))/2^k$ where k is the number of distinct prime divisors of n. When n is an odd prime we prove $s(n) \leq \frac{e^2}{\pi} n((n-1)/e)^{\frac{n-1}{2}}$. For the second problem, we prove $2^{(n-1)/2}.(\frac{n-1}{2})! \leq t(n) \leq 2^k.(n-1)!/\phi(n)$ when n is odd.

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1 Introduction

For $n \in \mathbb{Z}$, let $\mathbb{Z}_n$ denote the ring $\{0, \ldots, n-1\}$ with $+$ and . as addition and multiplication modulo n respectively. Let $\mathcal{S}(\mathbb{Z}_n)$ denote the set of all permutations of the set $\mathbb{Z}_n$. We are interested in obtaining bounds on the maximum size of a subset $\mathcal{P}$ of $\mathcal{S}(\mathbb{Z}_n)$ in the case when two distinct permutations in $\mathcal{P}$ sum up to a permutation (Section 3), and in the case when two distinct permutations in $\mathcal{P}$ sum up to a non-permutation (Section 4). As far as we know the problems considered above are new, though a similar looking problem for difference of permutations is well studied, in the form of mutually orthogonal *orthomorphisms* of finite groups. For the sake of completeness, we discuss the connection between difference of permutations problem with the orthomorphisms problem in Section 5. The families of permutations we consider have similarities to *reverse free* and *reverse full* families of permutations as considered by Füredi et al. [4] and Cibulka [2].

2 Preliminaries

We review some well known facts and basic results from linear algebra, elementary number theory and combinatorics which will be used in later sections. A comprehensive reference for linear algebra notions discussed below is [1].

Definition 2.1 (Vector Spaces, Bilinear Forms, Inner Products)**.** Let V be an n-dimensional vector space over the field F, which is naturally identified with the set F^n, with component-wise addition and scalar multiplication. A *bilinear form* on V is a function $f : V \times V \to F$ which is linear in both arguments, i.e, $f(\alpha u + u', \beta v + v') = \alpha\beta f(u, v) + \alpha f(u, v') + \beta f(u', v) + f(u', v')$ for $u, u', v, v' \in V$ and $\alpha, \beta \in F$. A *symmetric* bilinear form is a bilinear form f with $f(u, v) = f(v, u)$ for all $u, v \in V$. We will call a symmetric bilinear form on V as an *inner product* on V. The inner product on V given by $\langle u, v\rangle = u_1v_1 + \cdots + u_nv_n$ where, $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ will be called the *standard* inner product on V.

Definition 2.2 (Totally isotropic subspaces)**.** Let V be an n-dimensional vector space with the standard inner product $\langle\ ,\rangle$. We say that vectors $u, v \in V$ are *orthogonal*, denoted by $u \perp v$, if $\langle u, v\rangle = 0$. For a subspace S of V and a vector $v \in V$, we say $v \perp S$ if $v \perp u$ for all $u \in S$. For a subspace S of V, we define $S^\perp = \{v \in V : v \perp S\}$ which is also a subspace of V. We call a subspace S of V *totally isotropic* if $S \subseteq S^\perp$, or in other words, any two vectors in S are orthogonal.

We know the following fact from linear algebra.

Lemma 2.3. *Let V be an n-dimensional vector space with the standard inner product. Then for a subspace S of V, we have* $\dim(S) + \dim(S^\perp) = n$.

Next we discuss some basic notions from elementary number theory that will be used in the paper. An element $s \in \mathbb{Z}_n$ is said to be *invertible* if there exists $t \in \mathbb{Z}_n$ such that $st = 1$ (recall that multiplication is in $\mathbb{Z}_n$). The set of all invertible elements of $\mathbb{Z}_n$ is called the *unit group* of $\mathbb{Z}_n$ and is denoted by $\mathbb{Z}_n^\times$. It is easily seen that $\mathbb{Z}_n^\times$ is a group under multiplication. We know that $k \in \mathbb{Z}_n$ is invertible if and only if $\gcd(k, n) = 1$. The cardinality of the set $\{k \in \mathbb{Z} : 1 \le k \le n-1,\ \gcd(k, n) = 1\}$ is denoted by $\phi(n)$, also known as *Euler's totient function* in literature. The following results are well known.

Lemma 2.4. *Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_1, \ldots, p_k$ are distinct prime divisors of n. Then $\phi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$.*

Lemma 2.5 (Chinese Remainder Theorem)**.** *Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_1, \ldots, p_k$ are distinct prime divisors of n. Then we have the following isomorphism,*

$$\begin{aligned} \mathbb{Z}_n &\longrightarrow \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_k^{\alpha_k}} \\ s &\longmapsto (s \bmod p_1^{\alpha_1}, \ldots, s \bmod p_k^{\alpha_k}). \end{aligned} \tag{1}$$

From the above lemma we see that $s = (s_1, \ldots, s_k)$ is invertible in $\mathbb{Z}_n$ if and only if s_i is invertible in $\mathbb{Z}_{p_i^{\alpha_i}}$ for all $i = 1, \ldots, k$.

Finally we note the following combinatorial result.

Lemma 2.6. *Suppose the edges of the complete graph K_n, $n \geq 2$ are colored using k colors, such that the subgraph consisting of the edges of any given color is bipartite. Then $n \leq 2^k$.*

Proof. Let V denote the vertex set of the complete graph K_n. We show an injection from V into the set $\{0, 1\}^k$. For each i, choose a partition $\{A_i, B_i\}$ of V such that all edges of color i are between the sets A_i and B_i. By hypothesis, such a bi-partition exists for all $i = 1, \dots, k$. For $v \in V$, define $\tilde{v} = (v_1, \dots, v_k)$ by,

$$v_i = \begin{cases} 0 & \text{if } v \in A_i \ , \\ 1 & \text{if } v \in B_i \ . \end{cases}$$

We now show that the map $v \mapsto \tilde{v}$ is injective. Let v, w be two distinct vertices in V. Let i be the color of the edge vw. Then we see that $\tilde{v}, \tilde{w}$ differ in the i^{th} component. This proves the injectivity of the map $v \mapsto \tilde{v}$ and the lemma follows. $\square$

Notation 2.7. We will denote permutations of $\mathbb{Z}_n$ as n-tuples $(\sigma_1, \dots, \sigma_n)$ where each $\sigma_i \in \mathbb{Z}_n$ and is distinct. This is not to be confused with the cycle representation of a permutation, as is customary in algebra. Since we do not use cycle representation of permutations in this paper, we hope there is no confusion. Let $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\tau = (\tau_1, \dots, \tau_n)$, be n-tuples over $\mathbb{Z}_n$. Then $\sigma \pm \tau$ denotes the tuple $(\sigma_1 \pm \tau_1, \dots, \sigma_n \pm \tau_n)$. For $c \in \mathbb{Z}_n$, $c.\sigma$ will denote the tuple $(c\sigma_1, \dots, c\sigma_n)$, and $c + \sigma$ will denote the tuple $(c + \sigma_1, \dots, c + \sigma_n)$.

Lemma 2.8. *Let n be even and $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ be two permutations of $\mathbb{Z}_n$. Then $a + b$ is not a permutation of $\mathbb{Z}_n$.*

Proof. Let $c = a + b = (c_1, \dots, c_n)$. Treating a, b, c as n-tuples over $\mathbb{Z}$ we have, $c_i \equiv a_i + b_i \pmod{n}$. Summing up over all i, we have,

$$\begin{aligned} \sum_{i=1}^{n} c_i &\equiv \sum_{i=1}^{n} (a_i + b_i) \pmod{n} \\ \text{or, } \sum_{i=1}^{n} (i-1) &\equiv \sum_{i=1}^{n} 2(i-1) \pmod{n} \\ \text{or, } \frac{n(n-1)}{2} &\equiv n(n-1) \equiv 0 \pmod{n} \end{aligned}$$

which is a contradiction as $(n-1)/2$ is not an integer when n is even. This proves the lemma. $\square$

Lemma 2.9. *Let $a = ({}_0, \dots, a_{n-1})$ and $b = (b_0, \dots, b_{n-1})$ be distinct permutations of the set $\{0, \dots, n-1\}$ such that the component-wise sums $c_i = a_i + b_i$ are all distinct. Then there exist $0 \leq j, k \leq n-1$ such that $c_j = c_k + 1$.*

Proof. Without loss of generality assume $c_i = a_i + b_i$ satisfy the ordering $c_0 < c_1 < \cdots < c_{n-1}$. Now suppose the claim is not true. Then we have, $c_i - c_{i-1} \geq 2$ for $1 \leq i \leq n-1$. Summing up we get $2n - 2 \leq \sum_{i=1}^{n-1} (c_i - c_{i-1}) = c_{n-1} - c_0 \leq 2n - 2$.

Thus $c_i - c_{i-1} = 2$ for all $1 \le i \le n-1$, which implies $c_i = 2i$ for all $0 \le i \le n-1$. Now $a_0 + b_0 = c_0 = 0$ implies $a_0 = b_0 = 0$. Now $c_1 = a_1 + b_1 = 2$, therefore we must have $a_1 = b_1 = 1$. Continuing this way, we conclude $a_i = b_i = i$ for all $0 \le i \le n-1$, contradicting the fact that the permutations were distinct. $\square$

3 Sum of Permutations being Permutation

In this section, we consider the maximum size of a collection of permutations of $\mathbb{Z}_n$ such that sum of any two distinct permutations in the collection is again a permutation (not necessarily in the collection). A collection $\mathcal{P}$ of permutations of $\mathbb{Z}_n$ is said to satisfy property (**P1**) if for any two distinct permutations σ, τ in $\mathcal{P}$, $\sigma + \tau$ is again a permutation of $\mathbb{Z}_n$. Let $s(n) = \max\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}(\mathbb{Z}_n), \mathcal{P} \text{ satisfies } (\mathbf{P1})\}$.

From Lemma 2.8, we note that $s(n) = 1$ when n is even. Now we present a construction for the lower bound on $s(n)$ when n is odd.

Lemma 3.1. *Let n be an odd number ≥ 3. Then $s(n) \ge (n\phi(n))/2^k$ where $\phi(n)$ is the Euler's totient function and k is the number of distinct prime divisors of n.*

Proof. Our construction is based on the following observations.

(a) For a permutation τ of $\mathbb{Z}_n$, $k.\tau$ is a permutation of $\mathbb{Z}_n$ if and only if k is invertible in $\mathbb{Z}_n$.

(b) For a permutation τ of $\mathbb{Z}_n$, $k + \tau$ is a permutation for all $k \in \mathbb{Z}_n$.

(c) Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where p_i's are distinct prime divisors of n. Then there exists a subset S of invertible elements of $\mathbb{Z}_n$ with $|S| = \prod_{i=1}^{k} p^{\alpha_i - 1}(p_i - 1)/2$ such that for any $x, y \in S$, $x + y$ is invertible in $\mathbb{Z}_n$. To describe the set S, we use the isomorphism in Lemma 2.5. Let $S = \{s \in \mathbb{Z}_n : s = (s_1, \ldots, s_k) \text{ where } s_i \equiv c_i \bmod p_i \text{ for some } 1 \le c_i \le (p_i - 1)/2\}$. Note that in the description of S, each s_i has $p_i^{\alpha_i - 1}(p_i - 1)/2$ choices, and hence the set S has the desired cardinality. Further each element of S is invertible in $\mathbb{Z}_n$. Now for $s, t \in S$, we have $s + t = (s_1 + t_1, \ldots, s_k + t_k)$ where $s = (s_1, \ldots, s_k)$ and $t = (t_1, \ldots, t_k)$. By definition of S, observe that $s_i + t_i \not\equiv 0 \pmod{p_i}$ for all $1 \le i \le k$. Thus $s + t$ is invertible in $\mathbb{Z}_n$.

Now consider the set $\mathcal{P} = \{s.(x + 0, x + 1, \ldots, x + n - 1) : s \in S, x \in \mathbb{Z}_n\}$. By observations (a),(b) and (c), we see that $\mathcal{P}$ consists of permutations of $\mathbb{Z}_n$. Let σ, τ be two distinct permutations in $\mathcal{P}$ with $\sigma = s.(x + 0, \ldots, x + n - 1)$ and $\tau = t.(y + 0, \ldots, y + n - 1)$. Then $\sigma + \tau = (sx + ty) + (s + t).(0, 1, \ldots, n - 1)$. Since $s + t$ is invertible, by observations (a) and (b), we conclude that $\sigma + \tau$ is a permutation. Thus $\mathcal{P}$ satisfies (**P1**). Finally we observe that $|\mathcal{P}| = n.|S| = n.\prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)/2 = (n.\phi(n))/2^k$. This proves the lemma. $\square$

Remark 3.2. We note that when n is a prime number, the bound in Lemma 3.1 reduces to $n(n-1)/2$.

We now establish an upper bound on $s(n)$ in the case when n is an odd prime.

Theorem 3.3 (Cauchy-Davenport)**.** *Let n be any prime number and let $A, B \subset \mathbb{Z}_n$. Then $|A+B| \geq \min\{n, |A|+|B|-1\}$, where $A+B = \{a+b : a \in A, b \in B\}$.*

Lemma 3.4. *Let n be an odd prime. Then $s(n) \leq \frac{e^2}{\pi} n \left(\frac{n-1}{e}\right)^{\frac{n-1}{2}}$.*

Proof. We prove the case $n = 3$ seperately. It suffices to prove $s(3) \leq 3$. This follows from the observation that among any four permutations $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ of $\mathbb{Z}_3$, there exist permutations σ_i, σ_j such that σ_j is obtained from σ_i by interchanging exactly one pair, and thus $\sigma_i + \sigma_j$ is not a permutation.

From now on we assume $n > 3$. Let $\mathcal{P}$ with $|\mathcal{P}| = m$ be a collection of permutations of $\mathbb{Z}_n$ satisfying (**P1**). Let $A^i = (a_{i1}, \ldots, a_{in})$ be the permutations in $\mathcal{P}$ for $i = 1, \ldots, m$. Let M denote the $m \times n$ matrix with $(i,j)^{th}$ entry a_{ij}. When n is prime the ring $\mathbb{Z}_n$ is infact a field, and thus we can regard the permutations A^i as vectors in n-dimensional vector space $(\mathbb{Z}_n)^n$. Let $\langle \, , \rangle$ be the standard inner product on $(\mathbb{Z}_n)^n$. Then, we note that for a permutation A of $\mathbb{Z}_n$ we have,

$$\langle A, A \rangle = \sum_{i=0}^{n-1} i^2 = \frac{n}{6}(n-1)(2n-1) \equiv 0 \pmod{n}. \tag{2}$$

Claim: $\mathrm{rank}(M) \leq (n-1)/2$. Let u, v be vectors corresponding to two different rows in M. Then from (2), we have $\langle u, u \rangle = \langle v, v \rangle = 0$. Further since $u + v$ is also a permutation, we have $\langle u+v, u+v \rangle = 0$. Using these relations, we conclude $\langle u, v \rangle = 0$ for all $u, v \in M$. Let S be the subspace spanned by the rows of M. Then, by earlier observation S is totally isotropic, i.e, $S \subseteq S^{\perp}$. Combining this with the relation $\dim(S) + \dim(S^{\perp}) = n$, we have $\mathrm{rank}(M) = \dim(S) \leq \lfloor n/2 \rfloor = (n-1)/2$ as n is odd.

Let $r = \mathrm{rank}(M)$. We may assume without loss of generality, by permuting the columns of M if necessary, that the first r columns of M are linearly independent. Thus there exist linear functions $f_i : (\mathbb{Z}_n)^r \to \mathbb{Z}_n$ for $i = 1, \ldots, n-r$ such that a row of M can be written as $(x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_{n-r})$ where $y_i = f_i(x_1, \ldots, x_r)$. Thus any two rows in M have different projection on the first r positions.

Claim: For any odd number $k \in \{1, r\}$, let M' be an $l \times n$ submatrix of M in which all the rows agree on the first k positions. Then at most $(n-k)/2$ distinct elements of $\mathbb{Z}_n$ appear in any column of M' beyond k. Suppose, for the sake of contradiction, that the j-th column of M' for some $j > k$ contains more than $(n-k)/2$ elements of $\mathbb{Z}_n$. Let B be the set of elements that appear in the j-th column of M'. By Theorem 3.3, $|B + B| \geq n - k + 1$. Hence $B + B$ contains some $2x$ where x is the common entry in some i-th column, $i \leq k$, of M'. Thus either $x \in B$, which violates the property that every row of M' is a permutation or $\exists x', x'' \in B,\ x' \neq x''$ such that $x' + x'' = 2x$ which violates the property that sum of any two rows of M' is a permutation.

Since M is completely determined by its first r rows and the above claim, the total number of rows m of M is at most $n \times \prod_{i=2}^{r} \lceil (n-i+1)/2 \rceil$. Hence $m \leq n \left(\prod_{j=0}^{\lfloor r/2 \rfloor - 1} ((n-1)/2 - j) \right)^2$ and since $r \leq (n-1)/2$, we get $m \leq n \left(\frac{((n-1)/2)!}{(\lceil (n-1)/4 \rceil)!} \right)^2$ which by Stirling's approximation gives us $m \leq \frac{e^2}{\pi} n \left(\frac{n-1}{e}\right)^{\frac{n-1}{2}}$. $\square$

4 Sum of Permutations being Non-Permutation

We consider another variant of the previous problem. We now seek a collection of permutations of $\mathbb{Z}_n$ such that any two distinct permutations sum up to a non-permutation. We say that a collection $\mathcal{P}$ of permutations of $\mathbb{Z}_n$ satisfies property (**P2**) if any two distinct permutations in $\mathcal{P}$ sum up to a non-permutation. Let $t(n) = \max\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}(\mathbb{Z}_n), \mathcal{P} \text{ satisfies } (\mathbf{P2})\}$. From Lemma 2.8, we see that $t(n) = n!$ when n is even. We consider the case when n is odd in the following lemma.

Lemma 4.1. *Let n be an odd number. Then $2^{(n-1)/2}.(\frac{n-1}{2})! \leq t(n) \leq 2^k(n-1)!/\phi(n)$, where k is the number of distinct prime divisors of n.*

Proof. We prove $t(n) \geq 2^{(n-1)/2}.(\frac{n-1}{2})!$. We say the pair of permutations $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ of the set $\{0, \ldots, n-1\}$ are *admissible* if the sums $a_i + b_i$, $0 \leq i \leq m-1$, are not all distinct (the addition being in $\mathbb{Z}$). We construct a collection $\mathcal{P}$ of mutually admissible permutations of $\{0, \ldots, n-1\}$. Note that $\mathcal{P}$ when viewed as a set of permutations of $\mathbb{Z}_n$, satisfies (**P2**). Let $m = (n-1)/2$. For $0 \leq i \leq m-1$, let $B_i = (2i, 2i+1)$ and $\overline{B}_i = (2i+1, 2i)$. Let $c = c_0c_1 \ldots c_{m-1}$ be a binary string of length m, and σ be a permutation of $\{0, \ldots, m-1\}$. Define, $P_{\sigma,c} = (x_0, x_1, \ldots, x_{n-1})$ such that,

$$(x_{2i}, x_{2i+1}) = \begin{cases} B_{\sigma(i)} & \text{if } c_i = 0, \\ \overline{B}_{\sigma(i)} & \text{if } c_1 = 1 \end{cases}$$

for all $i = 0, \ldots, m-1$. It is easily observed that $P_{\sigma,c}$ is a permutation of the set $\{0, \ldots, n-1\}$ and $P_{\sigma,c} \neq P_{\sigma',c'}$ for $(\sigma, c) \neq (\sigma', c')$. Let $\mathcal{P}$ denote the collection of permutations $\{P_{\sigma,c}\}$. We now show that any two permutations in the collection $\mathcal{P}$ are admissible. Let $P_{\sigma,c}$ and $P_{\sigma',c'}$ be two distinct permutations in $\mathcal{P}$. We consider two cases:

Case: $c \neq c'$. Let i be such that $c_i \neq c'_i$. Without loss of generality assume $c_i = 0, c'_i = 1$. Let $P_{\sigma,c} = (x_0, \ldots, x_{n-1})$ and $P_{\sigma',c'} = (x'_0, \ldots, x'_{n-1})$. Then we have, $(x_{2i}, x_{2i+1}) = B_{\sigma(i)} = (a, a+1)$ and $(x'_{2i}, x'_{2i+1}) = \overline{B}_{\sigma'(i)} = (b+1, b)$ for some a, b. Thus $x_{2i} + x'_{2i} = x_{2i+1} + x'_{2i+1} = a+b+1$, and hence the permutations are admissible.
Case: $c = c'$. In this case we must have $\sigma \neq \sigma'$. In the block $B_i = (2i, 2i+1)$, let us call $2i$ as the *little* end and $2i+1$ as the *big* end. Then $c = c'$ implies that in the component-wise addition of $P_{\sigma,c}$ and $P_{\sigma',c'}$ the little ends are summed with little ends, and big ends are summed with big ends. Let L be the set of sums of little ends, i.e, $L = \{2\sigma(i) + 2\sigma'(i) : 0 \leq i \leq m-1\}$ and B be the sums of big ends, i.e., $B = \{2\sigma(i) + 2\sigma'(i) + 2 : 0 \leq i \leq m-1\}$. We show that $L \cap B$ is non-empty which will prove that $P_{\sigma,c}$ and $P_{\sigma',c'}$ are admissible. By Lemma 2.9, there exist $0 \leq j, k \leq m-1$ such that $\sigma(j) + \sigma'(j) = \sigma(k) + \sigma'(k) + 1$, or $2\sigma(j) + 2\sigma'(j) = 2\sigma(k) + 2\sigma'(k) + 2$. We see that the left side of the identity is in L and the right side is in B. Thus $L \cap B$ is non-empty.

Hence the collection of permutations $P_{\sigma,c}$, when viewed as permutations of $\mathbb{Z}_n$ satisfy (**P2**). Finally we note that the number of permutations is $2^m.m! = 2^{(n-1)/2}.(\frac{n-1}{2})!$.

Next we prove $t(n) \leq 2^k(n-1)!/\phi(n)$. Let $n = \prod_{i=1}^k p_i^{\alpha_i}$. We define a relation $\sim$ on the set of permutations of $\mathbb{Z}_n$ by $\sigma \sim \tau$ if $\sigma = t + s.\tau$ for some $t \in \mathbb{Z}_n$ and $s \in \mathbb{Z}_n^\times$ (recall that $\mathbb{Z}_n^\times$ denotes the set of invertible elements of $\mathbb{Z}_n$). We show that $\sim$ is an equivalence relation on $\mathcal{S}(\mathbb{Z}_n)$. Note that $\sigma \sim \sigma$ as $\sigma = 0 + 1.\sigma$. To show symmetry, assume $\sigma \sim \tau$. Then $\sigma = t + s.\tau$ for some $t \in \mathbb{Z}_n$ and $s \in \mathbb{Z}_n^\times$. Rearranging we have, $\tau = -t + s^{-1}.\sigma$. Thus $\tau \sim \sigma$. To show transitivity assume $\sigma \sim \tau$ and $\tau \sim \gamma$. Then we have, $\sigma = t_1 + s_1.\tau$ and $\tau = t_2 + s_2.\gamma$ for some $t_1, t_2 \in \mathbb{Z}_n$ and $s_1, s_2 \in \mathbb{Z}_n^\times$. From the previous relations we see that $\sigma = (t_1 + s_1 t_2) + s_1 s_2.\gamma$, and therefore $\sigma \sim \gamma$. Thus $\sim$ is an equivalence relation on $\mathcal{S}(\mathbb{Z}_n)$. We now make the following claim:

Claim: Let $\mathcal{P}$ be a collection of permutations of $\mathbb{Z}_n$ satisfying (**P2**), and $\sim$ be the equivalence relation on $\mathcal{S}(\mathbb{Z}_n)$ as defined. Then each equivalence class of $\sim$ contains atmost 2^k permutations from $\mathcal{P}$.

Let $\sigma_1, \ldots, \sigma_m$ be permutations from $\mathcal{P}$ which belong to the same equivalence class. Let σ be a permutation such that $\sigma_i \sim \sigma$ for all $i = 1, \ldots, m$. Let s_i, t_i be such that $\sigma_i = t_i + s_i.\sigma$. Now $\sigma_i + \sigma_j = (t_i + t_j) + (s_i + s_j).\sigma$. From observations (a) and (b) in the proof of Lemma 3.1, we see that $\sigma_i + \sigma_j$ is a non-permutation if and only if $s_i + s_j$ is not invertible in $\mathbb{Z}_n$. Thus $p_r | (s_i + s_j)$ for some r, $1 \leq r \leq k$. Consider the edge coloring of the complete graph K_m on vertex set $\{1, \ldots, m\}$ using k colors as follows. We color the edge ij with the color $\min\{r \in \{1, \ldots, k\} : p_r | (s_i + s_j)\}$.

Subclaim: For each color $i \in \{1, \ldots, k\}$, the edges of color i span a bipartite subgraph of K_m. We prove this by showing there are no monochromatic odd cycles. Assume $v_1 v_2 \ldots v_l$ is a monochromatic odd cycle. Let i be the color of the edges in the cycle. Then $p_i | (s_1 + s_2), p_i | (s_2 + s_3), \ldots, p_i | (s_{l-1} + s_l), p_i | (s_l + s_1)$. Adding the above we get $p_i | 2(s_1 + \cdots + s_l)$ or $p_i | (s_1 + \cdots + s_l)$ as p_i is odd. Now, from the $(l-1)/2$ relations $p_i | (s_1 + s_2), p_i | (s_3 + s_4), \ldots, p_i | (s_{l-2} + s_{l-1})$ we get $p_i | (s_1 + \cdots + s_{l-1})$. Combining this with $p_i | (s_1 + \cdots + s_l)$ we get $p_i | s_l$ which contradicts the fact that s_l in invertible in $\mathbb{Z}_n$. Thus there are no monochromatic odd cycles, and hence edges of a fixed color span a bipartite subgraph of K_m.

The subclaim allows us to apply Lemma 2.6 to the coloring of K_m, and hence we conclude that $m \leq 2^k$. This proves the claim.

Finally we notice that each equivalence class of $\sim$ contains $n\phi(n)$ elements, and hence there are $n!/(n\phi(n)) = (n-1)!/\phi(n)$ equivalence classes. Now from the claim, each equivalence class contains at most 2^k elements from $\mathcal{P}$, and hence $|\mathcal{P}| \leq 2^k(n-1)!/\phi(n)$. □

Remark 4.2. We note that for a prime number n, the upper bound $\sim (n/e)^n \sqrt{n}$ whereas the lower bound $\sim (n/e)^{n/2} \sqrt{n}$. Thus there is a quadratic gap between the upper and lower bounds. We also mention that one way to obtain permutations summing up to a non permutation is to consider a family of permutations with mutual reverse. Two permutations σ, τ of $\{0, \ldots, n-1\}$ are said to have a *mutual reverse* if there exist $i \neq j$ such that $\sigma(i) = \tau(j)$ and $\sigma(j) = \tau(i)$. We call such a family of permutations as *reverse full*. Note that a reverse full family of permutations satisfies (**P2**). From a result of Cibulka [2] on size of reverse-free families of permutations, the maximum size of a reverse full family of permutations on n symbols is at most

$n^{n/2+O(\log n)}$, though we are unaware of any non-trivial lower bound for the same. Our construction achieves $\sim (n/e)^{n/2}\sqrt{n}$, but under a much more flexbile constraint.

5 Differences of Permutations being Permutation

In this section, we wish to obtain upper and lower bounds on the maximum size of set $\mathcal{P} \subseteq \mathcal{S}(\mathbb{Z}_n)$ such that for any two distinct permutations $\sigma, \tau \in \mathcal{P}$, $\sigma - \tau$ is also a permutation of $\mathbb{Z}_n$. We say that $\mathcal{P} \subseteq \mathcal{S}(\mathbb{Z}_n)$ satisfies property (**P3**) if for any two $\sigma, \tau \in \mathcal{P}$, $\sigma \neq \tau$, $\sigma - \tau$ is a permutation of $\mathbb{Z}_n$. Let $f(n) = \max\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}(\mathbb{Z}_n)$ satisfies (**P3**)$\}$.

Question 5.1. What is a good estimate for $f(n)$ as defined above ?

The above question is only superficially different from the well studied problem of finding mutually orthogonal orthomorphisms of finite groups (See [3]). For a finite group G, a bijection $\theta : G \to G$ is called an *orthomorphism* of G if the map $x \mapsto \theta(x) - x$ is a bijection. Two orthomorphisms θ, ϕ are called *orthogonal* if $\theta - \phi$ is a bijection. It is not hard to see that a set of k permutations of $\mathbb{Z}_n$ satisfying **(P3)** gives a set of $k-1$ mutually orthogonal orthomorphisms of $\mathbb{Z}_n$ and vice versa. Orthogonal orthomorphisms have been used in construction of mutually orthogonal latin squares (MOLS). If we write the permutations of $\mathcal{P}$ as rows of a matrix, we get a $|\mathcal{P}| \times n$ matrix over $\mathbb{Z}_n$ with the property that the difference of any two distinct rows is a permutation of $\mathbb{Z}_n$. We point out that such matrices have been used in connection with constructions of Latin Squares and Orthogonal Arrays (cf. [5, Chapter 22]). Well known bounds for mutually orthogonal orthomorphisms yield the following lemma.

Lemma 5.2. *For $n \geq 2$, we have,*

(a) $f(n) \leq n-1$,

(b) $f(n) = 1$, *when n is an even number,*

(c) $f(n) = n-1$, *when n is a prime number.*

Question 5.3. Determine $f(n)$ for odd composite number n.

From results in [3] if follows that for an odd number $n > 3$ and n not divisible by 9, we have $f(n) \geq 3$.

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