On the Average of Triangular Numbers

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Abstract

The problem we are dealing with is the following: find two sequences a_n and b_n such that the average of the first b_n triangular numbers (starting with the triangular number 1) is still a triangular number, precisely the a_n-th triangular number. We get also some side results: for instance one of the sequence instrumental to finding the asked for sequences turns out to be a bisection of the sequence of the numerators of continued fraction convergents to $\sqrt{3}$.

The present note has been suggested by a problem proposed in the "Student Problems" section of *The Mathematical Gazette* (see [2]). The problem we tackle is the following: find two sequences a_n and b_n such that the average of the first b_n triangular numbers (starting with the triangular number 1) is still a triangular number, precisely the a_n-th triangular number.
If we want that the average of the first s triangular number be a triangular number it has to hold

$$\frac{1}{s}\sum_{k=1}^{s}\frac{k(k+1)}{2}=\frac{r(r+1)}{2},$$

for some positive integer r. This becomes

$$\frac{(s+1)(2s+4)}{12}=\frac{r(r+1)}{2},$$

that is, clearing the fractions,

$$s^2+3s+2=3r^2+3r. \tag{1}$$

Solving for s and considering that s has to be a positive quantity we have the solution

$$s = \frac{-3 + \sqrt{1 + 12r + 12r^2}}{2}.$$

Now because s has to be an integer we need r such that $1 + 12r + 12r^2$ is a perfect square; also $\sqrt{1 + 12r + 12r^2} - 3$ has to be even.

Consider the nonhomogeneous recurrence $w_n = w(k,\, r,\, s)$ defined by

$$w_n = 4w_{n-1} - w_{n-2} + k, \quad w_0 = r,\, w_1 = s.$$

Let α and β be the zeros of the polynomial x^2-4x+1, that is $\alpha = 2+\sqrt{3}$, $\beta = 2 - \sqrt{3}$. Note that $\alpha + \beta = 4$, $\alpha\beta = 1$, $\alpha - \beta = 2\sqrt{3}$, $\alpha^2 + \beta^2 = 14$.

It is useful to introduce the homogeneous sequence $L_n = w(0,\, 2, 4)$ given by

$$L_n = 4L_{n-1} - L_{n-2}.$$

This sequence is the analogous in this context of the Lucas numbers. The closed form of L_n is

$$L_n = \alpha^n + \beta^n.$$

The generating function of w_n is given by

$$g(x) = \frac{r + (s - 5r)x + (k + 4r - s)x^2}{(1 - x)(1 - 4x + x^2)},$$

from which we get the closed form

$$w_n = -\frac{k}{2} + \frac{s\alpha + (k + 4r - s)\beta + k - 2r}{12}\alpha^n + \frac{s\beta + (k + 4r - s)\alpha + k - 2r}{12}\beta^n. \tag{2}$$

It may be useful to write the RHS in terms of the L_n sequence

$$w_n = -\frac{k}{2} + \frac{4s + k - 2r}{12}L_n + \frac{k + 4r - 2s}{12}L_{n-1}. \tag{3}$$

Now introduce the sequence $a_n = w(1,\, 0,\, 1)$. a_n is sequence A061278 in [1]: the first values are 0, 1, 5, 20, 76, 285, ... Using Equation 3 we get

$$\begin{aligned} a_n &= \frac{5}{12}L_n - \frac{1}{12}L_{n-1} - \frac{1}{2} \\ &= \frac{L_n + 4L_n - L_{n-1} - 6}{12} \\ &= \frac{L_n + L_{n+1} - 6}{12}. \end{aligned}$$

Note that

$$\begin{aligned}
L_n^2 &= (\alpha^n + \beta^n)^2 \\
&= \alpha^{2n} + \beta^{2n} + 2 \\
&= L_{2n} + 2,
\end{aligned}$$

$$\begin{aligned}
L_n L_{n+1} &= (\alpha^n + \beta^n)(\alpha^{n+1} + \beta^{n+1}) \\
&= \alpha^{2n+1} + \alpha^n\beta^{n+1} + \alpha^{n+1}\beta^n + \beta^{2n+1} \\
&= L_{2n+1} + \alpha + \beta \\
&= L_{2n+1} + 4.
\end{aligned}$$

Then

$$\begin{aligned}
1 + 12a_n + 12a_n^2 &= \frac{-12 + L_{2n} + 2L_{2n+1} + L_{2n+2}}{12} \\
&= \frac{-12 + 6L_{2n+1}}{12} \\
&= -1 + \frac{L_{2n+1}}{2},
\end{aligned}$$

where we used the recurrence defining L_n. Now we are going to show that this expression is a perfect square. Write

$$\begin{aligned}
-1 + \frac{L_{2n+1}}{2} &= \frac{1}{2}\alpha^{2n+1} + \frac{1}{2}\beta^{2n+1} - 1 \\
&= \left(\frac{1}{\sqrt{2}}\alpha^{n+\frac{1}{2}} - \frac{1}{\sqrt{2}}\beta^{n+\frac{1}{2}}\right)^2.
\end{aligned}$$

Hence it remains to prove that

$$u_n = \frac{1}{\sqrt{2}}\alpha^{n+\frac{1}{2}} - \frac{1}{\sqrt{2}}\beta^{n+\frac{1}{2}}$$

is a positive integer. This will be done by strong induction. First of all note that

$$\begin{aligned}
\left(\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right)^2 &= \alpha + \beta - 2 \\
&= 2,
\end{aligned}$$

so that we will take $\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}} = \sqrt{2}$. Then $u_0 = 1$. Next assume that u_n is a positive integer for $n \leq n_0$. It follows that $u_{n_0}(\alpha + \beta) = 4u_{n_0}$ is a positive integer. But

$$\begin{aligned}
u_{n_0}(\alpha+\beta) &= \frac{1}{\sqrt{2}}\left(\alpha^{n_0+\frac{1}{2}}-\beta^{n_0+\frac{1}{2}}\right)(\alpha+\beta) \\
&= \frac{1}{\sqrt{2}}\alpha^{n_0+1+\frac{1}{2}}+\frac{1}{\sqrt{2}}\alpha^{n_0+\frac{1}{2}}\beta-\frac{1}{\sqrt{2}}\alpha\beta^{n_0+\frac{1}{2}}-\frac{1}{\sqrt{2}}\beta^{n_0+1+\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\left(\alpha^{n_0+1+\frac{1}{2}}-\beta^{n_0+1+\frac{1}{2}}\right)+\frac{1}{\sqrt{2}}\left(\alpha^{n_0-\frac{1}{2}}-\beta^{n_0-\frac{1}{2}}\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha^{n_0+1+\frac{1}{2}}-\beta^{n_0+1+\frac{1}{2}}\right)+\frac{1}{\sqrt{2}}\left(\alpha^{n_0-1+\frac{1}{2}}-\beta^{n_0-1+\frac{1}{2}}\right) \\
&= u_{n_0+1}+u_{n_0-1}.
\end{aligned}$$

Because by the induction hypothesis u_{n_0-1} is a positive integer, it follows that u_{n_0+1} is a positive integer. This ends the induction proof.
As a side result we get that u_n obeys the recurrence

$$u_n = 4u_{n-1} - u_{n-2}, \quad u_0 = 1,\, u_1 = 5,$$

since $\alpha^{\frac{3}{2}} - \beta^{\frac{3}{2}} = \sqrt{50} = 5\sqrt{2}$. u_n is sequence A001834 in [1].
It remains to prove that u_n is odd which is the same as to prove that $\frac{L_{2n+1}}{2}$ is even, which means that L_{2n+1} is a multiple of 4. Again this will be done by strong induction. For $n = 0$ we have $L_1 = 4$. Next assume that L_{2n+1} is a multiple of 4 for $n \leq n_0$, that is $L_{2n+1} = 4c_n$, where c_n is a positive integer. Then

$$\begin{aligned}
L_{2n_0+1}(\alpha^2+\beta^2) &= 4c_{n_0}\cdot 14 \\
&= \alpha^{2n_0+3}+\alpha^{2n_0+1}\beta^2+\alpha^2\beta^{2n_0+1}+\beta^{2n_0+3} \\
&= \alpha^{2(n_0+1)+1}+\beta^{2(n_0+1)+1}+\alpha^{2(n_0-1)+1}+\beta^{2(n_0-1)+1} \\
&= L_{2(n_0+1)+1}+L_{2(n_0-1)+1} \\
&= L_{2(n_0+1)+1}+4c_{n_0-1}.
\end{aligned}$$

Then

$$L_{2(n_0+1)+1} = 4c_{n_0}\cdot 14 - 4c_{n_0-1},$$

that is, a multiple of 4.
It follows that

$$b_n = \frac{u_n-3}{2}.$$

Hence

$$\begin{aligned}
4b_{n-1} - b_{n-2} + 3 &= 2u_{n-1} - 6 - \frac{1}{2}u_{n-2} + \frac{3}{2} + 3 \\
&= \frac{4u_{n-1} - u_{n-2} - 3}{2} \\
&= \frac{u_n - 3}{2} \\
&= b_n.
\end{aligned}$$

The initial values are $b_0 = -1$, $b_1 = 1$. It follows that $b_n = w(3, -1, 1)$. The initial values are $-1, 1, 8, 34, 131, 493, \ldots$.
If we solve Equation 1 for r, being r a positive quantity, we obtain

$$r = \frac{\sqrt{3}\sqrt{11 + 12s^2 + 4s^2} - 3}{6}.$$

If we define

$$v_n^2 = 3(11 + 12b_n + 4b_n^2),$$

after insertion of the value of b_n we get

$$v_n^2 = 3(u_n^2 + 2),$$

that is

$$v_n^2 = \frac{3}{2}(L_{2n+1} + 2).$$

Essentially repeating the previous reasoning we find that v_n is a positive integer $\forall\, n$ and we have the recurrence

$$v_n = 4v_{n-1} - v_{n-2}, \quad v_0 = 3,\ v_1 = 9,$$

that is $v_n = w(0, 3, 9)$. Also we get the closed form

$$v_n = \sqrt{\frac{3}{2}}\left(\alpha^{n+\frac{1}{2}} + \beta^{n+\frac{1}{2}}\right).$$

Furthermore

$$v_n \equiv 3 \pmod{6}.$$

This can be proved easily by strong induction, noting that $v_0 = 3$ and assuming that for all $n \leq n_0$ the former equation holds. Writing $v_{n_0} = 6k_1 + 3$, $v_{n_0-1} = 6k_2 + 3$ we have

$$\begin{aligned}
v_{n_0+1} &= 4v_{n_0} - v_{n_0-1} \\
&= 4(6k_1 + 3) - (6k_2 + 3) \\
&= 6(4k_1 - k_2) + 9,
\end{aligned}$$

so that

$$
\begin{aligned}
v_{n_0+1} - 3 &= 6(4k_1 - k_2) + 6 \\
&= 6(4k_1 - k_2 + 1).
\end{aligned}
$$

And finally we have

$$a_n = \frac{v_n - 3}{6}.$$

It is interesting the fact that the sequence u_n is a bisection of sequence A002531 in [1] that gives the numerators of continued fraction convergents to $\sqrt{3}$, a result that as far as we know is new. Denoting by z_n the sequence of the convergents we are going to prove

$$u_n = z_{2n+1}.$$

From the comments to A002531 in [1] we have that

$$z_{2n+1} = 2z_{2n} + z_{2n-1}, \quad z > 0, \tag{4}$$

$$z_{2n} = \frac{\alpha^n + \beta^n}{2}. \tag{5}$$

Now

$$
\begin{aligned}
u_n - u_{n-1} &= \frac{1}{\sqrt{2}}\left(\alpha^{n+\frac{1}{2}} - \beta^{n+\frac{1}{2}}\right) - \frac{1}{\sqrt{2}}\left(\alpha^{n-\frac{1}{2}} - \beta^{n-\frac{1}{2}}\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha^n(\alpha^{\frac{1}{2}} - \alpha^{-\frac{1}{2}}) + \beta^n(\beta^{-\frac{1}{2}} - \beta^{\frac{1}{2}})\right) \\
&= \frac{1}{\sqrt{2}}\left(\alpha^n(\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}) + \beta^n(\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}})\right) \\
&= \frac{1}{\sqrt{2}}(\alpha^n + \beta^n)\left(\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right).
\end{aligned}
$$

Now from

$$\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right)^2 = \alpha + \beta + 2 = 6$$

we get

$$\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}} = \sqrt{6}.$$

Also from

$$\left(\alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}\right)\left(\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}\right) = \alpha - \beta = 2\sqrt{3}$$

we get

$$\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}} = \sqrt{2},$$

so that

$$u_n - u_{n-1} = L_n = 2z_{2n} = z_{2n+1} - z_{2n-1}.$$

So we can write

$$u_n - z_{2n+1} = u_{n-1} - z_{2n-1}.$$

Setting

$$d_n = u_n - z_{2n+1}$$

we have the recurrence

$$d_n = d_{n-1}$$

with $d_0 = u_0 - z_1 = 1 - 1 = 0$. Hence $d_n = 0$ and $u_n = z_{2n+1}$.
In a similar way we obtain

$$v_n - v_{n-1} = 6F_n,$$

where $F_n = w(0, 0, 1)$, the analogous of the Fibonacci numbers for this type of recurrences. F_n is sequence A001353 in [1].

References

[1] N.J.A. Sloane, Editor (2003), The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/.

[2] "Problem 2003.2." *The Mathematical Gazette* **87**.508: 175.