Generalized Brouncker's continued fractions and their logarithmic derivatives

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Abstract

In this paper, we study the continued fraction $y(s,r)$ which satisfies the equation $y(s,r)y(s+2r,r) = (s+1)(s+2r-1)$ for $r > \frac{1}{2}$. This continued fraction is a generalization of the Brouncker's continued fraction $b(s)$. We extend the formulas for the first and the second logarithmic derivatives of $b(s)$ to the case of $y(s,r)$. The asymptotic series for $y(s,r)$ at ∞ are also studied. The generalizations of some Ramanujan's formulas are presented.

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1 Introduction

The Brouncker's continued fraction $b(s) = s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right)$ still attracts the attention of researchers due to its role in the theory of orthogonal polynomials and its relations to the Gamma and Beta functions (see [3], [5]–[7]). Recall the following theorem of Brouncker describing the properties of $b(s)$ (see [5], p. 145, Theorem 3.16).

Theorem 1.1 (Brouncker) *Let $b(s)$ be a function on $(0,+\infty)$ satisfying the functional equation $b(s)b(s+2) = (s+1)^2$ and the inequality $s < b(s)$ for $s > C$, where C is some constant. Then*

$$b(s) = (s+1)\prod_{n=1}^{\infty}\frac{(s+4n-3)(s+4n+1)}{(s+4n-1)^2} = s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right)$$

for every positive s.

Ramanujan discovered the formula, expressing the Brouncker's continued fraction in terms of the Gamma function (see [5], p. 153, Theorem 3.25).

Theorem 1.2 (Ramanujan) *For every $s > 0$*

$$b(s) = s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right) = 4\left[\frac{\Gamma(\frac{3+s}{4})}{\Gamma(\frac{1+s}{4})}\right]^2.$$

The following extension of Brouncker's theorem (Theorem 1) was obtained by Euler (see [5], p. 180, Theorem 4.17).

Theorem 1.3 (Euler) *Let $y(s,r)$ be a positive continuous function satisfying the inequality $s < y(s,r)$ and the equation*

$$y(s,r)y(s+2r,r) = (s+1)(s+2r-1)$$

for any $s > 0$, $r > \dfrac{1}{2}$. Then

$$y(s,r) = (s+1)\prod_{n=0}^{\infty}\frac{(s+2r-1+4nr)(s+4r+1+4nr)}{(s+2r+1+4nr)(s+4r-1+4nr)} =$$

$$= s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{(2n-1)^2r^2-(r-1)^2}{2s}\right).$$

In [5] Ramanujan's theorem (Theorem 2) was extended to the case of the continued fraction $y(s,r)$ (see [5], p. 220, ex. 4.22).

Theorem 1.4 *For every $s > 0$, $r > \frac{1}{2}$*

$$y(s,r) = s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{(2n-1)^2r^2-(r-1)^2}{2s}\right) = 4r\frac{\Gamma(\frac{s+2r+1}{4r})\Gamma(\frac{s+4r-1}{4r})}{\Gamma(\frac{s+1}{4r})\Gamma(\frac{s+2r-1}{4r})}.$$

The following exact continued fraction representation for the first logarithmic derivative of $b(s)$

$$\frac{b'}{b}(s) = \frac{1}{s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2}{s}\right)}$$

allows one to obtain the exponential representation for $b(s)$ (see [5], p. 192, Theorem 4.25).

Theorem 1.5 *For $s > 0$*

$$s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right) = \frac{8\pi^2}{\Gamma^4(\frac{1}{4})}\exp\left\{\int_0^s \frac{dt}{t + \operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2}{t}\right)}\right\}.$$

In this paper, we represent the first logarithmic derivative of $y(s,r)$ in the form of the sum of two continued fractions (see Section 4, Corollary 3). For $s > |r-1|$, $r > \dfrac{1}{2}$

$$\frac{\partial}{\partial s}(\ln y)(s,r) = f_1(s,r) + f_2(s,r),$$

where

$$f_1(s,r) = \frac{1}{2-2r+2s+2\operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2r^2}{1-r+s}\right)} \tag{1}$$

and

$$f_2(s,r) = \frac{1}{2r-2+2s+2\operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2r^2}{r-1+s}\right)}. \tag{2}$$

Then we extend Theorem 5 to the case of $y(s,r)$ (see Section 5, Theorem 9).

Theorem 1.9 *For $s > |r-1|$, $r > \frac{1}{2}$*

$$y(s,r) = s + \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2r^2-(r-1)^2}{2s}\right) =$$

$$= 8\pi r 2^{1-\frac{1}{r}}\frac{\Gamma^2(\frac{1}{2r})}{\Gamma^4(\frac{1}{4r})}\cot(\frac{\pi}{4r})\exp\left\{\int_0^s (f_1(t,r)+f_2(t,r))dt\right\},$$

where $f_1(t,r)$ and $f_2(t,r)$ are given by formulas (1) and (2), respectively.

There is also an exact integral representation of $\frac{b'}{b}(s)$ (see [5], p. 191, Formula 4.71). For $s > 0$

$$\frac{b'}{b}(s) = \frac{1}{s + \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{n^2}{s}\right)} = 2\int_0^{+\infty}\frac{e^{-sx}dx}{\cosh x}. \tag{3}$$

Theorem 5 together with (3) imply the following asymptotic relation, which holds for $b(s)$ as $s \to +\infty$ (see [5], p. 192, Corollary 4.26).

$$b(s) = s + \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right) \sim s\exp\left\{-\sum_{k=1}^{\infty}\frac{E_{2k}}{2ks^{2k}}\right\},$$

where E_{2k} are the Euler's numbers. Here the asymptotic power series $\exp\left\{-\sum_{k=1}^{\infty}\frac{E_{2k}}{2ks^{2k}}\right\}$ arises from replacing x in the formal power series $\exp(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ by $-\sum_{k=1}^{\infty}\frac{E_{2k}}{2ks^{2k}}$ and combining coefficients afterwards (on the possibility of such a substitution see [9], p. 15, Theorem 124, see also [2], p. 15).

We obtain exact integral representations for both the continued fractions (1) and (2) (see Section 3, Lemma 4). For $s > |r-1|$, $r > \frac{1}{2}$

$$\frac{1}{2-2r+2s+2\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{n^2r^2}{1-r+s}\right)} = \frac{1}{2r}\int_0^{+\infty}\frac{e^{-x\frac{1-r+s}{r}}dx}{\cosh x}.$$

$$\frac{1}{2r-2+2s+\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{n^2r^2}{r-1+s}\right)} = \frac{1}{2r}\int_0^{+\infty}\frac{e^{-x\frac{r-1+s}{r}}dx}{\cosh x}.$$

These two formulas together with Theorem 9 allows us to obtain the asymptotic expansion for $y(s,r)$ at infinity (see Section 6, Theorem 10), using Euler's methods.

$$y(s,r) = s + \mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{(2n-1)^2r^2-(r-1)^2}{2s}\right) \sim$$

$$\sim s\exp\left\{-\sum_{n=1}^{\infty}\frac{\sum_{k=0}^{n}\binom{2n}{2k}(r-1)^{2k}r^{2(n-k)}E_{2(n-k)}}{2ns^{2n}}\right\}.$$

Let us introduce the notation:

$$s^2-1+\frac{4\times 1^2}{1}\,_+\frac{4\times 1^2}{s^2-1+}\frac{4\times 2^2}{1}\,_+\frac{4\times 2^2}{s^2-1+}\cdots=s^2-1+\mathop{\mathrm{K}'}_{n=1}^{\infty}\left(\frac{4n^2}{1}\,_+\frac{4n^2}{s^2-1}\right).$$

Ramanujan stated the following formula for the second logarithmic derivative of $b(s)$, which was proved later by Perron (see [5], p. 231, Formula (5.6), see also [8]).

Theorem 1.6 (Ramanujan's formula) *For $s>1$*

$$(\ln b)''(s)=-\int_0^\infty \frac{xe^{-sx}}{\cosh x}dx=-\frac{1}{s^2-1+\mathop{\mathrm{K}'}\limits_{n=1}^{\infty}\left(\frac{4n^2}{1}\,_+\frac{4n^2}{s^2-1}\right)}. \tag{4}$$

We obtain the corresponding formula for the second logarithmic derivative of $y(s,r)$.

Theorem 1.12 *For $s>\max(1,\ 2r-1)$, $r>\dfrac{1}{2}$*

$$\frac{\partial^2}{\partial s^2}(\ln y)(s,r)=-\frac{1}{2r^2}\int_0^\infty \frac{x(e^{-\frac{1-r+s}{r}x}+e^{-\frac{r-1+s}{r}x})}{\cosh x}dx=-h_1(s,r)-h_2(s,r),$$

where

$$h_1(s,r)=\frac{1}{2(1-2r+s)(1+s)+2\mathop{\mathrm{K}'}\limits_{n=1}^{\infty}\left(\frac{4n^2r^2}{1}\,_+\frac{4n^2r^2}{(1-2r+s)(1+s)}\right)},$$

$$h_2(s,r)=\frac{1}{2(2r-1+s)(s-1)+2\mathop{\mathrm{K}'}\limits_{n=1}^{\infty}\left(\frac{4n^2r^2}{1}\,_+\frac{4n^2r^2}{(2r-1+s)(s-1)}\right)}.$$

2 Functional equations for logarithmic derivatives of $y(s,r)$

Let us recall the following statement, which will be used later (see [5], p. 152, Lemma 3.23).

Lemma 2.1 *Let $g(s)$ be a monotonic function on $(0,\infty)$, vanishing at infinity, and $a>0$ be a positive number. Then the functional equation $f(s)+f(s+a)=g(s)$ has a unique solution, vanishing at infinity, given by the formula*

$$f(s)=\sum_{n=0}^{\infty}(-1)^n g(s+na).$$

Let us prove two following statements for the first and the second logarithmic derivatives of $y(s,r)$.

Lemma 2.2 *The functional equation*

$$f(s,r)+f(s+2r,r)=\frac{1}{s+1}+\frac{1}{s+2r-1}=\frac{2(s+r)}{(s+1)(s+2r-1)} \tag{5}$$

has a unique solution, satisfying $\lim_{s\to\infty} f(s,r)=0$*, which is*

$$f(s,r)=\frac{\partial}{\partial s}(\ln y)(s,r).$$

Proof. The equality $y(s,r)y(s+2r,r)=(s+1)(s+2r-1)$ implies

$$\ln(y(s,r)y(s+2r,r))=\ln((s+1)(s+2r-1));$$

$$\ln y(s,r)+\ln y(s+2r,r)=\ln(s+1)+\ln(s+2r-1)).$$

Differentiating by s, we obtain

$$\frac{\frac{\partial}{\partial s}y(s,r)}{y(s,r)}+\frac{\frac{\partial}{\partial s}y(s+2r,r)}{y(s+2r,r)}=\frac{1}{s+1}+\frac{1}{s+2r-1}. \tag{6}$$

The function $f(s)=\frac{\frac{\partial}{\partial s}y}{y}(s,r)$ satisfy the conditions of Lemma 1 with $a=2r$ and $g(s)=\frac{2(s+r)}{(s+1)(s+2r-1)}$. Applying Lemma 1, we complete the proof. $\square$

Let us examine two equations:

$$f_1(s,r)+f_1(s+2r,r)=\frac{1}{s+1} \tag{7}$$

and

$$f_2(s,r)+f_2(s+2r,r)=\frac{1}{s+2r-1}. \tag{8}$$

Both of them satisfy the conditions of Lemma 1 with $a=2r$, $g(s)=\frac{1}{s+1}$ and $g(s)=\frac{1}{s+2r-1}$, respectively. So, applying Lemma 1, we obtain, that the solution $f_1(s,r)$ of equation (7) which satisfies $\lim_{s\to\infty} f_1(s,r)=0$ is unique. The solution $f_2(s,r)$ of equation (8) which satisfies $\lim_{s\to\infty} f_2(s,r)=0$ is also unique. Since their sum $f_1(s,r)+f_2(s,r)$ satisfies equation (5), we have from Lemma 2, that

$$\frac{\partial}{\partial s}(\ln y)(s,r)=f_1(s,r)+f_2(s,r),$$

where $f_1(s,r)$ and $f_2(s,r)$ are the solutions of (7) and (8), respectively, vanishing as $s\to+\infty$.

Lemma 2.3 *The functional equation*

$$f(s,r)+f(s+2r,r)=-\frac{1}{(s+1)^2}-\frac{1}{(s+2r-1)^2} \tag{9}$$

has a unique solution, satisfying $\lim_{s\to\infty} f(s,r)=0$*, which is*

$$f(s,r)=\frac{\partial^2}{\partial^2 s}(\ln y)(s,r).$$

Proof. Differentiate equation (6) once again by s:

$$\frac{\partial}{\partial s}\left(\frac{\frac{\partial}{\partial s}y(s,r)}{y(s,r)}\right)+\frac{\partial}{\partial s}\left(\frac{\frac{\partial}{\partial s}y(s+2r,r)}{y(s+2r,r)}\right)=-\frac{1}{(s+1)^2}-\frac{1}{(s+2r-1)^2}.$$

Applying Lemma 1 with $a = 2r$ and $g(s) = -\frac{1}{(s+1)^2} - \frac{1}{(s+2r-1)^2}$, we complete the proof. □

Repeating the above reasoning, we obtain that

$$-\frac{\partial^2}{\partial^2 s}(\ln y)(s,r) = h_1(s,r) + h_2(s,r), \tag{10}$$

where $h_1(s)$ is the unique solution of the equation

$$h_1(s,r) + h_1(s+2r,r) = \frac{1}{(s+1)^2}, \tag{11}$$

satisfying $\lim\limits_{s\to\infty} h_1(s,r) = 0$ and $h_2(s)$ is the unique solution of the equation

$$h_2(s,r) + h_2(s+2r,r) = \frac{1}{(s+2r-1)^2}, \tag{12}$$

satisfying $\lim\limits_{s\to\infty} h_2(s,r) = 0$.

3 Exact integral representation for certain type continued fractions

To begin, we formulate the following result by Euler (see [4] and [5], p. 191, Theorem 4.24).

Theorem 3.1 *For $s > 0$*

$$\frac{1}{s + \operatorname*{K}\limits_{n=1}^{\infty}\left(\frac{n^2}{s}\right)} = 2\int_0^1 \frac{x^s\,dx}{1+x^2}. \tag{13}$$

Corollary 3.1 *For $s > 0$*

$$\frac{1}{s + \operatorname*{K}\limits_{n=1}^{\infty}\left(\frac{n^2}{s}\right)} = \int_0^{+\infty} \frac{e^{-sx}\,dx}{\cosh x}. \tag{14}$$

Let us formulate and prove the following lemma.

Lemma 3.1 *Let $\varphi(s,r)$ be an arbitrary real-valued function of s and r. Then for $r > 0$, $\varphi(s,r) > 0$*[1]

$$\frac{1}{2\varphi(s,r) + 2\operatorname*{K}\limits_{n=1}^{\infty}\left(\frac{n^2 r^2}{\varphi(s,r)}\right)} = \frac{1}{r}\int_0^1 \frac{x^{\frac{\varphi(r,s)}{r}}\,dx}{1+x^2}. \tag{15}$$

[1] Actually we have the condition $\frac{\varphi(s,r)}{r} > 0$ but since in the conditions of Theorem 3 $r > \frac{1}{2}$ we restrict ourselves to the case $r > 0$.

Proof. Examine equality (13). Using the substitution $s := \dfrac{\varphi(s,r)}{r}$, where $\varphi(s,r)$ is an arbitrary real-valued function of s and r, we obtain the equality, which is correct for all $s,\ r$ satisfying $\varphi(s,r) > 0,\ r > 0$.

$$\frac{1}{\frac{\varphi(s,r)}{r} + \mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2}{\frac{\varphi(s,r)}{r}}\right)} = 2\int_0^1 \frac{x^{\frac{\varphi(s,r)}{r}}dx}{1+x^2}.$$

$$\frac{1}{2\varphi(s,r) + 2r\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2}{\frac{\varphi(s,r)}{r}}\right)} = \frac{1}{r}\int_0^1 \frac{x^{\frac{\varphi(s,r)}{r}}dx}{1+x^2}.$$

Let us apply the equivalence transform with the parameters $r_0 = 1$, $r_n = r$, $n = 1, 2, \ldots$ to the continued fraction on the left-hand side. This results the formula:

$$\frac{1}{2\varphi(s,r) + 2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{\varphi(s,r)}\right)} = \frac{1}{r}\int_0^1 \frac{x^{\frac{\varphi(s,r)}{r}}dx}{1+x^2}.$$

□

Corollary 3.2 *For $r > 0$, $\varphi(s,r) > 0$*

$$\frac{1}{\varphi(s,r) + \mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{\varphi(s,r)}\right)} = \frac{1}{r}\int_0^{+\infty} \frac{e^{-\frac{\varphi(r,s)}{r}x}dx}{\cosh x}.$$

Proof. It is enough for the proof to use the substitution $x := e^{-x}$. □

Example. Let $\varphi(s,r) = s + \sin r,\ s = 1,\ r = \dfrac{\pi}{2}$. Then

$$\frac{1}{2 + \mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2\frac{\pi^2}{4}}{2}\right)} = \frac{2}{\pi}\int_0^{+\infty} \frac{e^{-\frac{4}{\pi}x}dx}{\cosh x}.$$

Using the equivalence transformation with the parameters $r_0 = 1,\ r_n = 2$, $n = 1,\ 2,\ \ldots$, we get

$$\frac{1}{4 + \mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2\pi^2}{4}\right)} = \frac{1}{\pi}\int_0^{+\infty} \frac{e^{-\frac{4}{\pi}x}dx}{\cosh x}.$$

4 Functional equations for certain type continued fractions

Now let us formulate and prove the following theorem.

Theorem 4.1 *Let $\varphi(s,r) = s + \psi(r)$, where $\psi(r)$ is an arbitrary real-valued function of r. Then for $r > 0$, $s > -\psi(r)$ the continued fraction of the form*

$$f(s,r) = \frac{1}{2\varphi(s,r) + 2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{\varphi(s,r)}\right)}$$

is the unique solution of the functional equation

$$f(s,r)+f(s+2r,r)=\frac{1}{\varphi(s,r)+r}, \tag{16}$$

satisfying $\lim\limits_{s\to\infty} f(s,r)=0$.

Proof. Examine the series expansion for the right-hand side of equation (15):

$$\frac{1}{r}\int_0^1 \frac{x^{\frac{\varphi(s,r)}{r}}dx}{1+x^2}=\frac{1}{r}\sum_{n=0}^{\infty}(-1)^n\int_0^1 x^{2n+\frac{\varphi(s,r)}{r}}dx=\frac{1}{r}\sum_{n=0}^{\infty}\frac{(-1)^n}{\frac{\varphi(s,r)}{r}+2n+1}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{\varphi(s,r)+2rn+r}.$$

It follows from Lemma 1, that the unique solution of equation (16) satisfying $\lim\limits_{s\to\infty} f(s,r)=0$ is given by the formula:

$$f(s,r)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\varphi(s+2rn,r)+r}.$$

Since $\varphi(s,r)=s+\psi(r)$, we have

$$f(s,r)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\varphi(s+2rn,r)+r}=\sum_{n=0}^{\infty}\frac{(-1)^n}{\varphi(s,r)+2rn+r}=$$

$$=\frac{1}{r}\int_0^1 \frac{x^{\frac{\varphi(s,r)}{r}}dx}{1+x^2}=\frac{1}{2\varphi(s,r)+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{\varphi(s,r)}\right)},$$

by Lemma 4. $\square$

Corollary 4.1 *For $s>|r-1|$, $r>0$ functional equation (7) has a unique solution satisfying* $\lim\limits_{s\to 0} f_1(s,r)=0$ *which is*

$$f_1(s,r)=\frac{1}{2-2r+2s+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{1-r+s}\right)}. \tag{17}$$

Functional equation (8) also has a unique solution satisfying $\lim\limits_{s\to 0} f_2(s,r)=0$ *which is*

$$f_2(s,r)=\frac{1}{2r-2+2s+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{r-1+s}\right)}. \tag{18}$$

Proof. From the equality $\dfrac{1}{s+1}=\dfrac{1}{\varphi_1(s,r)+r}$ we obtain $\varphi_1(s,r)=s+1-r$. Since $\varphi_1(s,r)$ satisfies the conditions of Theorem 8, we obtain, that the continued fraction $f_1(s,r)$ is the solution of (7). By analogy, from the equality $\dfrac{1}{s+2r-1}=\dfrac{1}{\varphi_2(s,r)+r}$ we obtain $\varphi_2(s,r)=s+r-1$, which also satisfies the conditions of Theorem 8. Applying Theorem 8 again, we obtain that the continued fraction $f_2(s,r)$ is the solution of (8). $\square$

5 The exponential formula for generalized Brouncker's continued fraction

Theorem 5.1 *For $s > |r-1|$, $r > \dfrac{1}{2}$*

$$y(s,r) = s + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{(2n-1)^2 r^2 - (r-1)^2}{2s} \right) =$$

$$= 8\pi r 2^{1-\frac{1}{r}} \frac{\Gamma^2(\frac{1}{2r})}{\Gamma^4(\frac{1}{4r})} \cot(\frac{\pi}{4r}) \exp\left\{ \int_0^s f_1(t,r)dt \right\} \exp\left\{ \int_0^s f_2(t,r)dt \right\},$$

where the continued fractions $f_1(s,r)$ and $f_2(s,r)$ are defined by equations (17) and (18), respectively.

Proof. According to Corollary 3, the continued fractions $f_1(s,r)$ and $f_2(s,r)$ satisfy equations (7) and (8), respectively. Hence applying Lemma 2 we obtain, that

$$\frac{\partial}{\partial s}(\ln y)(s,r) = \frac{1}{2 - 2r + 2s + 2\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{n^2 r^2}{1-r+s} \right)} + \frac{1}{2r - 2 + 2s + 2\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{n^2 r^2}{r-1+s} \right)}.$$

Integrating the obtained differential equation, we get

$$\ln y(s,r) = \int_0^s (f_1(t,r) + f_2(t,r))dt + C(r)$$

$$y(s,r) = C(r) \exp\left\{ \int_0^s (f_1(t,r) + f_2(t,r))dt \right\},$$

where $C(r)$ is a function of r.

It is easy to see, that $C(r) = y(0,r)$. Let us calculate $y(0,r)$, using Theorem 4. At first let us recall some well-known formulas for the Gamma function. These are the diplication formula

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z),$$

and the Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}.$$

Since by definition $\Gamma(z+1) = z\Gamma(z)$, we have $\Gamma(1-z) = -z\Gamma(-z)$ and rewrite the Euler's reflection formula in the following form

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}. \tag{19}$$

Using simple calculations we obtain, that

$$y(0,r) = 4r \frac{\Gamma(\frac{2r+1}{4r})\Gamma(\frac{4r-1}{4r})}{\Gamma(\frac{1}{4r})\Gamma(\frac{2r-1}{4r})} = 4r \frac{\Gamma(\frac{1}{4r} + \frac{1}{2})\Gamma(1 - \frac{1}{4r})}{\Gamma(\frac{1}{4r})\Gamma(\frac{1}{2} - \frac{1}{4r})} = \ldots$$

Since

$$\Gamma(\frac{1}{2}+\frac{1}{4r})\Gamma(\frac{1}{4r}) = 2^{1-\frac{1}{2r}}\sqrt{\pi}\Gamma(\frac{1}{2r})$$

and

$$\Gamma(\frac{1}{2}-\frac{1}{4r})\Gamma(-\frac{1}{4r}) = 2^{1+\frac{1}{2r}}\sqrt{\pi}\Gamma(-\frac{1}{2r}),$$

we have

$$\ldots = 4r\frac{2^{1-\frac{1}{2r}}\sqrt{\pi}\Gamma(\frac{1}{2r})\Gamma(1-\frac{1}{4r})\Gamma(-\frac{1}{4r})}{\Gamma^2(\frac{1}{4r})2^{1+\frac{1}{2r}}\sqrt{\pi}\Gamma(-\frac{1}{2r})} = 4r\frac{\Gamma(\frac{1}{2r})\Gamma(1-\frac{1}{4r})\Gamma(\frac{1}{4r})\Gamma(-\frac{1}{4r})}{\Gamma^3(\frac{1}{4r})2^{\frac{1}{r}}\Gamma(-\frac{1}{2r})} =$$

$$= 4r\frac{\Gamma(\frac{1}{2r})\pi\Gamma(-\frac{1}{4r})}{\Gamma^3(\frac{1}{4r})2^{\frac{1}{r}}\Gamma(-\frac{1}{2r})\sin(\frac{\pi}{4r})} = \ldots$$

Using (19) we obtain

$$\ldots = 4r\frac{\Gamma(\frac{1}{2r})\pi\Gamma(-\frac{1}{4r})\Gamma(\frac{1}{4r})}{\Gamma^4(\frac{1}{4r})2^{\frac{1}{r}}\Gamma(-\frac{1}{2r})\sin(\frac{\pi}{4r})} = -4r\frac{\Gamma(\frac{1}{2r})\pi^2}{\Gamma^4(\frac{1}{4r})2^{\frac{1}{r}}\Gamma(-\frac{1}{2r})\sin(\frac{\pi}{4r})\frac{1}{4r}\sin(\frac{\pi}{4r})} =$$

$$= -16r^2\frac{\Gamma^2(\frac{1}{2r})\pi^2}{\Gamma^4(\frac{1}{4r})2^{\frac{1}{r}}\Gamma(-\frac{1}{2r})\Gamma(\frac{1}{2r})\sin(\frac{\pi}{4r})\sin(\frac{\pi}{4r})} = 16r^2\frac{\Gamma^2(\frac{1}{2r})\pi^2\frac{1}{2r}\sin(\frac{\pi}{2r})}{\Gamma^4(\frac{1}{4r})2^{\frac{1}{r}}\pi\sin(\frac{\pi}{4r})\sin(\frac{\pi}{4r})} =$$

$$= 8\pi r\frac{\Gamma^2(\frac{1}{2r})2\sin(\frac{\pi}{4r})\cos(\frac{\pi}{4r})}{\Gamma^4(\frac{1}{4r})2^{\frac{1}{r}}\sin(\frac{\pi}{4r})\sin(\frac{\pi}{4r})} = 8\pi r\frac{\Gamma^2(\frac{1}{2r})}{\Gamma^4(\frac{1}{4r})2^{\frac{1}{r}-1}}\cot(\frac{\pi}{4r}).$$

$\square$

Corollary 5.1 *For $s > 0$*

$$s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{(2n-1)^2}{2s}\right) = \frac{8\pi^2}{\Gamma^4(\frac{1}{4})}\exp\left\{\int_0^s \frac{dt}{t+\operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2}{t}\right)}\right\}.$$

Proof. Just put $r = 1$ and observe that

$$\frac{1}{2s+2\operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2}{s}\right)} + \frac{1}{2s+2\operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2}{s}\right)} = \frac{2}{2s+2\operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2}{s}\right)} = \frac{1}{s+\operatorname*{K}_{n=1}^{\infty}\left(\frac{n^2}{s}\right)}.$$

$\square$

Example. Putting $r = 2$ into the statement of Theorem 9 and calculating $\cot(\frac{\pi}{8}) = \dfrac{\sin(\frac{\pi}{4})}{1-\cos(\frac{\pi}{8})} = \sqrt{2}+1$, we obtain for $s > 1$

$$s + \operatorname*{K}_{n=1}^{\infty}\left(\frac{4(2n-1)^2-1}{2s}\right) = 16\pi(2+\sqrt{2})\frac{\Gamma^2(\frac{1}{4})}{\Gamma^4(\frac{1}{8})}\times$$

$$\times\exp\left\{\int_0^s \frac{dt}{2t-2+2\operatorname*{K}_{n=1}^{\infty}\left(\frac{4n^2}{t-1}\right)}\right\}\exp\left\{\int_0^s \frac{dt}{2t+2+2\operatorname*{K}_{n=1}^{\infty}\left(\frac{4n^2}{t+1}\right)}\right\}.$$

6 Generalized Brouncker's continued fraction and its asymptotic series

Let us recall the following lemma (see [1], p. 614, also [5], p. 150, Lemma 3.21).

Lemma 6.1 (Watson) *Let f be a function on $(0,+\infty)$, such that $|f(t)|<M$ for $t>\epsilon$ and $f(t)=\sum\limits_{k=0}^{\infty} c_k t^k$, $0<t<2\epsilon$. Then*

$$\int_0^{+\infty} f(t)e^{-st}dt \sim \sum_{k=0}^{\infty} \frac{k!c_k}{s^{k+1}}, \qquad s\to+\infty$$

is the asymptotic expansion for the Laplace transform of f.

Let us write the asymptotic expansions for both continued fractions (17) and (18). Applying Corollary 2, we get the following formulas for $s>|r-1|$, $r>0$:

$$\frac{1}{2-2r+2s+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{1-r+s}\right)} = \frac{1}{2r}\int_0^{+\infty}\frac{e^{-x\frac{1-r+s}{r}}dx}{\cosh x}; \tag{20}$$

$$\frac{1}{2r-2+2s+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{r-1+s}\right)} = \frac{1}{2r}\int_0^{+\infty}\frac{e^{-x\frac{r-1+s}{r}}dx}{\cosh x}. \tag{21}$$

Examine equation (20). Write the right-hand side of equation (20) in the following form:

$$\frac{1}{2r}\int_0^{+\infty}\frac{e^{-x\frac{1-r+s}{r}}dx}{\cosh x} = \frac{1}{2r}\int_0^{+\infty} e^{\frac{r-1}{r}x}\frac{1}{\cosh x}e^{-\frac{s}{r}x}dx. \tag{22}$$

Repeating the reasoning from [5], p. 92, we obtain:

$$\frac{1}{\cosh x} = \sum_{n=0}^{\infty}\frac{E_n}{n!}x^n,$$

where E_n are the Euler's numbers;

$$e^{\frac{r-1}{r}x} = \sum_{n=0}^{\infty}\frac{(r-1)^n}{r^n n!}x^n.$$

Using the rules of series multiplication, we get:

$$\frac{e^{\frac{r-1}{r}x}}{\cosh x} = \left(\sum_{n=0}^{\infty}\frac{(r-1)^n}{r^n n!}x^n\right)\left(\sum_{n=0}^{\infty}\frac{E_n}{n!}x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(r-1)^k}{r^k k!}\frac{E_{n-k}}{(n-k)!}\right)x^n.$$

Applying Watson's lemma 5 to (22) with $f(x)=\dfrac{e^{\frac{r-1}{r}x}}{\cosh x}$, we obtain:

$$\frac{1}{2r}\int_0^{+\infty}\frac{e^{-x\frac{1-r+s}{r}}dx}{\cosh x} \sim \frac{1}{2r}\sum_{n=0}^{\infty} n!\left(\sum_{k=0}^{n}\frac{(r-1)^k}{r^k k!}\frac{E_{n-k}}{(n-k)!}\right)\frac{r^{n+1}}{s^{n+1}}$$

as $s \to \infty$.

Since $\dfrac{n!}{k!(n-k)!} = \binom{n}{k}$, we have

$$\frac{1}{2-2r+2s+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{1-r+s}\right)} \sim \frac{1}{2}\sum_{n=0}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(r-1)^k r^{n-k}E_{n-k}}{s^{n+1}} \tag{23}$$

as $s \to \infty$.

Analogically, we obtain the following asymptotic expansion for (21):

$$\frac{1}{2r-2+2s+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{r-1+s}\right)} \sim \frac{1}{2}\sum_{n=0}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(1-r)^k r^{n-k}E_{n-k}}{s^{n+1}} \tag{24}$$

as $s \to \infty$.

Theorem 6.1 *The following asymptotic relation holds as $s \to +\infty$:*

$$s+\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{(2n-1)^2r^2-(r-1)^2}{2s}\right) \sim$$
$$\sim s\exp\left\{-\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{2n}{2k}(r-1)^{2k}r^{2(n-k)}E_{2(n-k)}}{2ns^{2n}}\right\}. \tag{25}$$

Proof. By Theorem 3, the left-hand side of (25) is divisible by $(s+1)$. Theorem 9 implies, that the continued fraction $y(s,r)$ can be written as

$$y(s,r) = (s+1)y(0,r)\exp\left\{\int_0^{+\infty}\gamma_1(t,r)dt\right\}\exp\left\{-\int_s^{+\infty}\gamma_1(t,r)dt\right\}\times$$
$$\times\exp\left\{\int_0^{+\infty}\gamma_2(t,r)dt\right\}\exp\left\{-\int_s^{+\infty}\gamma_2(t,r)dt\right\},$$

where

$$\gamma_1(t,r) = \frac{1}{2-2r+2t+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{1-r+t}\right)} - \frac{1}{2(1+t)},$$
$$\gamma_2(t,r) = \frac{1}{2r-2+2t+2\mathop{\mathrm{K}}\limits_{n=1}^{\infty}\left(\frac{n^2r^2}{r-1+t}\right)} - \frac{1}{2(1+t)}.$$

Using asymptotic expansions (23), (24) and the expansion

$$\frac{1}{(1+t)} = \frac{1}{t(1+\frac{1}{t})} \sim \frac{1}{t}\sum_{n=0}^{\infty}(-1)^n\frac{1}{t^n} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{t^{n+1}} \qquad t \to +\infty$$

we obtain

$$\gamma_1(t,r) \sim \frac{1}{2}\sum_{n=0}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(r-1)^k r^{n-k}E_{n-k}}{t^{n+1}} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{t^{n+1}} = \dots.$$

Since the numerator of the null's term in the first sum is equal to $E_0 = 1$,

$$\ldots = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(r-1)^k r^{n-k}E_{n-k}-(-1)^n}{t^{n+1}}. \qquad t\to+\infty$$

Analogically,

$$\gamma_2(t,r)\sim\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(1-r)^k r^{n-k}E_{n-k}-(-1)^n}{t^{n+1}}. \qquad t\to+\infty$$

Integrating this over $(s,\ +\infty)$, we obtain

$$\int_s^{+\infty}\gamma_1(t,r)dt\sim\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(r-1)^k r^{n-k}E_{n-k}-(-1)^n}{ns^n}. \qquad t\to+\infty$$

$$\int_s^{+\infty}\gamma_2(t,r)dt\sim\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(1-r)^k r^{n-k}E_{n-k}-(-1)^n}{ns^n}. \qquad t\to+\infty$$

Since $y(s,r)\sim s$ as $s\to+\infty$, we conclude that

$$y(0,r)\exp\left\{\int_0^{+\infty}\gamma_1(t,r)dt\right\}\exp\left\{\int_0^{+\infty}\gamma_2(t,r)dt\right\}=1$$

and

$$y(s,r)\sim(s+1)\exp\left\{-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(r-1)^k r^{n-k}E_{n-k}-(-1)^n}{ns^n}\right\}\times$$

$$\times\exp\left\{-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}(1-r)^k r^{n-k}E_{n-k}-(-1)^n}{ns^n}\right\}.$$

Using the equality $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{ns^n}=-\ln\left(\dfrac{s+1}{s}\right)$, as $s>1$, we obtain that

$$y(s,r)\sim s\exp\left\{-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{n}\binom{n}{k}((r-1)^k+(1-r)^k)r^{n-k}E_{n-k}}{ns^n}\right\}=$$

$$=\ [\text{ since}(1-r)^k=(-1)^k(r-1)^k]\ =$$

$$=s\exp\left\{-\sum_{n=1}^{\infty}\frac{\sum\limits_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}(r-1)^{2k}r^{n-2k}E_{n-2k}}{ns^n}\right\}.$$

The proof is completed by observing that all the Euler's numbers with odd parameters E_1, E_3, E_5, ... are equal to zero. $\square$

Example. Putting $r = 2$ we obtain

$$s + \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{4(2n-1)^2 - 1}{2s} \right) \sim s \exp \left\{ - \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{2n}{2k} 2^{2(n-k)} E_{2(n-k)}}{2ns^{2n}} \right\},$$

as $s \to +\infty$. Computations with the first few Euler's numbers $E_0 = 1$, $E_1 = 0$, $E_2 = -1$, $E_3 = 0$, $E_4 = 5$, $E_5 = 0$, $E_6 = -61$ shows that $\sum_{k=0}^{1} \binom{2}{2k} 2^{2(1-k)} E_{2(1-k)} = -3$ for $n = 1$, $\sum_{k=0}^{2} \binom{4}{2k} 2^{2(2-k)} E_{2(2-k)} = 57$ for $n = 2$ and $\sum_{k=0}^{3} \binom{6}{2k} 2^{2(3-k)} E_{2(3-k)} = -2763$ for $n = 3$.

So we have

$$s + \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{4(2n-1)^2 - 1}{2s} \right) \sim s \exp \left\{ \frac{3}{2s^2} - \frac{57}{4s^4} + \frac{2763}{6s^6} + O\left(\frac{1}{s^8}\right) \right\}.$$

Writing the first terms of the expansion of e^x

$$e^x \sim 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$$

and substituting $x = \frac{3}{2s^2} - \frac{57}{4s^4} + \frac{2763}{6s^6} + O\left(\frac{1}{s^8}\right)$ we obtain the first terms of the expansion:

$$s + \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{4(2n-1)^2 - 1}{2s} \right) \sim \left(1 + \frac{3}{2s^2} - \frac{105}{8s^4} + \frac{7035}{16s^6} + O\left(\frac{1}{s^8}\right) \right) =$$

$$= s + \frac{3}{2s} - \frac{105}{8s^3} + \frac{7035}{16s^5} + O\left(\frac{1}{s^7}\right).$$

7 Ramanujan's formula and its generalization

Our generalization of Ramanujan's formula (4) requires some preliminary results. The first of them is the following theorem.

Theorem 7.1 *Let $\varphi(s,r)$ be an arbitrary real-valued function of s and r. Then for $r > 0$, $\varphi(s,r) > r$*

$$\frac{1}{\varphi(s,r) - r^2 + \mathop{\mathrm{K}'}_{n=1}^{\infty} \left(\frac{4n^2}{1} \, + \, \frac{4n^2}{\varphi(s,r) - r^2} \right)} = \frac{1}{r^2} \int_0^{\infty} \frac{x e^{-x \frac{\varphi(s,r)}{r}}}{\cosh x} dx. \tag{26}$$

Proof. Examine equality (4) with the substitution $s := \frac{\varphi(s,r)}{r}$, where $\varphi(s,r)$ is an arbitrary real-valued function of s and r. Then we obtain the following formula for $\varphi(s,r) > r$, $r > 0$:

$$\frac{1}{\frac{\varphi^2(s,r)}{r^2} - 1 + \overset{\infty}{\underset{n=1}{\mathrm{K}'}} \left(\frac{4n^2}{1} + \frac{4n^2}{\frac{\varphi^2(s,r)}{r^2} - 1} \right)} = \int_0^\infty \frac{xe^{-x\frac{\varphi(s,r)}{r}}}{\cosh x} dx$$

$$\frac{1}{\varphi^2(s,r) - r^2 + r^2 \overset{\infty}{\underset{n=1}{\mathrm{K}'}} \left(\frac{4n^2}{1} + \frac{4n^2}{\frac{\varphi^2(s,r)}{r^2} - 1} \right)} = \frac{1}{r^2} \int_0^\infty \frac{xe^{-x\frac{\varphi(s,r)}{r}}}{\cosh x} dx.$$

Apply the equivalence transform with the parameters $r_0 = 1$, $r_n = r^2$, $n = 1, 2, \ldots$ to the continued fraction on the left-hand side. This results the formula:

$$\frac{1}{\varphi^2(s,r) - r^2 + \overset{\infty}{\underset{n=1}{\mathrm{K}'}} \left(\frac{4n^2r^4}{r^2} + \frac{4n^2r^4}{\varphi^2(s,r) - r^2} \right)} = \frac{1}{r^2} \int_0^\infty \frac{xe^{-x\frac{\varphi(s,r)}{r}}}{\cosh x} dx.$$

Using simple calculations:

$$\frac{1}{\varphi^2(s,r) - r^2 + \overset{\infty}{\underset{n=1}{\mathrm{K}'}} \left(\frac{4n^2r^2}{1} + \frac{4n^2r^2}{\varphi^2(s,r) - r^2} \right)} = \frac{1}{r^2} \int_0^\infty \frac{xe^{-x\frac{\varphi(s,r)}{r}}}{\cosh x} dx.$$

$\square$

Let us prove the following lemma, which describes the derivative of the continued fraction

$$f(s,r) = \frac{1}{\varphi(s,r) + \overset{\infty}{\underset{n=1}{\mathrm{K}}} \left(\frac{n^2r^2}{\varphi(s,r)} \right)}.$$

Lemma 7.1 *Let $\varphi(s,r) = s + \psi(r)$, where $\psi(r)$ is an arbitrary real-valued function of r. Then for $r > 0$, $s > r - \psi(r)$*

$$\frac{\partial}{\partial s} f(s,r) = -\frac{1}{\varphi^2(s,r) - r^2 + \overset{\infty}{\underset{n=1}{\mathrm{K}'}} \left(\frac{4n^2r^2}{1} + \frac{4n^2r^2}{\varphi^2(s,r) - r^2} \right)},$$

where

$$f(s,r) = \frac{1}{\varphi(s,r) + \overset{\infty}{\underset{n=1}{\mathrm{K}}} \left(\frac{n^2r^2}{\varphi(s,r)} \right)}.$$

Proof. Using Corollary 2, we obtain the equality

$$f(s,r) = \frac{1}{\varphi(s,r) + \overset{\infty}{\underset{n=1}{\mathrm{K}}} \left(\frac{n^2r^2}{\varphi(s,r)} \right)} = \frac{1}{r} \int_0^{+\infty} \frac{e^{-x\frac{\varphi(r,s)}{r}} dx}{\cosh x}.$$

Differentiating this equality by s and changing the sign, we obtain:

$$-\frac{\partial}{\partial s} f(s,r) = \frac{1}{r^2} \int_0^{+\infty} \frac{xe^{-\frac{\varphi(r,s)}{r}x} dx}{\cosh x},$$

which exactly coincide with the right-hand side of (26). $\square$

Corollary 7.1 *For $r>0$, $s>\max(1,\ 2r-1)$*

$$\frac{\partial}{\partial s}f_1(s,r)=-\frac{1}{2(1-2r+s)(1+s)+2\underset{n=1}{\overset{\infty}{\mathrm{K}'}}\left(\frac{4n^2r^2}{1}+\frac{4n^2r^2}{(1-2r+s)(1+s)}\right)},$$

where

$$f_1(s,r)=\frac{1}{2-2r+2s+2\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2r^2}{1-r+s}\right)}.$$

$$\frac{\partial}{\partial s}f_2(s,r)=-\frac{1}{2(2r-1+s)(s-1)+2\underset{n=1}{\overset{\infty}{\mathrm{K}'}}\left(\frac{4n^2r^2}{1}+\frac{4n^2r^2}{(2r-1+s)(s-1)}\right)},$$

where

$$f_2(s,r)=\frac{1}{2r-2+2s+2\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2r^2}{r-1+s}\right)}.$$

Example. Put $\varphi(s,r)=s+\sin r$, $r=\frac{\pi}{2}$. Then for $s>\frac{\pi}{2}-1$ we have

$$f'(s)=-\frac{1}{(s+1)^2-\frac{\pi^2}{4}+\underset{n=1}{\overset{\infty}{\mathrm{K}'}}\left(\frac{n^2\pi^2}{1}+\frac{n^2\pi^2}{(s+1)^2-\frac{\pi^2}{4}}\right)},$$

where

$$f(s)=\frac{2}{2s+2+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{n^2\pi^2}{2(s+1)}\right)}.$$

Theorem 7.2 *For $s>\max(1,\ 2r-1)$, $r>\frac{1}{2}$*

$$\frac{\partial^2}{\partial s^2}(\ln y)(s,r)=-\frac{1}{2r^2}\int_0^\infty\frac{x(e^{-\frac{1-r+s}{r}x}+e^{-\frac{r-1+s}{r}x})}{\cosh x}dx=-h_1(s,r)-h_2(s,r),$$

where

$$h_1(s,r)=\frac{1}{2(1-2r+s)(1+s)+2\underset{n=1}{\overset{\infty}{\mathrm{K}'}}\left(\frac{4n^2r^2}{1}+\frac{4n^2r^2}{(1-2r+s)(1+s)}\right)},$$

$$h_2(s,r)=\frac{1}{2(2r-1+s)(s-1)+2\underset{n=1}{\overset{\infty}{\mathrm{K}'}}\left(\frac{4n^2r^2}{1}+\frac{4n^2r^2}{(2r-1+s)(s-1)}\right)}.$$

Proof. The proof comes out from Equality 10 and Corollary 5. □

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