Computational Macroeconomics
Exercises

Week 2

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Contents

1 RBC model: preprocessing and steady-state in Dynare

Consider the basic Real Business Cycle (RBC) model with leisure. The representative household maximizes present as well as expected future utility

$$\max E_t \sum_{j=0}^{\infty} \beta^j U_{t+j}$$

with $\beta < 1$ denoting the discount factor and E_t is expectation given information at time t. The contemporaneous utility function

$$U_t = \gamma \log(C_t) + \psi \log(1 - L_t)$$

is additively separable and has two arguments: consumption C_t and labor L_t. The marginal utility of consumption is positive, whereas more labor reduces utility. Accordingly, γ is the consumption utility parameter and ψ the labor disutility parameter. In each period the household takes the real wage W_t as given and supplies perfectly elastic labor service to the representative firm. In return, she receives real labor income in the amount of $W_t L_t$ and, additionally, real profits DIV_t from the firm as well as revenue from lending capital K_{t-1} at interest rate R_t to the firms, as it is assumed that the firm and capital stock are owned by the household. Income and wealth are used to finance consumption C_t and investment I_t. In total, this defines the (real) budget constraint of the household:

$$C_t + I_t = W_t L_t + R_t K_{t-1} + DIV_t$$

The law of motion for capital K_t at the end of period t is given by

$$K_t = (1 - \delta) K_{t-1} + I_t$$

and δ is controling depreciations. [1] Assume that the transversality condition is full-filled.

Productivity A_t is the driving force of the economy and evolves according to

$$\log A_t = \rho_A \log A_{t-1} + \varepsilon_t^A$$

where ρ_A denotes the persistence parameter and ε_t^A is assumed to be normally distributed with mean zero and variance σ_A^2.

Real profits DIV_t of the representative firm are revenues from selling output Y_t minus costs from labor $W_t L_t$ and renting capital $R_t K_{t-1}$:

$$DIV_t = Y_t - W_t L_t - R_t K_{t-1}$$

The representative firm maximizes expected profits

$$\max E_t \sum_{j=0}^{\infty} \beta^j Q_{t+j} DIV_{t+j}$$

subject to a Cobb-Douglas production function

$$f(K_{t-1}, L_t) = Y_t = A_t K_{t-1}^{\alpha} L_t^{1-\alpha}$$

The discount factor takes into account that firms are owned by the household, i.e. $\beta^j Q_{t+j}$ is the present value of a unit of consumption in period $t+j$ or, respectively, the marginal utility of an additional unit of profit; therefore $Q_{t+j} = \frac{\partial U_{t+j}/\partial C_{t+j}}{\partial U_t/\partial C_t}$.

Finally, we have the non-negativity constraints $K_t \geq 0$, $C_t \geq 0$ and $0 \leq L_t \leq 1$ and clearing of the labor as well as goods market in equilibrium, i.e.

$$Y_t = C_t + I_t$$

[1] Note that we use the end-of-period timing for capital, i.e. K_t instead of K_{t+1}, because the investment decision is done in period t and hence capital is also determined in t. In older papers you will often find beginning-of-period timing for capital, so always think about when it is decided and determined.

1. Briefly provide intuition behind the transversality condition.
2. Show that the first-order conditions of the representative household are given by

$$U_t^C = \beta E_t \left[U_{t+1}^C \left(1 - \delta + R_{t+1}\right) \right]$$
$$W_t = -\frac{U_t^L}{U_t^C}$$

where $U_t^C = \gamma C_t^{-1}$ and $U_t^L = \frac{-\psi}{1-L_t}$. Interpret these equations in economic terms.

3. Show that the first-order conditions of the representative firm are given by

$$W_t = f_L$$
$$R_t = f_K$$

where $f_L = (1-\alpha) A_t \left(\frac{K_{t-1}}{L_t}\right)^{\alpha}$ and $f_K = \alpha A_t \left(\frac{K_{t-1}}{L_t}\right)^{1-\alpha}$. Interpret these equations in economic terms.

4. Derive the steady-state of the model, in the sense that there is a set of values for the endogenous variables that in equilibrium remain constant over time.
5. Discuss how to calibrate the following parameters α, β, δ, γ, ψ, ρ_A and σ_A.
6. Write a script for this RBC model with a feasible calibration for an OECD country that computes the steady-state of the model.
7. Write a Dynare mod file for this RBC model with a feasible calibration for an OECD country and compute the steady-state of the model by using a `steady_state_model` block. Compare this to the steady-state computed above.
8. Now assume a contemporaneous utility function of the CRRA (constant Relative Risk Aversion) type: [2]

$$U_t = \gamma \frac{C_t^{1-\eta_C} - 1}{1-\eta_C} + \psi \frac{(1-L_t)^{1-\eta_L} - 1}{1-\eta_L}$$

   a) Derive the model equations and steady-state analytically.
   b) Write a script to compute the steady-state for this model.
   c) Write a Dynare mod file and compute the steady-state for this model by using a helper function in the `steady_state_model` block.

Readings

- McCandless (2008).

[2] Note that due to L'Hopital's rule $\eta_C = \eta_L = 1$ implies the original specification, $U_t = \gamma \log C_t + \psi \log(1 - L_t)$.

2 Fiscal Policy in General Equilibrium: preprocessing and steady-state

Model description

Consider a version of the Baxter and King (1993) model.

Households Let C_t denote real consumption, N_t real labor supply and K_t private capital. The representative household maximizes its expected life-time utility

$$\max_{C_t,N_t,K_{t-1}} E_t \sum_{t=0}^{\infty} \beta^t \left[log(C_t) + \theta_l log(1-N_t) + \Gamma(G_t^B, I_t^B) \right]$$

subject to

$$C_t + I_t = (1-\tau_t)(w_t N_t + r_t K_{t-1}) + TR_t$$

where β denotes the discount rate, θ_l the Frisch elasticity of labor, w_t the real wage, r_t the real interest rate and TR_t real lump-sum transfers. $\Gamma(G_t^B, I_t^B)$ is a general function of public consumption G_t^B and public investment I_t^B such that it is non-decreasing in each of its arguments. Optimality is given by the consumption-leisure choice

$$(1-\tau_t)w_t = \theta_l \frac{C_t}{1-N_t} \tag{1}$$

and the savings decision

$$\lambda_t = \beta E_t \left\{ \lambda_{t+1} \left[(1-\delta) + (1-\tau_{t+1}) r_{t+1} \right] \right\} \tag{2}$$

where

$$\lambda_t = \frac{1}{C_t} \tag{3}$$

denotes marginal utility of consumption. The private and public capital stocks evolve according to

$$K_t = (1-\delta)K_{t-1} + I_t \tag{4}$$

$$K_t^B = (1-\delta)K_{t-1}^B + I_t^B \tag{5}$$

with δ denoting the depreciation rate.

Firms Firms maximize profits by choosing factor inputs according to

$$\max_{N_t,K_{t-1}} Y_t - w_t N_t - r_t K_{t-1}$$

subject to

$$Y_t = z_t (K_{t-1}^B)^{\eta} (K_{t-1})^{\alpha} (N_t)^{1-\alpha} \tag{6}$$

where η denotes productivity of public capital and α the share of capital in production. Taking factor prices as given, factor demand is determined by

$$w_t N_t = (1-\alpha) Y_t \tag{7}$$

$$r_t K_{t-1} = \alpha Y_t \tag{8}$$

Productivity evolves according to

$$\log\left(\frac{z_t}{\bar{z}}\right) = \rho_z \log\left(\frac{z_{t-1}}{\bar{z}}\right) + \varepsilon_t^z \tag{9}$$

where ρ_z is a smoothing parameter and $\varepsilon_t^z \overset{iid}{\sim} N(0, \sigma_z^2)$.

Fiscal authority The fiscal authority faces the budget constraint

$$G_t^B + I_t^B + TR_t = \tau_t(w_t N_t + r_t K_{t-1}) \tag{10}$$

and its behavior is described by exogenous AR(1) processes

$$G_t^B - \bar{G^B} = \rho_{G^B}\left({G^B}_{t-1} - \bar{G^B}\right) + \varepsilon_t^{G^B} \tag{11}$$

$$I_t^B - \bar{I^B} = \rho_{I^B}\left({I^B}_{t-1} - \bar{I^B}\right) + \varepsilon_t^{I^B} \tag{12}$$

$$\log\left(\frac{\tau_t}{\bar{\tau}}\right) = \rho_\tau \log\left(\frac{\tau_{t-1}}{\bar{\tau}}\right) + \varepsilon_t^\tau \tag{13}$$

where $\rho_{G^B}, \rho_{I^B}, \rho_\tau$ are smoothing parameters and

$$\varepsilon_t^{G^B} \overset{iid}{\sim} N(0, \sigma_{G^B}^2), \qquad \varepsilon_t^{I^B} \overset{iid}{\sim} N(0, \sigma_{I^B}^2), \qquad \varepsilon_t^\tau \overset{iid}{\sim} N(0, \sigma_\tau^2)$$

Notice that the inclusion of TR_t implies a balanced budget rule, i.e. there is no debt in the model.

Market clearing Market clearing implies that whatever is consumed by households must be produced

$$Y_t = C_t + I_t + G_t^B + I_t^B \tag{14}$$

Summary Overall, the Baxter and King (1993) model can be summarized through equations (1)-(23).

Exercises

Willi, a fellow student, wants to assess how changes in fiscal policy (taxation & spending) affect the real economy.

1. From his class on introductory macroeconomics Willi remembers that as a first step it is always important to distinguish between endogenous and exogenous variables as well as model parameters. Can you help him with that?

2. Willi is quite clever about calibrating the model parameters. In particular, he is interested in targeting steady-state values of the model:

Target	Symbol	Value
steady-state output level	$\bar{Y}$	1
steady-state public consumption	$\bar{G^B}$	$0.2\bar{Y}$
steady-state public investment	$\bar{I^B}$	$0.02\bar{Y}$
steady-state transfers	$\bar{TR}$	0
steady-state real wage	$\bar{w}$	2
steady-state labor supply	$\bar{N}$	1/3

   Furthermore, he thinks that public-capital productivity should be **lower** than the capital share in production. Regarding the exogenous processes he would like mild persistence (ρ's equal to 0.75) and small shock standard errors of 1%. Can you provide a calibration for all model parameters meeting his targets and economic intuition?
   Hint: First, set some reasonable values for β, δ and η. Then begin with the target values and try to derive the steady-states of all other endogenous variables and the implied parameter values.

3. Willi has heard of the powerful toolbox DYNARE, so he asks you to help him set up this model in DYNARE. Write a DYNARE mod-file for this model, commenting each step such that Willi clearly understands each block.

Readings

- Baxter and King (1993)

3 Matrix Algebra

Let

$$A = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.1 & 0.1 & 0.3 \\ 0 & 0.2 & 0.3 \end{pmatrix} \qquad \Sigma_u = \begin{pmatrix} 2.25 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0.5 & 0.74 \end{pmatrix} \qquad R = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

1. Compute the eigenvalues of A. What would this imply for the system $y_t = c + Ay_{t-1} + u_t$ with u_t being white noise?
2. Consider the matrices D: $m \times n$, E: $n \times p$ and F: $p \times k$. Show that

   $$vec(DEF) = (F' \otimes D)\, vec(E),$$

   where $\otimes$ is the Kronecker product and *vec* the vectorization operator.
3. Show that R is an orthogonal matrix. Why is this matrix called a rotation matrix?
4. Compute a regular lower triangular matrix $W \in \mathbb{R}^{3\times3}$ and a diagonal matrix $\Sigma_\varepsilon \in \mathbb{R}^{3\times3}$ such that $\Sigma_u = W\Sigma_\varepsilon W'$.
   Hint: Use the Cholesky factorization $\Sigma_u = PP' = W\Sigma_\varepsilon^{\frac{1}{2}}(W\Sigma_\varepsilon^{\frac{1}{2}})'$.
5. Solve the discrete Lyapunov matrix equation $\Sigma_y = A\Sigma_y A' + \Sigma_u$ using
   a) the Kronecker product and vectorization
   b) the following iterative algorithm:

   $$\begin{aligned} \Sigma_{y,0} &= I, A_0 = A, \Sigma_{u,0} = \Sigma_u \\ \Sigma_{y,i+1} &= A_i \Sigma_{y,i} A_i' + \Sigma_{u,i} \\ \Sigma_{u,i+1} &= A_i \Sigma_{u,i} A_i' + \Sigma_{u,i} \\ A_{i+1} &= A_i A_i \end{aligned}$$

   Write a loop until either a maximal number of iterations (say 500) is reached or each element of $\Sigma_{y,i+1} - \Sigma_{y,i}$ is less than 10^{-25} in absolute terms.
   c) Compare both approaches for A and Σ_u given above.

Readings:

- E. W. Anderson et al. (1996, Ch. 4.2)
- B. D. O. Anderson and Moore (2005, Ch. 6.7)
- Uribe and Schmitt-Grohe (2017, Ch. 4.10)

4 Law Of Large Numbers

Let $Y_1, Y_2, \ldots$ be an i.i.d. sequence of arbitrarily distributed random variables with finite variance σ_Y^2 and expectation μ. Define the sequence of random variables

$$\overline{Y}_T = \frac{1}{T}\sum_{t=1}^{T} Y_t$$

1. Briefly outline the intuition behind the "law of large numbers".
2. Write a program to illustrate the law of large numbers for uniformly distributed random variables (you may also try different distributions such as normal, gamma, geometric, student's t with finite or infinite variance). To this end, do the following:
   - Set $T = 10000$ and initialize the $T \times 1$ output vector u.
   - Choose values for the parameters of the uniform distribution. Note that $E[u] = (a + b)/2$, where a is the lower and b the upper bound of the uniform distribution.
   - For $t = 1, ..., T$ do the following computations:
     – Draw t random variables from the uniform distribution with lower bound a and upper bound b.
     – Compute and store the mean of the drawn values in your output vector at position t.
   - Plot your output vector and add a line to indicate the theoretical mean of the uniform distribution.
3. Now suppose that the sequence $Y_1, Y_2, \ldots$ is an $AR(1)$ process:

   $$Y_t = \phi Y_{t-1} + \varepsilon_t$$

   where $\varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$ is not necessarily normally distributed and $|\phi| < 1$. Illustrate numerically that the law of large numbers still holds despite the intertemporal dependence.

Readings

- Ploberger (2010)
- White (2001, Ch. 3)

References

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