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The Jermann (1998)-model[1]

1 Firms

1.1 Technology growth

Technology X_t grows deterministically with $X_t/X_{t-1} = \gamma$. In the following, we will abuse notation and not employ a different notation for detrended variables.

1.2 Firm Problem

Firm maximizes the sum of dividends

$$D_t = A_t K_t^\alpha (X_t N_t)^{1-\alpha} - W_t N_t - I_t \tag{1}$$

subject to the LOM

$$K_{t+1} = (1-\delta) K_t + \left(\frac{b}{1-a} \left(\frac{I_t}{K_t} \right)^{1-a} + c \right) K_t \tag{2}$$

The parameters a and c need to determined to satisfy steady state relationships. Detrending yields:

$$D_t = A_t K_t^\alpha N_t^{1-\alpha} - W_t N_t - I_t \tag{3}$$

and

$$\gamma K_{t+1} = (1-\delta) K_t + \left(\frac{b}{1-a} \left(\frac{I_t}{K_t} \right)^{1-a} + c \right) K_t \tag{4}$$

Using the households SDF, the firm maximizes

$$\max E_t \sum_{t=0}^{\infty} (\beta^*)^k \frac{\Lambda_{t+k}}{\Lambda_t} \begin{pmatrix} A_{t+k} K_{t+k}^\alpha N_{t+k}^{1-\alpha} - W_{t+k} N_{t+k} - I_{t+k} \\ -q_{t+k} \left(\gamma K_{t+k+1} - (1-\delta) K_{t+k} - \left(\frac{b}{1-a} \left(\frac{I_{t+k}}{K_{t+k}} \right)^{1-a} + c \right) K_{t+k} \right) \end{pmatrix} \tag{5}$$

FOC Capital

$$\begin{aligned} \frac{\partial L}{\partial K_{t+k+1}} =& (\beta^*)^k \frac{\Lambda_{t+k}}{\Lambda_t} (-q_{t+k}\gamma) \\ &+ E_t (\beta^*)^{k+1} \frac{\Lambda_{t+k+1}}{\Lambda_t} \begin{bmatrix} \alpha A_{t+k+1} K_{t+k+1}^{\alpha-1} N_{t+k+1}^{1-\alpha} \\ -q_{t+k+1} \left(-(1-\delta) - \frac{ba}{1-a} I_{t+k+1}^{1-a} K_{t+k+1}^{a-1} - c \right) \end{bmatrix} = 0 \end{aligned} \tag{6}$$

Simplified

[1] I want to thank Stefan Schaefer and Roland von Campe for pointing out errors in previous versions.

$$\gamma\Lambda_{t+k}q_{t+k}=\beta^*\Lambda_{t+k+1}\left(\alpha A_{t+k+1}K_{t+k+1}^{\alpha-1}N_{t+k+1}^{1-\alpha}+E_t q_{t+k+1}\left((1-\delta)+\frac{ba}{1-a}I_{t+k+1}^{1-a}K_{t+k+1}^{a-1}+c\right)\right) \tag{7}$$

Investment FOC

$$\frac{\partial L}{\partial I_{t+k}}=(\beta^*)^k\frac{\Lambda_{t+k}}{\Lambda_t}\left(1-q_{t+k}\left(-bI_{t+k}^{-a}K_{t+k}^{a}\right)\right)=0 \tag{8}$$

Simplified

$$1=q_{t+k}bI_{t+k}^{-a}K_{t+k}^{a} \tag{9}$$

or

$$q_t=\frac{1}{b}\left(\frac{I_t}{K_t}\right)^a \tag{10}$$

Labor FOC

$$W_t=(1-\alpha)\,A_tK_t^{\alpha}N_t^{1-\alpha} \tag{11}$$

Pricing follows

$$1=\beta^*E_t\frac{\Lambda_{t+1}}{\Lambda_t}x_t\ , \tag{12}$$

where x_t is the payoff of any asset. For capital, from (7)

$$1=\beta^*E_t\frac{\Lambda_{t+1}}{\Lambda_t}\frac{1}{\gamma q_t}\left(\alpha A_{t+1}K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha}+q_{t+1}\left((1-\delta)+\frac{ba}{1-a}I_{t+1}^{1-a}K_{t+1}^{a-1}+c\right)\right)=0 \tag{13}$$

2 Household

The household maximizes

$$\max E_t\sum_{k=0}^{\infty}\beta^k\left\{\frac{(C_{t+k}-hC_{t+k-1})^{1-\tau}}{1-\tau}\right\}=E_t\sum_{k=0}^{\infty}\underbrace{\left(\beta\gamma^{1-\tau}\right)^k}_{\beta^*}\left\{\frac{\left(C_{t+k}-\frac{h}{\gamma}C_{t+k-1}\right)^{1-\tau}}{1-\tau}\right\} \tag{14}$$

subject to

$$\left[W_{t+k}+a'_{t+k}\left(V_{t+k}^a+D_{t+k}^a\right)=C_{t+k}+a'_{t+k+1}V_{t+k}^a\right], \tag{15}$$

where a is the amount of assets held, V the value of the asset and D its dividend. This equation is already in detrended form. The FOCs are

$$\frac{\partial L}{\partial C_t}=\left(C_t-\frac{h}{\gamma}C_{t-1}\right)^{-\tau}-\Lambda_t-\beta^*\frac{h}{\gamma}\left(C_{t+1}-\frac{h}{\gamma}C_t\right)^{-\tau}=0 \tag{16}$$

$$\Lambda_t=\left(C_t-\frac{h}{\gamma}C_{t-1}\right)^{-\tau}-\beta^*\frac{h}{\gamma}\left(C_{t+1}-\frac{h}{\gamma}C_t\right)^{-\tau} \tag{17}$$

$$\frac{\partial L}{\partial a_{t+k}}=-(\beta^*)^k\Lambda_{t+k}V_{t+k}^a+(\beta^*)^{k+1}E_{t+k}\Lambda_{t+k+1}\left(V_{t+k+1}^a+D_{t+k+1}^a\right)=0 \tag{18}$$

$$\Lambda_{t+k}V_{t+k}^a=\beta\Lambda_{t+k+1}\left(V_{t+k+1}^a+D_{t+k+1}^a\right)\ , \tag{19}$$

where $N=1$ has been imposed.

For an one-period risk free bond, this implies

$$1 = \beta^* E_t \frac{\Lambda_{t+1}}{\Lambda_t} (1 + r^*) \tag{20}$$

For a firm share, this implies

$$\Lambda_t = \beta^* E_t \Lambda_{t+1} \frac{\left(V_{t+1}^k + D_{t+1}\right)}{V_t^k} \tag{21}$$

while for a consol that in perpetuity pays the risk free rate, we have

$$\Lambda_t = \beta^* E_t \Lambda_{t+1} \frac{(V_{t+1}^c + r^*)}{V_t^c} \tag{22}$$

3 Market Clearing

As usual

$$Y = C + I \tag{23}$$

All assets except for firm shares are in zero net supply, i.e. a=0 except for a^k which must be 1 $\forall t$, implying

$$W_t + \left(V_t^k + D_t^k\right) = C_t + V_t^k \Rightarrow W_t + D_t = C_t \tag{24}$$

4 Steady State

In steady state, we have

$$K = \frac{1}{\gamma} (1 - \delta) K + \frac{1}{\gamma} \underbrace{\left(\frac{b}{1-a} \left(\frac{I}{K} \right)^{1-a} + c \right) K}_{I} \tag{25}$$

Thus, we need

$$\frac{I}{K} = \delta + \gamma - 1 \,, \tag{26}$$

i.e. the investment capital ratio can be computed. From the investment FOC with Tobin's $q = 1$ imposed, we have

$$b I_t^{-a} K_t^a = 1 \Rightarrow b = \left(\frac{I}{K} \right)^a \tag{27}$$

Moreover, the LOM implies

$$\left(\frac{b}{1-a} \left(\frac{I}{K} \right)^{1-a} + const \right) = \frac{I}{K} \tag{28}$$

which allows solving for *const*:

$$const = \frac{I}{K} - \frac{b}{1-a} \left(\frac{I}{K} \right)^{1-a} \tag{29}$$

Finally, the capital FOC implies

$$\Lambda_t q_t \gamma = \beta^* \Lambda_{t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + q_{t+1} \left((1-\delta) + c + \frac{ba}{1-a} I_{t+1}^{1-a} K_{t+1}^{a-1} \right) \right) \tag{30}$$

$$\gamma = \beta^* \left(\alpha K^{\alpha-1} + \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) \right) \tag{31}$$

$$\frac{\gamma}{\beta^*} = \left(\alpha K^{\alpha-1} + \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) \right) \tag{32}$$

$$\alpha K^{\alpha-1} = \frac{\gamma}{\beta^*} - \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) \tag{33}$$

$$K = \left[\frac{1}{\alpha} \left(\frac{\gamma}{\beta^*} - \left((1-\delta) + c + \frac{ba}{1-a} \left(\frac{I}{K} \right)^{1-a} \right) \right) \right]^{\frac{1}{\alpha-1}} \tag{34}$$

From I/K investment immediately follows. Moreover,

$$W = (1-\alpha) K^{\alpha} \tag{35}$$

$$D = K^{\alpha} - W - I \tag{36}$$

and

$$\Lambda_{t+k} V_{t+k}^a = \beta^* \Lambda_{t+k+1} \left(V_{t+k+1}^a + D_{t+k+1}^a \right) \tag{37}$$

$$\frac{1}{\beta} = \left(1 + \frac{D}{V} \right) \tag{38}$$

$$\left(\frac{1}{\beta^*} - 1 \right) V = D \tag{39}$$

$$\Lambda = \left(C - \frac{h}{\gamma} C \right)^{-\tau} \left(1 - \beta \frac{h}{\gamma} \right) \tag{40}$$

$$W_t + D_t = C_t \tag{41}$$

$$\left(C_t \left(1 - \frac{h}{\gamma} \right) \right)^{-\tau} = \Lambda_t \tag{42}$$

5 Mapping a and ξ

ξ is the elasticity of

$$q = \frac{1}{b} \left(\frac{I}{K} \right)^a \tag{43}$$

$$\log q = \log \left(\frac{1}{b} \right) + a \log \left(\frac{I}{K} \right) \tag{44}$$

$$\log \left(\frac{I}{K} \right) = -\frac{1}{a} \log \left(\frac{1}{b} \right) + \frac{1}{a} \log q \tag{45}$$

$$\frac{\partial \log \left(\frac{I}{K} \right)}{\partial \log q} = \frac{1}{a} = \xi \tag{46}$$