Sorting with a stack

(math proof)

All numbers in this document are integers

Let L be an ordered list of N books numbered from 1 to N based on their size in increasing order.

Example : $[1, 3, 4, 2]$ (which is different from $[1, 4, 3, 2]$)

Thus, there are $N!$ different possible lists.

We dispose of three lists of books to sort a list of books:

I, the starting list, M, the middle list and F, the final list.

The goal is to have a sorted F list by starting with a list L, respecting the following rules:

-We can only put a book from the right end of I to the right end of M or to the right end of F.

-We can only put a book from the right end of M to the right end of F.

Example: Let's sort this list $L = [2, 3, 1]$:

Beginning : $I = [2, 3, 1]; M = [\,]; F = [\,]$
Step 1 : $I = [2, 3]; M = [\,]; F = [1]$
Step 2 : $I = [2]; M = [3]; F = [1]$
Step 3 : $I = [\,]; M = [3]; F = [1, 2]$
Step 4 : $I = [\,]; M = [\,]; F = [1, 2, 3]$

Question I : Which types of lists can be sorted ?

Notation: Let $L_{x\rightarrow}$ be the sub-list of L holding all books to the right of x (excluded). Let $L_{\leftarrow x}$ be the sub-list of L holding all books to the left of x (excluded).

Furthermore, let $L_{x\leftrightarrow y}$ be the sub-list of L holding all books in between x and y (both excluded)

Example: Let $L = [1, 4, 3, 5, 2]$, we thus have $L_{\leftarrow 3} = [1, 4]$ and $L_{3\rightarrow} = [5, 2]$

Conjecture: Criteria of non-sortability:

Let L be a list of N books,
L is not sortable ⇔ $\exists\ A, B, C \in L$ such that $A \in L_{\leftarrow B},\ C \in L_{B\rightarrow}$ and that $A < C < B$.

Corollary of the conjecture: Criteria of sortability:

Let L be a list of N books,
L is sortable ⇔ $\nexists\ A, B, C \in L$ such that $A \in L_{\leftarrow B},\ C \in L_{B\rightarrow}$ and that $A < C < B$:

In other words: $\forall\ B\ \in L,\ \ \forall\ C\ \in L_{B\rightarrow}$,

We have either $C > B$,

or $C < B$ ***AND*** $C < \min\ (L_{\leftarrow B})$. (Because if $\forall\ A\ \in L_{\leftarrow B}, C < A \Rightarrow\ C < \min\ (L_{\leftarrow B})$)

Proof of sufficiency:

For $N < 3$: The lists $[\,]$, $[1]$, $[1, 2]$ and $[2, 1]$ are sortable.

For $N \geq 3$: Let L be a list N books, and $A, B, C\ \in L$, such that $A \in L_{\leftarrow B},\ C \in L_{B\rightarrow}$ and that $A < C < B$.

We can split L in 7 different regions:

$L_{\leftarrow A},\ A,\ L_{A\leftrightarrow B},\ B,\ L_{B\leftrightarrow C}, C$, et $L_{C\rightarrow}$ ($L_{\leftarrow A},\ L_{A\leftrightarrow B},\ L_{B\leftrightarrow C}$, et $L_{C\rightarrow}$ can be empty)

$L_{\leftarrow A}$	A	$L_{A\leftrightarrow B}$	B	$L_{B\leftrightarrow C}$	C	$L_{C\rightarrow}$

Let's now sort L :

Beginning: $I = [L_{\leftarrow A},\ A,\ L_{A\leftrightarrow B},\ B,\ L_{B\leftrightarrow C}, C, L_{C\rightarrow}\ ]; M = [\ ]; F = [\ ]$

First, we sort $L_{C\rightarrow}$. If $1 \in L_{C\rightarrow}$, we can put at least one book in F, otherwise, we put all books in M. (Because our goal is to have a sorted list in increasing order starting at 1 at the end, 1 has to be the first book to go into F)

Step 1: $I = [L_{\leftarrow A},\ A,\ L_{A\leftrightarrow B},\ B,\ L_{B\leftrightarrow C}, C,]; M = [L_{C\rightarrow}]; F = [\ 1?, \dots]$

Afterwards, we sort C. $C > A$ thus C must at least wait for A to be in F so that C can be put into F. Thus, C goes into M.

Step 2: $I = [L_{\leftarrow A},\ A,\ L_{A\leftrightarrow B},\ B,\ L_{B\leftrightarrow C}]; M = [L_{C\rightarrow},\ C]; F = [\ 1?, \dots]$

Next, we sort $L_{B\leftrightarrow C}$. We sort it in the same way we sorted $L_{C\rightarrow}$.

Step 3 : $I = [L_{\leftarrow A},\ A,\ L_{A\leftrightarrow B},\ B]; M = [L_{C\rightarrow},\ C,\ L_{B\leftrightarrow C}]; F = [\ 1?, \dots]$

Next, we sort B. $B > A$, thus like C, B has to wait for A to be in F.

Step 4 : $I = [L_{\leftarrow A},\ A,\ L_{A\leftrightarrow B}]; M = [L_{C\rightarrow},\ C,\ L_{B\leftrightarrow C},\ B]; F = [\ 1?, \dots]$

Next, we sort $L_{A\leftrightarrow B}$ de la même manière que $L_{B\leftrightarrow C}$.

Step 5 : $I = [L_{\leftarrow A},\ A]; M = [L_{C\rightarrow},\ C,\ L_{B\leftrightarrow C},\ B,\ L_{A\leftrightarrow B}]; F = [\ 1?, \dots]$

Next, we sort A. $A < B$ and $A < C$, thus A doesn't have to wait for neither B, or C. Thus, we can put it into F if the book put last into F is $(A - 1)$ or if $A = 1$. Otherwise, we put A into M.

Step 5: $I = [L_{\leftarrow A}]; M = [L_{C\rightarrow},\ C,\ L_{B\leftrightarrow C},\ B,\ L_{A\leftrightarrow B},\ A]; F = [\ 1?, \dots]$

or

Step 5: $I = [L_{\leftarrow A}]; M = [L_{C\rightarrow},\ C,\ L_{B\leftrightarrow C},\ B,\ L_{A\leftrightarrow B}]; F = [\ 1?, \dots\ ,\ A]$

In either case, we now only have to sort $L_{\leftarrow A}$. We'll sort it in the same way we sorted $L_{A\leftrightarrow B}$.

Step 6 : $I = [\ ]; M = [L_{C\rightarrow},\ C,\ L_{B\leftrightarrow C},\ B,\ L_{A\leftrightarrow B,}\ A,\ L_{\leftarrow A}]; F = [\ 1?, \dots]$

or

Step 6 : $I = [\ ]; M = [L_{C\rightarrow},\ C,\ L_{B\leftrightarrow C},\ B,\ L_{A\leftrightarrow B},\ L_{\leftarrow A}]; F = [\ 1?, \dots\ ,$ $A, \dots\ ]$

Now that I is empty, the only possible move is to put all books from the right end of M at the right end of F.

Final step $I = [\,]; M = [\,]; F = [\,1, \dots, A, \dots, B, \dots, C, \dots\,]$

However, at the end, we still have the triplet A, B, C in the same order. But we know that $A < C < B$. Thus the list F isn't ordered, and thus L isn't sortable.

Proof of necessity:

Let's suppose $\exists\ L'$ non-sortable and not obeying the criteria of non-sortability. It then obeys its opposite, in other words:
$\forall\ B \in L', \ \forall\ C \in L'_{B\rightarrow}$,
we have either $C > B$,
or $C < B$ $\boldsymbol{AND}$ $C < \min\,(L'_{\leftarrow B})$.

With $\min(L'_{\leftarrow B}) = +\infty$ if $L_{\leftarrow B}$ is empty for convenience.

Let B be the book directly to the left of the rightmost book, named C, we can split L' into 3 parts:

$L'_{\leftarrow B}$	B	C

Let's now try to sort L'.

Beginning : $I = [L'_{\leftarrow B},\ B,\ C\,]; M = [\,]; F = [\,]$

First, we sort C.

if $C > B$, C has to be put into M, because B must be in F so that C can be put there.

If $C < B$ $\boldsymbol{AND}$ $C < \min\,(L'_{\leftarrow B})$, we can put it directly into F because it doesn't have to wait for any book in $L'_{\leftarrow C}$. This also means that the rightmost book with this property (aka the first book to be sorted with this property) will be $C = 1$. And thus, $\forall\, X \in L'_{1\rightarrow},\ X < L'_{X\rightarrow}$ et $X > L'_{1\leftrightarrow X}$. (Every number to the right of 1 is in increasing order). Next, we check if the rightmost book of M is the next book $(C + 1)$. If so, we can put it into F, and then we check again with the next book of M. Otherwise, we change to the next C. This also means that until now, all books put into M are in decreasing order.

Step 1 :

If $C = 1,\ I = [L'_{\leftarrow B},\ B]\,; M = [\,]; F = [1]$

Otherwise, $I = [L'_{\leftarrow B},\ B]\,; M = [C]; F = [\,]$

Next, we sort B. If we redo step 1 (now with $C = B$ and B the book directly to the left of C) until we have $C < B$ ***ET*** $C < \min\,(L'_{\leftarrow B})$ (aka $C = 1$), we have $I = [L'_{\leftarrow B},\ B]\,; M = [\ldots(ordre\ décroissante)]; F = [1]$ because all books to the right are in decreasing order. Next, if we continue redoing step 1, we have at the end the following:

$I = [\,]\,; M = [\,N, \ldots(ordre\ décroissante)]; F = [1, \ldots(ordre\ croissante)]$

Next, if we put all books of M into F, we thus have

$F = [1, 2, 3 \ldots, N]\ (increasing\ order)$. Thus, at the end, the list is sorted. However, L' is supposed to be non-sortable, which is absurd. Thus, the supposition from the start is false, $\nexists\, L'$ non sortable not obeying the criteria of non-sortability. The criteria of non-sortability is thus necessary.

By consequent, we also proved that its opposite is the criteria of sortability:

L is sortable $\Leftrightarrow \forall\, B \in L,\ \ \forall\, C \in L_{B\rightarrow}$,
we have either $C > B$,
or $C < B$ ***AND*** $C < \min\,(L_{\leftarrow B})$. With $\min([\,]) = +\infty$

Question II : How many lists of N books can be sorted?

Notation: Let L_N be the number of sortable lists of length N. Let L_N^K be the number of sortable lists of length N ending with K,with $1 \leq K \leq N$

We thus have $L_0 = 1$ because there is only one list with 0 books (the empty list), and it is sortable. $L_1 = 1$ for the same reason.

A list of length N will always end with a K such that $1 \leq K \leq N$, we thus have :
$L_N = L_N^1 + L_N^2 + \cdots + L_N^{N-1} + L_N^N = \sum_{i=1}^{N} L_N^i$

Let L be a list of length N,

Let M_k be the book at the position k, we thus have:

$$L = \boxed{L_{\leftarrow M_{N-1}}} \boxed{M_{N-1}} \boxed{M_N}$$

To respect the criteria of sortability proven in Question I, we have either:

$M_{N-1} < M_N$, or $M_{N-1} > M_N$ et $M_N < \min(L_{\leftarrow M_{N-1}})$.

If $M_N > 1$, $M_{N-1} < M_N$ because if $M_{N-1} > M_N$, $1 \in L_{\leftarrow M_{N-1}}$ and thus $\min\left(L_{\leftarrow M_{N-1}}\right) = 1$, which is impossible ($L$ is sortable).

If $M_{N-1} = 1$, and $M_N > 1$, thus $M_{N-2} \in [2; N] \setminus M_N$.

In this case, if $M_{N-2} > M_N$, $\exists\, K \in L_{\leftarrow M_{N-2}}$ such that $K < M_N$.
This would mean that the list wouldn't be sortable. (Criteria of non-sortability, with
$A = K$, $B = M_{N-2}$, $C = M_N$.

This would mean that if L ends with K, $L_{M_{N-(K-1)} \leftrightarrow K}$ has to be a sortable list of length $K-1$. Thus, there are L_{K-1} of such possible lists.

With a similar reasoning, $L_{\leftarrow M_{N-(K-1)}}$ is a sortable list of length $N-K$ with its smallest book being $K+1$. Furthermore, if we subtract K from all of these books, we have a normal sortable list of length $N-K$. Thus there are L_{N-K} of such possible lists.

$$L = [L_{\leftarrow M_{N-(K-1)}}, L_{M_{N-K} \leftrightarrow M_N}, M_N]$$

Thus, $L_N^K = L_{N-K} \times L_{K-1}$

And so, $L_N = L_N^1 + L_N^2 + \cdots + L_N^{N-1} + L_N^N = \sum_{i=1}^{N} L_N^i$

$$= L_{N-1} \times L_0 + L_{N-2} \times L_1 + \cdots + L_1 \times L_{N-2} + L_0 \times L_{N-1}$$

$$= \sum_{i=1}^{N} L_{N-1-i} \times L_i$$

If we instead work with $N+1$ as it is usual to do when working with series, we have: $L_{N+1} = \sum_{i=0}^{N} L_{N-i} \times L_i$

And this is simply the recursive formula for Catalan's number, $C(n)$

Thus, $L_N = C(n)$

Question III : What happens when we allow sorting in series (sorting a list after we sort it once) ?

If L is sortable, by sorting it, we'll get a sorted list. And a sorted list is always sortable, simply by putting every book $K \in L$ into M, and then every K into F.

If L isn't sortable, it means it obeys the criteria of non-sortability ($A < C < B$). Furthermore, we can notice that in the first proof of the Question I, when we sort it, we still have the triplet in the same order at the end. Thus, a non-sortable list will never be sortable, no matter how many times we sort it.

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