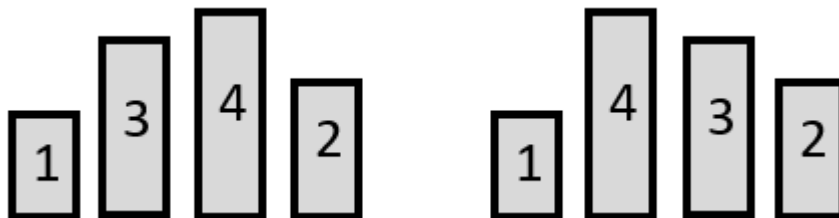


Sorting with a stack (math proof)

All numbers in this document are integers

Let L be an ordered list of N books numbered from 1 to N based on their size in increasing order.

Example : $[1, 3, 4, 2]$ (which is different from $[1, 4, 3, 2]$)



Thus, there are $N!$ different possible lists.

We dispose of three lists of books to sort a list of books:

I , the starting list, M , the middle list and F , the final list.

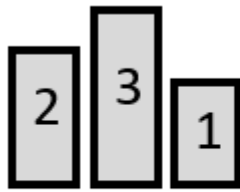
The goal is to have a sorted F list by starting with a list L , respecting the following rules:

- We can only put a book from the right end of I to the right end of M or to the right end of F .

- We can only put a book from the right end of M to the right end of F .

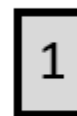
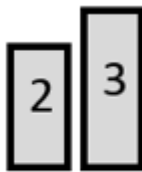
Example: Let's sort this list $L = [2, 3, 1]$:

Beginning : $I = [2, 3, 1]; M = []; F = []$



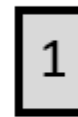
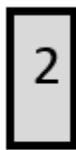
I	M	F
-----	-----	-----

Step 1 : $I = [2, 3]; M = []; F = [1]$



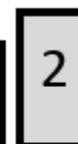
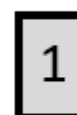
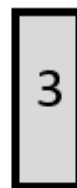
I	M	F
-----	-----	-----

Step 2 : $I = [2]; M = [3]; F = [1]$



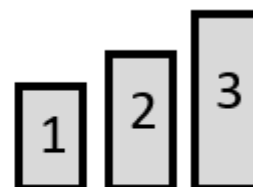
I	M	F
-----	-----	-----

Step 3 : $I = []; M = [3]; F = [1, 2]$



I	M	F
-----	-----	-----

Step 4 : $I = []; M = []; F = [1, 2, 3]$



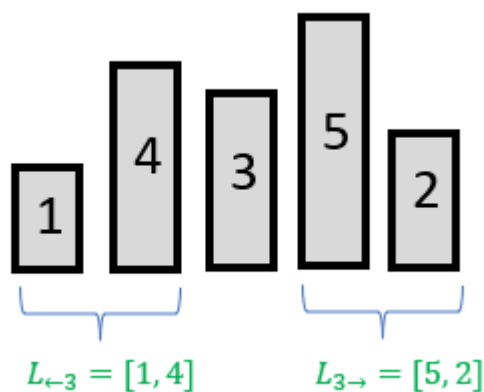
I	M	F
-----	-----	-----

Question I : Which types of lists can be sorted ?

Notation: Let $L_{x \rightarrow}$ be the sub-list of L holding all books to the right of x (excluded). Let $L_{\leftarrow x}$ be the sub-list of L holding all books to the left of x (excluded).

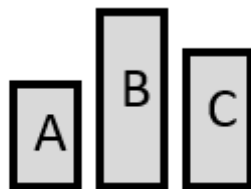
Furthermore, let $L_{x \leftrightarrow y}$ be the sub-list of L holding all books in between x and y (both excluded)

Example: Let $L = [1, 4, 3, 5, 2]$, we thus have $L_{\leftarrow 3} = [1, 4]$ and $L_{3 \rightarrow} = [5, 2]$



Conjecture: Criteria of non-sortability:

Let L be a list of N books,
 L is not sortable $\Leftrightarrow \exists A, B, C \in L$ such that $A \in L_{\leftarrow B}$, $C \in L_{B \rightarrow}$, and that $A < C < B$.



Corollary of the conjecture: Criteria of sortability:

Let L be a list of N books,
 L is sortable $\Leftrightarrow \nexists A, B, C \in L$ such that $A \in L_{\leftarrow B}$, $C \in L_{B \rightarrow}$ and that $A < C < B$:

In other words: $\forall B \in L, \forall C \in L_{B \rightarrow}$,

We have either $C > B$,

or $C < B$ **AND** $C < \min(L_{\leftarrow B})$. (Because if $\forall A \in L_{\leftarrow B}, C < A \Rightarrow C < \min(L_{\leftarrow B})$)

Proof of sufficiency:

For $N < 3$: The lists $[], [1], [1, 2]$ and $[2, 1]$ are sortable.

For $N \geq 3$: Let L be a list N books, and $A, B, C \in L$, such that $A \in L_{\leftarrow B}$, $C \in L_{B \rightarrow}$ and that $A < C < B$.

We can split L in 7 different regions:

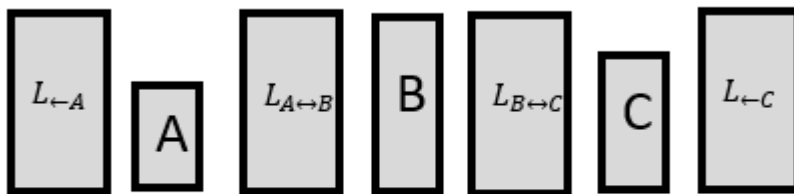
$L_{\leftarrow A}$, A , $L_{A \leftrightarrow B}$, B , $L_{B \leftrightarrow C}$, C , et $L_{C \rightarrow}$ ($L_{\leftarrow A}$, $L_{A \leftrightarrow B}$, $L_{B \leftrightarrow C}$, et $L_{C \rightarrow}$ can be empty)

$L_{\leftarrow A}$	A	$L_{A \leftrightarrow B}$	B	$L_{B \leftrightarrow C}$	C	$L_{C \rightarrow}$
--------------------	-----	---------------------------	-----	---------------------------	-----	---------------------

Let's now sort L :

Beginning: $I = [L_{\leftarrow A}, A, L_{A \leftrightarrow B}, B, L_{B \leftrightarrow C}, C, L_{C \rightarrow}]; M = []; F = []$

I :



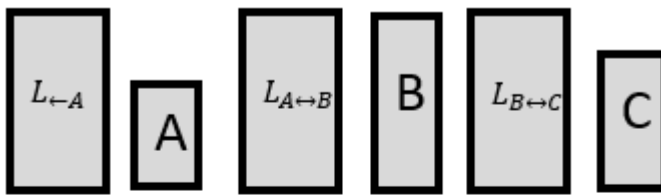
M :

F :

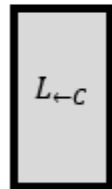
First, we sort $L_{C \rightarrow}$. If $1 \in L_{C \rightarrow}$, we can put at least one book in F , otherwise, we put all books in M . (Because our goal is to have a sorted list in increasing order starting at 1 at the end, 1 has to be the first book to go into F)

Step 1: $I = [L_{\leftarrow A}, A, L_{A \leftrightarrow B}, B, L_{B \leftrightarrow C}, C]; M = [L_{C \rightarrow}]; F = [1?, \dots]$

I:



M:



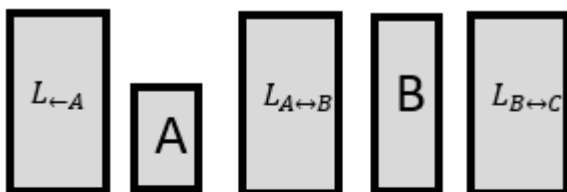
F:



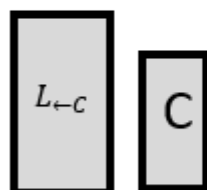
Afterwards, we sort C . $C > A$ thus C must at least wait for A to be in F so that C can be put into F . Thus, C goes into M .

Step 2: $I = [L_{\leftarrow A}, A, L_{A \leftrightarrow B}, B, L_{B \leftrightarrow C}]; M = [L_{C \rightarrow}, C]; F = [1?, \dots]$

I:



M:



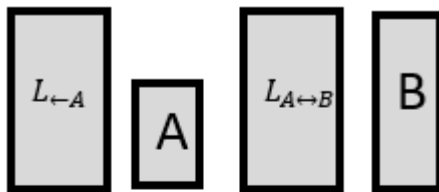
F:



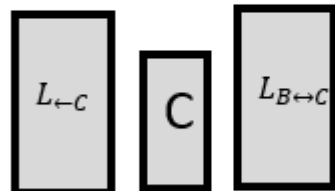
Next, we sort $L_{B \leftrightarrow C}$. We sort it in the same way we sorted $L_{C \rightarrow}$.

Step 3: $I = [L_{\leftarrow A}, A, L_{A \leftrightarrow B}, B]; M = [L_{C \rightarrow}, C, L_{B \leftrightarrow C}]; F = [1?, \dots]$

$I:$



$M:$



$F:$



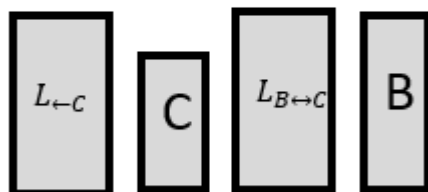
Next, we sort B . $B > A$, thus like C , B has to wait for A to be in F .

Step 4: $I = [L_{\leftarrow A}, A, L_{A \leftrightarrow B}]; M = [L_{C \rightarrow}, C, L_{B \leftrightarrow C}, B]; F = [1?, \dots]$

$I:$



$M:$



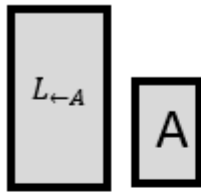
$F:$



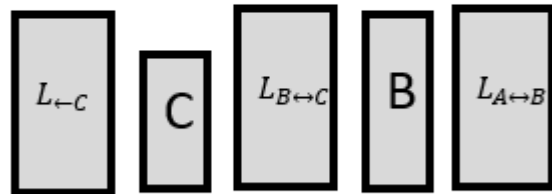
Next, we sort $L_{A \leftrightarrow B}$ de la même manière que $L_{B \leftrightarrow C}$.

Step 5: $I = [L_{\leftarrow A}, A]; M = [L_{C \rightarrow}, C, L_{B \leftrightarrow C}, B, L_{A \leftrightarrow B}]; F = [1?, \dots]$

I:



M:



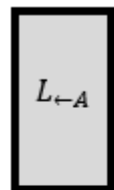
F:



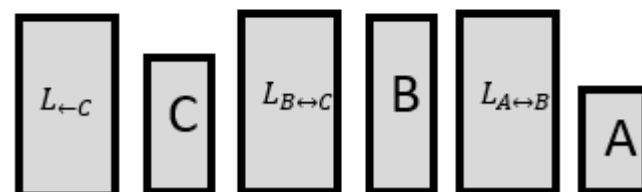
Next, we sort A . $A < B$ and $A < C$, thus A doesn't have to wait for neither B , or C . Thus, we can put it into F if the book put last into F is $(A - 1)$ or if $A = 1$. Otherwise, we put A into M .

Step 5: $I = [L_{\leftarrow A}]; M = [L_{C \rightarrow}, C, L_{B \leftrightarrow C}, B, L_{A \leftrightarrow B}, A]; F = [1?, \dots]$

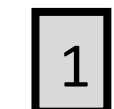
I:



M:



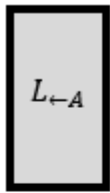
F:



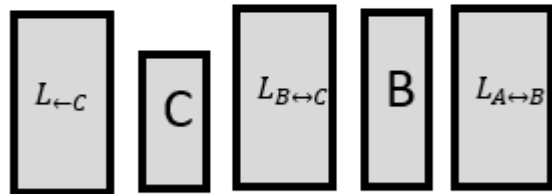
or

Step 5: $I = [L_{\leftarrow A}]; M = [L_{C \rightarrow}, C, L_{B \leftrightarrow C}, B, L_{A \leftrightarrow B}]; F = [1?, \dots, A]$

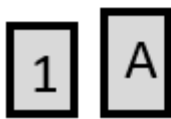
$I:$



$M:$



$F:$

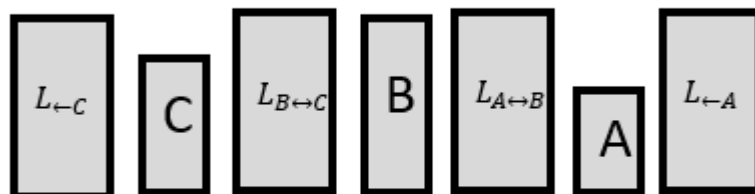


In either case, we now only have to sort $L_{\leftarrow A}$. We'll sort it in the same way we sorted $L_{A \leftrightarrow B}$.

Step 6 : $I = []; M = [L_{C \rightarrow}, C, L_{B \leftrightarrow C}, B, L_{A \leftrightarrow B}, A, L_{\leftarrow A}]; F = [1?, \dots]$

$I:$

$M:$



$F:$

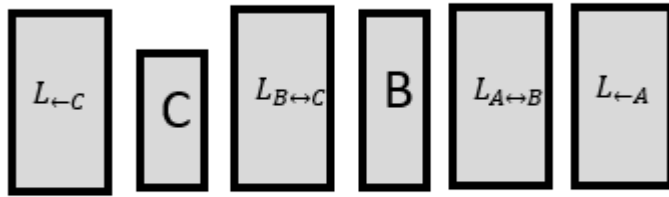


or

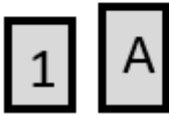
Step 6 : $I = []; M = [L_{C \rightarrow}, C, L_{B \leftrightarrow C}, B, L_{A \leftrightarrow B}, L_{\leftarrow A}]; F = [1?, \dots, A, \dots]$

I :

M:



F:



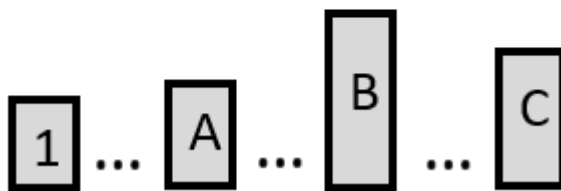
Now that *I* is empty, the only possible move is to put all books from the right end of *M* at the right end of *F*.

Final step $I = []; M = []; F = [1, \dots, A, \dots, B, \dots, C, \dots]$

I :

M:

F :



However, at the end, we still have the triplet A, B, C in the same order. But we know that $A < C < B$. Thus the list *F* isn't ordered, and **thus *L* isn't sortable**.

Proof of necessity:

Let's suppose $\exists L'$ non-sortable and not obeying the criteria of non-sortability. It then obeys its opposite, in other words:

$\forall B \in L', \forall C \in L'_{B \rightarrow},$

we have either $C > B$,

or $C < B$ **AND** $C < \min(L'_{\leftarrow B})$.

With $\min(L'_{\leftarrow B}) = +\infty$ if $L'_{\leftarrow B}$ is empty for convenience.

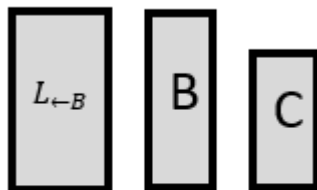
Let B be the book directly to the left of the rightmost book, named C , we can split L' into 3 parts:

$L'_{\leftarrow B}$	B	C
---------------------	-----	-----

Let's now try to sort L' .

Beginning : $I = [L'_{\leftarrow B}, B, C]; M = []; F = []$

$I :$



$M :$

$F :$

First, we sort C .

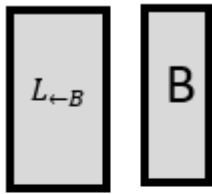
if $C > B$, C has to be put into M , because B must be in F so that C can be put there.

If $C < B$ **AND** $C < \min(L'_{\leftarrow B})$, we can put it directly into F because it doesn't have to wait for any book in $L'_{\leftarrow C}$. This also means that the rightmost book with this property (aka the first book to be sorted with this property) will be $C = 1$. And thus, $\forall X \in L'_{1 \rightarrow}, X < L'_{X \rightarrow}$ et $X > L'_{1 \leftarrow X}$. (Every number to the right of 1 is in increasing order). Next, we check if the rightmost book of M is the next book ($C + 1$). If so, we can put it into F , and then we check again with the next book of M . Otherwise, we change to the next C . This also means that until now, all books put into M are in decreasing order.

Step 1 :

If $C = 1$, $I = [L'_{\leftarrow B}, B]; M = []; F = [1]$

$I :$



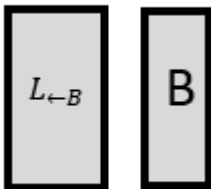
$M :$

$F :$

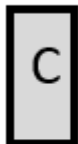


Otherwise, $I = [L' \leftarrow B, B]; M = [C]; F = []$

$I :$



$M :$



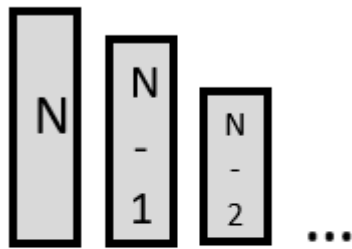
$F :$

Next, we sort B . If we redo step 1 (now with $C = B$ and B the book directly to the left of C) until we have $C < B$ **ET** $C < \min(L' \leftarrow B)$ (aka $C = 1$), we have $I = [L' \leftarrow B, B]; M = [... (ordre décroissante)]; F = [1]$ because all books to the right are in decreasing order. Next, if we continue redoing step 1, we have at the end the following:

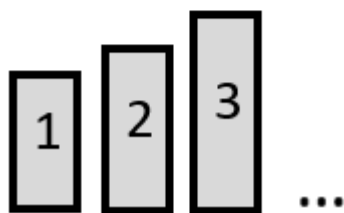
$I = []; M = [N, ... (ordre décroissante)]; F = [1, ... (ordre croissante)]$

$I :$

$M :$



$F :$



Next, if we put all books of M into F , we thus have

$F = [1, 2, 3 \dots, N]$ (increasing order). Thus, at the end, the list is sorted. However, L' is supposed to be non-sortable, which is absurd. Thus, the supposition from the start is false, $\nexists L'$ non sortable not obeying the criteria of non-sortability. The criteria of non-sortability is thus necessary.

By consequent, we also proved that its opposite is the criteria of sortability:

L is sortable $\Leftrightarrow \forall B \in L, \forall C \in L_{B \rightarrow},$
we have either $C > B,$
or $C < B$ **AND** $C < \min(L_{\leftarrow B})$. With $\min([\]) = +\infty$

Question II : How many lists of N books can be sorted?

Notation: Let L_N be the number of sortable lists of length N . Let L_N^K be the number of sortable lists of length N ending with K , with $1 \leq K \leq N$

We thus have $L_0 = 1$ because there is only one list with 0 books (the empty list), and it is sortable. $L_1 = 1$ for the same reason.

A list of length N will always end with a K such that $1 \leq K \leq N$, we thus have :

$$L_N = L_N^1 + L_N^2 + \dots + L_N^{N-1} + L_N^N = \sum_{i=1}^N L_N^i$$

Let L be a list of length N ,

Let M_k be the book at the position k , we thus have:

$$L = \boxed{L_{\leftarrow M_{N-1}} \mid M_{N-1} \mid M_N}$$

To respect the criteria of sortability proven in Question I, we have either:

$$M_{N-1} < M_N, \text{ or } M_{N-1} > M_N \text{ et } M_N < \min(L_{\leftarrow M_{N-1}}).$$

If $M_N > 1$, $M_{N-1} < M_N$ because if $M_{N-1} > M_N$, $1 \in L_{\leftarrow M_{N-1}}$ and thus $\min(L_{\leftarrow M_{N-1}}) = 1$, which is impossible (L is sortable).

If $M_{N-1} = 1$, and $M_N > 1$, thus $M_{N-2} \in [2; N] \setminus M_N$.

In this case, if $M_{N-2} > M_N$, $\exists K \in L_{\leftarrow M_{N-2}}$ such that $K < M_N$.

This would mean that the list wouldn't be sortable. (Criteria of non-sortability, with

$$A = K, B = M_{N-2}, C = M_N.$$

This would mean that if L ends with K , $L_{M_{N-(K-1)} \leftrightarrow K}$ has to be a sortable list of length $K - 1$. Thus, there are L_{K-1} of such possible lists.

With a similar reasoning, $L_{\leftarrow M_{N-(K-1)}}$ is a sortable list of length $N - K$ with its smallest book being $K + 1$. Furthermore, if we subtract K from all of these books, we have a normal sortable list of length $N - K$. Thus there are L_{N-K} of such possible lists.

$$L = [L_{\leftarrow M_{N-(K-1)}}, L_{M_{N-K} \leftrightarrow M_N}, M_N]$$

$$\text{Thus, } L_N^K = L_{N-K} \times L_{K-1}$$

$$\begin{aligned} \text{And so, } L_N &= L_N^1 + L_N^2 + \dots + L_N^{N-1} + L_N^N = \sum_{i=1}^N L_N^i \\ &= L_{N-1} \times L_0 + L_{N-2} \times L_1 + \dots + L_1 \times L_{N-2} + L_0 \times L_{N-1} \\ &= \sum_{i=1}^N L_{N-1-i} \times L_i \end{aligned}$$

If we instead work with $N + 1$ as it is usual to do when working with series, we have: $L_{N+1} = \sum_{i=0}^N L_{N-i} \times L_i$

And this is simply the recursive formula for Catalan's number, $C(n)$

$$\text{Thus, } L_N = C(n)$$

Question III : What happens when we allow sorting in series (sorting a list after we sort it once) ?

If L is sortable, by sorting it, we'll get a sorted list. And a sorted list is always sortable, simply by putting every book $K \in L$ into M , and then every K into F .

If L isn't sortable, it means it obeys the criteria of non-sortability ($A < C < B$). Furthermore, we can notice that in the first proof of the Question I, when we sort it, we still have the triplet in the same order at the end. Thus, a non-sortable list will never be sortable, no matter how many times we sort it.

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2023