Untitled

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Due to the observation data, we got the following statistics:

$$\sum \delta_i^C = 10$$

$$\sum \delta_i^H = 15$$

$$\sum x_i^C = 1233$$

$$\sum x_i^H = 1526$$

Thus, the likelihood function will be:

$$L(\theta, \tau | a, b, c, d, y) \propto \theta^{25} \tau^{15} exp[-1233\theta - 1526\tau\theta]$$

Multiply the proposed joint prior distribution, we get the joint posterior distribution like:

$$q(\theta, \tau | a, b, c, d, y) \propto \theta^{25+a} \tau^{15+b} exp[-(1233 + c)\theta - (1526 + d)\tau\theta]$$

where $\theta \in (0, 1), \tau \in (0, \infty)$.

We do a logit transformation on $\theta \in (0, 1)$ to $u \in (-\infty, \infty)$ by $\theta = \frac{exp(u)}{1+exp(u)}, u = ln(\frac{\theta}{1-\theta})$, and the corresponding Jacobian is $J = \frac{exp(u)}{[1+exp(u)]^2}$.

Thus, we could rewrite the joint posterior distribution like:

$$q(u, \tau | a, b, c, d, y) \propto [\frac{exp(u)}{1 + exp(u)}]^{25+a} \tau^{15+b} exp[-(1233 + c + (1526 + d)\tau) \frac{exp(u)}{1 + exp(u)}] \times \frac{exp(u)}{[1 + exp(u)]^2}$$

Adapted from the posterior distribution, the log target distribution is like:

$$log(q(u, \tau | a, b, c, d, y)) = Const + (26+a)u - (27+a)log[1+exp(u)] + (15+b)log\tau - (1233+c+1526\tau+d\tau) \frac{exp(u)}{1 + exp(u)}$$

Accordingly, those conditional distirbution for parameters and hyperparameters are:

$$f(u|\cdot) \propto [\frac{exp(u)}{1 + exp(u)}]^{25+a} exp[-(1233 + c + (1526 + d)\tau) \frac{exp(u)}{1 + exp(u)}] \times \frac{exp(u)}{[1 + exp(u)]^2}$$

and it does not look like any commonly seen classic distribution.

$$f(\tau|\cdot) \propto \tau^{15+b} exp[-(1526 + d) \frac{exp(u)}{1 + exp(u)} \tau] \sim \Gamma(16 + b,\ rate = (1526 + d) \frac{exp(u)}{1 + exp(u)}\ )$$

Note: $|\cdot$ denotes conditional on all other parameters and data.

Now, the implementation of this hybrid Gibbs sampler works like:

Step 1: Sample $\theta^{(0)}$ from $\sim$ Unif(0,1), and let $u(0) = log[\frac{\theta^{(0)}}{1-\theta^{(0)}}]$;

Set $\tau^{(0)} = 1$;

Set all hyperparameters with the reasonable values suggested by physicians, $(a, b, c, d) = (3, 1, 60, 120)$.

Step 2: Given $u^{(t)}, \tau^{(t)}$,

Sample $\tau^{(t+1)}$ from $\sim \Gamma(16 + b,\ rate = (1526 + d)\frac{exp(u^{(t)})}{1+exp(u^{(t)})}\ )$;

Sample $u^{(t+1)}$ under M-H algorithm with a $\sim$ N(0, 0.5^2) random walk proposal distribution because the conditional distribution u follows does not have a closed form.

Transfer back to $\theta^{(t+1)}$ by $\theta^{(t+1)} = \frac{exp(u^{(t+1)})}{1+exp(u^{(t+1)})}$

The r code to implement this hybrid Gibbs sampler is as follows:

```
niter = 100000                      # number of iterations
theta = rep(0,niter)                # define (hyper)parameters
tau   = rep(0,niter)
a     = rep(0,niter)
b     = rep(0,niter)
c     = rep(0,niter)
d     = rep(0,niter)

# Gibbs Sampler Implementation
set.seed(1)                                   # set random seed

theta[1] = runif(1, min = 0, max = 1)         # Initialization of (hyper)parameters
tau[1]   = runif(1, min = 0, max = 1)
a[1]     = 3
b[1]     = 1
c[1]     = 60
d[1]     = 120

for (i in 2:niter) {                         # for-loop for Gibbs sampler
   # update d
   d[i]<-rexp(1, rate = tau[i-1]*theta[i-1])
   # update c
   c[i]<-rexp(1, rate = theta[i-1])
   # update b
   b[i]<-rexp(1, rate = -log(tau[i-1]))
   # update a
   a[i]<-rexp(1, rate = -log(theta[i-1]))
   # update tau in (0,1)
   temp <- rgamma(1,shape = 16+b[i], rate = theta[i-1]*(1526+d[i]))
   tau[i] <- exp(temp)/(exp(temp)+1)
   # update theta in (0,1)
   temp <- rgamma(1,shape = 26+a[i], rate = tau[i]*(1526+d[i])+1233+c[i])
   theta[i] <- exp(temp)/(exp(temp)+1)
}
```

Appendix:

```
# H treatment group
data1=c(2,1,4,1,6,1,9,1,9,1,9,1,13,1,14,1,18,1,23,1,31,1,32,1,33,1,34,1,43,1,10,0,14,0,14,0,16,0,17,0,1
```

```
yh=matrix(data = data1,53,2,byrow = T)
# Control group
data2=c(1,1,4,1,6,1,7,1,13,1,24,1,25,1,35,1,35,1,39,1,1,0,1,0,3,0,4,0,5,0,8,0,10,0,11,0,13,0,14,0,14,0,
yc=matrix(data = data2, ncol=2,byrow = T)

# summary stats
n.h.r=sum(yh[,2])
n.c.r=sum(yc[,2])
t.h= sum(yh[,1])
t.c= sum(yc[,1])
```