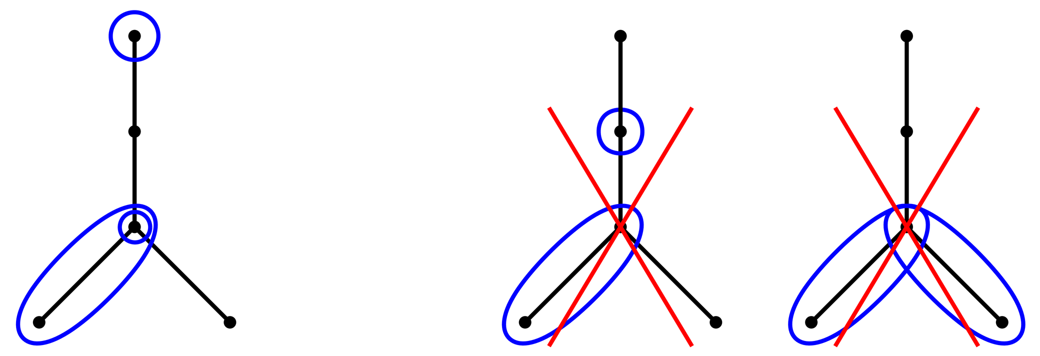
Generalizing Graph Associahedra: Hypercube Graph Associahedra

Jordan Almeter   North Carolina State University

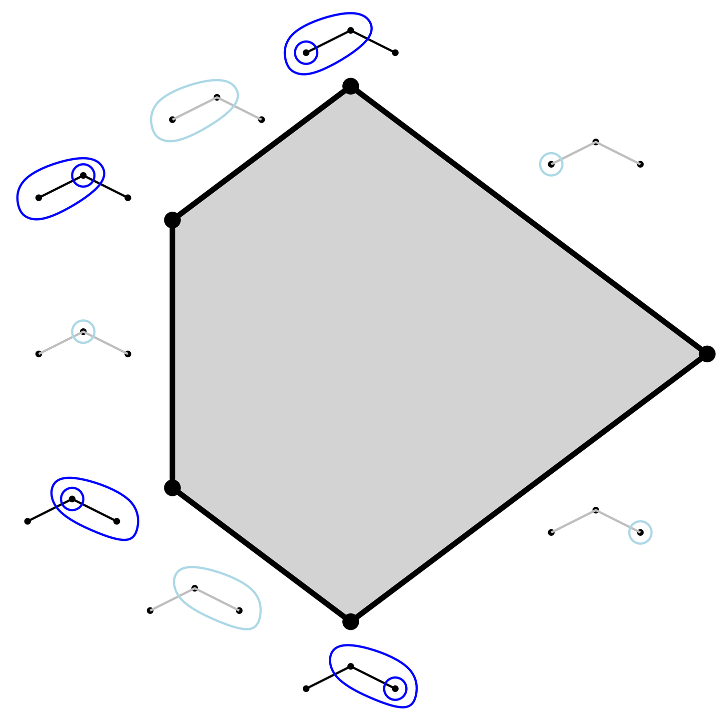
Graph Associahedra (C., D. 2006)

Given a graph G on $n+1$ vertices, a *tube* is a subset of vertices which induces a connected graph. Two tubes are *compatible* if they are either nested, or if they are disjoint and not adjacent.

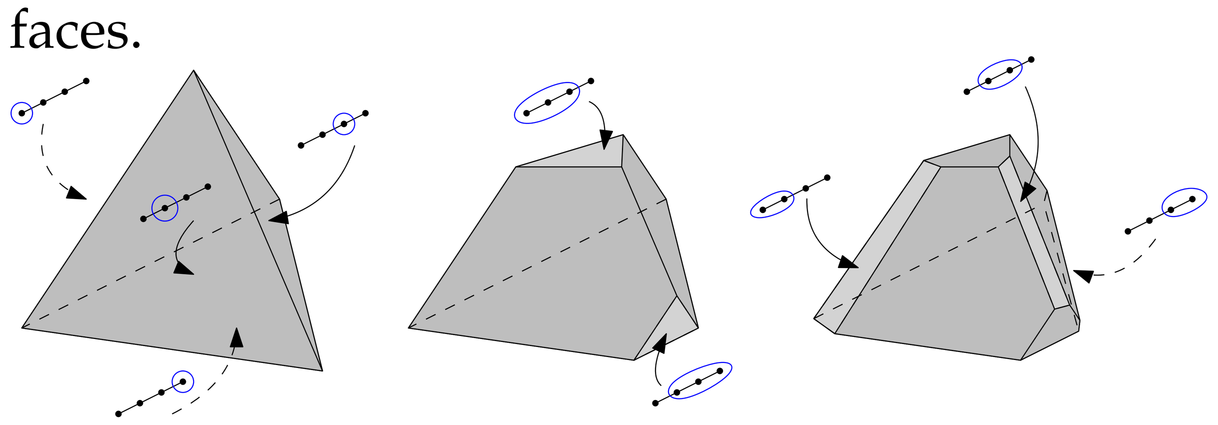


compatible tubes   incompatible tubes

A *tubing* is a collection of pairwise compatible tubes. The set of tubings forms a simplicial complex, and so is dual to a simple polytope called the *graph associahedron* of G.



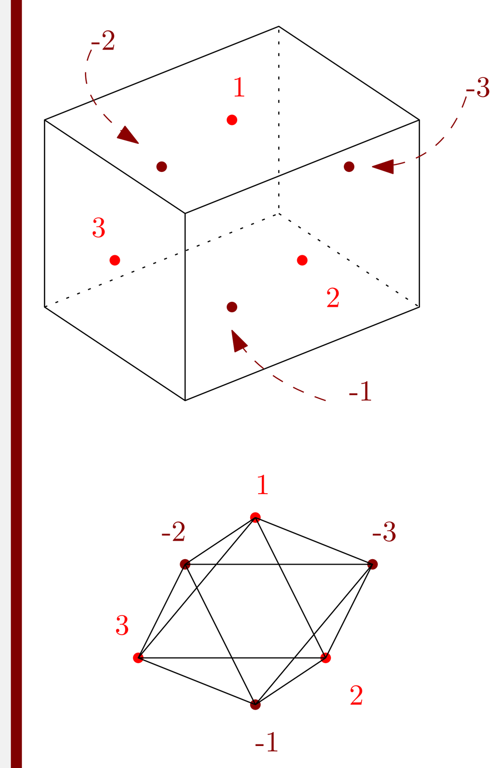
The graph associahedron can be found by repeatedly truncating faces of the simplex associated with tubes, starting with lower-dimensional faces.



Facets are in bijection with individual tubes, and vertices are in bijection with maximal tubings. When G is a path graph, the graph associahedron is called the *associahedron*.

Graph associahedra are special cases of *generalized permutahedra*, polytopes whose normal fans refine the braid arrangement.

$\mathcal{P}$-Graph Associahedra (A., 2020)

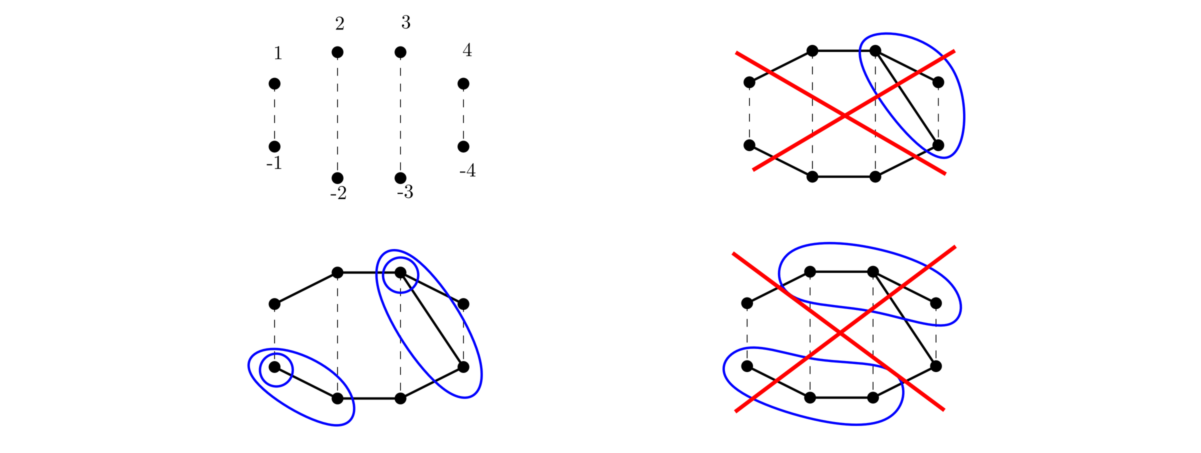


The face lattice of a simple polytope $\mathcal{P}$ is dual to a simplicial complex, whose 1-skeleton is the *adjacency graph* on facets of $\mathcal{P}$. Call any subgraph of this graph a $\mathcal{P}$-graph. Every face of $\mathcal{P}$ is uniquely defined as the intersection of a set of facets. If the intersection of the facets associated with a tube or a tubing is empty, that tube is not *valid*.

The complex of valid tubings is dual to a simple polytope, called the *$\mathcal{P}$-graph associahedron*, obtained by repeatedly truncating faces of $\mathcal{P}$ in ascending order of dimension, analogously to graph associahedra (Almeter, 2020).

Hypercube Graphs

In the case where $\mathcal{P}$ is an n-dimensional hypercube, the adjacency graph is the hyperoctahedral graph on vertices $\pm[n]$. Out of convention, we draw dashed lines between opposite nodes.



Empty graph. Valid tubing.   Invalid tube and tubing.

Valid tubes are connected induced subgraphs which do not contain dashed edges. Valid tubings are sets of compatible tubes which contain no pairs $\{i, -i\}$.

Examples of Hypercube Graph Associahedra

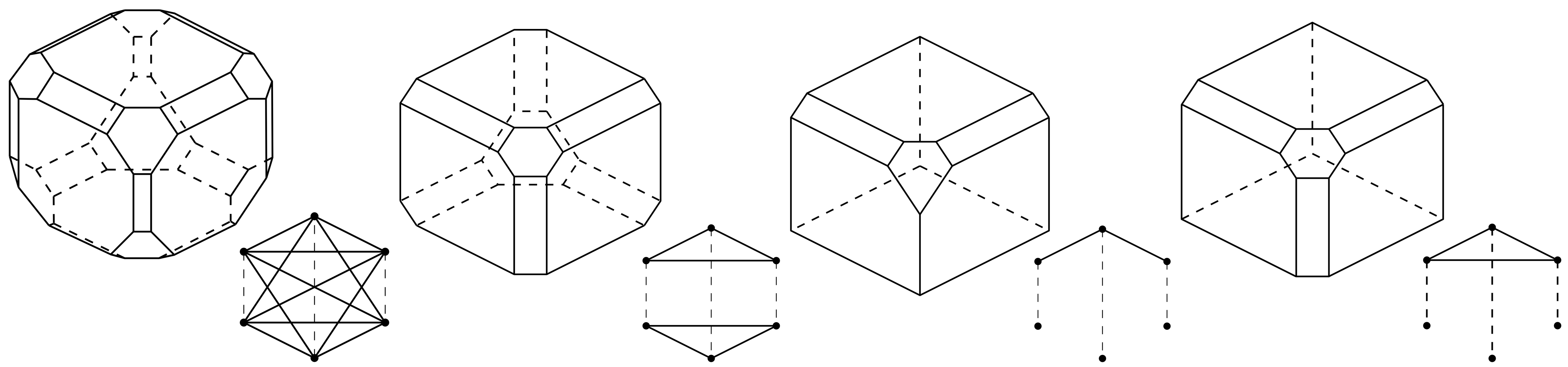
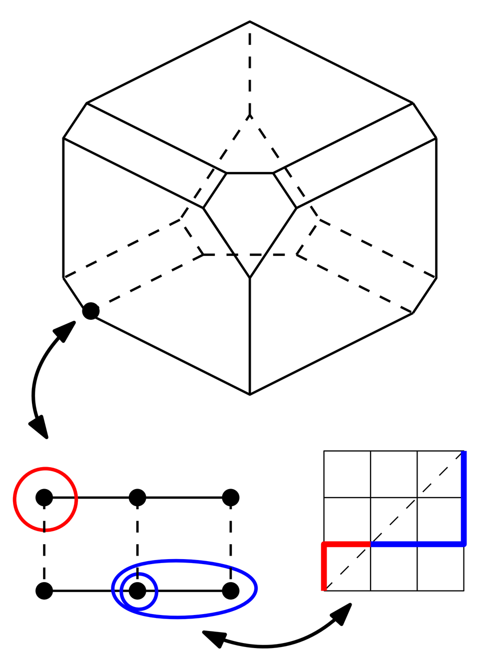


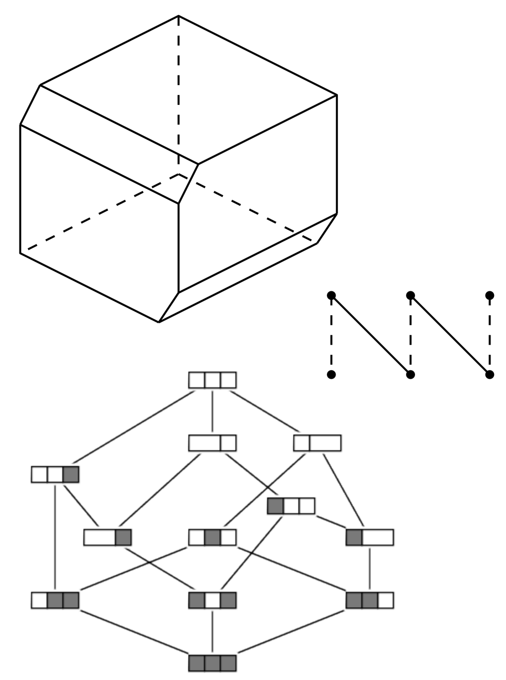
Figure 1: Type B_3 permutahedron, type A_3 permutahedron, type A_3 associahedron, stellohedron.

Linear Biassociahedron



The linear biassociahedron is normal to the refinement of a linear cluster fan with its antipodal fan (Barnard, Reading 2018). This polytope is realized by a dual path hypercube graph associahedron. This construction shows a bijection between its vertices and NE-lattice paths.

Pell Permutations



The graph associahedron of a 'zigzag' graph is shown here. By putting an ordering on the vertices, we obtain the Hasse diagram for the lattice of sashes (Law 2014) in $n \geq 7$ dimensions, and conjecture that this holds for all n.

Results

The original motivation of this research was to generalize graph associahedra from type A root systems to root systems of other types, and we have succeeded in the type B case as follows.

Theorem 1. *All hypercube graph associahedra have normal fans which coarsen the type B_n hyperplane arrangement, and refine the orthant hyperplane arrangement.*

Theorem 2. *There exist hypercube graph associahedra which cannot be expressed as graph associahedra, and there exist hypercube graph associahedra which cannot be expressed as graph associahedra.*

Theorem 3. *The type B_n permutahedron can be realized by the hypercube graph associahedron of a full adjacency graph, the type A_n permutahedron can be realized by the hypercube graph associahedron of two complete graphs on $[n] \cup -[n]$, the type A_n associahedron can be realized as a path hypercube graph associahedron on vertices $[n]$, and the stellohedron can be realized as a hypercube graph associahedron of a complete graph on $[n]$.*

Theorem 4. *Every face of a hypercube graph associahedron is the Cartesian product of a hypercube graph associahedron and several graph associahedra.*

The $\mathcal{P}$-graph associahedron generalizes to the $\mathcal{P}$-nestohedron associated with a $\mathcal{P}$-nested complex. Nested complexes of lattices are introduced in (Feichtner, Kozlov 2003).

Theorem 5. *The $\mathcal{P}$-nested complex is isomorphic to a lattice nested complex of the dual face lattice of $\mathcal{P}$.*

References

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[2] Emily Barnard, Nathan Reading, "Coxeter-biCatalan combinatorics." J. Algebraic Combin. 2018

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[4] Eva Maria Feichtner, Dmitry N. Kozlov, "Incidence combinatorics of resolutions." 2003

[5] Shirley Law, "Combinatorial Realization of the Hopf Algebra of Sashes." 2014