Introduction to Finite Elements and Algorithms

Group 3

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1 Poisson Equation 2D

Many physical phenomena can be modeled with the Poisson Equation.

$$\nabla\left(\kappa\nabla u\right)+\alpha u=f(x,y)\qquad\forall x,y\in\Omega \tag{1}$$

With boundary conditions:

$$\begin{gathered} q \quad \text{on} \quad \Gamma_N \\ T \quad \text{on} \quad \Gamma_D \\ \Gamma_N\cup\Gamma_D=\partial\Omega \\ \Gamma_N\cap\Gamma_D=0 \end{gathered} \tag{2}$$

We multiplicate or function with a test function v. And integrate over the domain.

$$\begin{gathered} \left(\nabla\left(\kappa\nabla u\right)+\alpha u\right)v=f(x,y)v\qquad\forall x,y\in\Omega \\ \int_\Omega\left(\nabla\left(\kappa\nabla u\right)+\alpha u\right)vd\Omega=\int_\Omega f(x,y)vd\Omega\qquad\forall x,y\in\Omega \end{gathered} \tag{3}$$

Using Green's theorem, we can change the divergence operator to the test function v.

$$\int_{\partial\Omega}\left(\kappa\nabla u\cdot\mathbf{n}_t\right)vd\Omega-\int_\Omega\kappa\nabla u\nabla vd\Omega+\int_\Omega\alpha uvd\Omega=\int_\Omega f(x,y)vd\Omega \tag{4}$$

Now, we discretize the domain into different elements Ω^e, the field on each element is approximate with shape functions $\varphi_i(x,y)$. The test function is selected to be the same as the shape function $\varphi_j(x,y)$ (Galerkin Formulation). For the following analysis, we will forget about the boundary conditions, which will be applied in the last steps of the formulation.

$$\begin{gathered} \tilde{u}\approx\sum_{i=1}^{n}u_i\varphi_i \\ -\int_{\Omega^e}\kappa(\nabla\tilde{u})(\nabla\varphi_j)d\Omega^e+\int_{\Omega^e}\alpha\tilde{u}\varphi_jd\Omega^e=\int_\Omega f(x,y)\varphi_jd\Omega^e \\ \nabla\tilde{u}=\nabla\left(\sum_{i=1}^{n}u_i\varphi_i\right)=\sum_{i=1}^{n}u_i\nabla(\varphi_i) \\ -\int_{\Omega^e}\kappa\left(\sum_{i=1}^{n}u_i\nabla(\varphi_i)\right)(\nabla\varphi_j)d\Omega^e+\int_{\Omega^e}\alpha\left(\sum_{i=1}^{n}u_i\varphi_i\right)\varphi_jd\Omega^e=\sum_{i=1}^{n}\left(\int_{\Omega^e}\alpha\varphi_i\varphi_jd\Omega^e-\int_{\Omega^e}\kappa(\nabla\varphi_i)(\nabla\varphi_j)d\Omega^e\right)u_i \\ \sum_{i=1}^{n}\left(\int_{\Omega^e}\alpha\varphi_i\varphi_jd\Omega^e-\int_{\Omega^e}\kappa(\nabla\varphi_i)(\nabla\varphi_j)d\Omega^e\right)u_i=\int_\Omega f(x,y)\varphi_jd\Omega^e \end{gathered} \tag{5}$$

For each element, we have n unknowns but only one equation. To get the missing $n-1$ equations, we vary the index $j=1,2..,n$ to get a linear equation system.

$$
\begin{aligned}
(-\mathbf{S}+\mathbf{M})u &= f \\
\mathbf{S} = [S_{ij}] &= \int_{\Omega^e} \kappa(\nabla\varphi_i)(\nabla\varphi_j)d\Omega^e \\
\mathbf{M} = [M_{ij}] &= \int_{\Omega^e} \alpha\varphi_i\varphi_j d\Omega^e \\
u = [u_i] &= u_i \\
f = [f_i] &= \int_{\Omega} f(x,y)\varphi_j d\Omega^e
\end{aligned} \tag{6}
$$

S is the element stiffness matrix, u is the unknown vector, and f is the element force vector.

If we discretize our domain in linear triangular elements, we will need to use natural coordinates to have a general formulation of the local matrices and vectors.

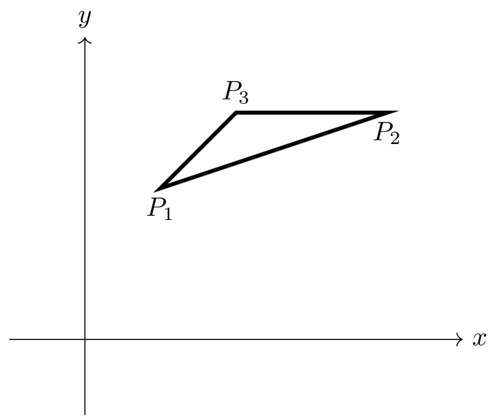

Figure 1: Triangular Element

Where x_i and y_i are the coordinates of the point P_i. We transform this element to the following natural coordinates.

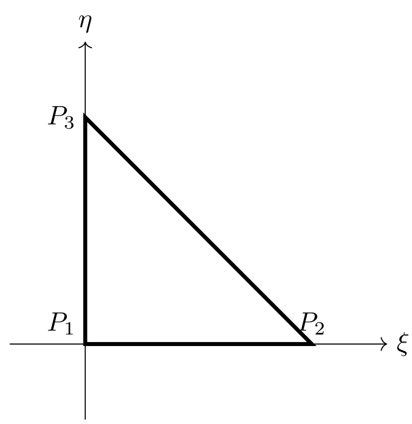

Figure 2: Triangular Element

The shape functions used for this element are linear Lagrange functions.

$$
\begin{aligned}
\varphi_1(\xi,\eta) &= 1-\xi-\eta \\
\varphi_2(\xi,\eta) &= \xi \\
\varphi_3(\xi,\eta) &= \eta
\end{aligned} \tag{7}
$$

And using an isoparametric approach, we can describe the geometry.

$$
\begin{aligned}
x &= \varphi_1 x_1 + \varphi_2 x_2 + \varphi_3 x_3 \\
y &= \varphi_1 y_1 + \varphi_2 y_2 + \varphi_3 y_3
\end{aligned}
\tag{8}
$$

The matrices in local coordinates should be the following.

$$
\begin{gathered}
\nabla\varphi_i = \frac{\partial\varphi_i}{\partial x} + \frac{d\varphi_i}{\partial y} \\
\frac{\partial\varphi_i}{\partial x} = \frac{\partial\varphi_i}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{d\varphi_i}{\partial\eta}\frac{\partial\eta}{\partial x} \\
\frac{\partial\varphi_i}{\partial y} = \frac{\partial\varphi_i}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{d\varphi_i}{d\eta}\frac{d\eta}{\partial y} \\
dx = \frac{\partial x}{\partial\xi}d\xi + \frac{\partial x}{d\eta}d\eta \\
dy = \frac{dy}{d\xi}d\xi + \frac{dy}{d\eta}d\eta \\
\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} \\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta} \end{bmatrix} \\
\mathbf{J}^{-1} = \begin{bmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial\xi} \end{bmatrix} \\
\begin{bmatrix} \frac{\partial\varphi_i}{\partial x} \\ \frac{\partial\varphi_i}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial\varphi_i}{\partial\xi} \\ \frac{\partial\varphi_i}{\partial\eta} \end{bmatrix}
\end{gathered}
\tag{9}
$$

The derivatives of the shape functions in local coordinates are the following:

$$
\begin{gathered}
\frac{\partial\varphi_1}{\partial\xi} = -1 \\
\frac{\partial\varphi_1}{\partial\eta} = -1 \\
\frac{\partial\varphi_2}{\partial\xi} = 1 \\
\frac{\partial\varphi_2}{\partial\eta} = 0 \\
\frac{\partial\varphi_3}{\partial\xi} = 0 \\
\frac{\partial\varphi_3}{\partial\eta} = 1
\end{gathered}
\tag{10}
$$

The global coordinates variables are:

$$\frac{\partial x}{\partial \xi} = -x_1 + x_2$$
$$\frac{\partial x}{\partial \eta} = -x_1 + x_3$$
$$\frac{\partial y}{\partial \xi} = -y_1 + y_2$$
$$\frac{\partial y}{\partial \eta} = -y_1 + y_3$$
$$\mathbf{J} = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$
$$\mathbf{J}^{-1} = \frac{1}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)} \begin{bmatrix} y_3 - y_1 & -(y_2 - y_1) \\ -(x_3 - x_1) & x_2 - x_1 \end{bmatrix} \tag{11}$$
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{J}^{-1} = \frac{1}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)} \begin{bmatrix} (y_3 - y_1) - (x_3 - x_1) & -(y_2 - y_1) + (x_2 - x_1) \end{bmatrix}$$
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial \varphi_1}{\partial \xi} \\ \frac{\partial \varphi_1}{\partial \eta} \end{bmatrix} = \frac{-y_3 + x_3 + y_2 - x_2}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}$$
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial \varphi_2}{\partial \xi} \\ \frac{\partial \varphi_2}{\partial \eta} \end{bmatrix} = \frac{y_3 - y_1 - x_3 + x_1}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}$$
$$\begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial \varphi_3}{\partial \xi} \\ \frac{\partial \varphi_3}{\partial \eta} \end{bmatrix} = \frac{-y_2 + y_1 + x_2 - x_1}{(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)}$$

The system 0f equations in local coordinates will be then:

$$\begin{gathered} \mathbf{S} = [S_{ij}] = \int_{\Omega^l} \kappa \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial \varphi_i}{\partial \xi} \\ \frac{\partial \varphi_i}{\partial \eta} \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial \varphi_j}{\partial \xi} \\ \frac{\partial \varphi_j}{\partial \eta} \end{bmatrix} \right) |det(\mathbf{J})| d\xi d\eta \\ \mathbf{M} = [M_{ij}] = \int_{\Omega^l} \alpha \varphi_i \varphi_j |det(\mathbf{J})| d\xi d\eta \\ u = [u_i] = u_i \\ f = [f_i] = \int_{\Omega^l} f(x(\xi, \eta), y(\xi, \eta)) \varphi_j |det(\mathbf{J})| d\xi d\eta \end{gathered} \tag{12}$$