Introduction into Finite Elements and Algorithms

Group 3

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1 Weak form of the problem

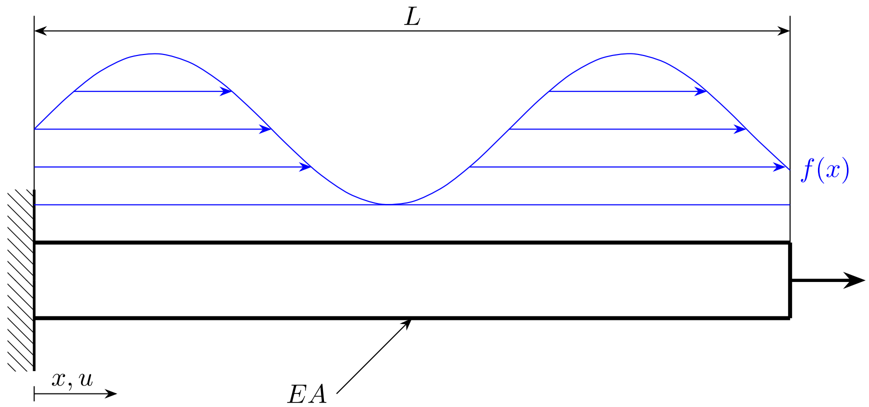

Figure 1: 1D Truss

Strong form of the problem:

$$\begin{cases} -u''(x) = f(x) \quad \forall x \in [0,1] \\ u(0) = 0 \\ u'(1) = 0 \end{cases} \tag{1}$$

This equation can be used to model a 1D truss/bar with linear elastic properties.
If we suppose that u is a solution for the strong form, given any function v, it is also true that:

$$-u''(x)v(x) = f(x)v(x) \quad \forall x \in [0,1]$$

If we integrate that expression in [0,1], we arrive at:

$$-\int_0^1 u''(x)v(x)dx = \int_0^1 f(x)v(x)dx$$

For that integrals to be well defined, we assume that $u \in H^2([0,1])$, $v \in H^1([0,1])$ and $f \in L^2([0,1])$. Then, we integrate by parts:

$$-[u'(x)v(x)]_{x=0}^{x=1} - \int_0^1 u'(x)v(x)' = dx \int_0^1 f(x)v(x)dx \quad \forall v \in H^1([0,1])$$

$$-u'(1)v(1) + u'(0)v(0) + \int_0^1 u'(x)v(x)' = dx \int_0^1 f(x)v(x)dx \quad \forall v \in H^1([0,1])$$

To force a solution that verifies the Dirichlet bound conditions, we define:

$$V = \{v \in H^1([0,1]) \,:\, v(0) = 0\}$$

Using that functional space and given that u'(1)=0, we then arrive at the weak form of the problem:

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx \quad \forall v \in V \tag{2}$$

The model can be extended to take into consideration a diffusion coefficient β, a mass coefficient α, and an inhomogeneous density $c(x)$.

$$\frac{d}{dx}\left(c(x)\frac{du(x)}{dx}\right) + \alpha\frac{du(x)}{dx} + \beta u(x) = f(x) \tag{3}$$

The derivation of the weak form is analogous to the previous one, leading to the following result:

$$\big(-K + \alpha D + \beta M\big)\mathbf{u} = \mathbf{F} \tag{4}$$

where K, D, and M are the stiffness matrix, the diffusion matrix, and the mass matrix, respectively.

$$K_{ij} = \int_\Omega \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}c(x)\,dx$$
$$D_{ij} = \int_\Omega \frac{d\phi_i}{dx}\phi_j\,dx$$
$$M_{ij} = \int_\Omega \phi_i\phi_j\,dx$$

The force vector has the same expression as before:

$$F_i = \int_\Omega f(x)\phi_i\,dx \tag{5}$$

2 FEM with first order Lagrange elements

Solve:

$$\mathbf{A}x = b$$

Where:

$$\mathbf{A} = [a_{ij}]$$

$$\begin{gathered}
a_{ij} = \int_a^b \varphi_j'\varphi_i' dx \quad \text{if } j \in \{i-1, i, i+1\} \\
a_{ij} = 0 \quad \text{if } j \notin \{i-1, i, i+1\} \\
b = [b_i], \quad b_i = \int_a^b f\varphi_i dx \quad i = 1, ..., N+1 \\
b_1 = b_{N+1} = 0
\end{gathered} \tag{6}$$

The functions φ_i are defined as:

$$\varphi_i = \begin{cases} \frac{x - x_{i-1}}{x_{i+1} - x_i} & \text{if } x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if } x \in [x_i, x_{i+1}] \\ 0 & \text{if } x \notin [x_{i-1}, x_{i+1}] \end{cases}$$

The previous linear system can be assembled with the help of an element stiffness matrix [2x2] and an element force vector [2x1]. For the element i, which nodes are $[x_i, x_{i+1}]$, the element stiffness matrix and force vector are defined as follows.

$$K^e = [K^i_{lk}] = \int_{x_i}^{x_{i+1}} \varphi'_{i-1+l}\varphi'_{i-1+k} dx \quad l, k = 1, 2$$

$$b^e = [b_l^i] = \int_{x_i}^{x_{i+1}} \varphi_{i-1+l} f dx \quad l = 1, 2$$

The assembly of the stiffness matrix in global coordinates is:

$$K = \begin{pmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & \cdots & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & \cdots & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & K_{22}^N \end{pmatrix}$$

And the assembly of b terms has the structure:

$$b = \begin{pmatrix} b_1^1 \\ b_2^1 + b_1^2 \\ b_2^2 + b_1^3 \\ \vdots \\ b_2^N \end{pmatrix}$$

As there is no mass matrix in this problem nor bound conditions to be added, we take $A = K$. And so, the FEM problem to solve is:

$$Ac = b$$

Which leads to the solution:

$$u^h = \sum_{i=1}^{N+1} c_i \psi_i(x)$$