Introduction into Finite Elements and Algorithms

Group 3

November 2023

We aim to solve the bi-harmonic equation for the 1D problem:

$$(P)\begin{cases} u^{4)} = f \quad \text{in } [0,1] \\ u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}$$

Weak form

$$u^{4)} = f \Rightarrow u^{4)}v = fv \quad \forall v \in V \Rightarrow \int_0^1 u^{4)}v\,dx = \int_0^1 fv\,dx \quad \forall v \in V$$

We integrate by part the integral on the left side:

$$\int_0^1 fv\,dx = [u'''v]_{x=0}^1 - \int_0^1 u'''v'dx = u'''(1)v(1) - u'''(0)v(0) - \int_0^1 u'''v'\,dx \quad \forall v \in V$$

As u(0)=0, we impose $v \in V = \{w \in H^2 : w(0) = 0,\, w'(0) = 0\}$. Then $u'''(0)v(0) = 0$, and we also have enough regularity to integrate by parts again.

$$\int_0^1 fv\,dx = -[u''v']_{x=0}^1 + \int_0^1 u''v''\,dx = -u''(1)v(1) + u''(0)v'(0) + \int_0^1 u''v''\,dx = \int_0^1 u''v''\,dx \quad \forall v \in V$$

Therefore, the weak problem is, given $f \in L^2([0,1])$, to find $u \in V$ such as for every $v \in V$ it is verified that:

$$\int_0^1 u''v''\,dx = \int_0^1 fv\,dx \tag{1}$$

FEM with Hermite elements

We use shape functions $\psi_i \in \mathcal{C}^1$ such as for every element E_i $\psi_{i|_{E_i}}$ is a third degree polynomial. The local basis in the interval $[-1,1]$ is:

$$\psi_1 = \frac{1}{4}[-x^3 + 3x + 2]$$
$$\psi_2 = \frac{1}{4}[x^3 - x^2 - x + 1]$$
$$\psi_3 = \frac{1}{4}[x^3 - 3x + 2]$$
$$\psi_4 = \frac{1}{4}[x^3 + x^2 - x - 1]$$

This shape functions have the following properties:

$$\begin{array}{llll} \psi_1(-1) = 1; & {\psi_1}'(-1) = 0; & \psi_1(1) = 0; & {\psi_1}'(-1) = 0 \\ \psi_2(-1) = 0; & {\psi_2}'(-1) = 1; & \psi_2(1) = 0; & {\psi_2}'(-1) = 0 \\ \psi_3(-1) = 0; & {\psi_3}'(-1) = 0; & \psi_3(1) = 1; & {\psi_3}'(-1) = 0 \\ \psi_4(-1) = 0; & {\psi_4}'(-1) = 0; & \psi_4(1) = 0; & {\psi_4}'(-1) = 1 \end{array}$$

The second derivative of this shape functions is:

$$\psi_j''(x) = \begin{cases} (-1)^{\frac{j+1}{2}} \frac{3}{2}x & \text{if } j \in \{1,3\} \\ \frac{3}{2}x + 2 \cdot (-1)^{\frac{j}{2}} & \text{if } j \in \{2,4\} \end{cases}$$

We now discretize the weak form in elements, and change variables to make $[x_i, x_{i+1}] \to [-1, 1]$. Then, we substitute the expression of the shape functions in order to obtain the coefficients of the FEM linear system:

OPTION 1: If $j, l \in 1, 3$:

$$\frac{2}{h}\int_{-1}^{1} \psi_j{}''(x)\psi_l{}''(x)\,dx = \frac{2}{h}\int_{-1}^{1} (-1)^{\frac{j+1}{2}}\frac{3}{2}x(-1)^{\frac{l+1}{2}}\frac{3}{2}x\,dx = (-1)^{\frac{j+l}{2}+1}\frac{2}{h}\int_{-1}^{1} x^2 dx = (-1)^{\frac{j+l}{2}+1}\frac{2}{h}\Big[\frac{x^3}{3}\Big]_{x=-1}^{1} =$$
$$= (-1)^{\frac{j+l}{2}+1}\frac{4}{3}$$

OPTION 2: If $j, l \in 2, 4$:

$$\frac{2}{h}\int_{-1}^{1} \psi_j{}''(x)\psi_l{}''(x)\,dx = \frac{2}{h}\int_{-1}^{1} \big(\frac{3}{2}x + 2\cdot(-1)^{\frac{j}{2}}\big)\big(\frac{3}{2}x + 2\cdot(-1)^{\frac{l}{2}}\big)dx =$$
$$= \frac{2}{h}\Big[\frac{9}{4}\int_{-1}^{1} x^2 dx + 2\big((-1)^{\frac{j}{2}} + (-1)^{\frac{l}{2}}\big)\int_{-1}^{1} x dx + 4(-1)^{\frac{j+l}{2}}\int_{-1}^{1} dx\Big] = \frac{2}{h}\Big[\frac{3}{2} + 2\big((-1)^{\frac{j}{2}} + (-1)^{\frac{l}{2}}\big) + 8(-1)^{\frac{j+l}{2}}\Big]$$

OPTION 3: If $j \in 1, 3$ and $l \in 2, 4$:

$$\frac{2}{h}\int_{-1}^{1} \psi_j{}''(x)\psi_l{}''(x)\,dx = \frac{2}{h}\int_{-1}^{1} (-1)^{\frac{j-1}{2}}\frac{3}{2}x\Big(\frac{3}{2}x + 2\cdot(-1)^{\frac{l}{2}}\Big)dx = \frac{2}{h}\Big[(-1)^{\frac{j-1}{2}}\frac{9}{4}\int_{-1}^{1} x^2 dx + 3\cdot(-1)^{\frac{j+l-1}{2}}\int_{-1}^{1} x dx\Big] =$$
$$= \frac{2}{h}\Big[(-1)^{\frac{j-1}{2}}\frac{3}{2} + 3\cdot(-1)^{\frac{j+l-1}{2}}\Big]$$

As a consequence, the elementary stiffness matrix is written as follows:

$$K^i = \frac{1}{h}\begin{pmatrix} -4/3 & -3 & 4/3 & 9 \\ -3 & 11 & 3 & -13 \\ 4/3 & 3 & -4/3 & -3 \\ 9 & -13 & -3 & 27 \end{pmatrix}$$