Introduction into Finite Elements and Algorithms

Group 3

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Given the bi-harmonic equation for the 1D problem:

$$(P)\begin{cases} u^{4)} = f \quad \text{in } [0,1] \\ u(0) = 0 \\ u'(0) = 0 \\ u''(1) = 0 \\ u'''(1) = 0 \end{cases}$$

We approach the problem by defining the function $w = u''$, which lead us to the breakdown in the following problems:

$$(P1)\begin{cases} w'' = f \quad \text{in } [0,1] \\ w(1) = 0 \\ w'(1) = 0 \end{cases}$$

$$(P2)\begin{cases} u'' = w \quad \text{in } [0,1] \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

Weak form for (P1):

$$w'' = f \Rightarrow w''v = fv \quad \forall v \in V \Rightarrow \int_0^1 w''v\,dx = \int_0^1 fv\,dx \quad \forall v \in V$$

We now integrate the left side by parts:

$$[w'v]_{x=0}^{x=1} - \int_0^1 w'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V$$

$$w'(1)v(1) - w'(0)v(0) - \int_0^1 w'v'\,dx = \int_0^1 fv\,dx \quad \forall v \in V$$

$$w'(0)v(0) + \int_0^1 w'v'\,dx = \int_0^1 -fv\,dx \quad \forall v \in V$$

1^{st} **order Lagrange FEM for (P1):** Let $x_1, x_2, ..., x_{N+1}$ be the nodes of the one-dimensional mesh. According to the first order report, the vector b is written as:

$$b = \begin{pmatrix} b_1^1 \\ b_2^1 + b_1^2 \\ b_2^2 + b_1^3 \\ \vdots \\ b_2^N \end{pmatrix}, \quad b_l^i = \int_{x_i}^{x_{i+1}} \varphi_{i-1+l} f dx \quad l = 1; \quad 1 \le i \le N$$

We approximate the result using the trapezoidal rule:

$$b_l^i = \int_{x_i}^{x_{i+1}} \varphi_{i-1+l} f\, dx = \frac{f(x_i) + f(x_{i+1})}{2h}$$

On the other hand, the stiffness matrix is given by the expression:

$$K = \begin{pmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 & \cdots & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 & \cdots & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & K_{22}^N \end{pmatrix}$$

Where the coefficients K_{lk}^1 for $l, k = 1, 2$ and $1 \leq i \leq N$ are defined as:

$$K_{lk}^i = \int_{x_i}^{x_{i+1}} \varphi'_{i-1+l} \varphi'_{i-1+k} dx = \int_{x_i}^{x_{i+1}} (-1)^l \frac{1}{h} (-1)^k \frac{1}{h} dx = \frac{(-1)^{l+k}}{h}$$

As a consequence,

$$K = \frac{1}{h} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 1+1 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 1+1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \frac{1}{h} \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

There is no mass matrix in the problem, so we take A=K.

Weak form for (P2): As stated before, we approach the problem:

$$(P2) \begin{cases} u'' = w \quad \text{in } [0, 1] \\ u(0) = 0 \\ u'(0) = 0 \end{cases}$$

$$u'' = w \Rightarrow u''v = wv \quad \forall v \in V \Rightarrow \int_0^1 u''v\, dx = \int_0^1 wv\, dx \quad \forall v \in V$$

We now integrate the left side by parts:

$$[u'v]_{x=0}^{x=1} - \int_0^1 u'v'\, dx = \int_0^1 wv\, dx \quad \forall v \in V$$

$$u'(1)v(1) - u'(0)v(0) - \int_0^1 u'v'\, dx = \int_0^1 wv\, dx \quad \forall v \in V$$

$$-u'(1)v(1) + \int_0^1 u'v'\, dx = \int_0^1 -wv\, dx \quad \forall v \in V$$

1^{st} **order Lagrange FEM for (P2):** Analogous to the (P1) case, there is no mass matrix, and the problem matrix is the matrix A given in (P1).

Coupling of (P1) and (P2): We reduced the approximation of problems (P1) and (P2) to solving the two linear systems:

$$Aw = b$$

$$Au = w \Rightarrow Au - Iw = 0$$

This is equivalent to the formulation:

$$\begin{pmatrix} A & -I \\ 0 & A \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

We can then find $(u\, w)^T$ by solving the linear system, and then extract u from the solution vector, finding the approximate solution of the problem (P).