



# Robotics

CSE4316

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Lecture 3: **Robot Kinematics**

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## Lecture: 3

# Robot Kinematics

- Robot Workspace, Reference Frames.
- Robot Kinematics
- Frames, Points and Vectors
- Basic Transformations
  - ▶ Translation along X-Axis
  - ▶ Translation along XY-Axis

# Workspace

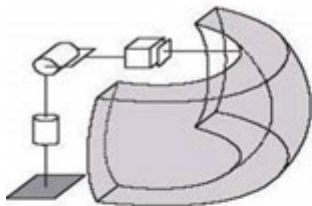
## Workspace (Work Envelope)

The Workspace of the manipulator is the total volume swept out by the end effector as the manipulator executes all possible motions.

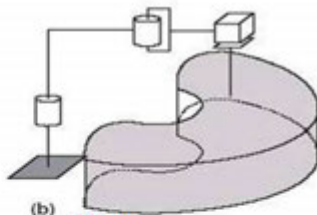
Workspace is constrained by:

- Geometry of the manipulator.
- Mechanical constraint of the joints (a revolute joint may be limited to less than 360 degrees)
- **Reachable** Workspace: the entire set of points reachable by the manipulator.
- **Dextrous** Workspace: consists of those points that the manipulator can reach with an arbitrary orientation of the end effectors.
  - ▶ Dextrous Workspace is a subset of Reachable Workspace.

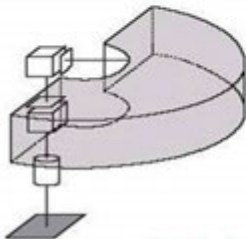
# Workspace



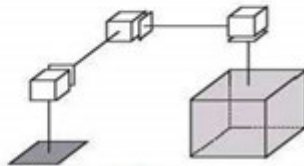
(a) **spherical**



(b) **SCARA**



(c) **cylindrical**



(d) **Cartesian**

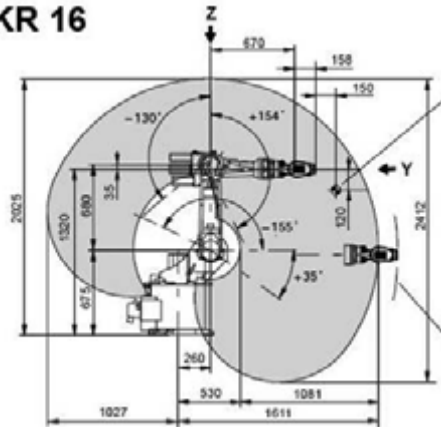
# Robot Characteristics and Performance Measure

- **Payload:** the max load a robot can handle
- **Reach:** The maximum distance a robot can reach within its work envelope.
- **Precision** (Accuracy, validity): how accurately a specified point can be reached.
- **Repeatability** (variability): how accurately the same position can be reached if the motion is repeated many times.

# Robot Characteristics and Performance Measure



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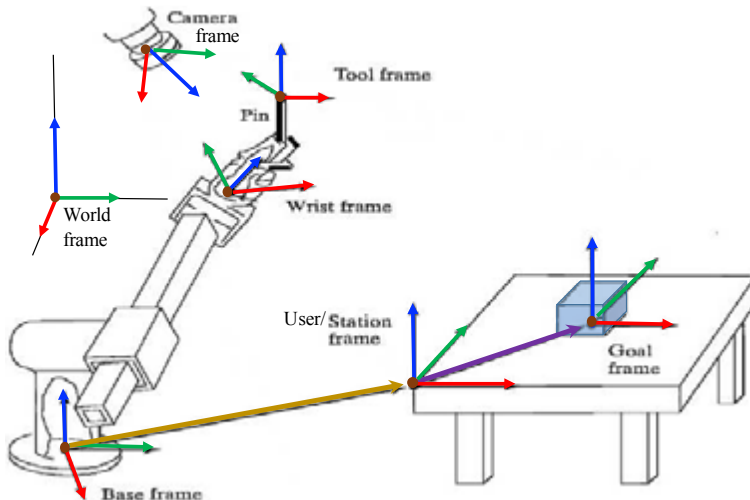


- The work envelope of the robot, as viewed from the side.
- payload at the wrist is 30 kg and repeatability is  $\pm 0.15$  mm.

# Reference Frames

## Frames with Standard Names

Most robots may be programmed to **move relative** to either of these **reference frames**.



# Reference Frames

## Frames with Standard Names

### World Reference Frame

A universal coordinate frame, as defined by the  $x - y - z$  axes. In this case the joints of the robot move simultaneously so as to create motions along the three major axes.

### Joint Reference Frame

used to specify movements of each individual joint of the Robot. In this case each joint may be accessed individually and thus only one joint moves at a time.

### Tool Reference Frame

specifies the movements of the robot tool-tip relative to the frame attached to the hand. All joints of the robot move simultaneously to create coordinated motions about the Tool frame.



# Robot Kinematics

# Robot Kinematics

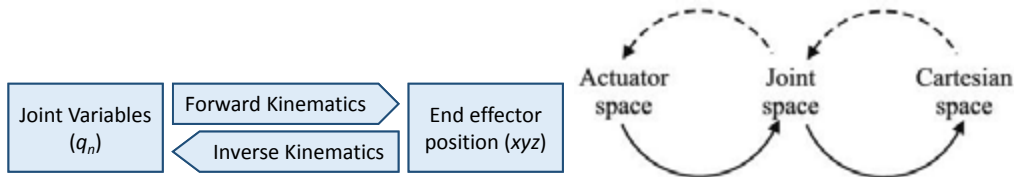
## Kinematics

The branch of classical mechanics that describes the motion of objects without consideration of the forces that cause it

- The study of movement
- Why do you need it?
  - ▶ Determine endpoint position and/or joint positions
  - ▶ Calculate mechanism velocities, accelerations, etc.
  - ▶ Calculate force-torque
- Allows you to move between Joint Space and Cartesian Space

# Robot Kinematics

- **Forward kinematics:** determines where the robot hand is (all joint variables are known)
  - ▶ you are given: length of each link, angle of each joint.
  - ▶ you find: position of any point (i.e. its (x, y, z) coordinates)

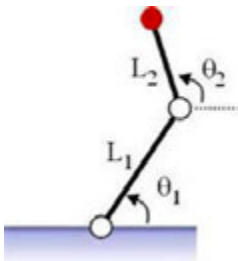


- **Inverse Kinematics:** to calculate what each joint variable is (If we desire that the hand be located at a particular point)
  - ▶ you are given: length of each link, position of some point on the robot
  - ▶ you find: The angles of each joint needed to obtain that position

# Robot Kinematics

## Joint Variables

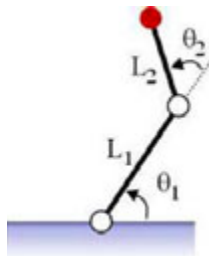
- In forward kinematics we convert from **Joint space** to **Cartesian Space**
- To reach a specific point in 3D space we need to get the required joint angles



- **Absolute** (Sometimes)

$$x = L_1 \cos(\theta_1) + L_2 \cos(\theta_2)$$

$$y = L_1 \sin(\theta_1) + L_2 \sin(\theta_2)$$



- **Relative** (usually)

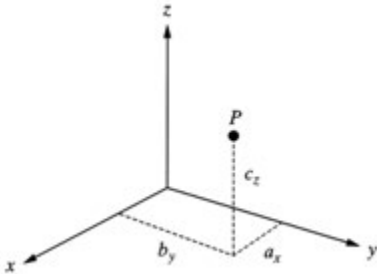
$$x = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)$$

$$y = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$$

# Description of position and orientation

## Frames, Points and Vectors

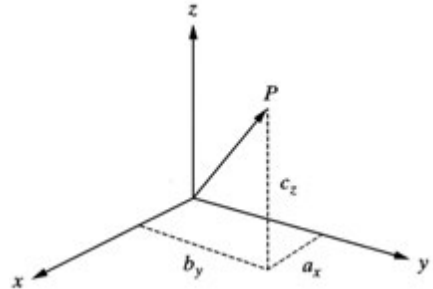
A **point** corresponds a particular location in the space.



3 coordinates relative to a reference frame:

$$P = a_x \hat{i} + b_y \hat{j} + c_z \hat{k}$$

A **vector** is defined by direction and magnitude.



3 coordinates of its tail and of its head:

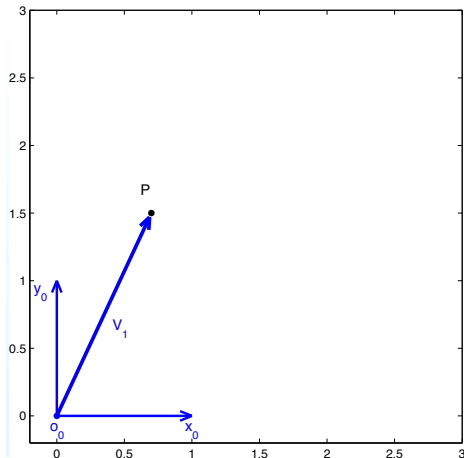
$$\vec{P} = a_x \hat{i} + b_y \hat{j} + c_z \hat{k}$$

# Description of position and orientation

## Frames

A coordinate frame in  $\mathbf{R}^2$ :

- the point  $P = [x_0^P, y_0^P]^T$  can be associated with the vector  $\vec{V}_1$ .
- Here  $x_0^P$  denotes x-coordinate of point  $P$  in  $\{x_0, o_0, y_0\}$ -frame (i.e. in the 0-frame).

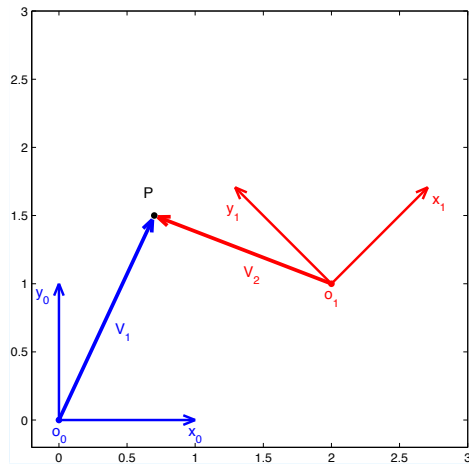


# Description of position and orientation

## Frames

Two coordinate frames in  $\mathbf{R}^2$ :

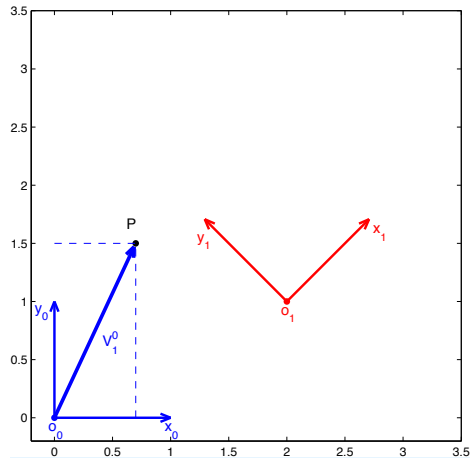
- the point  $P = [x_0^P, y_0^P]^T = [x_1^P, y_1^P]^T$  can be associated with vectors  $\vec{V}_1$  and  $\vec{V}_2$ .
- Here  $x_1^P$  denotes x-coordinate of point  $P$  in  $\{x_1, o_1, y_1\}$ -frame (i.e. in the 1-frame).



# Description of position and orientation

## Frames

- The coordinates of the vector  $\vec{V}_1$  in the 0-frame  $[x_0^p, y_0^p]^T$
- What would be coordinates of  $\vec{V}_1$  in the 1-frame?

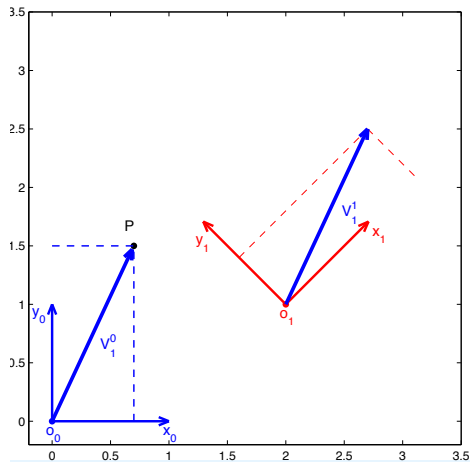




# Description of position and orientation

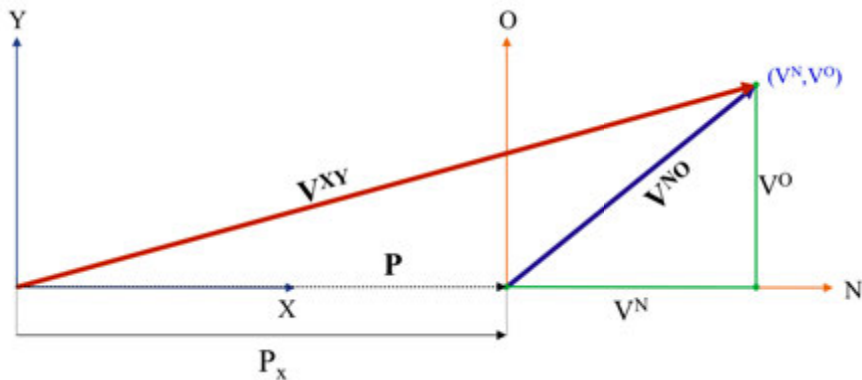
## Frames

- We need to consider the vector  $\vec{V}_1^1$  of the same direction and magnitude as  $\vec{V}_1^0$  but with the origin in **o1**.
- Conclusion: we can sum vectors only if they are expressed in parallel frames.



# Basic Transformations

## Translation along X-Axis

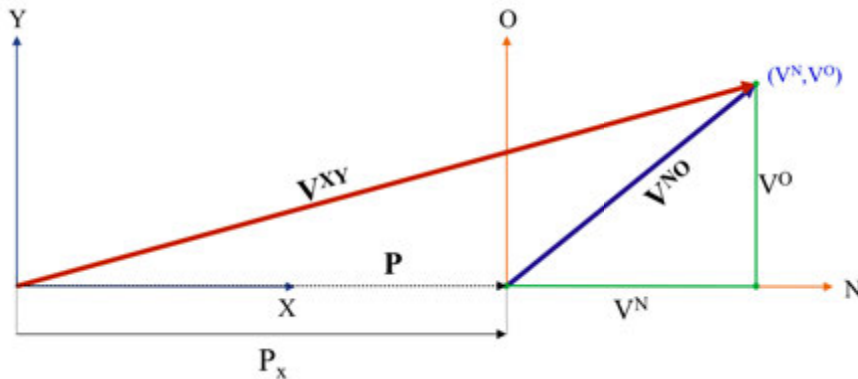


$p_x$  = distance between the XY and NO coordinate frames

$$V^{XY} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}, \quad V^{NO} = \begin{bmatrix} v_N \\ v_O \end{bmatrix}, \quad P = \begin{bmatrix} p_x \\ 0 \end{bmatrix}$$

# Basic Transformations

## Translation along X-Axis

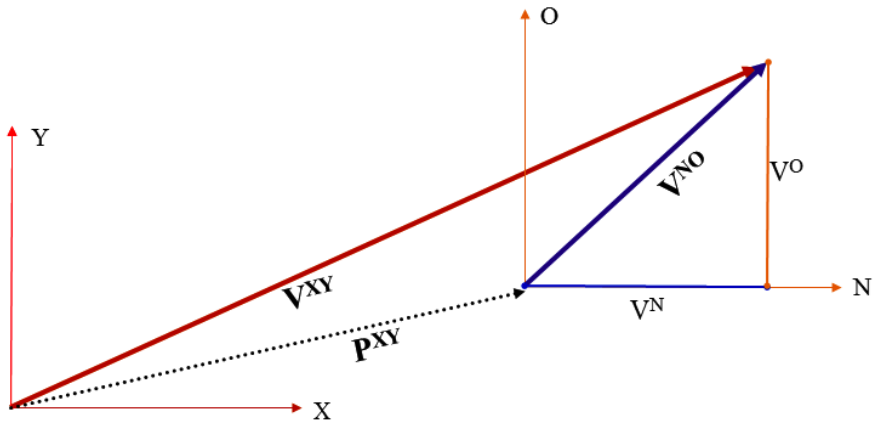


Writing  $v^{XY}$  in terms of  $v^{NO}$

$$v^{XY} = \begin{bmatrix} p_x + v_n \\ v_o \end{bmatrix} = P + v^{NO}$$

# Basic Transformations

## Translation along XY-Axis



$$P = \begin{bmatrix} p_x \\ p_y \end{bmatrix}, \quad v^{XY} = P + v^{NO} = \begin{bmatrix} p_x + v_n \\ p_y + v_o \end{bmatrix}$$

# Basic Transformations

## 2D Rotation

- Representing one coordinate frame in terms of another:

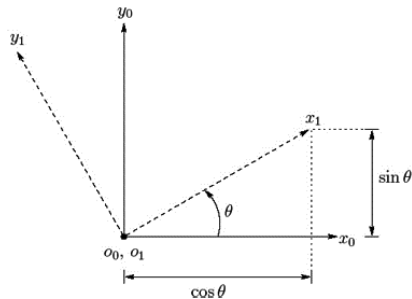
$$R_1^0 = \begin{bmatrix} x_1^0 & y_1^0 \end{bmatrix}$$

- unit vectors are defined as:

$$x_1^0 = |\vec{x}_0| = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \quad y_1^0 = |\vec{y}_0| = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- This is called a **rotation matrix**



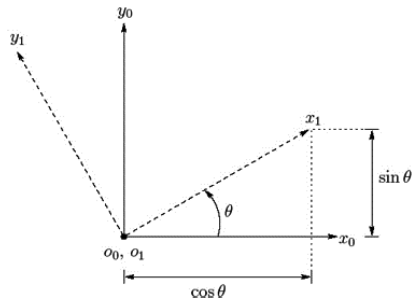
# Rotations

## Rotation matrices as projections

- Projecting the axes of  $O_1$  onto the axes of frame  $O_0$ :

$$x_1^0 = \begin{bmatrix} (\vec{x}_1 \cdot \vec{x}_0) \\ (\vec{x}_1 \cdot \vec{y}_0) \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} (\vec{y}_1 \cdot \vec{x}_0) \\ (\vec{y}_1 \cdot \vec{y}_0) \end{bmatrix}$$

$$\begin{aligned} R_1^0 &= [x_1^0(\theta) \mid y_1^0(\theta)] = \begin{bmatrix} (\vec{x}_1 \cdot \vec{x}_0) & (\vec{y}_1 \cdot \vec{x}_0) \\ (\vec{x}_1 \cdot \vec{y}_0) & (\vec{y}_1 \cdot \vec{y}_0) \end{bmatrix} \\ &= \begin{bmatrix} |\vec{x}_1| |\vec{x}_0| \cos \theta & |\vec{y}_1| |\vec{y}_0| \cos(\theta + \frac{\pi}{2}) \\ |\vec{x}_1| |\vec{x}_0| \cos(\theta - \frac{\pi}{2}) & |\vec{y}_1| |\vec{y}_0| \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$



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$${}^0\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\vec{a}| |\vec{b}| \cos(\widehat{\vec{a}, \vec{b}})$$

$${}^0\sin(\theta) = -\sin(-\theta) = -\cos(\theta + 90^\circ) = \cos(\theta - 90^\circ), \quad \cos(\theta) = \cos(\pm\theta) = \sin(\theta + 90^\circ) = -\sin(\theta - 90^\circ)$$

# Rotations

## Properties of rotation matrices

Rotation matrix has a number of interesting properties:

- Inverse Rotations:

$$(R_1^0)^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}^{-1} = \frac{1}{\det(R_1^0)} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = (R_1^0)^T$$

- using odd/even properties (another interpretation)

$$R_{01}^1 = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = (R_1^0)^T$$

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$${}^0\sin(\theta) = -\sin(-\theta) = -\cos(\theta + 90^\circ) = \cos(\theta - 90^\circ), \quad \cos(\theta) = \cos(\pm\theta) = \sin(\theta + 90^\circ) = -\sin(\theta - 90^\circ)$$

# Rotations

## Properties of rotation matrices

- Columns (rows) of  $R$  are mutually **orthogonal**
- Each column (row) of  $R$  is a unit vector
- The **determinant** of a rotation matrix is always  $\pm 1$ 
  - ▶  $+1$  if we only use **right-hand convention**

$$\det(R) = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos^2(\theta) + \sin^2(\theta) = 1$$

- The set of all  $n \times n$  matrices ( $X$ ) that satisfies these properties are called the **Special Orthogonal Group** of order  $n$

$$X^{-1} = X^T \Rightarrow XX^T = \mathbf{I}_n, \quad \det(XX^T) = 1 = \det(X)\det(X^T) = \det(X)^2$$
$$X \in \mathcal{O}(n). \text{ If } \det X = 1 \Rightarrow X \in \mathcal{SO}(n)$$

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$$\cos^2(\theta) + \sin^2(\theta) = 1$$

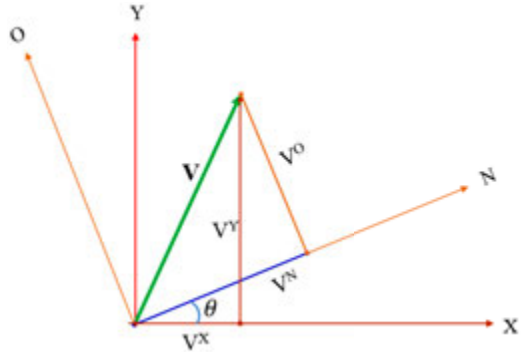


# Rotations

## Rotation around the Z-Axis

$\theta$  = Angle of rotation between the XY and NO frames

$$\begin{aligned} v^{XY} &= \begin{bmatrix} v_x \\ v_y \end{bmatrix}, \quad v^{NO} = \begin{bmatrix} v_N \\ v_O \end{bmatrix} \\ v^{XY} &= \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} v_N \\ v_O \end{bmatrix} \\ v^{XY} &= R_{NO}^{XY} v^{NO} \end{aligned}$$



# Rotations

## Rotation Around Major Axes

- Rotation around the Z-Axis

$$R_{z,\theta} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotation around the Y-Axis

$$R_{y,\theta} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

- Rotation around the X-Axis

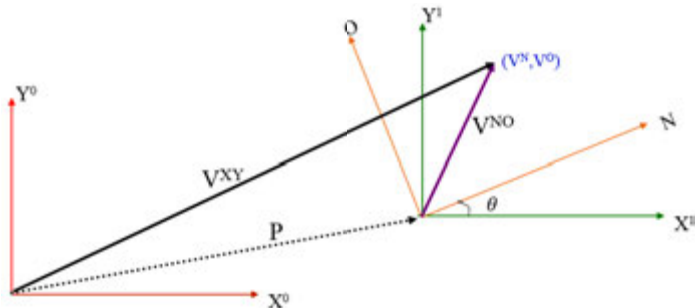
$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

# Frame Transformation

## Translation Followed by Rotation

- **Translation** along P followed by **rotation** by  $\theta$  around Z-axis

$$\begin{aligned} v^{XY} &= \begin{bmatrix} v_x \\ v_y \end{bmatrix} \\ &= \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} v_N \\ v_O \end{bmatrix} \end{aligned}$$



- $p_x, p_y$  are relative to the original coordinate frame.
- Translation followed by rotation is different than rotation followed by translation.
- knowing the coordinates of a point  $(V_N, V_O)$  in some coordinate frame  $\{NO\}$  you can find the position of that point relative to another coordinate frame  $\{X_0, Y_0\}$ .

# Thanks for your attention.

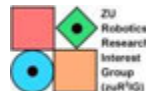
## Questions?

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Robotics Research Interest Group (zuR<sup>2</sup>IG)

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