MATH 590: Meshfree Methods

Chapter 38: Solving Elliptic Partial Differential Equations via RBF Collocation

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Fall 2010

Outline

In this chapter we discuss how the techniques for Lagrange and Hermite interpolation can be applied to the numerical solution of elliptic PDEs.

The resulting numerical method will be a collocation approach based on radial basis functions.

In the PDE literature this is also often referred to as a strong form solution.

To make the discussion transparent we will initially focus on the case of a time independent linear elliptic PDE in $\mathbb{R}^2$.

A now very popular non-symmetric method for the solution of elliptic PDEs with RBFs was suggested by Ed Kansa in [Kansa (1990b)].

In order to be able to clearly point out the differences between

- Kansa's method
- and a symmetric approach proposed in [Fasshauer (1997)]

we recall some of the basics of scattered data interpolation with RBFs in $\mathbb{R}^s$.

For scattered data interpolation we are given data $\{\boldsymbol{x}_i, f_i\}$, $i = 1, \ldots, N$, $\boldsymbol{x}_i \in \mathbb{R}^s$, where we think of the f_i being sampled from a function $f : \mathbb{R}^s \to \mathbb{R}$.

The goal is to find an interpolant of the form

$$\mathcal{P}_f(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \varphi(\|\boldsymbol{x} - \boldsymbol{x}_j\|), \qquad \boldsymbol{x} \in \mathbb{R}^s, \tag{1}$$

such that

$$\mathcal{P}_f(\boldsymbol{x}_i) = f_i, \qquad i = 1, \ldots, N.$$

The solution leads to a linear system $\mathsf{A}\boldsymbol{c} = \boldsymbol{f}$ with

$$\mathsf{A}_{ij} = \varphi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|), \qquad i, j = 1, \ldots, N. \tag{2}$$

The matrix A is non-singular for a large class of RBFs including (inverse) multiquadrics, Gaussians, and the CSRBFs of Wendland, Wu, Gneiting or Buhmann. In the case of strictly conditionally positive definite functions such as polyharmonic splines the problem needs to be augmented by polynomials.

We now switch to the collocation solution of PDEs.
Assume we are given

- a domain $\Omega \subset \mathbb{R}^s$,
- and a linear elliptic PDE of the form

$$\mathcal{L}u(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \text{ in } \Omega, \tag{3}$$

- with (for simplicity of description) Dirichlet boundary conditions

$$u(\boldsymbol{x}) = g(\boldsymbol{x}), \qquad \boldsymbol{x} \text{ on } \partial\Omega. \tag{4}$$

For Kansa's collocation method we then choose to represent the approximate solution $\hat{u}$ by an RBF expansion analogous to that used for scattered data interpolation, i.e.,

$$\hat{u}(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \varphi(\|\boldsymbol{x} - \boldsymbol{\xi}_j\|). \tag{5}$$

- As with Hermite interpolation we now formally distinguish in our notation between centers $\Xi = \{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N\}$ and collocation points $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset \Omega$.
- While formally different, these points will often physically coincide.
- A scenario with $\Xi \neq \mathcal{X}$ will be explored in Chapters 39 and 40.
- For the following discussion we assume the simplest possible setting, i.e., $\Xi = \mathcal{X}$ and no polynomial terms are added to the expansion (5).

The collocation matrix that arises when matching the differential equation (3) and the boundary conditions (4) at the collocation points $\mathcal{X}$ will be of the form

$$\mathsf{A} = \begin{bmatrix} \tilde{\mathsf{A}}_{\mathcal{L}} \\ \tilde{\mathsf{A}} \end{bmatrix}, \tag{6}$$

where the two blocks are generated as follows:

$$\begin{aligned} (\tilde{\mathsf{A}}_{\mathcal{L}})_{ij} &= \mathcal{L}\varphi(\|\boldsymbol{x} - \boldsymbol{\xi}_j\|)|_{\boldsymbol{x}=\boldsymbol{x}_i}, \qquad \boldsymbol{x}_i \in \mathcal{I},\ \boldsymbol{\xi}_j \in \Xi, \\ \tilde{\mathsf{A}}_{ij} &= \varphi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|), \qquad \boldsymbol{x}_i \in \mathcal{B},\ \boldsymbol{\xi}_j \in \Xi. \end{aligned}$$

Here the set $\mathcal{X}$ of collocation points is split into

- a set $\mathcal{I}$ of interior points,
- and a set $\mathcal{B}$ of boundary points.

The problem is well-posed if the linear system $\mathsf{A}\boldsymbol{c} = \boldsymbol{y}$, with $\boldsymbol{y}$ a vector consisting of entries $f(\boldsymbol{x}_i)$, $\boldsymbol{x}_i \in \mathcal{I}$, followed by $g(\boldsymbol{x}_i)$, $\boldsymbol{x}_i \in \mathcal{B}$, has a unique solution.

Remark

- *A change in the boundary conditions (4) is as simple as*
  - *making changes to a few rows of the matrix* A *in (6)*
  - *as well as on the right-hand side* $\boldsymbol{y}$.
- *The description here is rather general with no particular RBF in mind. However, Kansa specifically proposed to use multiquadrics in (5), and consequently the method is also called the multiquadric method.*
- *Kansa also suggests the use of varying shape parameters* ε_j, $j = 1, \ldots, N$.
  - *While the theoretical analysis of the resulting method is near intractable, Kansa shows that this technique improves the accuracy and stability of the method.*
  - *The literature contains very little on the theoretical aspects of varying shape parameters (see [Bozzini et al. (2002)] in the interpolation setting).*

Problem with Kansa's method:

- For a constant shape parameter ε the matrix A may be singular for certain configurations of the centers ξ_j.
- Originally, Kansa assumed that the non-singularity results established by Micchelli for interpolation matrices would carry over to the PDE case.
- As the numerical experiments of [Hon and Schaback (2001)] show, this is not so.
    - This fact is not really surprising since the matrix for the collocation problem is composed of rows that are built from different functions, which — depending on the differential operator $\mathcal{L}$ — might not even be radial.
    - The results for the non-singularity of interpolation matrices, however, are based on the fact that A is generated by a single function φ.

Nevertheless, an indication of the success of Kansa's method are the early papers [Dubal (1992), Dubal (1994), Golberg *et al.* (1996), Kansa (1992), Moridis and Kansa (1994)] and many more since.

Since the numerical experiments of Hon and Schaback show that Kansa's method cannot be well-posed for arbitrary center locations, it is now an open question to find sufficient conditions on the center locations that guarantee invertibility of the Kansa matrix.

One possible approach — built on the basic ideas of the greedy algorithm of Chapter 33 — is to adaptively select "good" centers from a large set of possible candidates.

Following this strategy it is possible to ensure invertibility of the collocation matrix throughout the iterative algorithm.

This approach is described in the recent paper [Ling *et al.* (2006)].

Remark

- *In [Moridis and Kansa (1994)] the authors suggest how Kansa's method can be applied to other types of partial differential equation problems such as*
  - *non-linear elliptic PDEs,*
  - *systems of elliptic PDEs,*
  - *and time-dependent parabolic or hyperbolic PDEs.*
- *We will also see in the next chapter that Kansa's method is well-suited for elliptic problems with variable coefficients.*
- *We will come back to the use of Kansa's method for time-dependent problems in Chapter 42.*
- *We next discuss an alternate approach (based on the symmetric Hermite interpolation method) which does ensure well-posedness of the resulting collocation matrix.*

Assume we are given the same linear elliptic PDE with Dirichlet boundary conditions (see (3),(4)) as before:

$$\begin{aligned} \mathcal{L}u(\boldsymbol{x}) &= f(\boldsymbol{x}), \qquad \boldsymbol{x} \text{ in } \Omega, \\ u(\boldsymbol{x}) &= g(\boldsymbol{x}), \qquad \boldsymbol{x} \text{ on } \partial\Omega. \end{aligned}$$

To apply the results from generalized Hermite interpolation that will ensure the non-singularity of the collocation matrix we use:

$$\hat{u}(\boldsymbol{x}) = \sum_{j=1}^{N_{\mathcal{I}}} c_j \mathcal{L}^{\xi} \varphi(\|\boldsymbol{x} - \xi\|)|_{\xi=\xi_j} + \sum_{j=N_{\mathcal{I}}+1}^{N} c_j \varphi(\|\boldsymbol{x} - \xi_j\|). \tag{7}$$

Here

$N_{\mathcal{I}}$: number of nodes in the interior of Ω,

$\mathcal{L}^{\xi}$: differential operator acting on φ viewed as a function of ξ, i.e., $\mathcal{L}\varphi$ is equal to $\mathcal{L}^{\xi}\varphi$ up to a possible difference in sign.

The linear functionals λ of Chapter 36 are given by

- $\lambda_j = \delta_{\xi_j} \circ \mathcal{L}$, $j = 1, \ldots, N_{\mathcal{I}}$, and
- $\lambda_j = \delta_{\xi_j}$, $j = N_{\mathcal{I}} + 1, \ldots, N$.

After enforcing the collocation conditions

$$\begin{aligned} \mathcal{L}\hat{u}(\boldsymbol{x}_i) &= f(\boldsymbol{x}_i), \qquad \boldsymbol{x}_i \in \mathcal{I}, \\ \hat{u}(\boldsymbol{x}_i) &= g(\boldsymbol{x}_i), \qquad \boldsymbol{x}_i \in \mathcal{B}, \end{aligned}$$

we end up with a collocation matrix A that is of the form

$$\mathsf{A} = \begin{bmatrix} \hat{\mathsf{A}}_{\mathcal{L}\mathcal{L}^\xi} & \hat{\mathsf{A}}_{\mathcal{L}} \\ \hat{\mathsf{A}}_{\mathcal{L}^\xi} & \hat{\mathsf{A}} \end{bmatrix}. \tag{8}$$

Here the four blocks are generated as follows:

$$\begin{aligned} (\hat{\mathsf{A}}_{\mathcal{L}\mathcal{L}^\xi})_{ij} &= \mathcal{L}\mathcal{L}^\xi \varphi(\|\boldsymbol{x} - \boldsymbol{\xi}\|)|_{\boldsymbol{x}=\boldsymbol{x}_i, \boldsymbol{\xi}=\boldsymbol{\xi}_j}, \qquad \boldsymbol{x}_i, \boldsymbol{\xi}_j \in \mathcal{I}, \\ (\hat{\mathsf{A}}_{\mathcal{L}})_{ij} &= \mathcal{L}\varphi(\|\boldsymbol{x} - \boldsymbol{\xi}_j\|)|_{\boldsymbol{x}=\boldsymbol{x}_i}, \qquad \boldsymbol{x}_i \in \mathcal{I},\ \boldsymbol{\xi}_j \in \mathcal{B}, \\ (\hat{\mathsf{A}}_{\mathcal{L}^\xi})_{ij} &= \mathcal{L}^\xi \varphi(\|\boldsymbol{x}_i - \boldsymbol{\xi}\|)|_{\boldsymbol{\xi}=\boldsymbol{\xi}_j}, \qquad \boldsymbol{x}_i, \in \mathcal{B},\ \boldsymbol{\xi}_j \in \mathcal{I}, \\ \hat{\mathsf{A}}_{ij} &= \varphi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|), \qquad \boldsymbol{x}_i, \boldsymbol{\xi}_j \in \mathcal{B}. \end{aligned}$$

Note that we have identified the two sets $\mathcal{X} = \mathcal{I} \cup \mathcal{B}$ of collocation points and Ξ of centers.

Remark

- *The matrix* A *is of the same type as the generalized Hermite interpolation matrices, and therefore non-singular as long as φ is chosen appropriately.*
- *Thus, viewed using the new expansion (7) for* $\hat{u}$*, the collocation approach is well-posed.*
- *The Hermite-based approach leads to a symmetric matrix (8) as opposed to the unstructured matrix (6) used in the non-symmetric approach.*
  - *The symmetry is of value when trying to devise an efficient implementation of the method.*
- *Although* A *now consists of four blocks, it still is of the same size, namely* $N \times N$*, as the collocation matrix for Kansa's approach.*
- *The symmetric matrix is more complicated to assemble:*
  - *It requires smoother basis functions than the non-symmetric Kansa method.*
  - *It does not lend itself very nicely to the solution of non-linear problems.*

Remark

One attempt to obtain an efficient implementation of the Hermite-based collocation method is a variation of the greedy algorithm described in Chapter 33.

We refer the reader to the original paper [Hon et al. (2003)] for details.

A convergence analysis for the symmetric collocation method was provided in [Franke and Schaback (1998a), Franke and Schaback (1998b)].

The error estimates established in those papers require the solution of the PDE to be very smooth.

Therefore, one should be able to use meshfree radial basis function collocation techniques especially well for (high-dimensional) PDE problems with smooth solutions on possibly irregular domains.

Due to the known counterexamples from [Hon and Schaback (2001)] for the non-symmetric method, a convergence analysis is still lacking for that method.

However, for an adaptive version of the non-symmetric method Schaback recently analyzed the convergence in [Schaback (2007)].

A convergence result for symmetric collocation [Wendland (2005a)]:

Theorem

Let $\Omega \subseteq \mathbb{R}^s$ be a polygonal and open region. Let $\mathcal{L} \neq 0$ be a second-order linear elliptic differential operator with coefficients in $C^{2(k-2)}(\bar{\Omega})$ that either vanish on $\bar{\Omega}$ or have no zero there. Suppose that $\Phi \in C^{2k}(\mathbb{R}^s)$ is a strictly positive definite function. Suppose further that the boundary value problem

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution $u \in \mathcal{N}_\Phi(\Omega)$ for given $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Let $\hat{u}$ be the approximate collocation solution of the form (7) based on $\Phi = \varphi(\|\cdot\|)$. Then

$$\|u - \hat{u}\|_{L_\infty(\Omega)} \leq Ch^{k-2}\|u\|_{\mathcal{N}_\Phi(\Omega)}$$

for all sufficiently small h, where h is the larger of the fill distances in the interior and on the boundary of Ω, respectively.

Remark

- *The proof uses the same techniques as in Chapter 14 and takes advantage of a "splitting theorem" that permits splitting the error into*
  - *a boundary error and*
  - *an error in the interior.*
- *As a consequence of the proof Wendland suggests that the collocation points and centers be chosen so that the fill distance on the boundary is smaller than in the interior since the approximation orders differ by a factor ℓ (for differential operators of order ℓ).*
  - *He suggests distributing the points so that*

$$h_{\mathcal{I},\Omega}^{k-\ell} \approx h_{\mathcal{B},\partial\Omega}^{k}.$$

- *Some numerical evidence for convergence rates of the symmetric collocation method is given by the examples in the next chapter, and in [Jumarhon et al. (2000), Power and Barraco (2002)].*

Since the methods described above were both originally used with globally supported basis functions, the same concerns about

- stability and
- numerical efficiency

apply as for interpolation problems.

The two recent papers [Ling and Kansa (2004), Ling and Kansa (2005)] address these issues.

In particular, the authors develop a preconditioner in the spirit of the one by Beatson and co-workers for the GMRES method (see Chapter 34).

They describe their experience with a domain decomposition algorithm.

Remark

- *Recently, [Miranda (2004)] showed that Kansa's method will be well-posed if it is combined with so-called R-functions.*
    - *R-functions were also used by Höllig and his co-workers in their development of web-splines (see, e.g., [Höllig (2003)]).*
- *In [Fedoseyev et al. (2002)] it is suggested that the collocation points on the boundary should also be used to satisfy the PDE.*
    - *The motivation for this is the well-known fact that both for interpolation and collocation with RBFs the error is largest near the boundary.*
    - *In order to prevent the collocation matrix from becoming trivially singular (by using duplicate columns, i.e., basis functions) it is suggested in [Fedoseyev et al. (2002)] that the corresponding centers lie outside the domain* Ω *(thus creating additional basis functions).*
    - *In various numerical experiments this strategy is shown to improve the accuracy of Kansa's non-symmetric method.*
    - *We implement this approach in the next chapter. However, once more there is no theoretical foundation for this modification of either the non-symmetric or the symmetric method.*

Remark (cont.)

- [Larsson and Fornberg (2003)] compare
  - Kansa's basic collocation method,
  - the modification just described, and
  - the Hermite-based symmetric approach.
- Using multiquadric basis functions in a standard implementation they conclude that the symmetric method is the most accurate, followed by the non-symmetric method with boundary collocation.
- The reason is the better conditioning of the symmetric system.

Remark (cont.)

- They also discuss an implementation of the three methods using the Contour-Padé method of Chapter 16.
  - With this technique stability problems are overcome.
  - Symmetric and non-symmetric collocation have comparable accuracy.
  - Boundary collocation of the PDE yields an improvement only if these conditions are used as additional equations, i.e., by increasing the problem size.
  - Often the most accurate results were achieved with values of the multiquadric shape parameter ε that would lead to severe ill-conditioning using a standard implementation.
  - Therefore these results could be achieved only using the complex integration method.
- They concluded that RBF collocation is much more accurate than standard second-order finite differences or even a standard Fourier-Chebyshev pseudospectral method.

Remark (cont.)

- [Leitão (2001)] applies the symmetric collocation method to a fourth-order Kirchhoff plate bending problem and emphasizes the simplicity of the implementation of the RBF collocation method.
- [Mai-Duy and Tran-Cong (2001)] suggest a collocation method for which the basis functions are taken to be anti-derivatives of the usual RBFs.
- In [Young *et al.* (2004)] the authors discuss the solution of 2D and 3D Stokes' systems by a self-consistent iterative approach based on Kansa's non-symmetric method.

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