# Stability Of Paths

**In what conditions is a path through a dynamic system stable?**

Mathematics Extended Essay

Colonel By Secondary School

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## Abstract

In this paper, we first reestablish the foundations of classical Lyapunov point stability. We then extend Lyapunov's arguments and establish a rigorous definition of Path Stability, in keeping with Lyapunov's original reasoning. Next, we note some important properties derived from the definition of Path Stability which allow the emergent behavior of the path to be decomposed and analyzed individually on each point. Finally, by linearizing the differential equation describing change of state, we arrive at insights linking the stability of the path to the Quadratic Form of the Jacobian of the system, another problem that has extensive literature in the field of linear algebra, demonstrating the concept of Path Stability has deep conceptual roots reaching into the heart of mathematics.

# Contents

# 1 Research Question

Given a particle travelling under a dynamic system, under what conditions will an arbitrarily small disturbance to the initial position always result in an arbitrarily small displacement to the new path, regardless of the time?

# 2 Introduction

In life, we can often describe the phenomenon we observe as a product of two factors: ever-present change and an element of randomness.

Firstly, the way in which things change often can be described as a dynamic system, which are systems whose development in time is dependent upon their current state. Dynamic systems are integral to many fields of study; through Hamiltonian mechanics, we can understand motion as a dynamic system whose state is defined as the combination of position and velocity. In ecological models, populations will grow if resources are abundant, and shrink if resources are scarce. Hence, the change from one year to the next is largely dependent upon the population of a species from the year before.

Secondly, no idealized model is ever perfectly accurate to its real world counterpart because of the prevalence of random disturbances of various magnitudes throughout every aspect of life. The work of the Russian mathematician Aleksandr Lyapunov [2] at the turn of the 20th century has given mathematicians frameworks and methods for understanding the stability of a system; in other words, to what degree the system is resilient to small initial disturbances. In the following century, this topic has been extensively studied with respect to stable *points*, but not much work exists in comparison in studying the stability of entire *paths*, studying how the state of a system develops over an arbitrarily large duration of time.

# 3 Dynamic Systems

For me, the consideration of the problem of stability arose from contemplating a ball in a rounded crater. The ball may start at any point along within the semi-circular cavity, but would always roll down and come to rest at the center. In this paper, we will be considering systems whose state in its entirety can be specified using a combination of $n$ real numbers.

# 4 Lyapunov Stability of the First Kind

We begin this section by defining the system $\mathcal{S}$ as $f_s(\boldsymbol{x}) : \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D} \subseteq \mathbb{R}^n$. We use $\boldsymbol{x}_0$ to represent a general vector in $\mathcal{D}$, and define $P_{x_0}(t) : [0, \infty) \to \mathcal{D}$ to be a path function with the following properties. $P_{x_0}(0) = \boldsymbol{x}_0$ and $\frac{d}{dt}\left(P_{x_0}(t)\right) = \dot{P}_{x_0}(t) = f_s(P_{x_0}(t))$.

**Definition 4.1** (Lyapunov Stability)**.** From Aleksandr Lyapunov [2], we say that the system $\mathcal{S}$ is stable at $\boldsymbol{x}_e$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $P_{x_0}(t)$ where $\|\boldsymbol{x}_e - P_{x_0}(0)\| < \delta$, we have $\|\boldsymbol{x}_e - P_{x_0}(t)\| < \epsilon$ for all $t \in [0, \infty)$.

*Remark.* Intuitively, a system is considered stable at the point $\boldsymbol{x}_e$ if, for every open ball $B_\epsilon$ around $\boldsymbol{x}_e$ of size $\epsilon$, there exists another open ball $B_\delta$ of size $\delta$ such that all paths that start in $B_\delta$ will stay within $B_\epsilon$.

# 5 Bounded Path Stability

Next, I used the method of defining stability in definition 4.1 and changed it to apply to paths as a generalization of point stability.

**Definition 5.1** (Bounded Stability)**.** A path function $P_{x_0}(t)$ is said to be bounded stable if, for *every* $\epsilon > 0$, there exists $\delta > 0$ such that for every

$\boldsymbol{x}_\delta$ where $\|\boldsymbol{x}_\delta\| < \delta$, we have $\|P_{x_0}(t) - P_{(x_0+x_\delta)}(t)\| < \epsilon$ for all $t \in [0, \infty)$

**Definition 5.2** (Local Attractiveness)**.** A point $\boldsymbol{x}_0 \in \mathcal{D}$ is said to be locally attractive if, for every $\epsilon > 0$, there exists $\delta \in (0, \epsilon]$ such that for every $\boldsymbol{x}_\delta$ where $\|\boldsymbol{x}_\delta\| < \delta$, there exists $\sigma > 0$ for which the following is true for all $\Delta t \in [0, \sigma)$.

$$\|P_{x_0}(0) - P_{(x_0+x_\delta)}(0)\| \geq \|P_{x_0}(\Delta t) - P_{(x_0+x_\delta)}(\Delta t)\|. \tag{1}$$

**Theorem 5.1** (Time Invariance)**.** *For any two paths $P_{\boldsymbol{x}_0}$ and $P_{\boldsymbol{x}_1}$, if there exists $t_0, t_1 \geq 0$ such that $P_{\boldsymbol{x}_0}(t_0) = P_{\boldsymbol{x}_1}(t_1)$, then $P_{\boldsymbol{x}_0}(t_0 + s) = P_{\boldsymbol{x}_1}(t_1 + s)$ for all $s \geq 0$.*

*Proof.* From the Taylor series expansion of a function $f(t)$ as found in Taylor [3],

$$\begin{aligned}
P_{\boldsymbol{x}_0}(t_0 + s) &= P_{\boldsymbol{x}_0}(t_0) + \sum_{n=1}^{\infty} \frac{s^n}{n!} P_{\boldsymbol{x}_0}^{(n)}(t_0) \\
&= P_{\boldsymbol{x}_0}(t_0) + \sum_{n=1}^{\infty} \frac{s^n}{n!} \left( \frac{d}{dt} P_{\boldsymbol{x}_0}^{(n-1)}(t) \bigg|_{t=t_0} \right) \\
&= P_{\boldsymbol{x}_0}(t_0) + \sum_{n=1}^{\infty} \frac{s^n}{n!} \left( \boldsymbol{f}_s^{(n-1)} \left( P_{\boldsymbol{x}_0}(t_0) \right) \right) \\
&= P_{\boldsymbol{x}_1}(t_1) + \sum_{n=1}^{\infty} \frac{s^n}{n!} \left( \boldsymbol{f}_s^{(n-1)} \left( P_{\boldsymbol{x}_1}(t_1) \right) \right) \\
&= P_{\boldsymbol{x}_1}(t_1 + s)
\end{aligned}$$

□

**Theorem 5.2.** *For some path $P_{\boldsymbol{x}_0}$, if every point on the path denoted by $\boldsymbol{x}_t = P_{\boldsymbol{x}_0}(t)$, where $t \in [0, \infty)$, is locally attractive, then the path $P_{\boldsymbol{x}_0}$ is bounded stable.*

*Proof.* Let $\tau > 0$ be an arbitrarily large length of time. For a given $\epsilon > 0$ and some $t \in [0, \tau]$, let $\delta_t \in (0, \epsilon]$ be equal to the corresponding $\delta$ value given in definition 5.2. We define $\delta_{min} = min\{\delta_t : 0 \le t \le \tau\}$, and let $P_{\boldsymbol{x}_\phi}(t)$ be any path such that $\|P_{\boldsymbol{x}_\phi}(0) - P_{\boldsymbol{x}_0}(0)\| < \delta_{min}$.

In the following, we use proof by contradiction to prove that, for all $\boldsymbol{x}_\phi$ where $\|\boldsymbol{x}_\phi - \boldsymbol{x}_0\| < \delta_{min}$ and $t \in [0, \tau]$, $\|P_{\boldsymbol{x}_\phi}(t) - P_{\boldsymbol{x}_0}(t)\| \le \|\boldsymbol{x}_\phi - \boldsymbol{x}_0\|$.

We let $D(t) = \|P_{\boldsymbol{x}_\phi}(t) - P_{\boldsymbol{x}_0}(t)\|$, and assume that there exists some $t_0 \in [0, \tau]$ such that $D(t_0) > D(0) = \|\boldsymbol{x}_\phi - \boldsymbol{x}_0\|$. By definition of a path function, $P_{\boldsymbol{x}_\phi}(t)$ and $P_{\boldsymbol{x}_0}(t)$ are continuous on domain $[0, \infty)$ and the norm $\|\boldsymbol{x}\|$ is continuous on domain $\mathcal{D}$. Therefore, $D(t)$ is also continuous on $t \in [0, \infty)$, and there must exist some $t_1 \in [0, t_0)$ such that $D(t_1) = D(0)$ and,

$$D(t_k) > D(t_1) = D(0) = \|\boldsymbol{x}_\phi - \boldsymbol{x}_0\|, \qquad \forall\ t_k \in (t_1, t_0)\,. \tag{2}$$

We let $\boldsymbol{x}_\alpha = P_{\boldsymbol{x}_0}(t_1)$ and $\boldsymbol{x}_\beta = P_{\boldsymbol{x}_\phi}(t_1)$. Then we have,

$$\begin{aligned} D(t_1) &= \|P_{\boldsymbol{x}_\phi}(t_1) - P_{\boldsymbol{x}_0}(t_1)\| \\ &= \|\boldsymbol{x}_\beta - \boldsymbol{x}_\alpha\| \\ &= D(0) \\ &= \|P_{\boldsymbol{x}_\phi}(0) - P_{\boldsymbol{x}_0}(0)\| \\ &= \|\boldsymbol{x}_\phi - \boldsymbol{x}_0\| \\ &< \delta_{min} \le \delta_{t_1}, \end{aligned} \tag{3}$$

from which there exists some $\sigma$ such that for all $\Delta t \in [0, \sigma)$, we have,

$$\|P_{\boldsymbol{x}_\alpha}(0) - P_{\boldsymbol{x}_\beta}(0)\| \ge \|P_{\boldsymbol{x}_\alpha}(\Delta t) - P_{\boldsymbol{x}_\beta}(\Delta t)\| \tag{4}$$

by local attractiveness.

Since $P_{\boldsymbol{x}_\alpha}(0) = \boldsymbol{x}_\alpha$, we have $P_{\boldsymbol{x}_0}(t_1 + s) = P_{\boldsymbol{x}_\alpha}(s)$ for all $s \ge 0$ by theorem 5.1. Similarly, we have $P_{\boldsymbol{x}_\phi}(t_1 + s) = P_{\boldsymbol{x}_\beta}(s)$ for all $s \ge 0$. By setting $s = 0$ on the left side of equation 4 and $s = \Delta t$ on the right side of equation 4, we obtain,

$$D(t_1) = \|P_{\boldsymbol{x}_0}(t_1) - P_{\boldsymbol{x}_\phi}(t_1)\| \geq \|P_{\boldsymbol{x}_0}(t_1 + \Delta t) - P_{\boldsymbol{x}_\phi}(t_1 + \Delta t)\|,$$

for some $\Delta t \in (0, t_0 - t_1)$. This is contradictory with equation 2. Therefore, $\|P_{\boldsymbol{x}_\phi}(t) - P_{\boldsymbol{x}_0}(t)\| \leq \|\boldsymbol{x}_\phi - \boldsymbol{x}_0\| < \delta_{min}$ for all $t \in [0, \tau]$. Since $\tau$ can be an arbitrarily large length of time, we can conclude that $P_{\boldsymbol{x}_0}(t)$ is bounded stable by denoting $\delta = \delta_{min}$ and $\boldsymbol{x}_\delta = \boldsymbol{x}_\phi - \boldsymbol{x}_0$. □

**Theorem 5.3.** *For some point $\boldsymbol{x}_0$, if there exists $\delta > 0$ such that for every $\boldsymbol{x}_\alpha$ where $\|\boldsymbol{x}_0 - \boldsymbol{x}_\alpha\| < \delta$, the following inequality holds.*

$$(\boldsymbol{x}_0 - \boldsymbol{x}_\alpha) \bullet \left(\dot{P}_{\boldsymbol{x}_0}(0) - \dot{P}_{\boldsymbol{x}_\alpha}(0)\right) < 0.$$

*Then $\boldsymbol{x}_0$ is locally attractive.*

*Proof.* Let $\boldsymbol{x}(t) : \mathbb{R} \to \mathbb{R}^n$ be a continuous and differentiable vector valued function. Let the function $d(t) = \|\boldsymbol{x}(t)\|$ be the norm of $\boldsymbol{x}(t)$ for $t \in \mathbb{R}$. For any time $t$, we re-write $\boldsymbol{x}(t)$ as $(x_1(t), x_2(t)..., x_n(t))$ where $x_i(t) : \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, \cdots, n$. By definition of the norm in $\mathcal{R}^n$,

$$\begin{aligned}
(d(t))^2 &= \sum_{i=1}^{n} x_i(t)^2 && \Longrightarrow \\
\frac{d}{dt}\left((d(t))^2\right) &= \frac{d}{dt}\left(\sum_{i=1}^{n} x_i(t)^2\right) && \Longrightarrow \\
2d(t)\dot{d}(t) &= 2\sum_{i=1}^{n} x_i(t)\dot{x}_i(t) && \Longrightarrow \\
d(t)\dot{d}(t) &= \boldsymbol{x}(t) \bullet \dot{\boldsymbol{x}}(t) && (5)
\end{aligned}$$

Given the paths $P_{\boldsymbol{x}_0}, P_{\boldsymbol{x}_\alpha}$ as defined in theorem 5.3, we define $\boldsymbol{x}_\delta(t) =$

$P_{\boldsymbol{x}_\alpha}(t) - P_{\boldsymbol{x}_0}(t)$, which implies $\dot{\boldsymbol{x}}_\delta(t) = \dot{P}_{\boldsymbol{x}_\alpha}(t) - \dot{P}_{\boldsymbol{x}_0}(t)$. Therefore,

$$\boldsymbol{x}_\delta(t) \bullet \dot{\boldsymbol{x}}_\delta(t) = (P_{\boldsymbol{x}_\alpha}(t) - P_{\boldsymbol{x}_0}(t)) \bullet \left(\dot{P}_{\boldsymbol{x}_\alpha}(t) - \dot{P}_{\boldsymbol{x}_0}(t)\right). \tag{6}$$

Note that while $\boldsymbol{x}_\delta(t)$ is a continuous and differentiable vector valued function because it is the sum of two continuous and differentiable path functions, $\boldsymbol{x}_\delta(t)$ itself is not necessarily a valid path.

From the inequality of theorem 5.3, equation 5, and equation 6, we have $d_\delta(0)\dot{d}_\delta(0) < 0$ where $d_\delta(t) = \|\boldsymbol{x}_\delta(t)\| = \|P_{\boldsymbol{x}_\alpha}(t) - P_{\boldsymbol{x}_0}(t)\|$ for all $t \in \mathbb{R}$. Since $d_\delta(t) \geq 0$ for all $t \in \mathbb{R}$ by the definition of the norm, $\dot{d}_\delta(0) < 0$. By the Mean Value Theorem [3] and the definition of the derivative, there exists $t_1 > t_0 = 0$ such that for the following is true for all $\Delta t \in (0, t_1)$.

$$\|P_{\boldsymbol{x}_\alpha}(\Delta t) - P_{\boldsymbol{x}_0}(\Delta t)\| = d_\delta(\Delta t) < d_\delta(0) = \|P_{\boldsymbol{x}_\alpha}(0) - P_{\boldsymbol{x}_0}(0)\|. \tag{7}$$

Therefore, for every $\epsilon > 0$, there exists $\hat{\delta} = \min\{\epsilon, \delta\} \in (0, \epsilon]$ such that for all $\boldsymbol{x}_\delta = \boldsymbol{x}_\alpha - \boldsymbol{x}_0$ with $\|\boldsymbol{x}_\delta\| < \hat{\delta}$, we always can find $\sigma = t_1 > 0$ for which the following is true for all $\Delta t \in (0, \sigma)$.

$$\|P_{\boldsymbol{x}_0}(0) - P_{\boldsymbol{x}_\alpha}(0)\| \geq \|P_{\boldsymbol{x}_0}(\Delta t) - P_{\boldsymbol{x}_\alpha}(\Delta t)\|. \tag{8}$$

By definition 5.2, the point $\boldsymbol{x}_0$ is locally attractive. $\square$

## 6 Linearization

To linearize the function $\boldsymbol{f}_s$, we evaluate its Jacobian as defined by Taylor [3].

**Definition 6.1** (Jacobian)**.** Given $\boldsymbol{x}_0$ and some unit vector $\boldsymbol{x}_\delta$ for which $\|\boldsymbol{x}_\delta\| = 1$, the Jacobian matrix of $\boldsymbol{f}_s$ evaluated at point $\boldsymbol{x}_0$ is the linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$ denoted by $\boldsymbol{J}_{\boldsymbol{f}_s}(\boldsymbol{x}_0)$ and given in the following equation.

$$\lim_{c \to 0} \frac{f_s(\boldsymbol{x}_0 + c\boldsymbol{x}_\delta) - f_s(\boldsymbol{x}_0)}{c} = \boldsymbol{J}_{\boldsymbol{f}_s}(\boldsymbol{x}_0)(\boldsymbol{x}_\delta). \tag{9}$$

**Definition 6.2** (Negative Definite)**.** A linear transformation denoted by $\mathscr{T}(\boldsymbol{x}) \in \mathcal{L}_{n,n}$ is said to be Negative Definite [1] if, for each non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T \mathscr{T} \boldsymbol{x} < 0$.

**Theorem 6.1.** *For a given path $P_{\boldsymbol{x}_0}(t)$, if for each point $\boldsymbol{x}_t = P_{\boldsymbol{x}_0}(t)$, the linear transformation led by $\boldsymbol{J}_{\boldsymbol{f}_s}(\boldsymbol{x}_t)$ is negative definite, then $P_{\boldsymbol{x}_0}(t)$ is bounded stable.*

*Proof.* From definition 6.2, the linear transformation led by $\boldsymbol{J}_{\boldsymbol{f}_s}(\boldsymbol{x}_t)$ is negative definite, if and only if for unit displacement vectors $\boldsymbol{x}_\delta$, we have,

$$(\boldsymbol{x}_\delta)^T \boldsymbol{J}_{\boldsymbol{f}_s}(\boldsymbol{x}_t)(\boldsymbol{x}_\delta) = (\boldsymbol{x}_\delta) \cdot \ (\boldsymbol{J}_{\boldsymbol{f}_s}(\boldsymbol{x}_t)(\boldsymbol{x}_\delta)) < 0. \tag{10}$$

The inner product $\boldsymbol{x} \cdot \ \boldsymbol{y}$ is defined to be continuous for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. If $\boldsymbol{x} \cdot \ \boldsymbol{y} < 0$, by definition of continuity, there exists some $\epsilon_1 > 0$ such that for all $\boldsymbol{z} \in \mathbb{R}^n$,

$$\|\boldsymbol{z} - \boldsymbol{y}\| < \epsilon_1 \implies \boldsymbol{x} \cdot \ \boldsymbol{z} < 0. \tag{11}$$

From equation 9, we know that for any $\epsilon_1 > 0$, we can find $\delta_1 > 0$ such that for scalar $c$,

$$|c| \in (0, \delta_1) \implies \|\frac{f_s(\boldsymbol{x}_t + c\boldsymbol{x}_\delta) - f_s(\boldsymbol{x}_t)}{c}\| < \epsilon_1 \tag{12}$$

Combining equation 11 and equation 12 into equation 10, we obtain,

$$c\boldsymbol{x}_\delta \cdot \left( \frac{f_s(\boldsymbol{x}_t + c\boldsymbol{x}_\delta) - f_s(\boldsymbol{x}_t)}{c} \right) < 0, \tag{13}$$

which gives rise to,

$$\boldsymbol{x}_\delta \cdot \ (f_s(\boldsymbol{x}_t + c\boldsymbol{x}_\delta) - f_s(\boldsymbol{x}_t)) < 0. \tag{14}$$

Because for all $\epsilon_1 > 0$, we can find $\delta_1 > 0$ such that, for all $\boldsymbol{x}_{\delta_1} = |c|\boldsymbol{x}_\delta$ satisfying $\|\boldsymbol{x}_{\delta_1}\| < \delta_1$, the following is true.

$$\boldsymbol{x}_{\delta_1} \cdot \ (f_s(\boldsymbol{x}_t + \boldsymbol{x}_{\delta_1}) - f_s(\boldsymbol{x}_t)) < 0. \tag{15}$$

By theorem 5.3, the point $\boldsymbol{x}_t$ is locally attractive. Because this applies to an arbitrary $\boldsymbol{x}_t = P_{\boldsymbol{x}_0}(t)$, all points on the path are locally attractive. By theorem 5.2, the path $P_{\boldsymbol{x}_0}(t)$ is bounded stable as defined in 5.1.
QUOD ERAT DEMONSTRANDUM. $\square$

# 7 Conclusion

This paper has explored the idea of stability and its standard mathematical representation. Taking Lyapunov's original idea, we have used the same epsilon-delta formalization to expand the concept to include entire paths. In determining if a path meets this definition of stability, it has been shown that if the set of all points on the path meet the criteria for local attractiveness, as defined in the paper, then the path overall must also be stable. In addition, by showing that points where the quadratic form of the Jacobian is negative determinant are always locally attractive, the paper presents a direct method of calculating the overall stability of a path, and provides areas where this concept can be developed even further, most namely into the complex vector space, as well as into stochastic or time-variant systems.

## References

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[3] Angus E. Taylor and W. Robert Mann. *Advanced Calculus*. John Wiley and Sons, Inc., 3rd edition, 1983.