What is the Maximal Set of All Functions $\langle \boldsymbol{x} | \boldsymbol{y} \rangle$ That Satisfy The Properties of An Inner Product?

Mathematics Essay

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Contents

1 Introduction

First, to define what is meant by a valid inner product:

Definition 1.1 (Inner Product)**.** Given a vector space $\mathbb{V}$ with a corresponding scalar field $\mathbb{S}$, it is said to posses an inner product if there exists a function $\langle \boldsymbol{x}|\boldsymbol{y}\rangle : \mathbb{V}\times\mathbb{V} \to \mathbb{S}$ defined over the entire vector space that satisfies the following conditions (Taylor and Mann (1983)):

1. Scalar Associative in Second Argument: $\langle \boldsymbol{x}|c\boldsymbol{y}\rangle = c\,\langle \boldsymbol{x}|\boldsymbol{y}\rangle\ \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}, c \in \mathbb{S}$
2. Distributive In Second Argument: $\langle \boldsymbol{x}|\ \boldsymbol{y}+\boldsymbol{z}\rangle = \langle \boldsymbol{x}|\boldsymbol{y}\rangle + \langle \boldsymbol{x}|\boldsymbol{z}\rangle\ \ \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{V}$
3. Conjugate Symmetry: $\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \overline{\langle \boldsymbol{y}|\boldsymbol{x}\rangle}\ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$
4. Positive Semi-Definite: $\langle \boldsymbol{x}|\boldsymbol{x}\rangle > 0 \leftrightarrow \boldsymbol{x} \neq \vec{0}, \langle \boldsymbol{x}|\boldsymbol{x}\rangle = 0 \leftrightarrow \boldsymbol{x} = \vec{0}$

In this essay, I will attempt to prove the following:

Theorem 1.1. *Assume vector space $\mathbb{V}$ has valid inner product $\langle|\rangle$. Let $\mathcal{L}_\circ$ denote the set of Hermitian positive-definite linear operators on $\mathbb{V}$.*

The set of valid inner products $\langle|\rangle'$ on $\mathbb{V}$ is given by the set

$$\{\,\langle\ |\ \rangle' \mid \langle \boldsymbol{x}|\boldsymbol{y}\rangle' = \langle \boldsymbol{x}|H\boldsymbol{y}\rangle\,, H \in \mathcal{L}_\circ\,\} \tag{1}$$

To prove this, I will first prove that every function $\langle|\rangle'$ generated in this way satisfies the properties of definition (1.1).

Then I will show that unique linear operator H generates a unique inner product $\langle|\rangle^H$.

Finally, I will prove that for any arbitrary function $f_i(\boldsymbol{x}, \boldsymbol{y})$ satisfying the properties of definition (1.1), we can always find an associated Hermitian operator $H_i \in \mathcal{L}_\circ$ such that

$$f_i(\boldsymbol{x}, \boldsymbol{y}) = \langle \boldsymbol{x}|H_i\boldsymbol{y}\rangle\ \ \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{V} \tag{2}$$

2 Inner Product Validity

To begin, first I will define what is meant by a Hermitian positive-definite linear operator. For the rest of this essay, let $\mathcal{L}$ denote the set of all linear operators on $\mathbb{V}$, with different subscripts denoting different subsets of $\mathcal{L}$.

Definition 2.1 (Adjoint)**.** Let $A \in \mathcal{L}$ be a linear operator defined in the vector space. Define the adjoint of A, denoted A^*, such that for any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$,

$$\langle \boldsymbol{x} | A\boldsymbol{y} \rangle = \langle A^* \boldsymbol{x} | \boldsymbol{y} \rangle \tag{3}$$

A linear operator H is then said to be Hermitian iff it is self-adjoint, or in other words:

$$\langle \boldsymbol{x} | H\boldsymbol{y} \rangle = \langle H \boldsymbol{x} | \boldsymbol{y} \rangle \tag{4}$$

Definition 2.2 (Positive Definite)**.** Let $A \in \mathcal{L}$ be a linear operator in the vector space. A is defined to be positive definite iff, for any vector $\boldsymbol{x} \neq \vec{0} \in \mathbb{V}$,

$$\langle \boldsymbol{x} | A\boldsymbol{x} \rangle > 0 \tag{5}$$

Consequently, the set $\mathcal{L}_\circ$ is defined to be the set of linear operators satisfying both conditions from equations (4) and (5).

Theorem 2.1 (Inner Product Validity)**.** *Define $H \in \mathcal{L}_\circ$ to be an arbitrary non-singular linear operator. Define the function $\langle \boldsymbol{x} | \boldsymbol{y} \rangle'$ from above, such that $\langle \boldsymbol{x} | \boldsymbol{y} \rangle' = \langle \boldsymbol{x} | H\boldsymbol{y} \rangle$. The function $\langle \boldsymbol{x} | \boldsymbol{y} \rangle'$ satisfies the properties of a valid inner product.*

Proof. Going down each item in definition (1.1), let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{V}$ be any arbitrary vectors and $c \in \mathbb{S}$ be any arbitrary scalar.

1. $\langle \boldsymbol{x} | c\boldsymbol{y} \rangle' = \langle \boldsymbol{x} | H(c\boldsymbol{y}) \rangle = c \langle \boldsymbol{x} | H\boldsymbol{y} \rangle = c \langle \boldsymbol{x} | \boldsymbol{y} \rangle'$

2. $\langle \boldsymbol{x} | \ \boldsymbol{y} + \boldsymbol{z} \rangle' = \langle \boldsymbol{x} | \ H(\boldsymbol{y} + \boldsymbol{z}) \rangle = \langle \boldsymbol{x} | H\boldsymbol{y} \rangle + \langle \boldsymbol{x} | H\boldsymbol{z} \rangle = \langle \boldsymbol{x} | \boldsymbol{y} \rangle' + \langle \boldsymbol{x} | \boldsymbol{z} \rangle'$

3. $\langle \boldsymbol{x} | \boldsymbol{y} \rangle' = \langle \boldsymbol{x} | H\boldsymbol{y} \rangle = \overline{\langle H\boldsymbol{y} | \boldsymbol{x} \rangle} = \overline{\langle \boldsymbol{y} | H\boldsymbol{x} \rangle} = \overline{(\langle \boldsymbol{y} | \boldsymbol{x} \rangle')}$

4. $(\langle \boldsymbol{x} | \boldsymbol{x} \rangle' > 0) \leftrightarrow (\langle \boldsymbol{x} | H\boldsymbol{x} \rangle > 0) \leftrightarrow (H\boldsymbol{x} \neq \vec{0}) \leftrightarrow (\boldsymbol{x} \neq \vec{0})$.

5. $\langle \vec{0} | \vec{0} \rangle' = \langle \vec{0} | H(\vec{0}) \rangle = \langle \vec{0} | \vec{0} \rangle = 0$

$\square$

3 Inner Product Completeness

Next, I will prove that every inner product can be associated with a Hermitian positive-definite linear operator.

Theorem 3.1 (Inner Product Completeness)**.** *Take $f(\boldsymbol{x},\boldsymbol{y})$ to be any function satisfying the definition of an inner product. We can always find an associated linear operator $H \in \mathcal{L}_\circ$ such that*

$$f(\boldsymbol{x},\boldsymbol{y}) = \langle \boldsymbol{x}|H\boldsymbol{y}\rangle \ \ \forall \boldsymbol{x},\boldsymbol{y} \in \mathbb{V} \tag{6}$$

Proof. Given $\langle|\rangle'$, take the function $f_{\boldsymbol{y}}(\boldsymbol{x}) = \langle \boldsymbol{y}|\boldsymbol{x}\rangle'$. This function is a linear transformation from $\mathbb{V}$ to $\mathbb{S}$:

1. $f_{\boldsymbol{y}}(c\boldsymbol{x}) = \langle \boldsymbol{y}|c\boldsymbol{x}\rangle' = c\,\langle \boldsymbol{y}|\boldsymbol{x}\rangle' = cf_{\boldsymbol{y}}(\boldsymbol{x})$
2. $f_{\boldsymbol{y}}(\boldsymbol{x}+\boldsymbol{z}) = \langle \boldsymbol{y}|\ \boldsymbol{x}+\boldsymbol{z}\rangle' = \langle \boldsymbol{y}|\boldsymbol{x}\rangle' + \langle \boldsymbol{y}|\boldsymbol{z}\rangle' = f_{\boldsymbol{y}}(\boldsymbol{x}) + f_{\boldsymbol{y}}(\boldsymbol{z})$

Because $\mathbb{V}$ forms an inner product space, each vector element $\boldsymbol{w}$ uniquely defines a linear transformation from $\mathbb{V} \to \mathbb{S}$ given by $\langle \boldsymbol{w}|\boldsymbol{x}\rangle = L(\boldsymbol{x})$. Moreover, the set of linear scalar maps over $\mathbb{V}$ is isomorphic to the set $\mathbb{V}$ itself, and any linear scalar transform $L_{\boldsymbol{w}}(\boldsymbol{x})$ can be uniquely assigned to a vector $\boldsymbol{w}$ such that $L_{\boldsymbol{w}}(\boldsymbol{x}) = \langle \boldsymbol{w}|\boldsymbol{x}\rangle\ \ \forall \boldsymbol{x} \in \mathbb{V}$ (Proved In Following Section).

For any given $\boldsymbol{y}$, $f_{\boldsymbol{y}}(\boldsymbol{x})$ defines a linear scalar transform, and therefore there exists $\boldsymbol{w}$ such that $f_{\boldsymbol{y}}(\boldsymbol{x}) = \langle \boldsymbol{w}|\boldsymbol{x}\rangle$. Therefore, for any $\boldsymbol{y} \in \mathbb{V}$, there exists unique $\boldsymbol{w} \in \mathbb{V}$ such that $\langle \boldsymbol{y}|\boldsymbol{x}\rangle' = \langle \boldsymbol{w}|\boldsymbol{x}\rangle\ \ \forall \boldsymbol{x} \in \mathbb{V}$. Define the function $H : \mathbb{V} \to \mathbb{V}$ by $\boldsymbol{w} = H(\boldsymbol{y})$, such that $\langle \boldsymbol{y}|\boldsymbol{x}\rangle' \triangleq \langle H(\boldsymbol{y})|\boldsymbol{x}\rangle$. By the properties of linearity, $H(\boldsymbol{y})$ must be linear:

$\langle H(c\boldsymbol{y})|\boldsymbol{x}\rangle = \langle c\boldsymbol{y}|\boldsymbol{x}\rangle' = \langle H(\boldsymbol{y})|\bar{c}\ \boldsymbol{x}\rangle = \langle cH(\boldsymbol{y})|\boldsymbol{x}\rangle\ \ \forall \boldsymbol{x},\boldsymbol{y}\ \ \therefore H(c\boldsymbol{y}) = cH(\boldsymbol{y})$

$\langle H(\boldsymbol{y}+\boldsymbol{z})|\boldsymbol{x}\rangle = \langle \boldsymbol{y}+\boldsymbol{z}|\boldsymbol{x}\rangle' = \langle H(\boldsymbol{y}) + H(\boldsymbol{z})|\boldsymbol{x}\rangle\ \ \therefore H(\boldsymbol{y}+\boldsymbol{z}) = H(\boldsymbol{y}) + H(\boldsymbol{z})$

So that $f(\boldsymbol{y},\boldsymbol{x}) = \langle \boldsymbol{y}|\boldsymbol{x}\rangle' = \langle H\boldsymbol{y}|\boldsymbol{x}\rangle$.

By the property of symmetry,

$$\langle \boldsymbol{y}|\boldsymbol{x}\rangle' = \langle H(\boldsymbol{y})|(\boldsymbol{x})\rangle = \overline{\langle(\boldsymbol{x})|H(\boldsymbol{y})\rangle}, \text{ and}$$
$$\langle \boldsymbol{y}|\boldsymbol{x}\rangle' = \overline{\langle \boldsymbol{x}|\boldsymbol{y}\rangle'} = \overline{\langle H(\boldsymbol{x})|(\boldsymbol{y})\rangle}$$
$$\therefore \langle H(\boldsymbol{x})|(\boldsymbol{y})\rangle = \langle(\boldsymbol{x})|H(\boldsymbol{y})\rangle \tag{7}$$

Equation (7) is true generally for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$, which is only possible if $H = H^*$, and consequently H must be Hermitian.

Finally, to show that $H(\boldsymbol{y})$ must be positive-definite, we see from condition (4) of definition (1.1) that

$$\boldsymbol{x} \neq \vec{0} \longleftrightarrow \langle \boldsymbol{x}|\boldsymbol{x}\rangle' > 0$$

and hence

$$\boldsymbol{x} \neq \vec{0} \longleftrightarrow \langle \boldsymbol{x}|H\boldsymbol{x}\rangle > 0 \tag{8}$$

Therefore, every valid inner product $\langle \boldsymbol{x}|\boldsymbol{y}\rangle'$ can be mapped to a Hermitian, positive definite linear operator such that

$$\langle \boldsymbol{x}|\boldsymbol{y}\rangle' = \langle \boldsymbol{x}|H\boldsymbol{y}\rangle \tag{9}$$

□

References

Taylor, A. E. and Mann, W. R. (1983). *Advanced Calculus*. John Wiley and Sons, Inc., 3rd edition.