Investigating numerical accuracy of the Runge-Kutta Method for simulating Newtonian gravity

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This report discusses results obtained from an orbital simulation program that uses the fourth order Runge-Kutta (RK4) method to solve newtons law of gravitation. It was found that RK4 effectively and accurately solves newtons law of gravitation for a range of orbital motions.

INTRODUCTION

The Runge-Kutta methods were developed by Carl Runge and Wilhelm Kutta during 1900. They are iterative numerical methods for approximating solutions to ordinary differential equations. This report focuses on the application of the fourth-order Runge-Kutta method (RK4) to solve Newton's law of gravitation and investigates the effectiveness of RK4 in this context.

THEORY

RK4 [1]

As mentioned, RK4 is iterative, it evaluates a given function four times within a single time step (h) and uses a weighted average of all four results to approximate a solution at time $t+h$ from a solution at time t. i.e:

$$t_{i+1} = t_i + h \qquad \boldsymbol{r}_{i+1} = \boldsymbol{r}_i + \frac{h}{6}(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4)$$

Where $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3$, $\boldsymbol{k}_4$ are given by:

$$\boldsymbol{k}_1 = f(\boldsymbol{r}_i) \qquad \boldsymbol{k}_2 = f(\boldsymbol{r}_i + \frac{hk_1}{2})$$

$$\boldsymbol{k}_3 = f(\boldsymbol{r}_i + \frac{hk_2}{2}) \qquad \boldsymbol{k}_4 = f(\boldsymbol{r}_i + hk_3)$$

Note: the above is provided the differential equation is first order where:

$$\frac{d\boldsymbol{r}}{dt} = f(\boldsymbol{r})$$

For second order differential equations a second set of k values are calculated for changing velocity.

Newtonian Gravity [2]

Newtons law of gravitation is a second order differential equation where the motion of an orbiting body of mass m around a central body of mass M is given by:

$$m\frac{d^2\boldsymbol{r}}{dt^2} = -\frac{GMm}{|\boldsymbol{r}|^3}\boldsymbol{r} \tag{1}$$

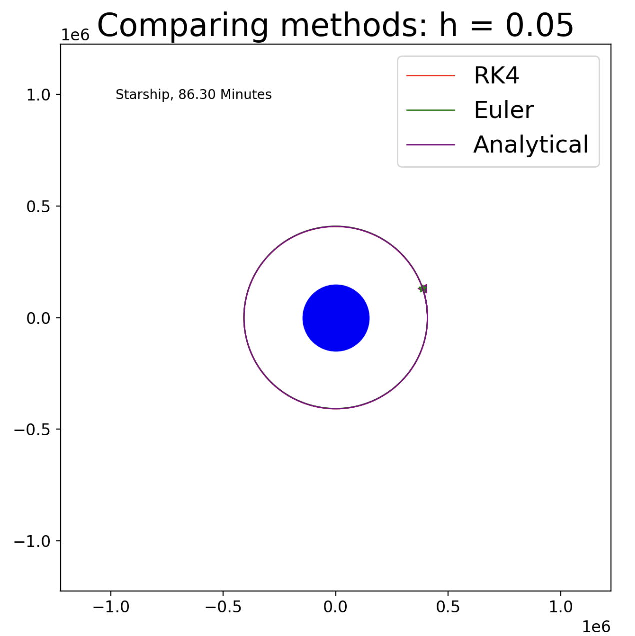

FIG. 1. A figure displaying a circular orbit around earth computed via three methods. The step size for both numerical methods is 0.05.

In order to place a body in a perfect circular orbit this equation can be equated to $F = \frac{mv^2}{r}$ and so:

$$\boldsymbol{v}_0 = \sqrt{\frac{GM}{|\boldsymbol{r}|}}\hat{\boldsymbol{r}}_\perp \tag{2}$$

With the introduction of an additional body. Equation 1 becomes:

$$\frac{d^2\boldsymbol{r}_1}{dt^2} = -Gm_2\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} - Gm_3\frac{\boldsymbol{r}_1 - \boldsymbol{r}_3}{|\boldsymbol{r}_1 - \boldsymbol{r}_3|^3} \tag{3}$$

This is known as the three-body problem see [3].

METHOD ANALYSIS

To investigate exactly how RK4 performs it is necessary to compare to an analytical solution of equation (1). However, finding a general solution requires relatively complex mathematics, a more simple method is to find the analytical solution

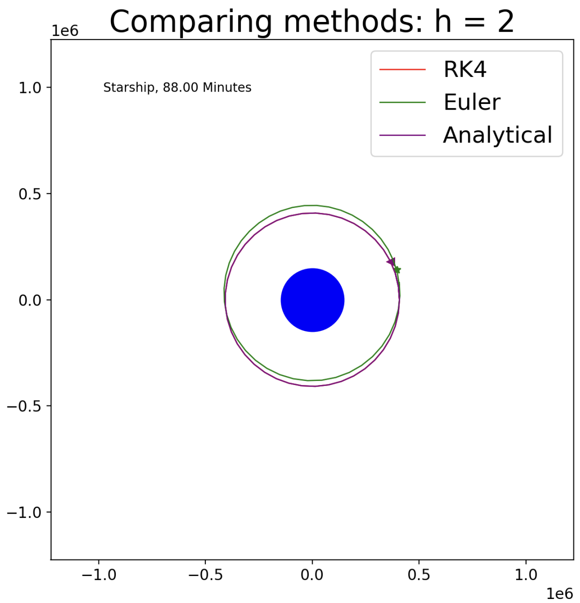

FIG. 2. A figure displaying a circular orbit around earth computed via three methods. The step size for both numerical methods is 2.

TABLE I.

Time Step (h)	Euler (χ^2)	RK4 (χ^2)
2.00	3.7×10^6	430
1.00	8.7×10^5	0.11
0.50	2.2×10^5	3.7×10^{-5}
0.10	8600	6.2×10^{-12}
0.05	2200	2.3×10^{-14}
0.01	105	1.3×10^{-20}

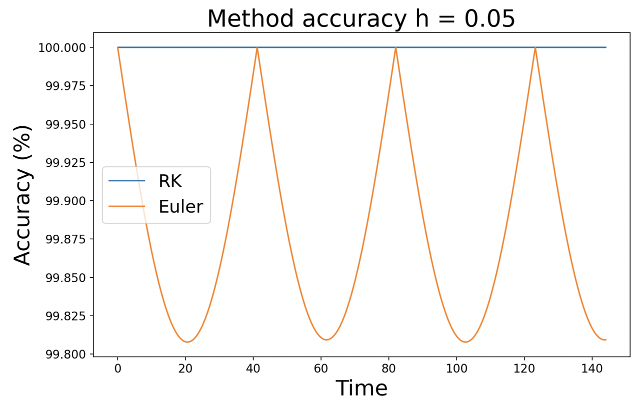

FIG. 3. An accuracy against time plot comparing the Euler and RK4 method for a circular orbit around earth where time step is 0.05.

for a circular orbit specifically. In such case the period of the orbit is defined by:

$$T = \sqrt{\frac{4\pi^2|\boldsymbol{r}|^3}{GM}} \tag{4}$$

and therefore:

$$x(t) = |\boldsymbol{r}|\cos\left(\frac{2\pi t}{T}\right) \quad y(t) = |\boldsymbol{r}|\sin\left(\frac{2\pi t}{T}\right)$$

Computing equation 1 for a circular orbit (where (2) is satisfied) using both the Euler and RK4 method and comparing directly to the analytical solution gave rise to the results displayed by FIG. 1 and 2.

Looking at both figures, visually the RK4 method produces near perfect orbit. Comparatively, the Euler method breaks down significantly when $h = 2$. To quantify accuracy of both methods an accuracy percentage (ϵ) was computed by:

$$\epsilon = 100\left(1 - \frac{|\boldsymbol{r}_m| - |\boldsymbol{r}_{an}|}{|\boldsymbol{r}_{an}|}\right)$$

Where $\boldsymbol{r}_m$ is the $\boldsymbol{r}$ value at a given time computed by a numerical method and $\boldsymbol{r}_{an}$ is the analytical solution at that time (constant for circular orbit). An accuracy against time plot for $h = 0.05$ is displayed by FIG. 3. In addition a table of χ^2 test results using a given time step h for each method is shown by table 1. Where the χ^2 test is defined by:

$$\chi^2 = \sum \frac{(0_i - E_i)^2}{E_i} \tag{5}$$

In this case the observed (O) is $\boldsymbol{r}_m$ and expected (E) is $\boldsymbol{r}_{an}$.

Observing figures 1, 2, 3 and table 1 the effectiveness of RK4 is made clear. In particular, the χ^2 test shows that at low step sizes this numerical method performs absurdly well. In comparison to the Euler method, it is a fair conclusion to say that RK4 is almost always worth the additional computing power, particularly when simulated systems become more complex as observed in the coming section. FIG. 4 is an additional plot of accuracy percentage (ϵ) against time for RK4 only. Notice the periodic nature of this plot and the decrease in accuracy with time, important to keep in mind for longer simulations.

MOON SHOT

Now that confidence in the RK4 method has been established the method can be applied to a three body system. Specifically, shooting a rocket from earths orbit towards the moon followed by a slingshot around the moon and back to earth. Throughout this section the distance from the earth to moon (R_M) was taken to be 382500km [4].

At this stage it is important to define a velocity factor α. Where the velocity of the launched rocket from orbit is defined by:

$$v = \alpha v_0 \tag{6}$$

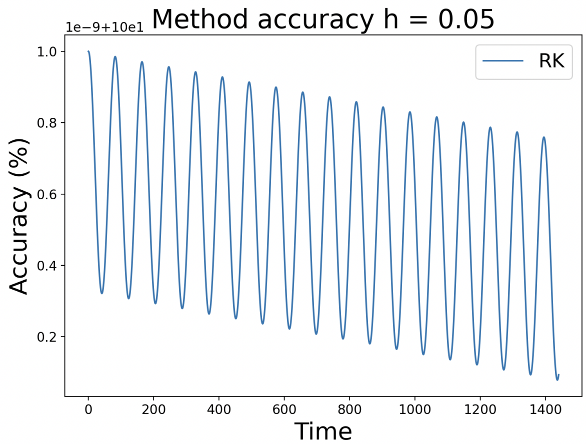

FIG. 4. An accuracy against time plot of the RK4 method for a step size of 0.05.

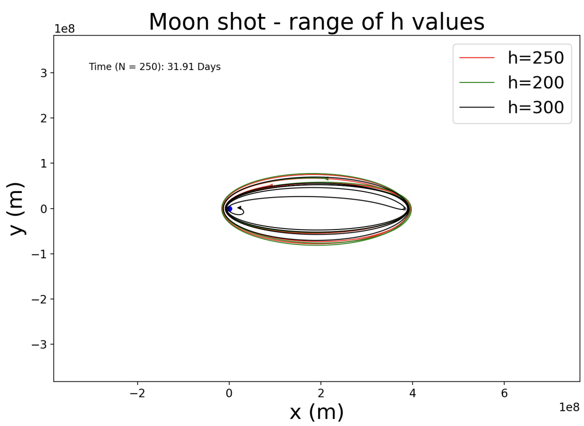

FIG. 5. A figure displaying a moon shot of three rockets with the same initial velocity and a stationary moon. Each rocket has been computed using RK4 with different step size. All have a velocity factor of 1.402.

Here v_0 is the velocity for a circular orbit that satisfies equation 2.

Through trial and error, with a stationary moon, it was found that a velocity factor (α) of ≈ 1.4 gave the perfect moon shot. However, in contrast to the circular orbit, the moon shot takes place over a number of days rather than minutes. Hence, to compute the simulation quickly a higher step size is required. To ensure confidence in the value α, trajectories for a range of step sizes is shown by FIG. 5. From this figure, it is observed that regardless of step size the moon shot successfully completes a moon shot a number of times, however as time progresses, the $h = 300$ rocket loses stability and crashes into the earth. Which is expected since the accuracy of RK4 reduces over time as seen in FIG 4. Since FIG. 5 is a little messy i have included a clear moon shot on a 3D axis (FIG. 6).

To further highlight the versatility of RK4, circular orbital motion was added to the moon. Again, through trial and error,

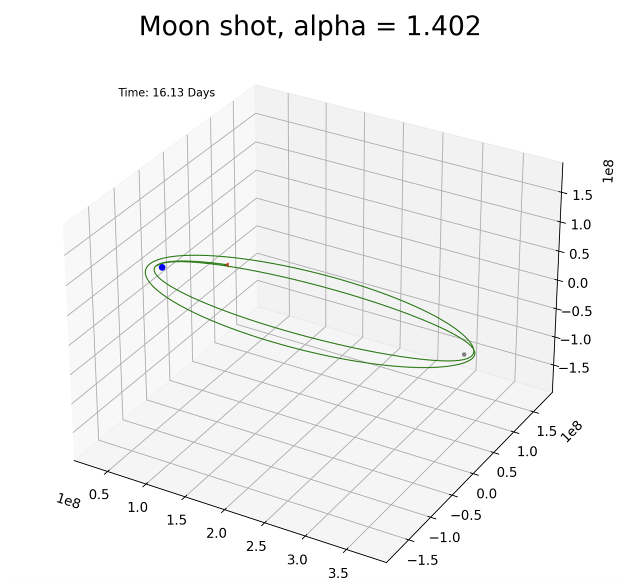

FIG. 6. A figure displaying a moon shot on a 3D axis where the velocity factor is 1.402.

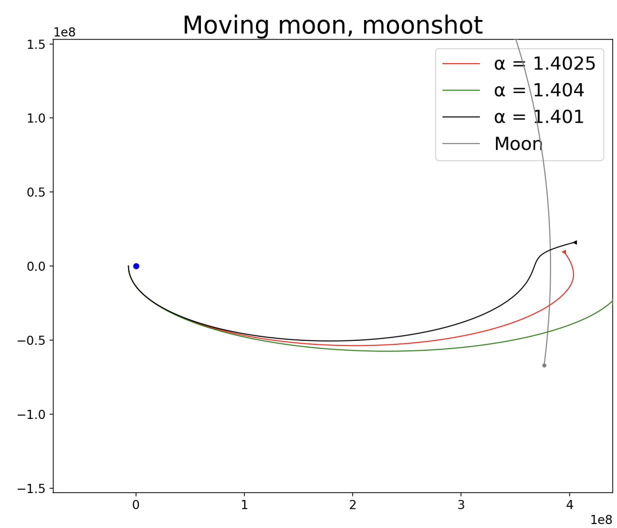

FIG. 7. A figure displaying a moon shot for a selection of rockets with different initial velocity factors. In this scenario the moon has a circular orbit around the earth. $M_E > M_M >> m$.

it was found that when the moon was moving, a slightly higher α gave a 'nicer' shot. It felt necessary to also display how such small changes in α can completely change the course of the rocket. All is displayed nicely by FIG 7.

CONCLUSIONS

In conclusion, it is clear that RK4 is an advanced, accurate method. Using it to display a range of orbital motions shown throughout figures one to seven brings to light it's ability to solve a second order differential equation accurately even at larger step sizes. By comparing to the Euler method, which requires $\frac{1}{4}$ of the computing power (approximately, since $\frac{1}{4}$ of the iterations), it has been confirmed that the accuracy RK4 is in almost all cases worth the additional computation. Aside from these advantages the method is not fool proof and can be limited by step size as seen in FIG 5.

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