Numerical Analysis Of Free Fall with Fixed and Varying Drag

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The aim of this paper is to accurately model the jump that Felix Baumgartner took from 39045m in 2012, along with demonstrating the mass and cross sectional area required to break the sound barrier. Modelling the jump, taking Felix's mass as 110kg and area as 0.9m^2 a mach number of 1.0077, this model correctly predicts that Felix broke the sound barrier.

INTRODUCTION

Humans have always been fascinated by the prospect of breaking the sound barrier. It was first broken in 1947 by Chuck Yeager in a Bell X-1 aircraft [1]. Not long after, in 1976, commercial aircraft went supersonic when Concorde took its first passengers [2]. As technology progressed the idea of a free-falling human breaking the sound barrier became a reality. In 2012, Felix Baumgartner did just that [3]. This report will attempt to model the jump using numerical methods and analyse which conditions are required to go supersonic as a free-falling object. In addition, it will explore the accuracy and limitations of the model.

THEORY

Air is a fluid and therefore, when an object moves through air, it will feel drag. This is an idea that dates to Aristotle. Drag is a form of friction and is proportional to velocity squared for high speeds. The force is therefore described by:

$$\boldsymbol{F} = -k|\boldsymbol{v}|\boldsymbol{v} \tag{1}$$

k is the drag factor where:

$$k = \frac{C_d \rho A}{2} \tag{2}$$

Where C_d is the Drag co-efficient, ρ is fluid density and A is cross-sectional area. The drag force will act opposite to the relative to the motion of any object. Therefore during free fall it will oppose the force due to gravity. Applying Newtons Second Law gives [4]:

$$\frac{dv_y}{dt} = -g - \frac{k}{m}|v_y|v_y \tag{3}$$

Constant Drag

If it is assumed that ρ in equation 2 is constant, (For air at low altitude $\rho \approx 1.2$ kg m^{-3}[5]) equation 3 is relatively simple to solve directly and yields the following equations :

$$y = y_0 - \frac{m}{k}\ln\left[\cosh\left(\sqrt{\frac{kg}{m}}t\right)\right] \tag{4}$$

$$v_y = -\sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}t\right) \tag{5}$$

Where m is mass, g is standard acceleration due to gravity and k is a constant drag factor.

However, free fall can be modelled without solving equation 3 directly using the Euler Method. The Euler Method is a numerical method to solve ordinary differential equations [6]. By using this method, the time at which an object reaches the ground (y=0) can be easily found. Along with that, a varying drag factor can be implemented into our model easily.

Following the Euler method produces these results:

$$t_{n+1} = t_n + \Delta t \tag{6}$$

$$y_{n+1} = y_n + \Delta t \cdot y_n \tag{7}$$

$$v_{y,n+1} = v_{y,n} - \Delta t \cdot \left(g + \frac{k}{m}|v_{y,n}|v_{y,n}\right) \tag{8}$$

Varying Drag

Modelling k as a constant is not entirely correct. This is because density of atmospheric air changes with altitude. By assuming air is an ideal gas (i.e. $pV = nRT$), the change can be modelled by the barometric equation [7].

The barometric equation for density is defined by:

$$\rho(y) = \rho_0\left(\exp\frac{-gMy}{RT}\right) \tag{9}$$

Subbing in a composite parameter H known as scale height where $H = \frac{RT}{Mg}$. placing into equation 9 gives a simple result:

$$\rho(y) = \rho_0\left(\exp\frac{-y}{H}\right) \tag{10}$$

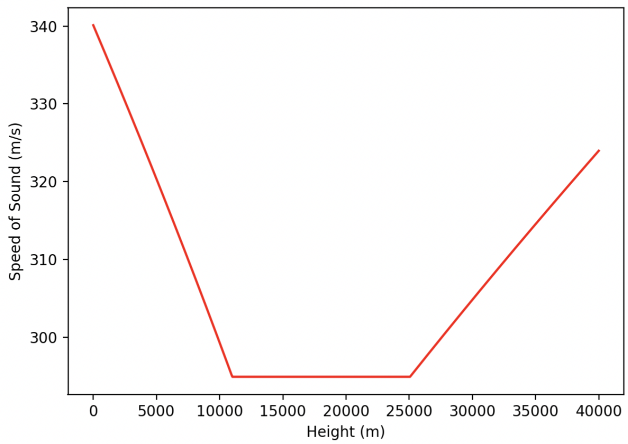

FIG. 1. A graph showing how Speed of sound changes with altitude. According to equation 12.

To acquire the final Euler equation of motion, equation 8 is combined with equation 10 and gives:

$$v_{y,n+1} = v_{y,n} - \Delta t \cdot (g + \frac{C_d A \rho_0 \exp\left(-\frac{y_n}{H}\right)}{m} |v_{y,n}| v_{y,n}) \quad (11)$$

Speed of Sound in Air

It is important to understand how the speed of sound changes throughout the earth's atmosphere in order to determine when or if the sound barrier is broken during a fall. The speed of sound will vary with temperature. The speed of sound can be described by the Newton-Laplace equation. Again, by modelling air as an ideal gas, the following equation is found [8]:

$$v_s = \sqrt{\frac{\gamma RT}{M}} \quad (12)$$

Where γ is the adiabatic index, R is the gas constant, M is molar mass and T is temperature. Where temperature varies with altitude according to:

$$T(K) = 288.0 - 0.0065H : H \leq 11000 \text{ m}$$

$$T(K) = 216.5 : 11000 < H \leq 25100 \text{ m}$$

$$T(K) = 141.3 + 0.0030H : H > 25100 \text{ m}$$

This relationship between altitude and speed of sound can be visualized by figure 1

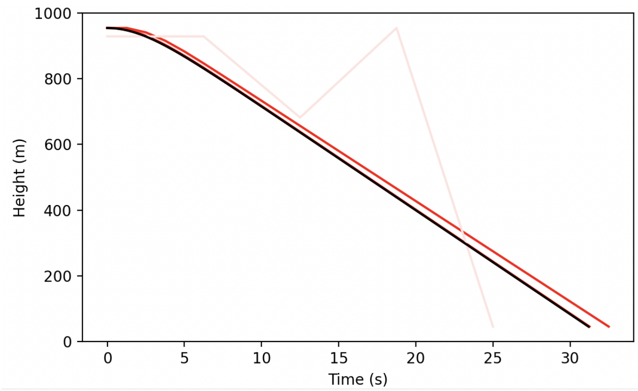

FIG. 2. A Distance-Time graph for an object free falling from a height of 1000m with constant drag. The black line represents a theoretical model. The red lines represented results obtained by the Euler numerical method. The light pink line is where Δt is largest (6.25s)

METHOD

The fall will be modelled using Python, accompanied by two libraries, matplotlib and numpy. Initially, equations 4 and 5 for constant drag will be modelled and compared to an equivalent model produced by equations 6, 7 and 8 in order to access the accuracy of the Euler method and find a suitable value of Δt.

Furthermore, varying drag will be taken into account to produce a suitable model for Felix's Jump. Equations 6, 7 and 11 will be used. Parameters A and m will be adjusted to analyse what was required to go supersonic from the height Felix jumped from (39045m) [9]. Determining whether the sound barrier is broken will require the use of equation 12

RESULTS

Accuracy of the Euler Method

As described in the method, free fall was modelled with constant drag in order to compare the results obtained by the theoretical model (4,6) and the Euler method (6,7,8). Five different values of Δt were used, where $\Delta t = 0.01 \cdot 5^n : n = 0, 1, 2, 3, 4$ The results are shown in by Figure 2 and 3.

Notice that once Δt reaches a value of 0.25 ($n < 3$ two visible lines are $n = 3, 4$) The red lines are hidden behind the black theoretical model - hence, validating the accuracy of the Euler method for low values of Δt.

Felix Baumgartner's Jump

Using a value of 0.01 for Δt, Felix's jump was then modelled. This model used equation 11 and is shown by figure 4

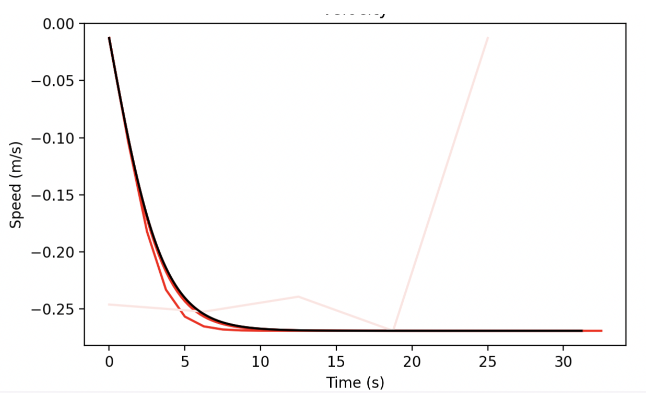

FIG. 3. A Velocity-Time graph for an object free falling from a height of 1000m with constant drag. Speed is in the vertical direction where upwards is positive and down is negative. The black line represents a theoretical model. The red lines represented results obtained by the Euler numerical method. The light pink line is where Δt is largest (6.25s)

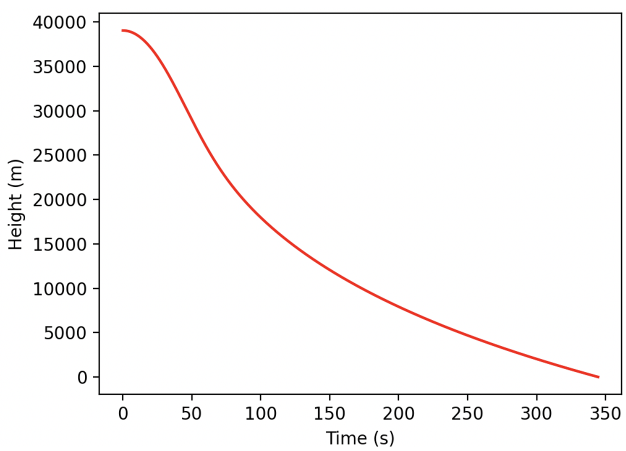

FIG. 4. A Distance-Time graph to model Felix Baumgartner's jump from 39400m with varying drag.

and 5. The chosen parameter for Felix's total mass, including his suite, was 118kg [9]. For cross sectional area, it was assumed to be half the surface area of an average male [10] (0.9m^2).

Observe from figure 5 that with these parameters, the sound barrier is only just broken. At max velocity the mach number was found to be 1.0077 (mach number is defined by $\frac{v}{v_s}$).

DISCUSSION

There are a few key results that require discussion. Initially, it was shown that the Euler method is accurate at Δt values below 0.25 and can be assumed to be perfect at around 0.01. However, it is important to note the limitation of this model. At the highest value used (6.25), the descriptions of how both distance and velocity evolves over time becomes extremely inaccurate. (FIG.2,FIG.3). This is due to the explicit nature of

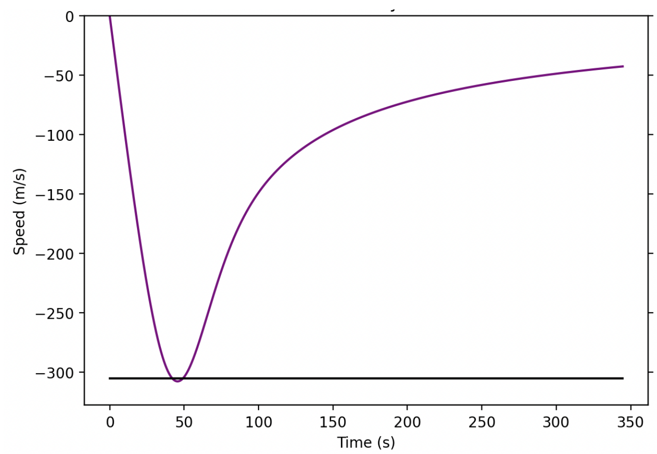

FIG. 5. A Velocity-Time graph to model Felix Baumgartner's jump from 39400m with varying drag. The black line represents the sound barrier at Felix's height when Felix has reached max velocity.

the Euler Method. The state of the object at $t + \Delta t$ only depends on the state of the object at t, which causes what seems to be a limitation in step size. Although not directly relevant to the aims of this paper, there is possible further investigation to find where exactly this limit appears.

Figure 5 presents another interesting result. In comparison to figure 3 in which maximum velocity is equal to terminal velocity, a maximum is reached before terminal velocity and is much larger. Since the altitude from which Felix jumps is so high, the effect produced by the exponential nature of equation 10 is in full force. Since the change in speed of sound with altitude follows a very different and less dynamic relationship (FIG.1), speed of sound can be reached by a relatively small and light free-falling object, such as Felix.

CONCLUSIONS

Overall, the model can be considered successful. The Euler method's accuracy was validated and a model for Felix's jump was produced, correctly predicting that Felix broke the sound barrier. However, the model is far from perfect. In reality, Felix exceeded a mach number of 1.25, showing that our model has some limitations and inaccurate assumptions. First of all, cross-sectional area was taken to be constant throughout the jump; this is unlikely. He may have purposely reduced his area initially in an attempt to break the barrier. One might be concerned that the assumption that air is an ideal gas may cause inaccuracy but overall this is actually a very good approximation. In spite of this, using equation 10 to model air density is likely a bad model. The earth's climate is extremely active and air density is affected by a variety of external factors [11] [12].

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