Effect of Enclosure Symmetry on Acoustic Resonance

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The following report covers a range of findings in an attempt to develop a well-rounded understanding of how symmetry of an enclosure effects the location and nature of resonant peaks. Data collected across three cavities, primarily a cuboid box, provides clear cut evidence of how modal degeneracy is broken. Furthermore changes to amplitude and width of peaks are observed and analysed , where a vital component of the analysis was a measurement of the speed of sound using each cavity in which a precise result of 343.4 $\pm$ 1.5 ms^{-1} was obtained.

INTRODUCTION

Resonance is a fascinating phenomena that was discovered during the mid seventeenth century by Galileo and ever since is known to arise in a variety of interesting places [1]. This report considers acoustic resonance within cuboid and cylindrical cavities, which is covered in substantial details in multiple textbooks such as 'Fundamentals of Acoustics' by L. E. Kinsler [2]. Nevertheless, studying observations made by 'modern-day' equipment can, as shown, produce a valuable insight and prospective.

THEORY

Acoustic Resonance [2]

When sound waves are passed into an enclosed space they are reflected off walls of the space. Interference between incident and reflected waves results in the formation of a standing wave where the nodes of air displacement and anti-nodes of pressure variation are located at the walls of the enclosure. It is possible to define the frequencies of stationary waves in terms of dimensions of the enclosure using a velocity potential Φ defined by:

$$\boldsymbol{v} = \boldsymbol{\nabla}\Phi \tag{1}$$

Where v is rate of change of air displacment (velocity). Φ satisfies the wave equation:

$$\boldsymbol{\nabla}^2\Phi = \frac{1}{u_s}\frac{\delta^2\Phi}{\delta t^2} \tag{2}$$

Where u_s is the speed of sound. To solve this partial differential equation Φ is separated into time dependant and position dependant functions $\Phi = \theta(t)U(r)$ and the position dependant function follows:

$$\boldsymbol{\nabla}^2 U = \frac{\omega^2}{u_s^2}U \tag{3}$$

Where ω is related to frequency f by $\omega = 2\pi f$. Solving in the Cartesian co-ordinate system gives:

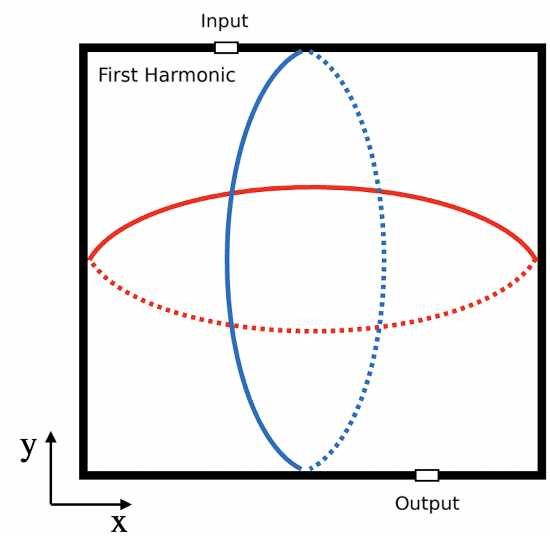

FIG. 1. A diagram showing first harmonic stationary sound waves forming inside a two dimensional box. The red and blue lines represent the displacement of air within the box.

$$f_{l,m,n} = \frac{u}{2}\sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}} \tag{4}$$

l, m and n are integers ($\geq$ 0) and a, b and c are the dimensions of the enclosed space in the x, y and z direction respectfully. A depiction of stationary waves in a 2D box (l,m,n = 1,0,0 and 0,1,0) is shown by FIG. 1.

For this report it is also important to define resonant frequencies by solving equation (3) in the cylindrical co-ordinate system:

$$f = u_s\sqrt{\frac{n^2}{4L^2} + \frac{\alpha_{i,j}^2}{4\pi^2 R^2}} \tag{5}$$

Where L is length of cavity, R is radius, n is an integer and $\alpha_{i,j}$ is the j th zero of the derivative of the i th-order Bessel function (Refer to [3]).

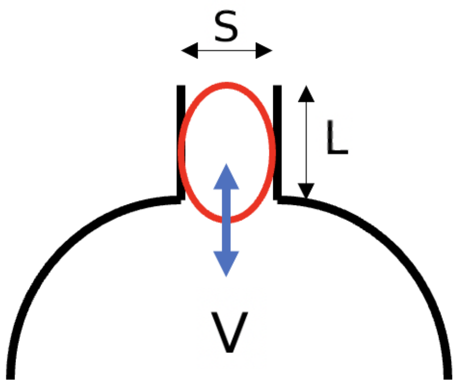

FIG. 2. A Helmholtz resonance resonator, where the red oval represents a pocket of air oscillating as a result of inward air flow and outward pressure.

Degeneracy [2]

FIG. 1. has additional significance, it shows a phenomenon known as modal degeneracy. Modal degeneracy occurs when two or more resonant modes have equal frequency. This occurs as a byproduct of enclosure symmetry, since FIG. 1 has cubic symmetry ($a = b$) if a signal is measured at the output, it is impossible to distinguish whether sound is primarily coming from the resonant mode (1,0,0) or (0,1,0). If the symmetry were to be broken, new, spread out frequencies can be measured. This is the key result this report will investigate, analysing the implications the phenomenon has on experimental error and measuring physical quantities.

Helmholtz Resonance [2]

Helmholtz resonance is a phenomenon that occurs when sound passes into an enclosure via an opening. As air passes into the cavity it feels an opposing force due increased pressure inside, just as a mass feels an opposing force from a spring. Therefore, an oscillation occirs within the neck of the opening as shown by FIG. 2. The resistance to the motion of the air (equivalent to 'stiffness' of a spring) is related to the size of the neck. With consideration of such information, Helmholtz resonance has a frequency defined by:

$$f = \frac{u_s}{2\pi}\sqrt{\frac{A}{VL}} \tag{6}$$

Where A is the area of opening (eg. $A = \pi S^2$, see FIG. 2) , V is volume of enclosure, L is length of the neck.

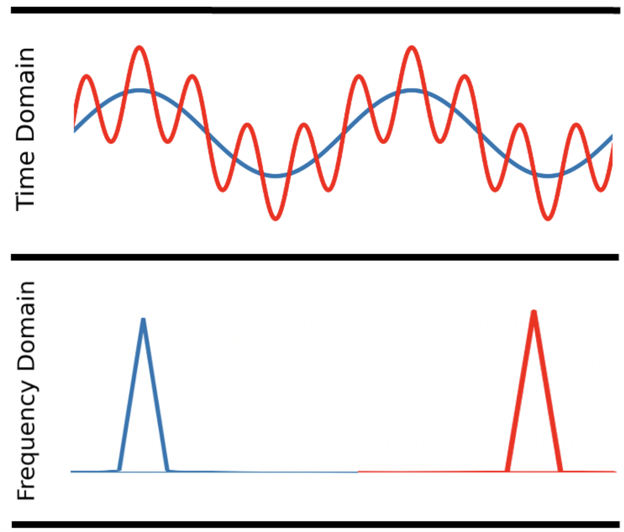

FIG. 3. This figure demonstrates the role of the Fourier transform, splitting a complex signal into it's frequency components.

The Fast Fourier Transform [3][4]

In the context of acoustics, the role of the Fourier transform is to convert a complex time domain signal into a frequency domain graph representing amplitude contributions of each frequency. This mathematical conversion is given by:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{t_1}^{t_2} f(t)e^{-i\omega t}dt \tag{7}$$

In order to compute the Fourier transform integral a recursive numerical method must be used. Directly computing equation (7) this way is known as a discrete Fourier transform (DFT), where the run time of the algorithm is proportional to the number of data points squared ($O(N^2)$). As a result, the DFT is slow and can be dramatically improved by evaluating even and odd parts of a function separately, this method is called the fast Fourier transform (FFT) and has order $O(N \log N)$. A diagram to represent the role of the Fourier transform is given by FIG. 3.

Experimental Method

The method carried out aimed to show how a signal changes when symmetry is broken. In particular, providing strong evidence of degeneracy and analysing how accuracy of results is related to symmetry via the determination of the speed of sound. With this in mind, a short duration pulse from a clicker was injected into a range of cavities and on the opposing side of the cavity a microphone recorded the output signal. The range of cavities used is displayed by FIG. 4. Cavity a), a cylindrical pipe was used to generate resonance in one dimension (across the length of the pipe). Cavity b), an almost per-

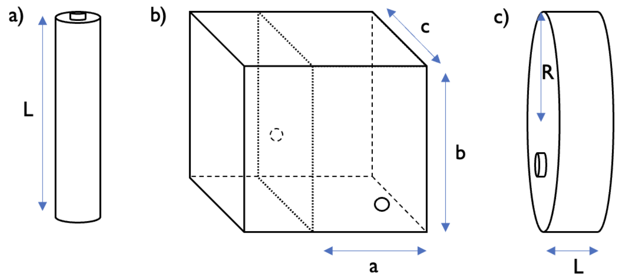

FIG. 4. A diagram showing the three cavities used to measure acoustic resonance. Cavity b) includes a dashed line to represent a plate placed inside the cube to reduce it's size in one dimension.

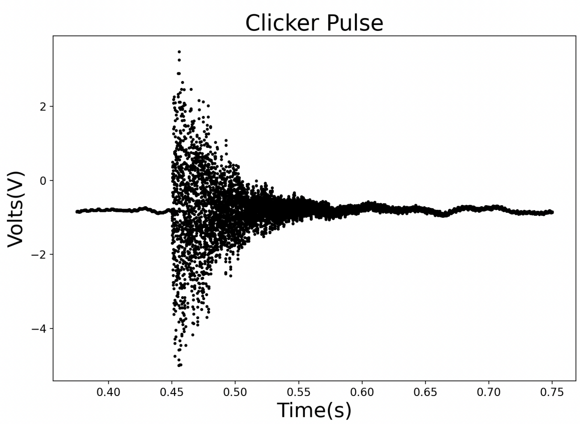

FIG. 5. A clickers acoustic pulse in the time domain.

fectly cubic box (a = 0.201m, b = 0.205m, c = 0.202m) was used as the primary comparison of frequency spectra from a symmetrical shape to a significantly less symmetrical shape created when a plate is placed inside the cube. For further investigation, cavity c), a cylindrical disk was used. The microphones recorded a time domain signal converted into the required frequency spectra by a software using the FFT, an example of the raw 'pulse' signal from the clicker is shown in FIG. 5.

Mitigation of Helmholtz resonance

For calculation of the speed of the sound it is important to recognise and remove a frequency peak as a result of Helmholtz resonance. To do so a piece of tape was placed over a the cavity opening, a simple yet effective method, shown by FIG. 6.

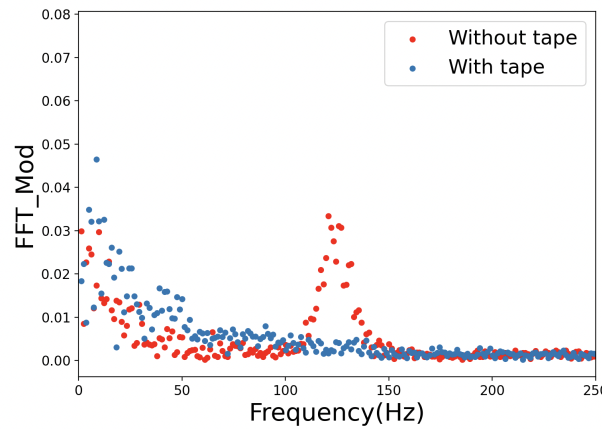

FIG. 6. A graph comparing observations made when tape is or is not placed over the cavities opening. The peak present is a result of Helmholtz resonance.

TABLE I.

(l, m, n)	a : 0.201m (Hz)	a : 0.154m (Hz)	a : 0.127m (Hz)
(0, 1, 0)	849.0	849.0	849.0
(1, 0, 0)	853.2	1113.6	1350.4

RESULTS

Demonstrating Degeneracy

FIG. 7 shows frequency spectra for a cubic box (Cavity b, FIG. 4) where a takes three values based on where the plate is placed inside the cube. The figure clearly demonstrates the reduction in modal degeneracy or 'spreading out' of resonant frequencies as symmetry is broken. The recorded spectra are compared to black lines representing expected frequency assuming $u_s = 343\text{ms}^{-1}$ (equation 4), table 1 shows these values.

Error propagation

In order to propagate error and further understand the effect of symmetry on frequency spectra a Lorentzian distribution is fitted across a range of spectral peaks. The Lorentzian distribution is given by:

$$l(x) = I_{\max}\frac{\frac{\Gamma}{2}^2}{(x-\mu)^2 + \frac{\Gamma}{2}^2} \tag{8}$$

Where Γ represents the FWHM, and μ represents the centre of the peak [6].

The width and maxima of these peaks were calculated using an algorithm within the scipy python library (see Appendix), FIG. 8 shows examples of the distribution fitted onto a frequency peak and highlights the performance of this algorithm.

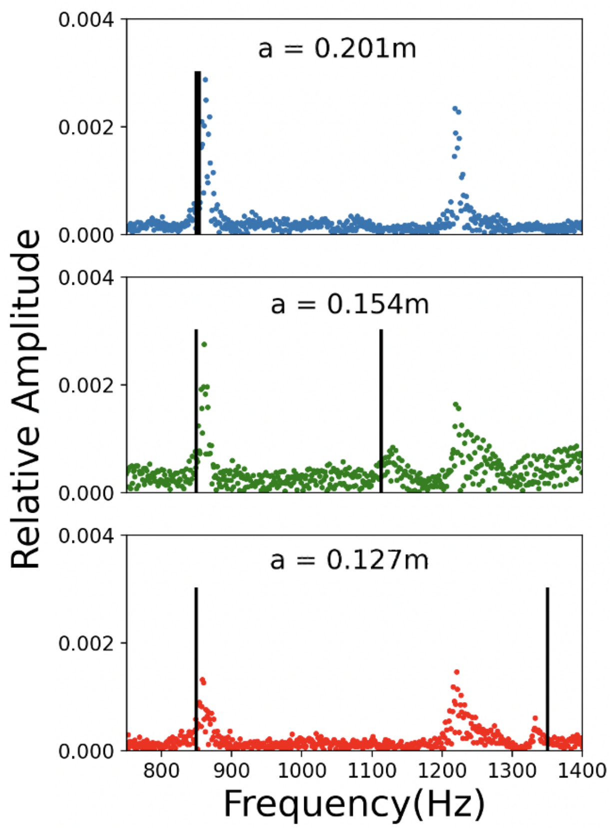

FIG. 7. Graphs displaying frequency domain signals from a cuboid cavity with different lengths. Where black lines fitted above results are predicted values according to theory.

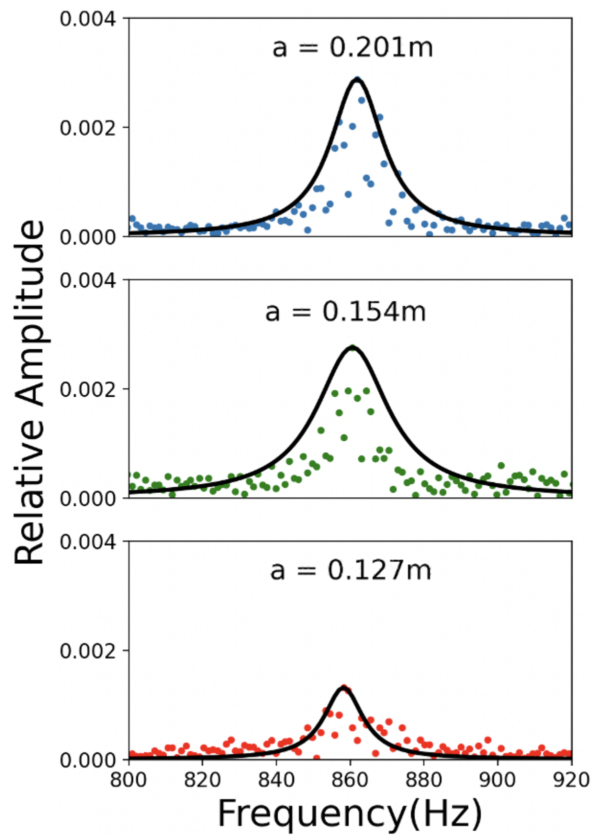

FIG. 8. Peaks of resonant frequency from a cuboid cavity with different lengths fitted with Lorentzian distributions.

TABLE II.

	Average probable error ($\Gamma/2$) (Hz)		
(l, m, n)	a : 0.201m	a : 0.154m	a : 0.127m
(0, 1, 0)	21.2	14.3	6.4
(0, 1, 1)	6.6	18.1	22.9
(0, 2, 0)	17.4	20.8	18.0

Table II shows probable error calculated this way for a small selection of l, m, n, each value is an average from two data sets with the same value of a. There seems to be no general trend in probable error in relation to symmetry. However, the (0,1,0) peak error follows a trend and in this case degeneracy has been made clear by FIG. 7.

In addition, FIG. 8 shows a quite signicant suppression in relative amplitude when a is reduced to 0.127, although, to conclude degeneracy is responsible the same suppression would be expected to occur when a = 0.154 which is not recorded.

TABLE III.

Cavity	Speed of sound, u_s (ms^{-1})	$\pm$ Error (ms^{-1})
a) Tube	341.6	0.3
b) Cube (a=0.201)	343.2	2.3
b) Cube (a=0.127)	343.41	1.5
c) Cylinder	345.7	1.8

Finding the speed of sound

The determination of the speed of sound and analysis of results provides further information into the effect of symmetry. By measuring the frequency of spectral peaks and producing a linear fit according to equations 4 and 5 the speed of sound is determined, this method was carried out for all three cavities, FIG 9. and 10 show the results when considering first seven clear peaks of each spectrum. The final results for the speed of sound are shown in table III

DISCUSSION

Referring to FIG. 7, it is shown that experimental results follow exactly what the theory predicts (taking 343 ms^{-1} as

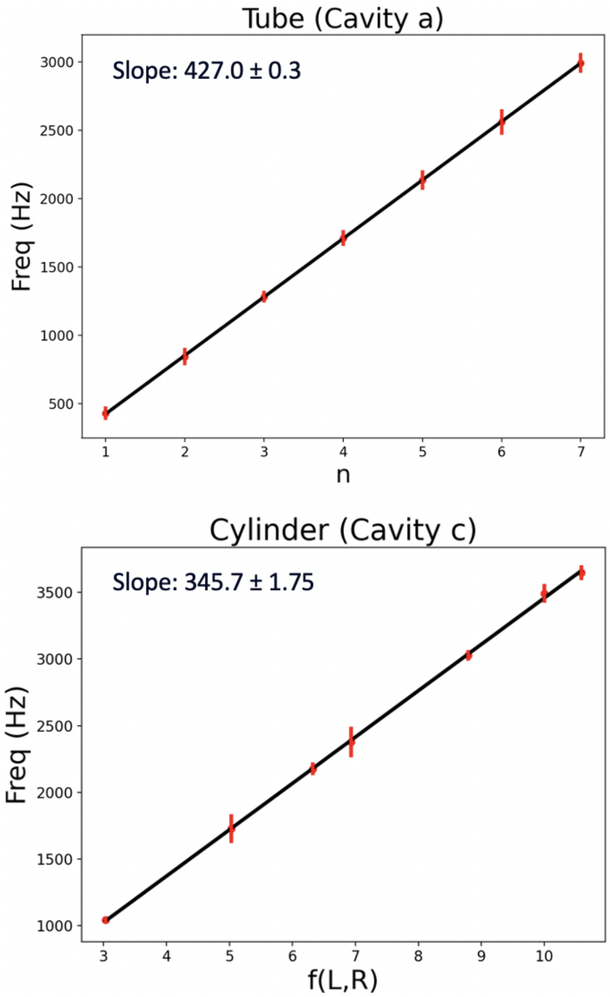

FIG. 9. Linear fits of resonant peaks for cylindrical cavities. The Tube is modeled as one dimensional and given x values (n = 1, 2, 3 ...). The cylinder is taken to be a three dimensional shape and therefore x values are computed by: $f(L, R) = \sqrt{\frac{n^2}{4L^2} + \frac{\alpha_{i,j}^2}{4\pi^2 R^2}}$

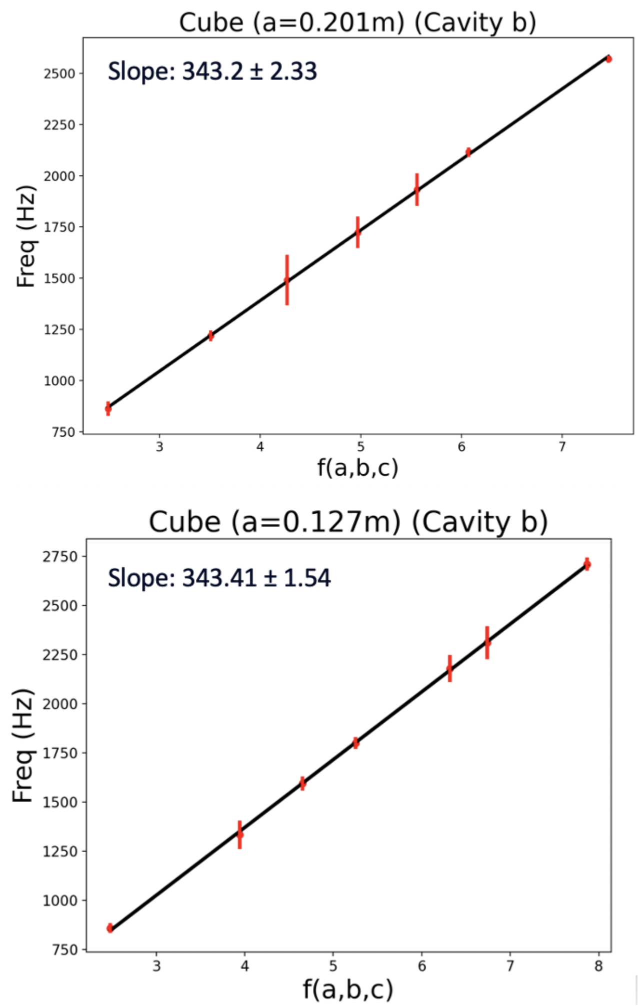

FIG. 10. Linear fits of resonant peaks for cubiod cavities, where x values are computed by: $f(a, b, c) = \frac{1}{2}\sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}$

u_s). The slight deviation from the theory appears to be a small shift to left or right of each peak. It is assumed that this arises as a result of over or under shooting measurements of the cubes dimension, since a constant shift is observed for the first peak (the same distance is used, b) and different shifts when degeneracy is broken.

The primary goal of FIG. 6 is to show degeneracy breaking within the box which is done successfully. However, the apparatus was limited to changing only one of three dimensions inside the box and therefore may not show the extent at which a spectrum may change. Regardless the diagram gives a clear insight into the effect of symmetry on a resonant frequency spectrum.

A further area of short discussion is the observation that when a is reduced from 0.154m to 0.127m a secondary peak is surpassed. Therefore concluding that somewhere in the range 0.127m $< a <$ 0.154m, these peaks will overlap causing less recognisable overlap and hence looking at a recorded spectrum alone (without considering theoretical values) would likely produce inaccurate results.

FIG. 8. analyses the left-most peaks in FIG. 7, showing how they change as degeneracy is broken. A large suppression in amplitude is observed when a is reduced from 0.201 to 0.127, along with a reduction in FWHM. The same cannot be said for the 0.154 peak, however, when taking an average probable error (FWHM/2) across two data sets a reduction in width is observed (table II). Further evidence for reduction in peak width as degeneracy breaks (therefore error) is observed within speed of sound measurements, shown by FIG. 10 and table III.

The mentioned 'expected' suppression does not occur in either data-set for 0.154, but interestingly, if amplitude is manually reduced ($I_{\max}$, equation (8)) The Lorentzian distribution seems to fit the data more 'effectively' (see FIG. 11). Although this gives possible evidence of a peak suppression, further investigation is required to make conclusions.

Considering all of the above, it is predicted that in general,

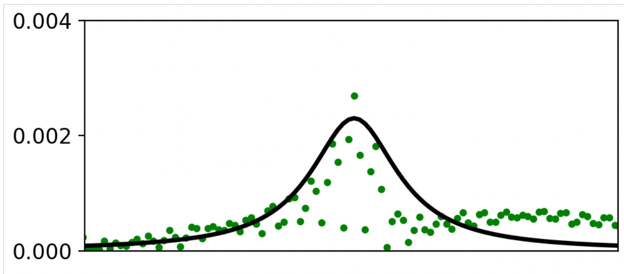

FIG. 11. A Lorentzian distribution fitted onto a resonant peak to demon state that when the height of the distribution is reduce it appears to fit data 'more effectively'.

breaking degeneracy reduces the width and hence probable error of any given peak.

Looking again at table III we see a disparity in speed of sound values insufficient to provide clear conclusions but impressive results, highlighting the accuracy and precision of the apparatus and FFT. The low error obtained measuring speed of sound using a tube, shows that modelling the tube as a one dimensional cavity is successful, no degeneracy is expected and according to conclusions made from cube data this results in the low error table III presents. Although, The current conclusion is valid, a key element of theory is unaccounted for.

When the Fourier transform is performed on a signal, the resulting resonant peaks acquire a width inversely proportional to the lifetime of the signal ($\Gamma \propto \frac{1}{t_2 - t_1}$, equation 7). The consequences of this may be dramatic and are hard to measure and mitigate. Since the same clicker is used throughout data collection it is assumed the lifetime of resonant modes is approximately the same, likely a poor approximation, requiring further investigation and may explain mis-alignment of results when comparing cylindrical cavities a and c to a cube, b.

Additional insight into cavities a and c is found when studying how exactly acoustic stationary waves form inside a cylinder, a much less intuitive phenomenon than in a cubiod (FIG. 1). FIG. 12 shows the three types of stationary waves that can form in a cylinder, it provides valuable information and indicates a possible explanation as to why there is such a difference in error between a and c. For example, it is possible that; longer, lower frequency longitudinal waves that have a longer lifetime form in cavity a producing sharp peaks, whereas short, high frequency longitudinal waves with short lifetimes form in cavity c producing broader peaks.

CONCLUSIONS

The results and discussion can be broken down into three key conclusions on the effect of symmetry on acoustic resonance:

i) As theory predicts when symmetry is broken, modal degeneracy is too, causing a dramatic change in recorded resonant frequencies and frequency spectra (FIG. 6)

ii) As a result of i) breaking symmetry reduces the width of

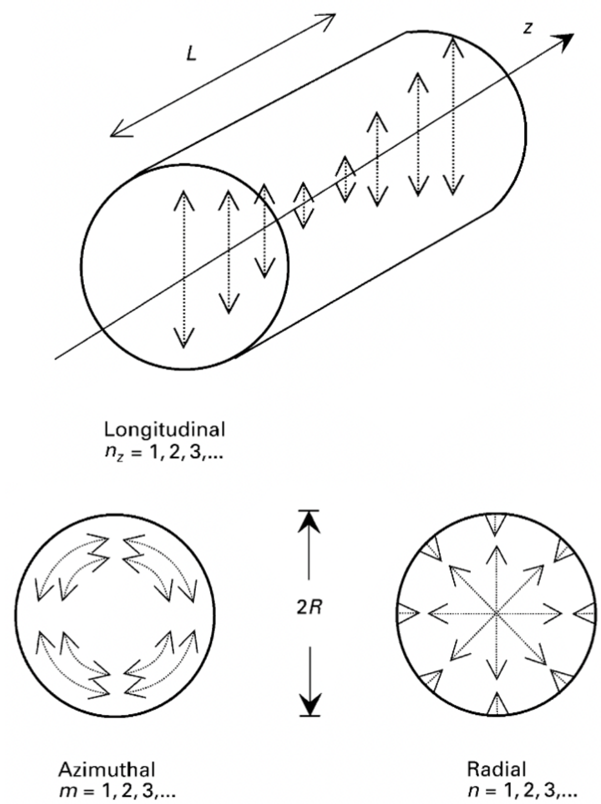

FIG. 12. Schematic showing types of acoustic modes inside a cylindical cavity. Taken from [5].

resonant peaks and consequently produces more precise measurements of physical quantities such a the speed of sound. However, the duration of the input and type of stationary waves formed hence cavity dimensions MUST be considered.

iii) Finally, indications of peak suppression and amplification due to non-degenerate and degenerate modes respectively have been made but with insufficient data to make a clear conclusion.

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TABLE IV. Sample output

Intensity	Peak Frequency	FWHM
0.0214	417	1.67
0.0768	**431**	**12.1**
0.0138	443	1.13

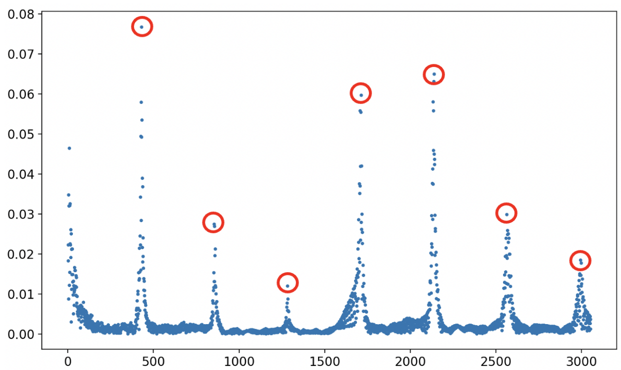

FIG. 13. Frequency spectrum for tube. Red circles mark the most obvious peaks.

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APPENDIX

A1. Selecting peaks

To plot the linear fits seen in FIG 9 and 10, the following method was used:

1. By eye, consider most obvious, stand out peaks (FIG. 13)

2. run scipy methods on experimental data and acquire location and width of peaks (FIG. 14).

3. Compare visual results to python output and select logical and obvious peaks (Table IV).

```
import scipy.signal as ss
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

data = pd.read_csv('results.csv')
FFT = np.array(data['FFT_Mod'])
freq = np.array(data['Freq(Hz)'])
peaks,peak_heights = ss.find_peaks(FFT,distance=10)

widths,width_heights,left_ips, right_ips= ss.peak_widths(FFT,peaks,

peaks = list(peaks)
peak_heights = list(peak_heights['peak_heights'])

freq_new = []
widths_new = []
intensity = []
for i in peaks:
    if widths[peaks.index(i)]:
        freq_new.append(freq[i])
        widths_new.append(widths[peaks.index(i)])
        intensity.append(peak_heights[peaks.index(i)])



data = {'Intensity': intensity, 'Peak Frequency': freq_new,'FWHM':
```

FIG. 14. Python code to return data set of height and widths of peaks