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# Weighted Voronoi region algorithms for political districting

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ABSTRACT

Political districting on a given territory can be modelled as bi-objective partitioning of a graph into connected components. The nodes of the graph represent territorial units and are weighted by populations; edges represent pairs of geographically contiguous units and are weighted by road distances between the two units. When a majority voting rule is adopted, two reasonable objectives are population equality and compactness. The ensuing combinatorial optimization problem is extremely hard to solve exactly, even when only the single objective of population equality is considered. Therefore, it makes sense to use heuristics. We propose a new class of them, based on discrete weighted Voronoi regions, for obtaining compact and balanced districts, and discuss some formal properties of these algorithms. These algorithms feature an iterative updating of the distances in order to balance district populations as much as possible. Their performance has been tested on randomly generated rectangular grids, as well as on real-life benchmarks; for the latter instances the resulting district maps are compared with the institutional ones adopted in the Italian political elections from 1994 to 2001.



## 1. Introduction

Soon after modern democracies were established, gerrymandering practices, consisting of partisan manipulation of electoral district boundaries, began to occur in several states and countries. From then on, political districting has become a critical determinant of the representation of parties in legislatures. The process of how the shape and the size of districts may influence the electoral outcome in terms of seat allocation has been widely studied in the literature by political scientists and economists [1–6], but also by mathematicians, statisticians and computer scientists [7–11].

In order to oppose such practices, when the electronic computer became available researchers started thinking of formal procedures for political districting, designed so as to be as neutral as possible. Commonly adopted criteria are:

*Integrity* — The territory to be subdivided into districts consists of territorial units (wards, townships, counties, etc.) and each unit cannot be split between two or more districts.

*Contiguity* — The units of each district should be geographically contiguous, that is, one can walk from any point in the district to any other point of it without ever leaving the district.

*Population equality* (or *population balance*) — Under the assumption that the electoral system is a plurality one, all districts should have the same portion of representation (according to the *One person – one vote* Principle); in particular, if they are single-member districts, they should have nearly the same populations.

*Compactness* — Each district should be compact, that is, "closely and neatly packed together" (Oxford Dictionary). Thus, a round-shaped district is deemed to be acceptable, while an octopus- or an eel-like one is not.

A broad survey of political districting algorithms is given in [10]. Later work focuses on local search [8,11]. It is also worth mentioning the branch-and-price approach in [9]. Here we propose a novel approach based on *weighted Voronoi regions* (or

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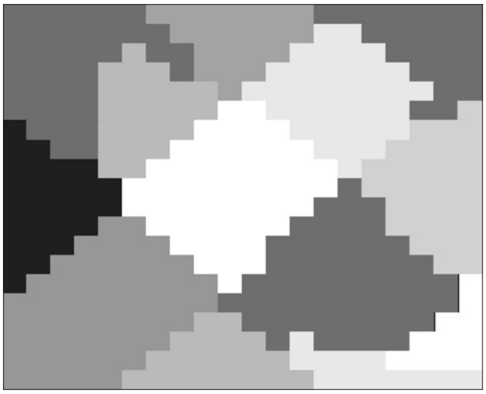

**Fig. 1.** A district map obtained by a WVR procedure for a $20 \times 20$ grid graph.

diagrams; WVR for short). This notion, although – to the best of our knowledge – introduced here for the first time in the context of political districting, is not new in the literature, especially in the area of computational geometry (see, e.g., [12]). Also the *discrete* version of the Voronoi regions is not new: for example, it was widely applied in network location problems [13]. What we believe to be new is our iterative updating of distances in order to balance district populations. As we will show later, in the specific application to political districting, the computation of the WVR with respect to these updated distances is a useful tool to achieve both compactness and population balance of the districts. WVR procedures – as implemented here – on the one hand guarantee that the integrity and contiguity criteria are always satisfied; on the other hand they provide a flexible approach to the districting problem, since solutions with different trade-offs between compactness and population balance can be easily found: when population balance improves, compactness inevitably worsens; but, by stopping the procedure at different time points, one can always control the current trade-off between these two basic districting criteria.

Fig. 1 gives an idea of the shape of the districts obtained by a WVR procedure for a $20 \times 20$ grid in the plane. The grid is represented as a chessboard whose squares correspond to the grid nodes and there is an arc between two nodes iff the corresponding squares are adjacent in the chessboard. A node $v$ in the grid can always be identified by two integral coordinates, $v(x)$ and $v(y)$ and, thus, a natural way to compute the distance between two nodes $u$ and $v$ of the grid is to consider the $L_1$– (or Manhattan –) distance:

$$d(u, v) = |u(x) - v(x)| + |u(y) - v(y)| .$$

The paper is organized as follows. In Section 2 we describe a general paradigm for our WVR algorithm and the specific features that characterize each different variant of it. In Section 3, after indicating some pathologies that may occur if some caution is not taken, we define some desirable properties to be met by WVR algorithms and we provide conditions under which such properties hold. We also give finite termination results based on the theory of Majorization introduced in [14]. In fact, we establish a pseudo-polynomial upper bound on the number of iterations. In Section 4 we present some preliminary computational experiments performed with a specific implementation of the algorithm on a sample of four Italian Regions and on a benchmark of rectangular grid graphs.

## 2. Weighted Voronoi region procedures

The input to the weighted Voronoi procedures to be described in this section is the following: a connected *contiguity graph* $G = (V, E)$, whose nodes represent the territorial units and where there is an edge between two nodes if the two corresponding units are neighbouring, with $|V| = n$ and $|E| = m$; a positive integer $r$, the number of districts; a subset $S \subset V$ of $r$ nodes, called *centers* (all the remaining nodes will be called *sites*); positive integral node weights $p_i$, $i \in V$, representing territorial unit *populations*; positive real *lengths* $l_{ij}$ for all edges $(i, j)$. Usually, edge-lengths represent road distances, so as to take into account the presence of rivers, lakes, mountains and any other geographical barrier located between territorial units. Given the edge-lengths, we can accordingly compute the *distance* $d_{ij}$ between any two nodes $i$ and $j$ as the length of any shortest path on $G$ with endpoints $i$ and $j$. A path on $G$ is called a *geodesic* if it is a shortest path between its two endpoints. By slightly perturbing edge-lengths, if necessary, we may, and shall, assume without loss of generality that:

(i) the distance function is injective, i.e., $d_{ij} \neq d_{i'j'}$ whenever $(i, j) \neq (i', j')$;
(ii) between any two nodes there is a unique geodesic.

In fact, let $0 < \varepsilon < \frac{1}{2} \min\{1, L\}$, where $L = \min\left\{ |d_{ij} - d_{i'j'}| : d_{ij} \neq d_{i'j'}, \forall i, j, i', j' \right\}$. After numbering the $m$ edges of $G$ from 1 to $m$, let us assign to the $k$-th edge $(i, j)$ the perturbed length

$$l_{ij}(\varepsilon) = l_{ij} + \varepsilon^{k+1}, \quad (k = 1, \dots, m).$$

For every geodesic $Q$, let us denote by $d_Q$ and $d_Q(\varepsilon)$ the total length of $Q$ relative to the original and to the perturbed edge-lengths, respectively, and let

$$\Delta_Q(\varepsilon) = d_Q(\varepsilon) - d_Q = \sum_{h \in Q} \varepsilon^{h+1}.$$

Finally, let $\chi(Q)$ be the characteristic vector of $Q$ in $E$: supposing that the edges of $G$ have been labelled from 1 to $m$, $\chi(Q)$ is a binary $m$-vector whose $i$-th component is 1 if edge $i$ belongs to $Q$ and 0 otherwise. Note that, since $0 < \varepsilon < 1/2$, $\Delta_Q(\varepsilon) = \varepsilon \sum_{h \in Q} \varepsilon^h < \varepsilon \left( \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{|Q|}} \right) < \varepsilon$.

Furthermore, $\Delta_Q(\varepsilon) < \Delta_{Q'}(\varepsilon)$ iff $\chi(Q)$ is lex-smaller than $\chi(Q')$. Hence, under the above assumptions on $\varepsilon$, we have $d_Q = d_{Q'} \Rightarrow d_Q(\varepsilon) \neq d_{Q'}(\varepsilon)$, and $d_Q < d_{Q'} \Rightarrow d_Q(\varepsilon) < d_{Q'}(\varepsilon)$.

It follows that the corresponding perturbed distance function $d(\varepsilon)$ is injective, and $d_{ij} < d_{i'j'} \Rightarrow d_{ij}(\varepsilon) < d_{i'j'}(\varepsilon)$. Moreover, with the above-defined perturbed edge-lengths there is a unique geodesic between any two nodes.

Under the injectivity assumption on $d$, for each site $i$ there is always a unique center that is closest to $i$. We denote by $P$ the total population and by $\bar{P}$ the mean district population ($=P/r$). In the remainder of the paper, for brevity, we will denote by $s$ both a center and a district centered in $s$ (when this does not cause any confusion).

The integrity criterion dictates that a district must be a subset of nodes; according to the contiguity criterion, such a subset must be connected. A *district map* is a partition of $V$ into $r$ connected subsets (the districts), each containing exactly one center. Given any district map, we denote by $D_s$ the unique district containing center $s$. We look for a district map such that, informally speaking, the district population imbalance is small and the districts are compact enough. Hence, the districting problem is modelled as a bi-objective partitioning of a graph into connected components. When the problem is explicitly formulated as an optimization model, the population imbalance objective function is typically computed as the sum of the imbalances of the districts, i.e., the absolute deviations of the district populations from the average one. On the other hand, several possible measures for compactness are available [10], but, generally, they are nonlinear functions of the model variables, or they are not even given by an explicit analytical expression, but, rather, by an oracle, i.e., a subroutine that returns the value of the objective function for each solution.

Actually, the heuristic approach is motivated by the fact that, even in the presence of a single objective function, our partitioning problem is difficult to solve. In particular, it can be shown that the problem of finding a partition of a graph $G$ into $r$ connected components that minimizes the above defined population imbalance is NP-hard even if the graph is a tree with at most one vertex of degree $\geq 3$ and diameter $\leq 4$ [15].

According to our approach, we first locate the subset $S \subset V$ of $r$ centers and then we define the district map by drawing on $G$ the Voronoi diagram w.r.t. the distances $d_{is}$, $\forall i \in V - S$, $\forall s \in S$ (initial *discrete* Voronoi regions), which can be seen as the graph-theoretic counterpart of the ordinary Voronoi diagram in continuous space. More precisely, the *Voronoi region* (or *diagram*) of center $s$ is the set of all nodes $i$ such that the closest center to $i$ is $s$. Under the injectivity assumption, such a closest center is necessarily unique; moreover, since all edge-lengths are positive, for any center $s \in S$, we have $d_{ss'} > 0$, $\forall s' \neq s$, thus $s$ always belongs to $D_s$. Hence the Voronoi regions form a partition of $G$. Furthermore, it can be shown (see Section 3) that Voronoi regions are connected.

The idea is that if we take as districts the Voronoi regions on $G$, the compactness of the districts will be good. Notice that the compactness of the district map strongly depends on the location of the $r$ centers. In order to get a good performance of the Voronoi approach w.r.t. compactness, we locate the $r$ centers by solving the following *unweighted r-center location problem* on $G$ (cfr. [13]). A graph $G = (V, E)$, with $|V| = n$ is given, together with a weighting function $w : V \to R^+$ that assigns a weight $w(v)$ to each vertex $v \in V$. For each pair of vertices $u$ and $v$ in $V$ the distance between $u$ and $v$ is computed as the length of the shortest path connecting $u$ and $v$ in $G$ (and, thus, in $G$ it corresponds to the number of edges belonging to such a path). Given a fixed $r \leq n$, select $r$ vertices in $V$ as centers, such that the maximum distance among the distances between any vertex in $V$ and its nearest center is minimized.

It is well known that the above optimization problem is NP-hard [16]. For this reason, here we apply a heuristic procedure to locate our $r$ centers.

By this approach, a good compactness level is usually achieved, but a poor population balance might ensue. In order to re-balance district populations, one would like to promote site migration out of "heavier" districts (population-wise) and into lighter ones. Then, the basic idea is to consider weighted distances $d'_{is} = w_s d_{is}$, where each weight $w_s$ is proportional to $P_s$, the population of district $D_s$, and to perform a Voronoi iteration (that is, the computation of the discrete Voronoi regions) w.r.t. the biased distances $d'_{is}$. Do this iteratively: at iteration $k$, $k = 1, 2, \ldots$, two different recursions may be taken into consideration, namely, a *static* one,

$$d_{is}^k = \frac{P_s^{k-1}}{\bar{P}} d_{is}^0, \quad i \in V - S, s \in S \tag{1}$$

and a *dynamic* one,

$$d_{is}^k = \frac{P_s^{k-1}}{\bar{P}} d_{is}^{k-1}, \quad i \in V - S, s \in S. \tag{2}$$

**WEIGHTED VORONOI REGION ALGORITHM**

**INPUT:** G=(V,E), $r$, $p_i$ $\forall i \in V$, $d_{ij}$ $\forall i, j \in V$

**OUTPUT:** a connected partition of G

**1.** Locate the set $S$ of the $r$ centers in G

**2.** $k = 0$;

**3.** Let $d_{is}^{o} = d_{is}$, $\forall i \in V - S$, $\forall s \in S$

**4.** Compute the discrete Voronoi regions w.r.t. [ $d_{is}$ ] (*initial district map*)

**5.** **repeat**

$k = k + 1$

update (using either (1) or (2) throughout) the distances $d_{is}^{k}$, $\forall i \in V - S$,

according to $P_s^{k-1}$, $\forall s \in S$

compute the subset $M'$ of sites that are candidates for migrating and perform the corresponding migrations

compute the discrete Voronoi regions w.r.t. [ $d_{is}^{k}$ ] (*current district map*)

**until** $M'$ is empty

**Fig. 2.** General paradigm of a WVR algorithm.

where, in both cases, $d_{is}^0 = d_{is}$, $P_s^0$ is the population of the initial Voronoi region containing center $s$, and $P_s^k$ is the population of $D_s$ after iteration $k = 1, 2, \ldots$, while $\bar{P}$ is the average district population given by the total population divided by $r$. Stop as soon as the districts become "stable", that is, the district map at some iteration coincides with the district map at the previous iteration. The above sketched algorithm will be called a *full transfer* one because, at each step, all the sites for which the closest center changes are transferred from their old district to the new one. Denote the set of these sites by $M \subseteq V - S$. In view of the possible finite termination difficulties of the full transfer algorithm (see Section 3), we also consider different versions for the WVR algorithm in which only a subset $M' \subset M$ of sites is actually allowed to migrate. Fig. 2 shows the general paradigm of a WVR algorithm. One may consider a *single transfer* version of the WVR algorithm, by letting sites migrate one at a time from one district to another; or a *partial transfer* version, in which only a particular subset of sites (suitably selected according to some rule) migrates at each iteration. Here too, one may adopt either the static or the dynamic recursion defined above. So, one altogether gets six variants of the weighted Voronoi algorithm (static/dynamic recursion; full/partial/single transfer).

In particular, the implementation of the single transfer algorithm is the following: at iteration $k$, some district $D_t$ with minimum population, $P_t^{k-1} = \min\left\{P_s^{k-1} : s = 1, \ldots, r\right\}$, is selected as the destination district. Then, a subset $M' \subseteq M$ of those sites that are candidates for migrating into $D_t$ is selected according to the following rule: site $i \notin D_t$ is a candidate for migrating into $D_t$ if $d_{it}^k = \min\left\{d_{is}^k : s = 1, \ldots, r\right\}$. Finally, site $i$ is chosen for migrating from $D_q$ (the district it belongs to) to $D_t$ if the following two conditions hold:

(i) $d_{it}^k = \min\left\{d_{jt}^k : j \in M', j \notin D_t\right\}$;

(ii) $P_t^k < P_q^k$.

Notice that if site $i$ belongs to district $D_q$ and it is a candidate for migrating to $D_t$, then $d_{it}^k < d_{iq}^k$. The algorithm stops when there is no $i$ in $M'$ that satisfies conditions (i) and (ii). Notice that (ii) holds if and only if the amount of population moved from $D_q$ to $D_t$ is smaller than $\left(P_q^{k-1} - P_t^{k-1}\right)/2$.

Other implementations of the single transfer algorithm may be thought of in order to prevent the algorithm from prematurely stopping. Actually, since the minimum population district may not be unique, one could try all such districts before stopping, or one may decide to give priority to districts with smaller population.

An intermediate approach between the single and the full transfer ones leads to the so called *path transfer* algorithm. The initial discrete Voronoi regions are calculated at the beginning. At each iteration $k$ an auxiliary network $\mathcal{N}$ is constructed such that nodes in $\mathcal{N}$ correspond to the current districts, while there is an arc between two nodes $D_q$ and $D_t$ of $\mathcal{N}$ if and only if there exists a site $j$ that can be moved from $D_q$ to $D_t$, that is, if and only if $d_{jt}^k < d_{jq}^k$ and $P_t^k < P_q^k$ hold. At each iteration a suitable path $P$ is selected in $\mathcal{N}$ and for each arc $(D_q, D_t)$ of $P$ a site migrates from $D_q$ to $D_t$. The path $P$ can be selected according to different criteria. One possibility could be, for example, to select a shortest available path (that is, one having as few arcs as possible).

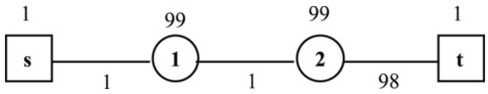


**Fig. 3.** Lack of termination for the dynamic full transfer WVR algorithm.

| distances | | sites 3 | sites 4 |
|---|---|---|---|
| centers | 1 | 1 | 2 |
| | 2 | 2 | 1 |

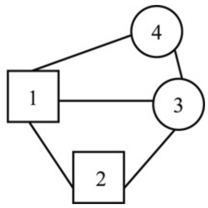


**Fig. 4.** An example of lack of contiguity, where all the nodes have the same population and the site-to-center distances are given in the table.

**Remark 1.** Even when the initial distances $d_{is}$ are injective, injectivity might not be preserved at subsequent iterations. Thus, at a certain iteration $k$, one might have $d_{is}^k = d_{jt}^k$ for some $i, j \in V - S$, $s, t \in S$. In this case, one has

$$
\begin{aligned}
d_{is}^k &= w_s^{(k)} d_{is}^k, \quad \text{where} \\
w_s^{(k)} &= \frac{P_s^{(k-1)}}{\bar{P}}, \quad \text{(static update)}, \\
w_s^{(k)} &= \frac{P_s^{(k-1)} P_s^{(k-2)} \dots P_s^{(0)}}{\left(\bar{P}\right)^k}, \quad \text{(dynamic update)},
\end{aligned}
$$

and similarly, $d_{jt}^k = w_t^{(k)} d_{jt}^k$. Hence,

$$\frac{d_{jt}}{d_{is}} = \frac{w_s^{(k)}}{w_t^{(k)}}. \tag{3}$$

If one assumes, as we do, that all the unit populations range in a finite integer interval $[0, p_{\max}]$, then the r.h.s. of (3) can take but a finite number of values. It follows that, in the cone of all real metrics $[d_{is}]$, the set of those satisfying (3) has measure zero. Therefore, if the algorithm terminates after a finite number of iterations, almost all metrics $[d_{is}]$ are such that their iterates $[d_{is}^k]$ remain injective for all $k$. In practice, if at a certain iteration it turns out that $d_{is}^k = d_{it}^k$, then one employs a tie-breaking rule to assign $i$ either to $D_s$ or to $D_t$.

## 3. Pathologies and desirable properties for WVR algorithms

The WVR algorithms described above may encounter some pathological situations, such as those discussed below. In Fig. 3, one case of lack of termination is presented in which the full transfer algorithm may loop. The numbers next to the nodes represent node-weights, those next to the edges edge-weights. At the beginning, both sites 1 and 2 are assigned to center $s$, thus generating a map with population equal to 199 and to 1 for districts $s$ and $t$, respectively. In this extremely unbalanced situation, the iterative distance updating results in the repeated transfer of both sites 1 and 2 from district $s$ to district $t$ and back without termination.

Fig. 4 shows an example where lack of contiguity might arise when the site-to-center distances are completely arbitrary. Suppose that all the nodes have the same population. It is easy to check that the Voronoi regions are given by the two subsets of nodes {1, 3} and {2, 4}. This district map is perfectly balanced, but not contiguous. Here, however, the distance function is neither metric nor injective.

In order to prevent these and other pathologies from occurring, we introduce four desirable properties to be met by weighted Voronoi algorithms — or at least by some variants of them.

(1) *order invariance*: at each step of the algorithm, the order relation on the sites w.r.t. their distances to any given center $s$ does not change. Formally, at iteration $k = 1, 2, \dots$, we have

$$d_{is}^k < d_{js}^k \Leftrightarrow d_{is} < d_{js}, \quad s \in S;\, i, j \in V - S. \tag{4}$$

(2) *re-balancing*:

$$\text{at iteration } k = 1, 2, \dots, \text{ site } i \text{ migrates from } D_q \text{ to } D_t \text{ only if } P_q^{k-1} > P_t^{k-1}. \tag{5}$$

(3) *geodesic consistency*: at any iteration, if site $j$ belongs to district $D_s$ and site $i$ lies on the geodesic between $j$ and $s$, then $i$ also belongs to $D_s$.
(4) *finite termination*: the algorithm stops after a finite number of iterations.

**Proposition 1.** *Order invariance holds for the full transfer WVR algorithms.*

**Proof.** In the static case, the statement directly follows from the updating formula (1); in the dynamic case, it inductively follows from (2). □

**Proposition 2.** *Re-balancing holds for the dynamic full transfer WVR algorithm.*

**Proof.** If at iteration $k$ site $i$ is assigned to center $s^*$, then one must necessarily have

$$d^k_{is^*} = \min_{s=1,2,\dots,r} d^k_{is},$$

that is, $d^k_{is^*} \le d^k_{is}$, $\forall s \ne s^*$, and in particular, for a given center $\bar{s} \ne s^*$,

$$d^k_{is^*} \le d^k_{i\bar{s}}. \tag{6}$$

Similarly, if at iteration $k+1$ site $i$ belongs to the district with center $\bar{s}$, it must be

$$d^{k+1}_{i\bar{s}} = \min_{s=1,2,\dots,r} d^{k+1}_{is},$$

and thus

$$d^{k+1}_{i\bar{s}} \le d^{k+1}_{is^*}. \tag{7}$$

If the dynamic updating formula (2) is adopted, one gets from (7)

$$d^k_{i\bar{s}} P^k_{\bar{s}} = d^{k+1}_{i\bar{s}} \le d^{k+1}_{is^*} = d^k_{is^*} P^k_{s^*}$$

so, since (6) holds, one must have

$$P^k_{\bar{s}} \le P^k_{s^*},$$

implying that at iteration $k$ the population of district $\bar{s}$ into which node $i$ migrates was no larger than that of district $s^*$ where $i$ migrates from. □

**Remark 2.** Notice that re-balancing is not guaranteed for the static full transfer WVR algorithm. In this case, however, reasoning as in the above proof one gets the following slightly different property: at iteration $k+1$, site $i$ migrates from district $s^*$ to district $\bar{s}$ only if

$$\frac{P^k_{\bar{s}}}{P^{k-1}_{\bar{s}}} \le \frac{P^k_{s^*}}{P^{k-1}_{s^*}}.$$

**Proposition 3.** *Geodesic consistency holds for the full transfer WVR algorithms.*

**Proof.** Let us firstly show that the property holds w.r.t. the initial distances. So, suppose that node $j$ is assigned to center $s$ and that $i$ lies on the geodesic from $j$ to $s$. If $i$ were assigned to center $s' \ne s$, then one would have $d_{is'} < d_{is}$ by the injectivity of $d$. But then one would have also $d_{js'} < d_{js}$, a contradiction. Then by order invariance the property must hold at each iteration $k$. □

**Proposition 4.** *Geodesic consistency implies contiguity.*

**Proof.** Suppose that vertex $i$ belongs to district $s$. By geodesic consistency, every vertex of the geodesic between $i$ and $s$ must also be assigned to $s$. Thus for every $i$ in district $s$ there is a path from $i$ to $s$ entirely contained in district $s$, implying that the district is connected. □

Although Propositions 1–4 provide good properties for the full transfer WVR algorithm (at least for the dynamic version) the example in Fig. 4 shows that finite termination does not hold in general for this class of algorithms. Hence, in our experiments, we have taken into consideration only the single and the partial transfer versions of WVR algorithms. As we shall see, the implementations of these algorithms satisfy order invariance, geodesic consistency, re-balancing and finite termination properties. The last two results are based on the notion of Majorization introduced in 1929 by Hardy, Littlewood and Pólya (cfr. [17,18]). They introduce the following definition of *transfer*:

Given a positive real vector $a = (a_1, a_2, \ldots, a_n)$, and given a pair $i$ and $j$ such that $a_i < a_j$, a *transfer* is an operation that transforms vector $a$ into a new vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ as follows:

$$\alpha_s = \begin{cases} a_s + \delta, & \text{if } s = i \\ a_s - \delta, & \text{if } s = j \\ a_s, & \text{if } s \neq i, j \end{cases}$$

where $0 < \delta < (a_j - a_i)/2$. A transfer operation involving the pair $(a_i, a_j)$, with $a_i < a_j$, returns the pair $(\alpha_i, \alpha_j)$ such that $\alpha_i < \alpha_j$, that is, the order relation between elements $i$ and $j$ does not change.

One says that a nonnegative vector $a$ is *strictly majorized* by another nonnegative vector $b$ (notation: $a \prec b$) if $a$ can be obtained from $b$ through a finite number of transfers. The relation is a strict preorder (i.e., an irreflexive, asymmetric and transitive relation) in $R^n_+$.

The integer counterpart of a transfer in the sense of Hardy, Littlewood and Pólya (HLP transfer for short) can be defined when $a = (a_1, a_2, \ldots, a_n)$ and $\delta$ are all integers. Since populations are integers anyhow, we shall always perform integral transfers in our WVR algorithms.

Let $P^k = (P^k_1, P^k_2, \ldots, P^k_r)$ be the integer vector of the populations of the $r$ districts at a generic iteration $k$. Then the following results hold.

**Proposition 5.** *At iteration k, the single transfer algorithm performs a HLP transfer on $P^{k-1}$.*

**Proof.** The result trivially follows from the definition of single transfer. □

Consider the path transfer algorithm and denote by $q_1, \ldots, q_h$ the sequence of $h$ consecutively adjacent districts that correspond to the nodes of a path in the auxiliary network $\mathcal{N}$. For every pair of consecutive districts, denote by $\delta_{q_i q_{i+1}}$ the total amount of population that migrates from $q_i$ to $q_{i+1}$, $i = 1, \ldots, h-1$, (that is, the population of the site that migrates from $q_i$ to $q_{i+1}$).

**Proposition 6.** *At iteration k, if each $\delta_{q_i q_{i+1}}$ is chosen so that $0 < \delta_{q_i q_{i+1}} < \delta = \min_{i=1,\ldots,h-1}(P^{k-1}_{q_i} - P^{k-1}_{q_{i+1}})/2$, $i = 1, \ldots, h-1$, then the path transfer algorithm performs a sequence of HLP transfers on $P^{k-1}$.*

**Proof.** According to the definition of path transfer, we must have $P^{k-1}_{q_1} > P^{k-1}_{q_2} > \cdots > P^{k-1}_{q_h}$. If $0 < \delta_{q_i q_{i+1}} < \delta = \min_{i=1,\ldots,h-1}(P^{k-1}_{q_i} - P^{k-1}_{q_{i+1}})/2$ holds for $i = 1, \ldots, h-1$, then for every pair of consecutive districts in the path we have $P^k_{q_i} > P^k_{q_{i+1}}$, thus implying the result. □

Notice that, if in a path transfer we have $\delta_{q_i q_{i+1}} = \delta > 0$ for all $i = 1, \ldots, h-1$, then the transfer modifies only the populations of the first and the last district in the path, respectively. On the other hand, one may also notice that if the path algorithm is designed so that if $\delta_{q_i q_{i+1}}$ satisfies the appropriate inequality initially for each $i$, that is, $\delta_{q_i q_{i+1}} < (P^{k-1}_{q_i} - P^{k-1}_{q_{i+1}})/2$, the algorithm performs a sequence of HLP transfers as well, but the amounts of population transferred from a district to another in the path $P$ may be different.

After Propositions 5 and 6 re-balancing is guaranteed by the very constructions for both the single and the path transfer algorithm, since HLP transfers always move a site from a heavier district into a lighter one. Moreover, order invariance and geodesic consistency hold by similar arguments to those in Propositions 1 and 3, respectively. Finally, the following result guarantees finite termination for these algorithms.

**Proposition 7.** *The single and path transfer weighted Voronoi algorithms halt after a finite number of steps.*

**Proof.** On the basis of the previous results, during the execution of these algorithms the vector of district populations, $P^k = (P^k_1, P^k_2, \ldots, P^k_r)$, decreases w.r.t. the strict preorder $\prec$, so it cannot be encountered twice. Since the total number of partitions of $V$ is finite, finite termination is guaranteed both for the single transfer and path transfer weighted Voronoi algorithm. □

Proposition 7 guarantees that our WVR algorithms terminate after a finite number of transfers. As a matter of fact, the next proposition provides an upper bound on them. Let

$$p_{\min} = \min_{i \in V} p_i, \qquad p_{\max} = \max_{i \in V} p_i, \qquad \rho = \frac{p_{\min}}{p_{\max}}.$$

**Proposition 8.** *In the single and path transfer weighted Voronoi algorithms the number of transfers is at most $2\rho n$.*

**Proof.** Consider an arbitrary partition $\pi$ of $V$ into $r$ components. At least one of its components, the first one, say, has population $P_1$ no smaller than $\bar{P}$. If one moves into the first component all the nodes of the remaining components, with populations $P_2, \ldots, P_r$, respectively, the imbalance of the first component increases at least by $P_2 + \cdots + P_r$, while the total imbalance of the remaining components decreases at most by the same amount. Hence the total imbalance of the partition does not decrease. It follows that the total imbalance of $\pi$ is bounded above by $(P - \bar{P}) + (r-1)\bar{P} = 2(r-1)\bar{P}$. At each

**Table 1**
Properties of the WVR algorithms

| Property | Static | | | Dynamic | | |
|---|---|---|---|---|---|---|
| | Single transfer | Path transfer | Full transfer | Single transfer | Path transfer | Full transfer |
| Order invariance | Yes | Yes | Yes | Yes | Yes | Yes |
| Re-balancing | Yes | Yes | ? | Yes | Yes | Yes |
| Geodesic consistency | Yes | Yes | Yes | Yes | Yes | Yes |
| Finite termination | Yes | Yes | ? | Yes | Yes | No |

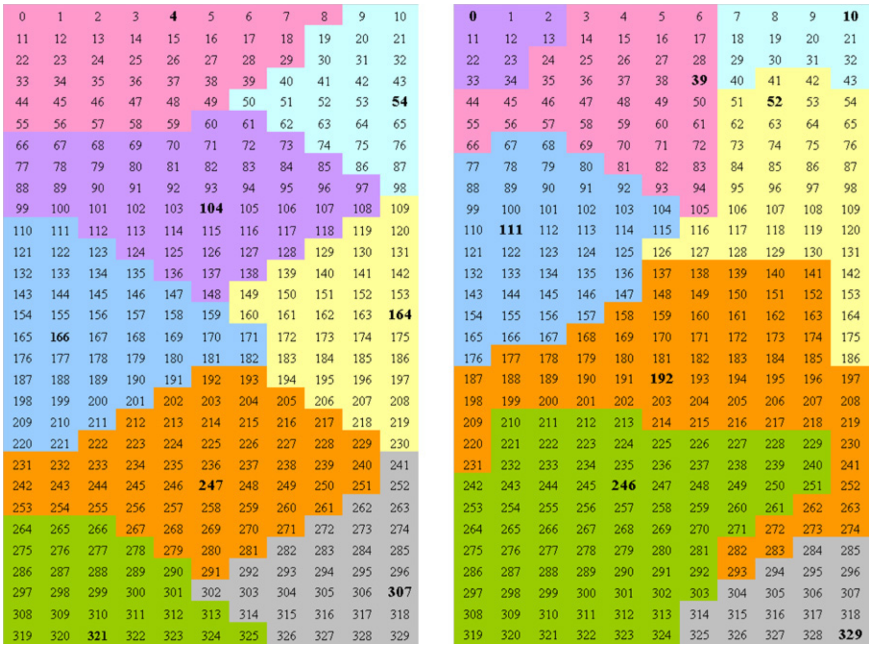


**Fig. 5.** Different district maps obtained on a rectangular 30 × 11 grid graph according to different procedures for the location of the $r$ centers.

iteration, our WVR procedures, starting from the initial partition, move at least a quantity of population equal to $p_{\min}$. If each transfer is a HLP transfer, at most

$$\frac{2(r-1)\bar{P}}{p_{\min}} \leq \frac{2(r-1)}{r}\,\frac{p_{\max}}{p_{\min}}n = 2\,(1-1/r)\;\rho\,n < 2\rho n$$

such transfers will be performed. □

Notice that the above proposition yields a *pseudo-polynomial* upper bound on the maximum number of iterations. For bounded $\rho$, such an upper bound is linear in the number of territorial units.

Table 1 summarizes the main theoretical results presented in this section. Here we denote the remaining open questions by "?".

## 4. Preliminary experimental results

In this section we present some preliminary results we obtained with one of our WVR algorithms, namely, the single transfer WVR.

Following the pseudocode presented in Fig. 2, the implementation of a WVR algorithm requires, first of all, the definition of the procedure for locating $r$ centers in $G$.

In our implementation we compared two different methods of locating the centers: on the one hand, the location was performed by solving an unweighted $r$-center problem on $G$; on the other hand, the centers were located as far apart from each other as possible (we refer to this particular location as "sparse centers"). Both approaches are heuristics, but they produce very different results (see, e.g., the example reported in Fig. 5), showing that the location of the centers at the beginning is a crucial step for a WVR algorithm. According to our experimental results, in general, the $r$-center approach performs better than the other. This behaviour is confirmed also by the numerical results shown in Tables 2–5. These tables show the results obtained with the single transfer algorithm on four contiguity graphs corresponding to a sample of Italian regions, both when the $r$-center and the sparse centers approaches are adopted. Population equality and compactness are

**Table 2**
Latium 374 nodes, 2012 arcs, 19 districts

| Sparse centers | Initial value | Final value | Variation (%) | Number of iterations |
|---|---|---|---|---|
| Population equality | 0.540 | 0.297 | −45 | 486 |
| Compactness | 0.295 | 0.419 | +42 | |
| $r$-center | Initial value | Final value | Variation (%) | Number of iterations |
| Population equality | 0.364 | 0.199 | −45 | 769 |
| Compactness | 0.081 | 0.263 | +2.25 | |

**Table 3**
Piedmont 1208 nodes, 7055 arcs, 28 districts

| Sparse centers | Initial value | Final value | Variation (%) | Number of iterations |
|---|---|---|---|---|
| Population equality | 0.444 | 0.264 | −40 | 3201 |
| Compactness | 0.506 | 0.612 | +21 | |
| $r$-center | Initial value | Final value | Variation (%) | Number of iterations |
| Population equality | 0.363 | 0.187 | −48 | 3087 |
| Compactness | 0.150 | 0.421 | +1.81 | |

**Table 4**
Abruzzi 305 nodes, 1694 arcs, 11 districts

| Sparse centers | Initial value | Final value | Variation (%) | Number of iterations |
|---|---|---|---|---|
| Population equality | 0.280 | 0.130 | 54 | 200 |
| Compactness | 0.338 | 0.451 | −54 | |
| $r$-center | Initial value | Final value | Variation (%) | Number of iterations |
| Population equality | 0.361 | 0.216 | −40 | 239 |
| Compactness | 0.134 | 0.305 | +1.28 | |

**Table 5**
Trentino 339 nodes, 1876 arcs, 8 districts

| Sparse centers | Initial value | Final value | Variation (%) | Number of iterations |
|---|---|---|---|---|
| Population equality | 0.357 | 0.165 | −54 | 239 |
| Compactness | 0.368 | 0.532 | +45 | |
| $r$-center | Initial value | Final value | Variation (%) | Number of iterations |
| Population equality | 0.412 | 0.306 | −26 | 314 |
| Compactness | 0.202 | 0.327 | +62 | |

measured by suitable indices varying between 0 and 1. For these indices, a value close to 0 corresponds to a very balanced and very compact district map, respectively, thus meaning that the result is very good. On the other hand, values close to 1 suggest that the performance of the algorithm was poor. In fact, both indices can be read as percentages of *lack of population equality* and *lack of compactness*, respectively. The former index is just a normalized version of the previously defined imbalance; the latter one, broadly speaking, is based on the comparison between each district and an "ideal", round one with the same center and a suitable radius (for details, see [11]). The "Initial value" refers to the partition obtained at the beginning by computing the discrete Voronoi regions in G. It is easy to recognize that very good values of the index of compactness are associated with these initial solutions, due to the fact that Voronoi diagrams generally produce "compact-shaped" regions. Unfortunately, the poor performance w.r.t. population equality – and the importance of such criterion in political districting – necessitates the search for better compromises through the application of a WVR procedure.

The final values found by our algorithm show that in the best case ($r$-center approach) we are able to improve population equality without worsening compactness too much. However, if, on the one hand, the values of compactness are very good, it must be noticed that the lack of population equality here remains still too high to be compared with the corresponding values typically associated to a political district map. Table 6 reports the values for the population equality and compactness indices associated to the institutional district map of the four Italian regions that was actually adopted in Italy for the political elections of the Chamber of Deputies until 2001.[1] While it is immediately recognized that our compactness values are definitively better than the institutional ones, we cannot say the same for population equality. It must be also pointed out that, according to the electoral law,[2] population equality is the main criterion in the design of the districts,

[1] This was the last election of the Chamber of Deputies performed in Italy with the old mixed system (Law n. 277/1993) for which a map of single-member districts was available. In 2005 the Italian electoral law was reformed and a proportional electoral system was adopted (Law n. 270/2005).

[2] Law n. 277/1993.

**Table 6**
Italian institutional district map

| Region | Population equality | Compactness |
|---|---|---|
| Latium | 0.06 | 0.68 |
| Piedmont | 0.1 | 0.88 |
| Abruzzi | 0.08 | 0.63 |
| Trentino | 0.04 | 0.70 |

**Table 7**
Rectangular grid graphs ($r$-center approach)

| Grid 20 × 20 (15 districts) | Initial value | Final value | Variation (%) |
|---|---|---|---|
| Population equality | 0.206 | 0.148 | −28 |
| Compactness | 0.098 | 0.157 | +59 |
| Grid 30 × 11 (8 districts) | Initial value | Final value | Variation (%) |
| Population equality | 0.185 | 0.157 | −15 |
| Compactness | 0.107 | 0.120 | +12 |

while compactness is not considered at all. Nevertheless, better results can be attained for population equality, even in combination with good values for compactness. Actually, in a previous experimental work [11] we obtained such results on the same territories by applying local search techniques. On these grounds, we believe that local search could be successfully combined with WVR to obtain an algorithm that is able to reach both a good compactness together with good balance in population. We shall leave this project to future work.

Additional results are provided in Table 7, showing the outcome of the application of the single transfer WVR algorithm to rectangular grids. A rectangular grid shares with the contiguity graph of a real territory the properties that it is planar and has low vertex degree.

Here we show two examples related to medium-size grids with a different number of districts. The results confirm the good performance of the single transfer WVR algorithm, combined with the $r$-center approach, since the values obtained for the population equality and compactness indices are all very low (between 12% and 15%).

To conclude, on the basis of our results, the class of WVR algorithms appears to be a useful tool for the design of impartial electoral district maps. Even if the (very preliminary) experimental results are still not fully satisfactory, we believe that better results can be achieved through the implementation of more sophisticated variants of WVR, such as the path transfer, also in view of the possibility of combining it with local search.

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