Extensions of GW processes - Point processes - Epidemic processes

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Outline

Section 1

Back to GW processes

GW processes

The structure of GW process

- Sample paths are trees generated as follows:
- The tree originates from an ancestor/root (generation-0 vertex)
- Every generation-i vertex v in the tree produces a random number Y_v of generation-$i+1$ vertices (the v-children).
- Numbers of children of different individuals are are i.i.d. random variables.
- the process stops if/when an empty generation is obtained

Section 2

Hawkes Processes

Possible extensions: Adding time

- GW can model epidemic processes but time reference is completely missing in GW processes (the time at which an individual gets infected)
- how can we add it? Any suggestion?

Adding time

- Time-stamp T_{root} of the root-vertex is conventionally set to 0 (or any other value).
- Time-stamp T_w, for every children w of v is obtained as:

$$T_w = T_v + \tau_{v,w}$$

where variables $\tau_{v,w} \geq 0$ are i.i.d arbitrarily distributed r.v. with assigned distribution $h(t)$. T_w can be interpreted as the time at which individual w is exposed to infection.

Hawkes processes and their intensity

- Ancestors are generated according to a Poisson process with intensity $\sigma \mathbf{1}_{(t \geq 0)}$ (or some generalization).
- Y_v is a discrete random variable with Poisson(m) distribution;
- $\tau_{v,w} \stackrel{L}{=} h(t)$ (fertility function);

- Widely used to model cascading processes (applications range from earthquake, to epidemics, social interactions, etc.)
- How to generate efficiently points of an Hawkes process over $[0, t]$?

Stochastic intensity

Given a point process, let $N(t)$ denotes the number of points in $[0, t)$. Assuming that $N(t)$ has unitary jumps and the process is not too pathological, we can define its stochastic intensity as:

$$\lambda(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{P}(N(t + \Delta t) - N(t) = 1).$$

Note that in general $\lambda(t)$ is a random variable, since it may depend on events happened in the past (i.e. the points in $[0, t)$).

Stochastic intensity for particular classes of process

- For homogeneous Poisson Processes $\lambda(t \mid H_t) = \lambda$
- For in-homogeneous Poisson Processes $\lambda(t \mid H_t) = \lambda(t)$
- For Hawkes processes:

$$\lambda(t \mid H_t) = \sigma + m \sum_{v: T_v < t} h(t - T_v)$$

Generating processes with an assigned stochastic intensity

A process with stochastic intensity $\lambda(\tau \mid H_\tau) \leq \gamma$ for $\tau \in [t, t + \Delta t]$ can be generated according to the following algorithm:

- Generate $\{T_i\}_i \in [t, t + \Delta t]$ points of a Poisson point with intensity γ.
- Thin independently points $\{T_i\}_i$, by retaining point T_i with a probability $p_i = \frac{\lambda(T_i|H_{T_i})}{\gamma}$

Section 3

SIR-SEIR

Limits of Hawkes processes

- Hawkes process model well the evolution of an epidemics at its early stage (exponential growth);
  - Typically $\sigma = \sigma(t)$ with support in $[0, \tau)$,
  - $m > 1$.
- However they assume population of susceptible to be practically infinite; they do not model saturation effects that take place after a while, due to presence of large fractions of immune individuals.

SIR models

- A population of N individuals is represented.
- Individuals are partitioned into three classes:

Classes

- *Susceptible:* the number of susceptible individuals at time t is denoted by $S(t)$
- *Infected/Contagious:* the number of infected individuals at time t is denoted by $I(t)$
- *Recovered/Immune/Dead:* the number of recovered individuals at time t is denoted by $R(t)$

Dynamics

Susceptible individuals gets infected according to a point process with stochastic intensity:

$$\lambda_{S\to I}(t \mid H_t) = \lambda S(t) I(t)$$

Infected individuals gets recovered according to a point process with stochastic intensity:

$$\gamma_{I\to R}(t \mid H_t) = \gamma I(t)$$

Reproducton number $R(t)$

- The reproduction number $R(t)$ is defined as the *average number of infected* that a contagious individual at t produces .
- it can be computed as, as long as dynamics of $S(t)$ are sufficiently slow:

$$R(t) = \frac{\lambda S(t)}{\gamma}$$

Indeed $\lambda S(t)$ is the rate at which a contagious individual at time t transmits the infection to the susceptible he/she come in contact with; while $1/\gamma$ represents the average time interval during which an individual is contagious.

Further extensions

How to represent the impact of non pharmaceutical interventions on the epidemic dynamics?

The impact of non pharmaceutical interventions

- by limiting mobility of individuals reduce the rate of human contacts through which epidemic spreads
- early detection of contagious people may further help since they are early quarantined

Modeling non pharmaceutical interventions

- We can model the impact of such interventions by reducing the intensity of infection spread:

$$\lambda_{R\to I}(t \mid H_t) = \lambda\rho(t)S(t)I(t)$$

  with $\rho(t) \in (0,1)$ characterizing the strength of restrictions.
- We can add the new class: *Quarantined* ($Q(t)$). Infected individuals may migrate to state Q or directly to R with intensities:

$$\gamma_{I\to Q}(t \mid H_t) = \gamma_Q I(t)$$

$$\gamma_{I\to R}(t \mid H_t) = \gamma I(t)$$

$$\gamma_{Q\to R}(t \mid H_t) = \gamma Q(t)$$

Section 4

Mean Field

Mean field SIR

We can further simply the model describing "average" dynamics through ODEs. We have:

- $\frac{\mathrm{d}I(t)}{\mathrm{d}t} = (\lambda\rho(t)S(t) - \gamma)I(t)$
- $\frac{\mathrm{d}S(t)}{\mathrm{d}t} = -\lambda\rho(t)I(t)S(t)$
- $\frac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma I(t)$

Under the initial condition $S(0), I(0), R(0)$. Typically $S(0) \approx N$, $R(0) = 0$.

Initial solution

As long as $S(t) \approx N$, we can approximately solve previous equations, it turns out:

$$I(t) \approx I(0) \exp\left(\gamma \int_0^t \left[\frac{\Lambda\rho(\theta)}{\gamma} - 1\right] \mathrm{d}\theta\right)$$

with $\Lambda = \lambda N$. Now approximating $\rho(t)$ with a constant in $[t, t + \Delta t)$ We have:

$$I(t + \Delta t) \approx I(t) \exp\left(\gamma \left[\frac{\Lambda\rho}{\gamma} - 1\right] \Delta t\right)$$

Therefore

- if $\frac{\Lambda\rho}{\gamma} > 1 \rightarrow$ **exponential grow**
- if $\frac{\Lambda\rho}{\gamma} < 1 \rightarrow$ **exponential decrease**
- if $\frac{\Lambda\rho}{\gamma} = 1 \rightarrow$ **constant behavior**

Possible extensions

Several extensions/refinements can be developed to take into account:

- spatial properties (spatial Hawkes/ compartimental models)
- population heterogeneity (sociability/ fragility)
- hospitalizations/intensive treatments
- vaccinations