Dynamical processes on graphs

Emilio Leonardi

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Outline

Section 1

Dynamical Processes on Graphs

Definition

Dynamical Process on Graphs

- Given a graph $G(\mathcal{V}, \mathcal{E})$
- Each vertex $v \in \mathcal{V}$ is provided with a dynamical variable (state variable) $x_v(t)$
- Vertex v wakes up at Poisson random times (the process of v-wakes-up is Poisson with intensity λ_v) and updates its state
  - the law according to which v-state is updated is arbitrary,
  - but the outcome should be a function of the current-state distribution of v-neighbors and its own state.
- state variables are arbitrarily initialized at $t = 0$.

Definition (cnt)

- In some cases it is more convenient to have wake-up timers associated to edges:
  - upon wake-up of an edge, both adjacent vertices may update their state.

state variable $x_v(t)$

- variables $x_v(t)$ can take values over:
  - a finite set (i.e. $\{0,1\}$, $\{0,1,2\cdots,K\}$)
  - an infinite discrete (countable) set (typically $\mathbb{N}$ or $\mathbb{Z}$)
  - a continuous set (i.e. $[0,1]$, $\mathbb{R}$).

Examples of dynamics: Majority Model

- $X_v(t) \in \{-1, 1\}$
- $\lambda_v = 1$
- Denote with $N_v^+(t)$ ($N_v^-(t)$) the number of neighbors of v in state $+1$ (-1) a time t
- Upon wake-up at time t_v
  - $X_v(t_v^+) = +1$ if $N_v^+(t_v^-) > N_v^-(t_v^-)$
  - $X_v(t_v^+) = -1$ if $N_v^-(t_v^-) > N_v^+(t_v^-)$
  - ties are randomly broken.

Examples of dynamics: Linear Threshold Model

- $X_v(t) \in \{0, 1\}$
- $\lambda_v = 1$
- Denote with $N_v(t)$ the number of neighbors of v in state $+1$ a time t
- Upon wake-up at time t_v
  - $X_v(t_v^+) = 1$ if $N_v(t_v^-) > r$
  - $X_v(t_v^+) = 0$ if $N_v(t_v^-) \leq r$
  - $r \in \mathbb{N}$ is a parameter

Examples of dynamics: Voter Model

- $X_v(t) \in \{-1, 1\}$
- either $\lambda_v = 1$ or $\lambda_v = \mathrm{Degree}(v)$
- Upon wake-up at time t_v, vertex v:
    - selects uniformly at random a neighbor w of v and set:
    - $x_v(t_v^+) = x_w(t_v^-)$, i.e. v copies the state of w ($v \overset{t_v}{\leftarrow} w$).

Examples of dynamics: Averaging Model

- $X_v(t) \in [0, 1]$;
- edges wake-up with $\lambda_{(v,w)} = 1$;
- upon wake-up at time $t_{(v,w)}$ vertices v and w set:
  - $x_v(t_v^+) = \alpha x_v(t_v^-) + (1-\alpha) x_w(t_v^-)$
  - $x_w(t_v^+) = \alpha x_w(t_v^-) + (1-\alpha) x_v(t_v^-)$
  - for $\alpha \in (0, 1)$

Examples of dynamics: Ising Model

- $x_v(t) \in \{-1, 1\}$;
- vertices wake-up with $\lambda = 1$;
- let $\mathcal{N}(v)$ denote the set of neighbors of v;
- upon wake-up at time t_v vertex v compute:
  - $H_+ = -J \sum_{w \in \mathcal{N}(v)} 1 \cdot x_v(t_w^-)$
  - $H_- = J \sum_{w \in \mathcal{N}(v)} 1 \cdot x_v(t_w^-) = -H_+$
  - Then set $x_v(t_c^+) = \pm 1$ with a probability $p^\pm = \frac{\exp(-\beta H_\pm)}{\exp(-\beta H_+) + \exp(-\beta H_-)}$

Section 2

The underlying graph

Possible classes of underlying graphs

$G(n, p)$

- a graph with n vertices
- For every pair of vertices (u, w) add edge (v, w) to the graph with probability p, independently from choices.

k-dimensional Grids: $\mathbb{Z}^k$

- a graph with with infinite vertices corresponding to points with integer coordinates $(z_1^{(v)}, z_2^{(v)}, \cdots, z_k^{(v)})$ where $z_i^{(v)} \in \mathbb{Z}$ for $1 \leq i \leq k$
- edge (v, w) belongs to the graph iff $\sum_{i=1}^{k} |z_i^{(v)} - z_i^{(w)}| = 1$

Section 3

Study of dynamics

Possible questions

- Does the dynamics of converge some-where?
- What are the properties of the equilibrium point (consensus)?
- What is the impact of the initial condition on the dynamics?
- What is the impact of the underlying graph?

Focusing on the Voter Model

- Time backward analysis is the key!
- we say that w is an ascendant of v (and v descends from w) at time t if we can find a chain $v \overset{t_v^{(v)}}{\leftarrow} w_1 \overset{t_{w_1}^{(v)}}{\leftarrow} w_2 \cdots w_k \overset{t_{w_k}^{(v)}}{\leftarrow} w \overset{t_w^{(v)}}{\leftarrow} \cdots$ with $t_w^{(v)} < t_{w_k}^{(v)} < t_{w_{k-1}}^{(v)} < \cdots t_{w_1}^{(v)} < t_v^{(v)} < t$
- Note that the ascendant chain from v is generated by a backward random walk
  - starting at v, at t;
  - jumping from time to time to a new randomly selected neighbor of the current vertex.
- We say that chains from v_1 and v_2 coalesce at (w, t_w) if:
  - w is an ascendant of both v_1 and v_2
  - $t_w = t_w^{(v_1)} = t_w^{(v_2)}$.

Voter Model

- if chains from v_1 and v_2 coalesce at (w, t_w) then necessarily $x_{v_1}(t) = x_{v_2}(t)$ for any $t > t_W$
- Therefore probability that two arbitrarily picked vertices v_1 and v_2 agree:

$$\lim_{t \to \infty} \mathbb{P}(x_{v_1}(t) = x_{v_2}(t))$$

  is lower bounded by the probability that the associated RWs coalesce (theory of RW is a well studied topic - key observation - the "difference" of two RWs is still a RW).
- It turns out that on finite graphs

$$\lim_{t \to \infty} \mathbb{P}(x_{v_1}(t) = x_{v_2}(t)) = 1$$

  for any v_1 and v_2 and consensus is reached.

Voter models on infinite graphs

- The things are more involved when the voter models runs on infinite graphs.
  - Consensus is reached on $\mathbb{Z}$ and $\mathbb{Z}^2$ (RWs are null recurrent)
  - Consensus is not reached on $\mathbb{Z}^d$ for $d \geq 3$ (RWs are transient).