Our ultimate goal is to understand the relationship between liquidity constraints and precautionary saving. There are three steps: consumption concavity increases prudence (Section **??**), liquidity constraints cause consumption concavity (Section **??**), and prudence affects precautionary saving (Section **??**).

Our analysis of consumption concavity and prudence in this section is couched in general terms and therefore applies whether the source of concavity is liquidity constraints or something else. This generality is useful, because there is a good candidate for the 'something else': uncertainty.

0.1 Definitions

The central issue in our approach will involve whether the value function exhibits a property we call consumption concavity (CC). We will first define property CC before we define a concept we call *counterclockwise concavification*, capturing a specific transformation of a consumption function that makes the modified function globally "more" concave.

Definition 1. *(Local Consumption Concavity).*
A function $V(w)$ has property CC (alternately, strict CC) over the interval between w_1 and $w_2 > w_1$ in relation to a utility function $u(c)$ with nonnegative, non-increasing prudence if

$$V'(w) = u'(c(w))$$

for some increasing function $c(w)$ that satisfies concavity (alternately, strict concavity) over the interval from w_1 to w_2.

Since $V'(w) = u'(c(w))$ holds by the envelope theorem, $V(w)$ having property CC (alternately, strict CC) is the same as having a concave (alternately, strictly concave) consumption function

$c(w)$.[1] Note that we allow for 'non-strict' concavity – that is, linearity – because we want to encompass cases like quadratic utility in which parts of the consumption function can be linear. Henceforth, unless otherwise noted, we will drop the cumbersome usage 'alternately, strict' – the reader should assume that what we mean always applies in the two alternate cases in parallel.

The definition of consumption concavity above only holds locally. If a function has property CC at every point, we define it as having property CC globally.

Definition 2. *(Global Consumption Concavity).*
A function $V(w)$ has property CC in relation to a utility function $u(c)$ with $u' > 0$, $u'' < 0$ if $V'(w) = u'(c(w))$ for some monotonically increasing concave function $c(w)$.

We are going to show how consumption concavity affects the prudence of the value function. To compare two consumption functions and their respective level of concavity, we need to define the concept that one function exhibits greater concavity than another.

Definition 3. *(More Consumption Concavity).*
Consider two functions $V(w)$ and $\hat{V}(w)$ that both exhibit property CC with respect to the same u' at a point w for some interval (w_1, w_2) such that $w_1 < w < w_2$. Then $\hat{V}(w)$ exhibits property greater CC than $V(w)$ if

$$\hat{c}(w) - \left(\frac{w_2 - w}{w_2 - w_1}\hat{c}(w_1) + \frac{w - w_1}{w_2 - w_1}\hat{c}(w_2)\right) \geq c(w) - \left(\frac{w_2 - w}{w_2 - w_1}c(w_1) + \frac{w \cdots}{w_2 \cdots}\right.$$

for all $w \in (w_1, w_2)$, and property strictly greater CC if (1) holds as a strict inequality.

[1] Remember that the envelope theorem depends only on being able to spend *current* wealth on *current* consumption, so it holds whether or not there is a liquidity constraint.

If c'' and $\hat{c}''$ exist everywhere between w_1 and w_2, this condition is equivalent to $\hat{c}''$ being weakly larger in absolute value than c'' everywhere in the range from w_1 to w_2. The strict version of the proposition would require the inequality to hold strictly over some interval between w_1 and w_2.

The next concept we introduce is 'counterclockwise concavification.' It captures an operation that makes the modified consumption function more concave than in the original situation. The idea is to think of the consumption function in the modified situation as being a twisted version of the consumption function in the baseline situation, where the kind of twisting allowed is a progressively larger increase in the MPC as the level of wealth gets lower. We call this a 'counterclockwise concavification' to capture the sense that at any specific level of wealth, we can think of the increase in the MPC at lower levels of wealth as being a counterclockwise rotation of the lower portion of the consumption function around that level of wealth.

Definition 4. *(Counterclockwise Concavification).*
Function $\hat{c}(w)$ is a counterclockwise concavification of $c(w)$ around $w^{\#}$ if the following conditions hold:

1. $\hat{c}(w) = c(w)$ *for* $w \geq w^{\#}$
2. $\lim_{w \uparrow w^{\#}} \left(\frac{\hat{c}'(w)}{c'(w)} \right) \geq 1$
3. $\lim_{v \uparrow w} \left(\frac{\hat{c}'(v)}{c'(v)} \right)$ *is weakly decreasing in* w *for* $w \leq w^{\#}$
4. *If* $\lim_{w \uparrow w^{\#}} \left(\frac{\hat{c}'(w)}{c'(w)} \right) = 1$, *then* $\lim_{w \uparrow w^{\#}} \left(\frac{\hat{c}''(w)}{c''(w)} \right) > 1$

The limits are necessary to allow for the possibility of discrete drops in the MPC at potential 'kink points' in the consumption functions. To understand the concept of counterclockwise concavification, it is useful to derive its implied properties.

Lemma 1. *(Properties of a Counterclockwise Concavification). If $\hat{c}(w)$ is a counterclockwise concavification of $c(w)$ around $w^{\#}$ and $c''(w) \leq 0$ for all w, then*

1. $\hat{c}(w) < c(w)$ *for* $w < w^{\#}$.
2. $\lim_{\upsilon \uparrow w} \hat{c}'(\upsilon) > \lim_{\upsilon \uparrow w} c'(\upsilon)$ *for* $w < w^{\#}$.
3. $\lim_{\upsilon \uparrow w} \hat{c}''(\upsilon) \leq \lim_{\upsilon \uparrow w} c''(\upsilon)$ *for* $w < w^{\#}$.

See Appendix **??** for the proof. A counterclockwise concavification thus reduces consumption, increases the MPC, and makes the consumption function more concave for all wealth levels below the point of concavification. Figure 1 illustrates two examples of counterclockwise concavifications: the introduction of a constraint and the introduction of a risk. In both cases, we start from the situation with no risk or constraints (solid line). The introduction of a constraint is a counterclockwise concavification around a kink point $w^{\#}$. Below $w^{\#}$, consumption is lower and the MPC is greater. The introduction of a risk also represents a counterclockwise concavification of the original consumption function, but this time around ∞. For all $w < \infty$, consumption is lower, the MPC is higher, and the consumption function is strictly more concave.

0.2 Consumption Concavity and Prudence

The section above established all the tools necessary to show the relationship between consumption concavity and prudence. Our method in this section is to compare prudence in a *baseline*

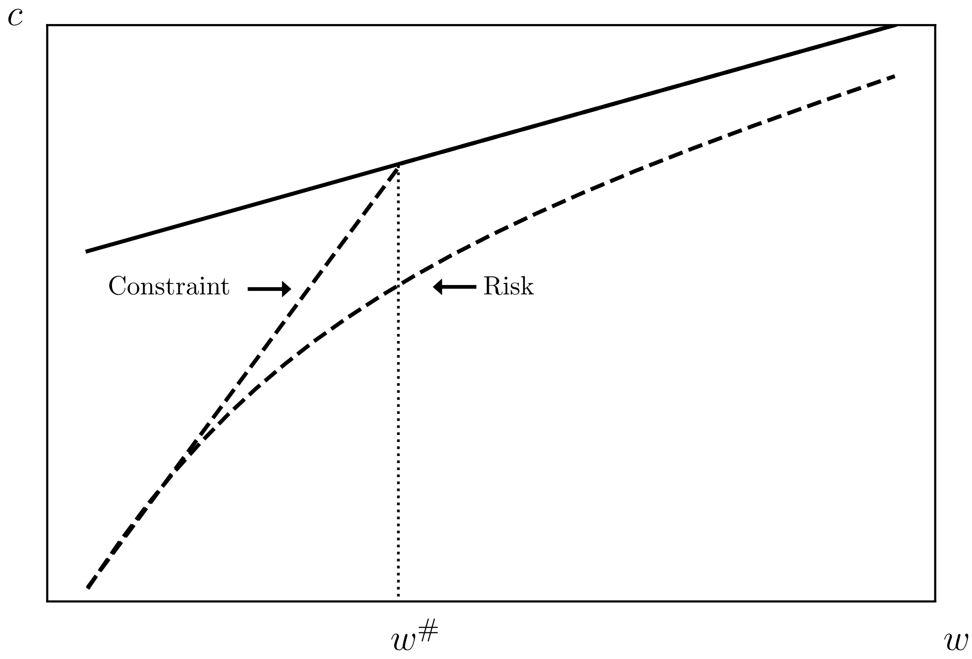

Figure 1 Examples of Counterclockwise Concavifications.

Notes: The solid line shows the linear consumption function in the case with no constraints and no risks. The two dashed line show the consumption function when we introduce a constraint and a risk, respectively. The introduction of a constraint is a counterclockwise concavification of the solid consumption function around $w^{\#}$, while the introduction of a risk is a counterclockwise concavification around ∞.

case where the consumption function is $c(w)$ to prudence in a *modified* situation in which the consumption function $\hat{c}(w)$ is a counterclockwise concavification of the baseline consumption function.

Theorem 1. *(Counterclockwise Concavification and Prudence). Consider an agent who has a utility function with $u' > 0$, $u'' < 0$, $u''' \geq 0$, and non-increasing absolute prudence $-u'''/u''$. If $c(w)$ is concave and $\hat{c}(w)$ is a counterclockwise concavification of $c(w)$, then the value function associated with $\hat{c}(w)$ exhibits greater prudence than the value function associated with $c(w)$ for all w.*

See Appendix **??** for the proof. There are two channels through which a counterclockwise concavification heightens prudence. First, if the absolute prudence of the utility function is non-increasing, then the reduction in consumption and increase in MPC from the counterclockwise concavification make households more prudent. Second, the increase in consumption concavity from the counterclockwise concavification itself also heightens prudence. The channels operate separately, implying that a counterclockwise concavification heightens prudence *even if absolute prudence is zero* as in the quadratic case.

Theorem 1 only provides conditions for when the value function exhibits greater prudence, but not strictly greater prudence. In particular, the value function associated with $\hat{c}(w)$ will in some cases exhibit equal prudence for many values of w and strictly greater prudence only for some values of w. In Corollary 1, we provide conditions for when the value function exhibits strictly greater prudence.

Corollary 1. *(Counterclockwise Concavification and Strictly Greater Prudence).*
Consider an agent who has a utility function with $u' > 0$, $u'' < 0$, $u''' \geq 0$, and non-increasing absolute prudence $-u'''/u''$. If $c(w)$ is concave and $\hat{c}(w)$ is a counterclockwise concavification of $c(w)$, then the value function associated with $\hat{c}(w)$ exhibits strictly greater prudence than the value function associated with $c(w)$ at w if the utility function satisfies $u''' > 0$ and $w < w^{\#}$ (the point of counterclockwise concavification) or the utility function is quadratic ($u''' = 0$) and $\frac{\hat{c}'(w)}{c'(w)}$ strictly declines at w.

See Appendix ?? for the proof. For prudent households ($u''' > 0$), the value function exhibits strictly greater prudence for all levels of wealth where the counterclockwise concavification affects consumption. This is a result of the fact that a reduction in consumption and higher marginal propensity to consume heighten prudence if the utility function has a positive third derivative and prudence is non-increasing. If the utility function instead is quadratic, the third derivative is zero and the absolute prudence of the utility function does not depend on the level of consumption or the marginal propensity to consume. In this case, the counterclockwise concavification only affects prudence at the kink points in the consumption function, i.e. where $\frac{\hat{c}'(w)}{c'(w)}$ strictly declines at w.

References