1 Proof of Lemma ??

Proof. First, condition 2 and 4 in Definition **??** implies that $\hat{c}'(w) > c'(w)$ for $w = w^{\#} - \epsilon$ for a small $\epsilon > 0$. Condition 3 then ensures that $\lim_{\upsilon \uparrow w} \hat{c}'(\upsilon) > \lim_{\upsilon \uparrow w} c'(\upsilon)$ holds for all $w \leq w^{\#} - \epsilon$ (equivalently $w < w^{\#}$). Second, condition 1 and the fact that $\lim_{\upsilon \uparrow w} \hat{c}'(\upsilon) > \lim_{\upsilon \uparrow w} c'(\upsilon)$ for $w < w^{\#}$ implies that $\lim_{\upsilon \uparrow w} \hat{c}(\upsilon) < \lim_{\upsilon \uparrow w} c(\upsilon)$ for $w < w^{\#}$. Third, condition 2 in Definition **??** implies that

$$\lim_{\upsilon \uparrow w} \hat{c}''(\upsilon) \leq \lim_{\upsilon \uparrow w} c''(\upsilon) \frac{\hat{c}'(\upsilon)}{c'(\upsilon)}$$

for $w < w^{\#}$. Then

$$\lim_{\upsilon \uparrow w} \hat{c}''(\upsilon) \leq \lim_{\upsilon \uparrow w} c''(\upsilon)$$

since $\lim_{\upsilon \uparrow w} \hat{c}'(\upsilon) > \lim_{\upsilon \uparrow w} c'(\upsilon)$ for $w < w^{\#}$. Note that the inequality is not strict since $c''(\upsilon)$ could be 0. □

2 Proof of Theorem ??

Proof. By the envelope theorem, we know that

$$V'(w) = u'(c(w))$$

Differentiating with respect to w yields[1]

$$V''(w) = u''(c(w))c'(w) \tag{1}$$

Taking another derivative can run afoul of the possible discontinuity in $c'(w)$ that we will show below can arise from liquidity

[1]Since $c(w)$ is concave, it has left-hand and right-hand derivatives at every point, though the left-hand and right-hand derivatives may not be equal. Equation (1) should be interpreted as applying the left-hand and right-hand derivatives separately. (Reading (1) in this way implies that $c'(w^-) \geq c'(w^+)$; therefore $V''(w^-) \leq V''(w^+)$).

constraints. We therefore consider two cases: (i) $c''(w)$ exists and (ii) $c''(w)$ does not exist.

Case I: ($c''(w)$ exists).
In the case where $c''(w)$ exists, we can take another derivative

$$V'''(w) = u'''(c(w))[c'(w)]^2 + u''(c(w))c''(w)$$

Absolute prudence of the value function is thus defined as

$$-\frac{V'''(w)}{V''(w)} = -\frac{u'''(c(w))[c'(w)]^2 + u''(c(w))c''(w)}{u''(c(w))c'(w)}$$
$$-\frac{V'''(w)}{V''(w)} = -\frac{u'''(c(w))}{u''(c(w))}c'(w) - \frac{c''(w)}{c'(w)} \tag{2}$$

From the assumption that $\hat{c}(w)$ is a counterclockwise concavification of $c(w)$, we know from Lemma **??** that $\hat{c}(w) \leq c(w)$ and $\hat{c}'(w) \geq c'(w)$. Furthermore, since $-\frac{u'''(c(w))}{u''(c(w))}$ is non-increasing, we know that $-\frac{u'''(\hat{c}(w))}{u''(\hat{c}(w))} \geq -\frac{u'''(c(w))}{u''(c(w))}$. As a result, $-\frac{u'''(\hat{c}(w))}{u''(\hat{c}(w))}\hat{c}'(w) \geq -\frac{u'''(c(w))}{u''(c(w))}c'(w)$.

The second part of the absolute prudence expression, $-\frac{c''(w)}{c'(w)}$, is a measure of the curvature of the consumption function. Since the consumption function is concave, it is then a measure of the degree of concavity. Formally, if we have two functions, $f(x)$ and $g(x)$, that both are increasing and concave functions, then the concave transformation $g(f(x))$ always has more curvature.[2] A counterclockwise concavification is an example of such a g.

[2]To see this, calculate

$$-\frac{\frac{d^2}{dx^2}g(f(x))}{\frac{d}{dx}g(f(x))} = -\frac{g''f'}{g'} - \frac{f''}{f'} \geq -\frac{f''}{f'}$$

where the inequality holds since $g' \geq 0$ and $g'' \leq 0$.

Hence, $-\frac{\hat{c}''(w)}{\hat{c}'(w)} \geq -\frac{c''(w)}{c'(w)}$. Then

$$
\begin{aligned}
-\frac{\hat{V}'''(w)}{\hat{V}''(w)} &= -\frac{u'''(\hat{c}(w))}{u''(\hat{c}(w))}\hat{c}'(w) - \frac{\hat{c}''(w)}{\hat{c}'(w)} \\
&\geq -\frac{u'''(c(w))}{u''(c(w))}c'(w) - \frac{c''(w)}{c'(w)} = -\frac{V'''(w)}{V''(w)}
\end{aligned}
$$

Case II: ($c''(w)$ does not exist).
Informally, if nonexistence is caused by a constraint binding at w, the effect will be a discrete decline in the marginal propensity to consume at w, which can be thought of as $c''(w) = -\infty$, implying positive infinite prudence at that point (see (2)). Formally, if $c''(w)$ does not exist, greater prudence of $\hat{V}$ than V is defined as $\frac{\hat{V}''(w)}{V''(w)}$ being a decreasing function of w. This is defined as

$$
\frac{\hat{V}''(w)}{V''(w)} \equiv \left(\frac{u''(\hat{c}(w))}{u''(c(w))}\right)\left(\frac{\hat{c}'(w)}{c'(w)}\right)
$$

The second factor, $\frac{\hat{c}'(w)}{c'(w)}$, is weakly decreasing in w by the assumption of counterclockwise concavity. At any specific value of w where $\hat{c}''(w)$ does not exist because the left and right hand values of $\hat{c}'$ are different, we say that $\hat{c}'$ is decreasing if

$$
\lim_{w^- \to w} \hat{c}'(w) \;>\; \lim_{w^+ \to w} \hat{c}'(w). \tag{3}
$$

As for the first factor, note that nonexistence of $\hat{V}'''(w)$ and/or $\hat{c}''(w)$ do not spring from nonexistence of either $u'''(c)$ or $\lim_{w \uparrow w} \hat{c}'(w)$ (for our purposes, when the left and right derivatives of $\hat{c}(w)$ differ at a point, the relevant derivative is the one coming from the left; rather than carry around the cumbersome limit notation, read the following derivation as applying to the left derivative). To discover whether $\frac{\hat{V}''(w)}{V''(w)}$ is decreasing we can simply differentiate:

$$\frac{d}{dw}\left(\frac{u''(\hat{c}(w))}{u''(c(w))}\right) = \frac{u'''(\hat{c}(w))\hat{c}'(w)u''(c(w)) - u''(\hat{c}(w))u'''(c(w))c'(w)}{[u''(c(w))]^2}.$$

Since the denominator is always positive, this will be negative if the numerator is negative, i.e. if

$$\begin{aligned} u'''(\hat{c}(w))u''(c(w))\hat{c}'(w) &\leq u''(\hat{c}(w))u'''(c(w))c'(w) \\ \frac{u'''(\hat{c}(w))}{u''(\hat{c}(w))}\hat{c}'(w) &\leq \frac{u'''(c(w))}{u''(c(w))}c'(w) \\ -\frac{u'''(\hat{c}(w))}{u''(\hat{c}(w))}\hat{c}'(w) &\geq -\frac{u'''(c(w))}{u''(c(w))}c'(w). \end{aligned} \tag{4}$$

Recall from Lemma **??** that $\hat{c}'(w) \geq c'(w)$ and $\hat{c}(w) \leq c(w)$ so non-increasing absolute prudence of the utility function ensures that $-\frac{u'''(\hat{c}(w))}{u''(\hat{c}(w))} \geq -\frac{u'''(c(w))}{u''(c(w))}$. Hence both terms on the LHS are greater than or equal to the corresponding terms on the RHS of equation (4).

□

3 Proof of Corollary ??

Proof. We prove each statement in Corollary **??** separately.

Case I: $(u''' > 0)$.

If $u''' > 0$, a counterclockwise concavification around $w^{\#}$ implies that $\hat{c}(w) < c(w)$ and $\hat{c}'(w) > c'(w)$ for all $w < w^{\#}$. Then

$$-\frac{u'''(\hat{c}(w))}{u''(\hat{c}(w)}\hat{c}'(w) > -\frac{u'''(c(w))}{u''(c(w))}c'(w) \text{ for } w < w^{\#}$$

Since we know that

$$-\frac{\hat{c}''(w)}{\hat{c}'(w)} \geq -\frac{c''(w)}{c'(w)} \text{ for } w < w^{\#}$$

from the proof of Theorem **??**, we know that

$$-\frac{\hat{V}'''(w)}{\hat{V}''(w)} = -\frac{u'''(\hat{c}(w))}{u''(\hat{c}(w)}\hat{c}'(w) - \frac{\hat{c}''(w)}{\hat{c}'(w)}$$
$$> -\frac{u'''(c(w))}{u''(c(w))}c'(w) - \frac{c''(w)}{c'(w)} = -\frac{V'''(w)}{V''(w)} \text{ for } w < w^{\#}$$

Case II: $(u''' = 0)$.
The quadratic case requires a different approach. Note first that the conditions in Corollary **??** hold only below the bliss point for quadratic utility. In addition, since $u'''(\cdot) = 0$, strict inequality between the prudence of $\hat{V}$ and the prudence of V hold only at those points where $\hat{c}(\cdot)$ is strictly concave.

Recall from the proof of Theorem **??** that greater prudence of $\hat{V}(w)$ than $V(w)$ occurs if $\frac{\hat{V}''(w)}{V''(w)}$ is decreasing in w. In the quadratic case

$$\frac{\hat{V}''(w)}{V''(w)} = \frac{u''(\hat{c}(w))}{u''(c(w))}\frac{\hat{c}'(w)}{c'(w)} = \frac{\hat{c}'(w)}{c'(w)} \tag{5}$$

where the second equality follows since $u''(\cdot)$ is constant with quadratic utility. Thus, prudence is strictly greater in the modified case only if $\frac{\hat{c}'(w)}{c'(w)}$ strictly declines in w. □

4 Proof of Theorem ??

Proof. First, to facilitate readability of the proof, we assume that $R = \beta = 1$ with no loss of generality. Our goal is to prove that $V(w_t) \in CC$ if $V_{t+1}(s_t + y_{t+1}) \in CC$ for all realization of y_{t+1}. The proof proceeds in two steps. First, we show that property CC is preserved through the expectation operator (vertical aggregation),

$$\Omega(s_t) = E_t[V_{t+1}(s_t + y_{t+1})] \in CC,$$

whenever $V_{t+1}(s_t + y_{t+1}) \in CC$ for all realization of y_{t+1}. Second, we show that property CC is preserved through the value function operator (horizontal aggregation),

$$V(w_t) = \max_s u(c_t(w_t - s)) + \Omega(s) \in CC,$$

whenever $\Omega(s) \in CC$. Throughout the proof, the first order condition holds with equality since no liquidity constraint applies at the end of period t.

Step 1: Vertical aggregation

We show that consumption concavity is preserved under vertical aggregation for three cases of the HARA utility function with $u''' \geq 0$ ($a \geq -1$) and non-increasing absolute prudence ($a \notin (-1, 0)$). The three cases are

$$u'(c) = \begin{cases} (ac+b)^{-1/a} & a > 0 \text{ (CRRA)} \\ e^{-c/b} & a = 0 \text{ (CARA)} \\ ac+b & a = -1 \text{ (Quadratic)} \end{cases} \tag{6}$$

Case I ($a > 0$, CRRA). We will show that concavity is preserved under vertical aggregation for $c^{-1/a}$ to avoid clutter, but the results hold for all affine transformations, $ac+b$, with strictly positive a. Concavity of $c_{t+1}(s_t + \chi y_{t+1})$ implies that

$$c_{t+1}(s_t + y_{t+1}) \geq pc_{t+1}(s_1 + y_{t+1}) + (1-p)c_{t+1}(s_2 + y_{t+1}) \tag{7}$$

for all $y_{t+1} \in [\underline{y}, \bar{y}]$ if $s_t = ps_1 + (1-p)s_2$ with $p \in [0, 1]$. Since this holds for all y_{t+1}, we know that

$$\left\{ E_t \left[c_{t+1}(s_t + \chi y_{t+1})^{-\frac{1}{a}} \right] \right\}^{-a} \geq \left\{ E_t \left[\{pc_{t+1}(s_1 + \chi y_{t+1}) + (1-p)c_{t+1}(s_2 +$$

By Minkowski's inequality, we know that for $a > 0$ ($a > -1, a \neq 0$)

$$\left\{E[(u+v)^{-\frac{1}{a}}]\right\}^{-a} \geq \left\{E[u^{-\frac{1}{a}}]\right\}^{-a} + \left\{E[v^{-\frac{1}{a}}]\right\}^{-a}$$

if $u \geq 0$ and $v \geq 0$. Thus

$$\left\{E_t\left[\{pc_{t+1}(s_1+\chi y_{t+1}) + (1-p)c_{t+1}(s_2+\chi y_{t+1})\}^{-\frac{1}{a}}\right]\right\}^{-a}$$
$$\geq \left\{E_t\left[\{pc_{t+1}(s_1+\chi y_{t+1})\}^{-\frac{1}{a}}\right]\right\}^{-a} + \left\{E_t\left[\{(1-p)c_{t+1}(s_2+\chi y_{t+1})\}^{-\frac{1}{a}}\right]\right\}$$
$$= p\left\{E_t\left[\{c_{t+1}(s_1+\chi y_{t+1})\}^{-\frac{1}{a}}\right]\right\}^{-a} + (1-p)\left\{E_t\left[\{c_{t+1}(s_2+\chi y_{t+1})\}^{-\frac{1}{a}}\right]\right.$$
$$= p(\Omega'(s_1))^{-a} + (1-p)(\Omega'(s_2))^{-a}$$

which implies that

$$(\Omega'(s_t))^{-a} \geq p(\Omega'(s_1))^{-a} + (1-p)(\Omega'(s_2))^{-a}$$

Thus, defining $\chi_t(s_t) = \{\Omega_t'(s_t)\}^{-a}$, we get

$$\chi_t(s_t) \geq p\chi_t(s_1) + (1-p)\chi_t(s_2)$$

for all s_t, where the inequality is strict if c_{t+1} is strictly concave for at least one realization of y_{t+1}.

Case II ($a = 0$, CARA). For the exponential case, property CC holds at s_t if

$$\exp(-\chi_t(s_t)/b) \;=\; E_t[\exp(-c_{t+1}(s_t+y_{t+1})/b)]$$

for some $\chi_t(s_t)$ which is strictly concave at s_t. We set $b = 1$ to reduce clutter, but results hold for $b \neq 1$. Consider first a case where c_{t+1} is linear over the range of possible values of $s_t + y_{t+1}$, then

$$\begin{aligned}\chi_t(s_t) &= -\log E_t[e^{-c_{t+1}(s_t+y_{t+1})}] \\ &= -\log E_t[e^{-(c_{t+1}(s_t+\bar{y})+(y_{t+1}-\bar{y})c_{t+1}')}]\end{aligned}$$

$$= \; c_{t+1}(s_t+\bar{y}) - \log E_t[e^{-(y_{t+1}-\bar{y})c'_{t+1}}] \tag{8}$$

which is linear in s_t since the second term is a constant.

Now consider a value of s_t for which $c_{t+1}(s_t+y_{t+1})$ is strictly concave for at least one realization of y_{t+1}. Global weak concavity of c_{t+1} tells us that for every y_{t+1}

$$\begin{aligned} -c_{t+1}(s_t+y_{t+1}) \; &\leq \; -((1-p)c_{t+1}(s_1+y_{t+1})+pc_{t+1}(s_2+y_{t+1})) \\ E_t[e^{-c_{t+1}(s_t+y_{t+1})}] \; &\leq \; E_t[e^{-((1-p)c_{t+1}(s_1+y_{t+1})+pc_{t+1}(s_2+y_{t+1}))}]. \end{aligned} \tag{9}$$

Meanwhile, the arithmetic-geometric mean inequality states that for positive u and v, if $\bar{u}=E_t[u]$ and $\bar{v}=E_t[v]$, then

$$E_t\left[(u/\bar{u})^p(v/\bar{v})^{1-p}\right] \leq E_t\left[p(u/\bar{u})+(1-p)(v/\bar{v})\right]=1,$$

implying that

$$E_t[u^p v^{1-p}] \leq \bar{u}^p\bar{v}^{1-p},$$

where the expression holds with equality only if v is proportional to u. Substituting in $u=e^{-c_{t+1}(s_1+y_{t+1})}$ and $v=e^{-c_{t+1}(s_2+y_{t+1})}$, this means that

$$E_t[e^{-pc_{t+1}(s_1+y_{t+1})-(1-p)c_{t+1}(s_2+y_{t+1})}] \; \leq \; \left\{E_t[e^{-c_{t+1}(s_1+y_{t+1})}]\right\}^p\left\{E_t[e^{-c_{t+1}(s_2+y_{t}}\right.$$

and we can substitute for the LHS from (9), obtaining

$$\begin{aligned} E_t[e^{-c_{t+1}(s_t+y_{t+1})}] \; &\leq \; \left\{E_t[e^{-c_{t+1}(s_1+y_{t+1})}]\right\}^p\left\{E_t[e^{-c_{t+1}(s_2+y_{t+1})}]\right\}^{1-p} \\ \log E_t[e^{-c_{t+1}(s_t+y_{t+1})}] \; &\leq \; p\log E_t[e^{-c_{t+1}(s_1+y_{t+1})}]+(1-p)\log E_t[e^{-c_{t+1}(s_2+y_{t}}] \end{aligned} \tag{10}$$

which holds with equality only when $e^{-c_{t+1}(s_1+y_{t+1})}/e^{-c_{t+1}(s_2+y_{t+1})}$ is a constant. This will only happen if $c_{t+1}(s_1+y_{t+1})-c_{t+1}(s_2+y_{t+1})$ is constant, which (given that the MPC is strictly positive everywhere) requires $c_{t+1}(s_t+y_{t+1})$ to be linear for $y_{t+1}\in(\underline{y},\bar{y})$. Hence,

$$\chi_t(s_t) \; \geq \; p\chi_t(s_1)+(1-p)\chi_t(s_2).$$

where the inequality is strict for an s_t from which c_{t+1} is strictly concave for some realization of y_{t+1}.

Case III ($a = -1$, Quadratic). In the quadratic case, linearity of marginal utility implies that

$$\begin{aligned} u'(\chi_t(s_t)) &= E_t[u'(c_{t+1}(s_t + y_{t+1}))] \\ \chi_t(s_t) &= E_t[c_{t+1}(s_t + y_{t+1})] \end{aligned}$$

so χ_t is simply the weighted sum of a set of concave functions where the weights correspond to the probabilities of the various possible outcomes for y_{t+1}. The sum of concave functions is itself concave. And if additionally the consumption function is strictly concave at any point, the weighted sum is also strictly concave.

Step 2: Horizontal aggregation:
We now proceed with horizontal aggregation, namely how concavity is preserved through the value function operation. Assume that $\Omega_t(s_t) \in CC$ at point s_t, then the first order condition implies that

$$\Omega_t'(s_t) = u'(\chi_t(s_t))$$

for some monotonically increasing $\chi_t(s_t)$ that satisfies

$$\chi_t(ps_1 + (1-p)s_2) \geq p\chi_t(s_1) + (1-p)\chi_t(s_2) \tag{11}$$

for any $0 < p < 1$, and $s_1 < s_t < s_2$.

In addition, we know that the first order condition holds with equality such that $\Omega_t'(s_t) = u'(c_t(w_t)) = u'(\chi_t(s_t))$ which implies that $s_t = \chi_t^{-1}(c_t)$. Using this equation, we get

$$\begin{aligned} \chi_t(ps_1 + (1-p)s_2) &\geq p\chi_t(s_1) + (1-p)\chi_t(s_2) \\ ps_1 + (1-p)s_2 &\geq \chi_t^{-1}(p\chi_t(s_1) + (1-p)\chi_t(s_2)) \\ p\chi_t^{-1}(c_1) + (1-p)\chi_t^{-1}(c_2) &\geq \chi_t^{-1}(pc_1 + (1-p)c_2) \end{aligned}$$

which implies that χ_t^{-1} is a convex function.

Use the budget constraint to define

$$w_t = s_t + c_t$$
$$\omega(c_t) = \chi^{-1}(c_t) + c_t$$

Now, since χ_t^{-1} is a convex function, and $\omega(c_t)$ is the sum of a convex and a linear function, it is also a convex function satisfying

$$\begin{aligned} p\omega(c_1) + (1-p)\omega(c_2) &\geq \omega(pc_1 + (1-p)c_2) \\ \omega^{-1}(p\omega(c_1) + (1-p)\omega(c_2)) &\geq pc_1 + (1-p)c_2 \\ c(pw_1 + (1-p)w_2) &\geq pc(w_1) + (1-p)c(w_2) \end{aligned} \qquad (12)$$

so c is concave.

Strict Consumption Concavity. When $V_{t+1}(w_{t+1})$ exhibits property strict consumption concavity for at least one $w_{t+1} \in [Rs_t + \underline{y}, Rs_t + \bar{y}]$, we know that $\chi_t(s_t)$ also exhibit property strict consumption concavity from the proof of vertical aggregation. Subsequently, equation (11) holds with strict inequality, and this strict inequality goes through the proof of horizontal aggregation, implying that equation (12) holds with strict inequality. Hence, $c_t(w_t)$ is strictly concave if $c_{t+1}(s_t + y_{t+1})$ is concave for all realizations of y_{t+1} and strictly concave for at least one realization of y_{t+1}. □

5 Proof of Theorem ??

We prove Theorem **??** by induction in two steps. First, we show that all results in Theorem **??** hold when we add the first constraint. The second step is then to show that the results hold when we go from n to $n+1$ constraints.

Lemma 1. $(c'_t < c'_{t+1})$
Consider an agent who has a utility function with $u' > 0$ and $u'' < 0$, faces constant income, is impatient $\beta R < 1$, and has a finite life. Then $c'_t < c'_{t+1}$.

Proof. The marginal propensity to consume in period t can be obtained from the MPC in period $t+1$ from the Euler equation

$$u'(c_t(w_t)) \;=\; \beta R u'(c_{t+1}(R(w_t - c_t(w_t)) + y)).$$

Differentiating both sides with respect to w_t and omitting arguments to reduce clutter we obtain

$$\begin{aligned}
u''(c_t)c'_t &= \beta R u''(c_{t+1})c'_{t+1}R(1-c'_t)\\
(u''(c_t) + \beta R u''(c_{t+1})c'_{t+1}R)c'_t &= \beta R u''(c_{t+1})Rc'_{t+1}\\
\frac{c'_{t+1}}{c'_t} &= \frac{u''(c_t) + \beta R u''(c_{t+1})c'_{t+1}R}{\beta R u''(c_{t+1})R}\\
\frac{c'_{t+1}}{c'_t} &= \frac{u''(c_t)}{\beta R u''(c_{t+1})R} + c'_{t+1}
\end{aligned}$$

Since $\beta R < 1$ ensures that $c_t > c_{t+1}$, we know that

$$\frac{u''(c_t)}{\beta R u''(c_{t+1})R} \geq \frac{u''(c_{t+1})}{\beta R u''(c_{t+1})R} = \frac{1}{\beta R R} > \frac{1}{R}$$

Furthermore, we know that

$$c'_t \geq \frac{R-1}{R}$$

since $\frac{R-1}{R}$ is the MPC for an infinitely-lived agent with $\beta R = 1$. Hence,

$$\frac{c'_{t+1}}{c'_t} = \left(\frac{u''(c_t)}{\beta R u''(c_{t+1})R} + c'_t\right) > \frac{1}{R} + \frac{R-1}{R} = 1$$

and it follows that $c'_t < c'_{t+1}$. □

Lemma 2. *(Consumption with one Liquidity Constraint).*

Consider an agent who has a utility function with $u' > 0$ and $u'' < 0$, faces constant income, y, and is impatient, $\beta R < 1$. Assume that the agent faces a set $\mathcal{T}$ of one relevant constraint. Then $c_{t,1}(w)$ is a counterclockwise concavification of $c_{t,0}(w)$ around $\omega_{t,1}$.

Proof. Define $\tau = \mathcal{T}[1]$, the time period of the constraint. Note first that consumption is unaffected by the constraint for all periods after τ, i.e. $c_{\tau+k,1} = c_{\tau+k,0}$ for any $k > 0$. For period τ we can calculate the level of consumption at which the constraint binds by realizing that a consumer for whom the constraint binds will save nothing, and that the maximum amount of consumption at which the constraint binds will satisfy the Euler equation (only points where the constraint is strictly binding violate the Euler equation; the point on the cusp does not). Thus, we define $c^{\#}_{\tau,1}$ as the level of consumption in period τ at which the constraint stops binding, we have

$$\begin{aligned} u'(c^{\#}_{\tau,1}) &= \beta R u'(c_{\tau+1,0}(y)) \\ c^{\#}_{\tau,1} &= (u')^{-1}\left(\beta R u'(c_{\tau+1,0}(y))\right), \end{aligned}$$

and the level of wealth at which the constraint stops binding can be obtained from

$$\omega_{\tau,1} = \left(V'_{\tau,1}\right)^{-1}(u'(c^{\#}_{\tau,1})). \tag{13}$$

Below this level of wealth, we have $c_{\tau,1}(w) = w$ so the MPC is one, while above it we have $c_{\tau,1}(w) = c_{\tau,0}(w)$ where the MPC equals the constant MPC for an unconstrained perfect foresight optimization problem with a horizon of $T-\tau$. Thus, $c_{\tau,1}$ satisfies our definition of a counterclockwise concavification of $c_{\tau,0}$ around $\omega_{\tau,1}$.

Further, we can obtain the value of period $\tau - 1$ consumption at which the period τ constraint stops impinging on period $\tau-1$ behavior from

$$u'(c^{\#}_{\tau-1,1}) \ = \ \beta R u'(c^{\#}_{\tau,1})$$

and we can obtain $\omega_{\tau-1,1}$ via the analogue to (13). Iteration generates the remaining $c^{\#}_{.,1}$ and $\omega_{.,1}$ values back to period t.

Now consider the behavior of a consumer in period $\tau-1$ with a level of wealth $w < \omega_{\tau-1,1}$. This consumer knows he will be constrained and will spend all of his resources next period, so at w his behavior will be identical to the behavior of a consumer whose entire horizon ends at time τ. As shown in step I, the MPC always declines with horizon. The MPC for this consumer is therefore strictly greater than the MPC of the unconstrained consumer whose horizon ends at $T > \tau$. Thus, in each period before $\tau+1$, the consumption function $c_{.,1}$ generated by imposition of the constraint constitutes a counterclockwise concavification of the unconstrained consumption function around the kink point $\omega_{.,1}$. □

We have now shown the results in Theorem ?? for $n = 0$. The last step is to show that they also hold for $n+1$ when it holds strictly for n. Consider imposing the $n+1$'th constraint and suppose for concreteness that it applies at the end of period τ. It will stop binding at a level of consumption defined by

$$u'(c^{\#}_{\tau,n+1}) = \beta R u'(c_{\tau+1,n}(y)) = RBu'(y)$$

where the second equatility follows because a consumer with total resources y, constant income, and $\beta R < 1$ will be constrained. But note that by definition of $c^{\#}_{\tau,n}$, we obtain

$$u'(c^{\#}_{\tau,n}) = (R\beta)^{\mathcal{T}[n]-\tau} u'(y) < R\beta u'(y) = u'(c^{\#}_{\tau,n+1})$$

where $\mathcal{T}[n]-\tau$ denotes the time remaining to the n'th constraint. Then, from the assumption of decreasing marginal utility, we know that

$$c^{\#}_{\tau,n} \;\geq\; c^{\#}_{\tau,n+1}.$$

This means that the constraint is relevant: The pre-existing constraint n does not force the consumer to do so much saving in period τ that the $n+1$'th constraint fails to bind.

The prior-period levels of consumption and wealth at which constraint $n+1$ stops impinging on consumption can again be calculated recursively from

$$\begin{aligned} u'(c^{\#}_{\tau,n+1}) &= R\beta u'(c_{\tau+1,n}(y)) \\ \omega_{\tau,n+1} &= \left(V'_{\tau,n}\right)^{-1}(u'(c^{\#}_{\tau,n+1})). \end{aligned}$$

Furthermore, once again we can think of the constraint as terminating the horizon of a finite-horizon consumer in an earlier period than it is terminated for the less-constrained consumer, with the implication that the MPC below $\omega_{\tau,n+1}$ is strictly greater than the MPC above $\omega_{\tau,n+1}$. Thus, the consumption function $c_{\tau,n+1}$ constitutes a counterclockwise concavification of the consumption function $c_{\tau,n}$ around the kink point $\omega_{\tau,n+1}$.

6 Proof of Theorem ??

Proof. We prove Theorem **??** by induction. We first show that it holds when we introduce the first constraint, before we show that it holds when we introduce constraint number $n+1$ when n constraints already hold.

Lemma 3. *(Precautionary Saving with one Liquidity Constraint).*

Consider an agent who has a utility function with $u' > 0$, $u'' < 0$, $u''' > 0$, and non-increasing absolute prudence $-u'''/u''$. Then

$$c_{t,1}(w) - \tilde{c}_{t,1}(w) \geq c_{t,0}(w) - \tilde{c}_{t,0}(w), \tag{14}$$

and the inequality is strict if $w_t < \bar{\omega}_{t,1}$.

Our proof proceeds by constructing behavior of consumers facing the risk from the behavior of the perfect foresight consumers. We consider matters from the perspective of some level of wealth w for the perfect foresight consumers. Because the same marginal utility function u' applies to all four consumption rules, the Compensating Precautionary Premia associated with the introduction of the risk ζ_{t+1}, $\kappa_{t,0}$ and $\kappa_{t,1}$, must satisfy

$$c_{t,0}(w) \;=\; \tilde{c}_{t,0}(w + \kappa_{t,0}) \tag{15}$$

$$c_{t,1}(w) \;=\; \tilde{c}_{t,1}(w + \kappa_{t,1}). \tag{16}$$

Define the amounts of precautionary saving induced by the risk ζ_{t+1} at an arbitrary level of wealth w in the two cases as

$$\psi_{t,0}(w) \;=\; c_{t,0}(w) - \tilde{c}_{t,0}(w) \tag{17}$$

$$\psi_{t,1}(w) \;=\; c_{t,1}(w) - \tilde{c}_{t,1}(w) \tag{18}$$

where the mnemonic is that the first two letters of the Greek letter psi stand for **p**recautionary **s**aving.

We can rewrite (16) (resp. (15)) as

$$c_{t,1}(w + \kappa_{t,1}) + \int_{w+\kappa_{t,1}}^{w} c'_{t,1}(v)dv \;=\; \tilde{c}_{t,1}(w + \kappa_{t,1})$$

$$c_{t,1}(w + \kappa_{t,1}) - \tilde{c}_{t,1}(w + \kappa_{t,1}) \;=\; \int_{w}^{w+\kappa_{t,1}} c'_{t,1}(v)dv = \psi_{t,1}(w + \kappa_{t,1})$$

$$c_{t,0}(w + \kappa_{t,0}) - \tilde{c}_{t,0}(w + \kappa_{t,0}) \;=\; \int_{w}^{w+\kappa_{t,0}} c'_{t,0}(v)dv = \psi_{t,0}(w + \kappa_{t,0})$$

and

$$\psi_{t,0}(w+\kappa_{t,1}) \;=\; \psi_{t,0}(w+\kappa_{t,0}) - \int_{w+\kappa_{t,0}}^{w+\kappa_{t,1}} (\tilde{c}'_{t,0}(v) - c'_{t,0}(v))dv$$

so the difference between precautionary saving for the constrained and unconstrained consumers at $w+\kappa_{t,1}$ is

$$\begin{aligned}
&\psi_{t,1}(w+\kappa_{t,1}) - \psi_{t,0}(w+\kappa_{t,1}) = \\
&\quad = \int_{w}^{w+\kappa_{t,0}} (c'_{t,1}(v) - c'_{t,0}(v))dv + \int_{w+\kappa_{t,0}}^{w+\kappa_{t,1}} \left(c'_{t,1}(v) + (\tilde{c}'_{t,0}(v) - c'_{t,0}(v))\right) d \\
&\quad = \int_{w}^{w+\kappa_{t,1}} (c'_{t,1}(v) - c'_{t,0}(v))dv + \int_{w+\kappa_{t,0}}^{w+\kappa_{t,1}} \tilde{c}'_{t,0}(v)dv \qquad (19)
\end{aligned}$$

If we can show that (19) is a positive number for all feasible levels of w satisfying $w < \bar{\omega}_{t,1}$, then we have proven Lemma 3. In particular, we know that the marginal propensity to consume is always strictly positive and that $\kappa_{t,1} \geq \kappa_{t,0} \geq 0$[3] so to prove that (19) is strictly positive, we need to show one of the following sufficient conditions:

1. $\kappa_{t,1} > 0$ or $\kappa_{t,0} > 0$, and $c'_{t,1}(v) > c'_{t,0}(v)$
2. $\kappa_{t,1} > \kappa_{t,0}$

Now, since $u''' > 0$, we know that $\kappa_{t,0} > 0$ from Jensen's inequality. The first integral in (19) is therefore strictly positive as long as $c'_{t,1} > c'_{t,0}$, which is true for $w < \omega_{t,1}$ (Theorem **??**).

For $w \geq \omega_{t,1}$, we know that $c'_{t,1} = c'_{t,0}$ so the first integral in (19) is always zero. For the second integral in (19) to be strictly positive, we need to show that $\kappa_{t,1} > \kappa_{t,0}$. Recall first the definition of $\kappa_{t,0}$ and $\kappa_{t,1}$:

$$u'(c_{t,0}) \;=\; E_t[u'(c(\kappa_{t,0}+\zeta))]$$

[3] Since we know that liquidity constraints increase prudence (Theorem **??**) and that prudence results in a positive precautionary premium (Lemma **??**).

$$u'(c_{t,1}) \;=\; E_t[u'(\hat{c}(\kappa_{t,1}+\zeta))],$$

where the two consumption functions are defined as

$$
\begin{aligned}
c(\kappa_{t,0}+\zeta) \;&=\; c_{t+1,0}(\overbrace{s_{t,0}}^{=s_{t,1}}+y+\kappa_{t,0}+\zeta) &(20)\\
\hat{c}(\kappa_{t,1}+\zeta) \;&=\; c_{t+1,1}(s_{t,1}+y+\kappa_{t,1}+\zeta). &(21)
\end{aligned}
$$

Now recall that Lemma **??** tells us that if absolute prudence of $u'(c(\kappa_{t,0}+\zeta))$ is identical to absolute prudence of $u'(\hat{c}(\kappa_{t,1}+\zeta))$ for every realization of ζ, then $\kappa_{t,0}=\kappa_{t,1}$. This can only be true if $w_{t+1}\geq\omega_{t+1,1}$ for all possible realizations of $\zeta\in(\underline{\zeta},\bar{\zeta})$. We defined this limit as $w_{t+1}\geq\bar{\omega}_{t+1,1}$. We therefore know that $\kappa_{t,1}=\kappa_{t,0}$ if $w\geq\bar{\omega}_{t+1,1}$ and $\kappa_{t,1}>\kappa_{t,0}$ if $w<\bar{\omega}_{t+1,1}$. Equation (19) is therefore strictly positive if $w<\bar{\omega}_{t+1,1}$ and we have proven Lemma 3.

The $n+1$'th constraint

Consider now the case where we have imposed n constraints and are considering imposing constraint $n+1$ and where constraint $n+1$ applies at the end of some future period. Similar to the introduction of the first constraint, we need to show that the following equation is strictly positive:

$$
\begin{aligned}
&\psi_{t,n+1}(w+\kappa_{t,n+1})-\psi_{t,n}(w+\kappa_{t,n+1}) =\\
&\qquad = \int_{w}^{w+\kappa_{t,n}}(c'_{t,n+1}(\upsilon)-c'_{t,0}(\upsilon))d\upsilon+\int_{w+\kappa_{t,n}}^{w+\kappa_{t,n+1}}\left(c'_{t,n+1}(\upsilon)+(\tilde{c}'_{t,n}(\upsilon)-c\right.\\
&\qquad = \int_{w}^{w+\kappa_{t,n+1}}(c'_{t,n+1}(\upsilon)-c'_{t,n}(\upsilon))d\upsilon+\int_{w+\kappa_{t,n}}^{w+\kappa_{t,n+1}}\tilde{c}'_{t,n}(\upsilon)d\upsilon
\end{aligned}
\qquad (22)
$$

The sufficient conditions for (22) to be strictly positive are

1. $\kappa_{t,n+1} > 0$ or $\kappa_{t,n} > 0$, and $c'_{t,n+1}(\upsilon) > c'_{t,n}(\upsilon)$

2. $\kappa_{t,n+1} > \kappa_{t,n}$

Now since $u''' > 0$, we know that $\kappa_{t,n} > 0$ from Jensen's inequality and Lemma **??**. The first integral in (19) is therefore strictly positive as long as $c'_{t,n+1} > c'_{t,n}$, which is true for $w < \omega_{t,n+1}$ (Theorem **??**).

For $w \geq \omega_{t,n+1}$, we know that $c'_{t,n+1} = c'_{t,n}$ so the first integral in (19) is always zero. For the second integral in (19) to be strictly positive, we need to show that $\kappa_{t,n+1} > \kappa_{t,n}$. Recall the definition of $\kappa_{t,n}$ and $\kappa_{t,n+1}$:

$$\begin{aligned} u'(c_{t,n}) &= E_t[u'(c(\kappa_{t,n} + \zeta))] \\ u'(c_{t,n+1}) &= E_t[u'(\hat{c}(\kappa_{t,n+1} + \zeta))], \end{aligned}$$

where the two consumption functions are defined as

$$c(\kappa_{t,n} + \zeta) = c_{t+1,n}(\overbrace{s_{t,n}}^{=s_{t,n+1}} + y + \kappa_{t,n} + \zeta) \tag{23}$$

$$\hat{c}(\kappa_{t,n+1} + \zeta) = c_{t+1,n+1}(s_{t,n+1} + y + \kappa_{t,n+1} + \zeta). \tag{24}$$

Now recall that Lemma **??** tells us that if absolute prudence of $u'(c(\kappa_{t,n}+\zeta))$ is identical to absolute prudence of $u'(\hat{c}(\kappa_{t,n+1}+\zeta))$ for every realization of ζ, then $\kappa_{t,n} = \kappa_{t,n+1}$. This can only be true if $w_{t+1} \geq \omega_{t+1,n+1}$ for any realization of $\zeta \in (\underline{\zeta}, \bar{\zeta})$. We defined this limit as $w_{t+1} \geq \bar{\omega}_{t,n+1}$. We therefore know that $\kappa_{t,n+1} = \kappa_{t,n}$ if $w \geq \bar{\omega}_{t,n+1}$ and $\kappa_{t,n+1} > \kappa_{t,n}$ if $w < \bar{\omega}_{t,n+1}$. Equation (19) is therefore strictly positive if $w < \bar{\omega}_{t,n+1}$ and we have proven Theorem **??**.

□

7 Resemblance Between Precautionary Saving and a Liquidity Constraint

In this appendix, we provide an example where the introduction of risk resembles the introduction of a constraint. Consider the second-to-last period of life for two CRRA utility consumers, and assume for simplicity that $R = \beta = 1$.

The first consumer is subject to a liquidity constraint $c_{T-1} \geq w_{T-1}$, and earns non-stochastic income of $y = 1$ in period T. This consumer's saving rule will be

$$s_{T-1,1}(w_{T-1}) = \begin{cases} 0 & \text{if } w_{T-1} \leq 1 \\ (w_{T-1} - 1)/2 & \text{if } w_{T-1} > 1. \end{cases}$$

The second consumer is not subject to a liquidity constraint, but faces a stochastic income process,

$$y_T = \begin{cases} 0 & \text{with probability } p \\ \frac{1}{1-p} & \text{with probability } (1-p). \end{cases}$$

If we write the consumption rule for the unconstrained consumer facing the risk as $\tilde{s}_{T-1,0}$, the key result is that in the limit as $p \downarrow 0$, behavior of the two consumers becomes the same. That is, defining $\tilde{s}_{T-1,0}(w)$ as the optimal saving rule for the consumer facing the risk,

$$\lim_{p \downarrow 0} \tilde{s}_{T-1,0}(w_{T-1}) = s_{T-1,1}(w_{T-1})$$

for every w_{T-1}.

To see this, start with the Euler equations for the two consumers given wealth w,

$$u'(w - s_{T-1,1}(w)) = u'(s_{T-1,1}(w) + 1) \tag{25}$$

$$u'(w - \tilde{s}_{T-1,0}(w)) = pu'(\tilde{s}_{T-1,0}(w)) + (1-p)u'(\tilde{s}_{T-1,0}(w) + 1) \tag{26}$$

Consider first the case where w is large enough that the constraint does not bind for the constrained consumer, $w > 1$. In this case the limit of the Euler equation for the second consumer is identical to the Euler equation for the first consumer (because for $w > 1$ savings are positive for the consumer facing the risk, implying that the limit of the first u' term on the RHS of (26) is finite), implying that the limit of (26) is (25) for $w > 1$.

Now consider the case where $w < 1$ so that the first consumer would be constrained. This consumer spends her entire resources w, and by the definition of the constraint we know that

$$u'(w) > u'(1). \tag{27}$$

Now consider the consumer facing the risk. If this consumer were to save exactly zero and then experienced the bad shock in period T, she would experience negative infinite utility (the Inada condition). Therefore we know that for any fixed p and any $w > 0$ the consumer will save some positive amount. For a fixed w, hypothesize that there is some positive amount δ such that no matter how small p became the consumer would always choose to save at least δ. But for any positive δ, the limit of the RHS of (26) is $u'(\delta + 1)$. But we know from concavity of the utility function that $u'(1 + \delta) < u'(1)$ and we know from (27) that $u'(w) > u'(1) > u'(1 + \delta)$, so as $p \downarrow 0$ there must always come a point at which the consumer can improve her total utility by shifting some resources from the future to the present, i.e. by saving less. Since this argument holds for any $\delta > 0$ it demonstrates that as p goes to zero there is no positive level of saving that would make the consumer better off. But saving of zero or a negative amount is ruled out by the Inada condition at $u'(0)$. Hence saving must approach, but never equal, zero as $p \downarrow 0$.

Thus, we have shown that for $w \leq 1$ and for $w > 1$ in the limit as $p \downarrow 0$ the consumer facing the risk but no constraint behaves identically to the consumer facing the constraint but no risk. This argument can be generalized to show that for the CRRA utility consumer, spending must always be strictly less than the sum of current wealth and the minimum possible value of human wealth. Thus, the addition of a risk to the problem can rule out certain levels of wealth as feasible, and can also render either future or past constraints irrelevant, just as the imposition of a new constraint can.

8 Proof of Theorem ??

Proof. To simplify notation and without loss of generality, we assume that when a household faces n constraints and m risks, there are one constraint and one risk for each time period. For example, if $c^m_{t,n}$ faces m future risks and n future constraints, then the next period consumption function is $c^{m-1}_{t+1,n-1}$ (and $m = n$). Note that we can transform any problem into this notation by filling in with degenerate risks and non-binding constraints. However, for Theorem **??** to hold with strict inequality, we need to assume that there is at least one relevant future risk and one relevant constraint.

Suppose we are in period t and want to understand the effect of introducing a risk that will be realized between period t and $t+1$. To prove Theorem **??**, we need to find the following set of values:

$$\mathcal{P}^{m-1}_{t,n} = \{w | c^{m-1}_{t,n}(w) - c^m_{t,n}(w) > c_{t,0}(w) - c^1_{t,0}(w)\} \tag{28}$$

We know that both the introduction of risk or a constraint result in a counterclockwise concavification of the original consumption function. However, this is only true when we introduce risks in the absence of constraints (see Carroll and Kimball, 1996) and when we introduce constraints in the absence of risk (see Theorem **??**). In this proof, we therefore need to show that the introduction of all risks and constraints is a counterclockwise concavification of the linear case with no risks and constraints.

The key to the proof is to understand that the introduction of risks or constraints will never fully reverse the effects of all other risks and constraints, even though they sometimes reduce absolute prudence for some levels of wealth because risks and constraints can mask the effects of future risks and constraints. Hence, the new consumption function must still be a counterclockwise concavification of the consumption function with no risks and constraints.

Since a counterclockwise concavification increases prudence by Theorem **??**, and higher prudence increases precautionary saving by Lemma **??**, our required set can be redefined as

$\mathcal{P}_{t,n}^{m-1} = \{w | c_{t,n}^{m-1}(w)$ is a counterclockwise concavification of $c_{t,0}(w)$ and c_t

where we add the last condition, $c_{t,n}^{m-1}(w) > w$ to avoid the possibility that some constraint binds such that the household does not increase precautionary saving. In words: $\mathcal{P}_{t,n}^{m-1}$ is the set where the consumption function is a counterclockwise concavification of $c_{t,0}(w)$ and no constraint is strictly binding. We construct the set recursively for two different cases: CARA and all other type of utility functions. We start with the non-CARA utility functions.

First add the last constraint. The set $\mathcal{P}_{T,1}$ is then

$$\mathcal{P}_{T,1} = \emptyset$$

since we know that $c_{T,1}(w)$ is a counterclockwise concavification of $c_{T,0}(w)$ around $\omega_{T,1}$, but wherever the constraint affects consumption, it holds strictly.

We next add the risk between period $T-1$ and T. To construct the new set, we note three things. First, by Theorem **??**, we know that (strict) consumption concavity is recursively propagated so we know that the set

$$\{w_{T-1}|\omega_{T-1} \in \big[w_{T-1} - c^1_{T-1,1}(w_{T-1}) + y_T + \underline{\zeta}, w_{T-1} - c^1_{T-1,1}(w_{T-1}) + y_T +$$

has property strict CC, while it has property CC for all possible values of w_{T-1}. Further, we know from Theorem **??** (rearrange equation (**??**)) that

$$\begin{aligned} c^1_{T-1,1}(w) &\leq c_{T-1,1}(w) - c_{T-1,0}(w) + c^1_{T-1,0}(w) \\ &\leq c_{T-1,0}(w) - c_{T-1,0}(w) + c_{T-1,0}(w) = c_{T-1,0}(w). \end{aligned}$$

Third, we know that $c^1_{T-1,1}(w)' \geq c'_{T-1,0}(w)$ since $c^1_{T-1,1}(w) < c_{T-1,0}(w)$ for $w \leq \omega^1_{T-1,1}$, $\lim_{w\to\infty} c^1_{T-1,1}(w) - c_{T-1,0}(w) = 0$, and that $c^1_{t,1}(w)$ is concave while $c_{t,0}(w)$ is linear. Hence, $c^1_{T-1,1}$ is a counterclockwise concavification of $c_{T-1,0}$ around the minimum value of wealth when the constraint will never bind and the new set is

$$\mathcal{P}^1_{T-1,1} = \{w_{T-1}|\omega_{T-1,1} \in \big[w_{T-1} - c^1_{T-1,1}(w_{T-1}) + y_T + \underline{\zeta}, w_{T-1} - c^1_{T-1,1}(w$$
$$\wedge c^1_{T-1,}$$

We can now add the next constraint. The consumption function now has two kink points, $\omega^1_{T-1,1}$ and $\omega^1_{T-1,2}$. We know again from Theorem **??** that consumption concavity is preserved under the addition of a constraint, and strict consumption concavity is preserved for all values of wealth at which a future constraint might bind. Further, we know from Theorem **??** that

$$
\begin{aligned}
c^1_{T-1,2}(w) &\leq c_{T-1,2}(w) - c_{T-1,1}(w) + c^1_{T-1,1}(w) \\
&\leq c_{T-1,1}(w) - c_{T-1,1}(w) + c_{T-1,1}(w) = c_{T-1,1}(w) \leq c_{T-1,0}(w).
\end{aligned}
$$

Third, $c^1_{T-1,2}(w) < c_{T-1,0}(w)$, $\lim_{w\to\infty} c^1_{T-1,2}(w) - c_{T-1,0}(w) = 0$, and that $c^1_{T-1,2}(w)$ is concave while $c_{T-1,0}(w)$ is linear, implies that $c'^1_{T-1,2}(w) \geq c'_{T-1,0}(w)$. $c^1_{T-1,2}(w)$ is then a counterclockwise concavification of $c_{T-1,0}(w)$ around the wealth level at which the first constraint starts impinging on time $T-1$ consumption, $\omega^1_{T-1,1}$, and the new set is

$$
\mathcal{P}^1_{T-1,2} = \{w_{T-1} | w_{T-1} \leq \omega^1_{T-1,1} \wedge c^1_{T-1,2}(w) > w_{T-1}\}.
$$

It is now time to add the next risk. The argument is similar. We still know that (strict) consumption concavity is recursively propagated and that $\lim_{w\to\infty} c^2_{T-2,2}(w) - c_{T-2,0}(w) = 0$. Further, we can think of the addition of two risks over two periods as adding one risk that is realized over two periods. Hence, the results from Theorem **??** must hold also for the addition of multiple risks so we have

$$
\begin{aligned}
c^2_{T-2,2}(w) &\leq c_{T-2,2}(w) - c_{T-2,1}(w) + c^2_{T-2,1}(w) \\
&\leq c_{T-2,1}(w) - c_{T-2,1}(w) + c_{T-2,1}(w) = c_{T-2,1}(w) \leq c_{T-2,0}(w).
\end{aligned}
$$

Hence, we again know that $c'^2_{T-2,2}(w) \geq c'_{T-2,0}(w)$. $c^2_{T-2,2}(w)$ is thus a counterclockwise concavification of $c_{T-2,0}(w)$ around the level of wealth at minimum value of wealth when the last constraint will never bind. The new set is therefore

$$
\mathcal{P}^2_{T-2,2} = \{w_{T-2} | w_{T-2} - c^2_{T-2,2}(w_{T-2}) + y_{T-1} + \zeta_{T-1} \in \mathcal{P}^1_{T-1,2} \wedge c^2_{T-2,2}(w) < w
$$

Doing this recursively and defining $\bar{\omega}^{m-1}_{t,n}$ as the minimum value of wealth at which the last constraint will never bind, the set of wealth levels at which Theorem **??** holds can be defined as

$$\mathcal{P}_{t,n}^{m-1} = \{w_t | w_t \leq \bar{\omega}_{t,n}^{m-1} \wedge c_{t,n}^{m-1}(w) > w\}$$

In words, precautionary saving is higher if there is a positive probability that some future constraint could bind and the consumer is not constrained today.

The last requirement is to define the set also for the CARA utility function. The problem with CARA utility is that $\lim_{w\to\infty} c_{t,n}^{m-1}(w) - c_{t,0}(w) = -k^{m-1} \leq 0$ where k^{m-1} is some positive constant. We can therefore not use the same arguments as in the preceding proof. However, by realizing that equation (**??**) in the CARA case can be defined as

$$c_{t,n+1}(w) - \tilde{c}_{t,n+1}(w) - \tilde{k} \geq c_{t,n}(w) - \tilde{c}_{t,n}(w) - \tilde{k} \geq 0$$

where the last inequality follows since precautionary saving is always higher than in the constant limit in the presence of constraints. We can therefore rearrange to get

$$\tilde{c}_{t,n+1}(w) \leq c_{t,n+1}(w) - \tilde{k} \leq c_{t,n}(w) - \tilde{k} \leq c_{t,0} - \tilde{k}$$

which implies that the arguments in the preceding section goes through even for CARA utility.

□

References

Carroll, Christopher D., and Miles S. Kimball (1996): "On the Concavity of the Consumption Function," *Econometrica*, 64(4), 981–992, http://econ.jhu.edu/people/ccarroll/concavity.pdf.