In the three previous sections, we have derived the relationships between liquidity constraints, consumption concavity, and prudence. It is now time to be explicit about the last step: the relationship between liquidity constraints and precautionary saving. We first explain the relationship between the precautionary premium and absolute prudence. In the next subsection, we use this result to show how the introduction of an additional constraint induces households to increase precautionary saving when they face a current risk. Next, we explain why the result cannot be generalized to the cases with risks in different time periods or earlier constraints. We end this section by showing our most general result on liquidity constraints and precautionary saving: The introduction of a risk has a greater precautionary effect on consumption in the presence of all future risks and constraints than in the absence of any future risks or constraints.

0.1 Notation

We begin by defining two marginal value functions $V'(w)$ and $\hat{V}'(w)$ which are convex, downward sloping, and continuous in wealth, w. We consider a risk ζ with support $[\underline{\zeta}, \bar{\zeta}]$, and follow Kimball (1990) by defining the Compensating Precautionary Premia (CPP) as the values κ and $\hat{\kappa}$ such that

$$
\begin{aligned}
V'(0) &= E[V'(\zeta + \kappa)] \qquad (1) \\
\hat{V}'(0) &= E[\hat{V}'(\zeta + \hat{\kappa})]. \qquad (2)
\end{aligned}
$$

The CPP can be interpreted as the additional resources an agent requires to be indifferent between accepting the risk and not accepting the risk. The relevant part of Pratt (1964)'s Theorem

1 as reinterpreted using Kimball (1990)'s Lemma (p. 57) can be restated as

Lemma 1. *Let $A(w)$ and $\hat{A}(w)$ be absolute prudence of the value functions V and $\hat{V}$ respectively at w,*[1] *and let κ and $\hat{\kappa}$ be the respective compensating precautionary premia associated with imposition of a given risk ζ as per (1) and (2). Then the following conditions are equivalent:*

1. *$\hat{A}(\zeta+\kappa) \geq A(\zeta+\kappa)$ for all $\zeta \in [\underline{\zeta}, \bar{\zeta}]$ and $\hat{A}(\zeta+\kappa) > A(\zeta+\kappa)$ for at least one [no] point $\zeta \in [\underline{\zeta}, \bar{\zeta}]$.*
2. *$\hat{\kappa} > [=]\kappa$ with repect to the same risk $\zeta \in [\underline{\zeta}, \bar{\zeta}]$.*

Note finally that precautionary premia are not equivalent to precautionary saving effects because precautionary premia apply at a given level of consumption, while precautionary saving applies at a given level of wealth.

We now take up the question of how the introduction of a risk ζ_{t+1} that will be realized between period t and $t+1$ affects consumption in period t in the presence and in the absence of a subsequent constraint. To simplify the discussion, consider a consumer for whom $\beta = R = 1$, with mean income y in $t+1$.

Assume that the realization of the risk ζ_{t+1} will be some value ζ with support $[\underline{\zeta},\bar{\zeta}]$, and signify a decision rule that takes account of the presence of the immediate risk by a $\sim$. Thus, the perfect foresight unconstrained consumption function is $c_{t,0}(w)$, the perfect foresight consumption function in the presence of the future constraint is $c_{t,1}(w)$, the consumption function with

[1] A small technicality: Absolute prudence of value functions is infinite at kink points in the consumption function, so if both $c(w)$ and $\hat{c}(w)$ had a kink point at exactly the same w but the amount by which the slope declined were different, the comparison of prudence would not yield a well-defined answer. Under these circumstances we will say that $\hat{A}(w) > A(w)$ if the decline in the MPC is greater for $\hat{c}$ at w than for w.

no constraints but with the risk is $\tilde{c}_{t,0}(w)$ and the consumption function with both risk and constraint is $\tilde{c}_{t,1}(w)$, and similarly for the other functions. We now define two wealth levels that describe the subsets where constraint $n+1$ affects households at time t. The wealth limits corresponds to the levels of wealth where liquidity constraint $n+1$ never binds (1) and always binds (2) for a consumer facing income risk.

Definition 1. *(Wealth Limits).*

1. $\underline{\omega}_{t,n+1}$ *is the level of wealth such that an agent who faces risk* ζ_{t+1} *and* $n+1$ *constraints save so little that constraint* $n+1$ *will always bind in period* $t+1$*, defined as*

$$\underline{\omega}_{t,n+1} = \left(\tilde{V}'_{t,n+1}\right)^{-1}(\tilde{\Omega}'_{t,n+1}(\omega_{t+1,n+1} - (y + \bar{\zeta}))). \quad (3)$$

How to read these limits: $\omega_{t+1,n+1}$ *is the level of wealth at which constraint* $n+1$ *starts binding in period* $t+1$. $\omega_{t+1,n+1} - (y + \bar{\zeta})$ *is then the level of wealth that ensures that constraint* $n+1$ *binds in period* $t+1$ *even with the best possible draw,* $\bar{\zeta}$.

2. $\bar{\omega}_{t,n+1}$ *is the level of wealth such that an agent who faces risk* ζ_{t+1} *and* $n+1$ *constraints save enough to guarantee that constraint* $n+1$ *will never bind in period* $t+1$*, defined as*

$$\bar{\omega}_{t,n+1} = \left(\tilde{V}'_{t,n+1}\right)^{-1}(\tilde{\Omega}'_{t,n+1}(\omega_{t+1,n+1} - (y + \underline{\zeta}))) \quad (4)$$

We must be careful to check that $\omega_{t+1,n+1} - (y + \underline{\zeta})$ is inside the realm of feasible values of s_t, in the sense of values that permit the consumer to guarantee that future levels of consumption will be within the permissible range (e.g. positive for consumers with CRRA utility). If this is not true for some level of wealth, then any constraint that binds at or below that level of wealth is irrelevant, because the restriction on wealth imposed by the risk is more stringent than the restriction imposed by the constraint.

0.2 Precautionary Saving with Liquidity Constraints

We are now in the position to analyze the relationship between precautionary saving and liquidity constraints. Our first result regards the effect of an additional constraint on the precautionary saving of a household facing risk between period t and $t+1$.

Theorem 1. *(Precautionary Saving with Liquidity Constraints). Consider an agent who has a utility function with $u' > 0$, $u'' < 0$, $u''' > 0$, and non-increasing absolute prudence $-u'''/u''$, and that faces the risk, ζ_{t+1}. Assume that the agent faces a set $\mathcal{T}$ of N relevant constraints and $n \leq N-1$. Then*

$$c_{t,n+1}(w) - \tilde{c}_{t,n+1}(w) \geq c_{t,n}(w) - \tilde{c}_{t,n}(w), \tag{5}$$

and the inequality is strict if $w_t < \bar{\omega}_{t,n+1}$.

See Appendix **??** for the proof. Theorem 1 shows that the introduction of the next constraint induces the agent to reduce consumption in response to an immediate risk. Theorem 1 can be generalized to period $s < t$ if there is no risk or constraint between period s and t. We just have to define $\bar{\omega}_{s,n+1}$ as the wealth level at which the agent will arrive in the beginning of period t with wealth $\bar{\omega}_{t,n+1}$.

Theorem 2. *(Precautionary Saving with Liquidity Constraints). Consider an agent in period $s < t$ who has a utility function with $u' > 0$, $u'' < 0$, $u''' > 0$, and non-increasing absolute prudence $-u'''/u''$, and that faces the risk, ζ_{t+1}. Assume that the agent faces a set $\mathcal{T}$ of N relevant constraints and $n \leq N-1$, and that* *no risk or constraint applies in the periods between s and t. Then*

$$c_{s,n+1}(w) - \tilde{c}_{s,n+1}(w) \geq c_{s,n}(w) - \tilde{c}_{s,n}(w), \tag{6}$$

and the inequality is strict if $w_s < \bar{\omega}_{s,n+1}$.

To illustrate the result in Theorem 2, Figure 1 shows an example of optimal consumption rules in period t under different combinations of an immediate risk (realized between t and $t+1$) and a future constraint (applying between periods $t+1$ and $t+2$).

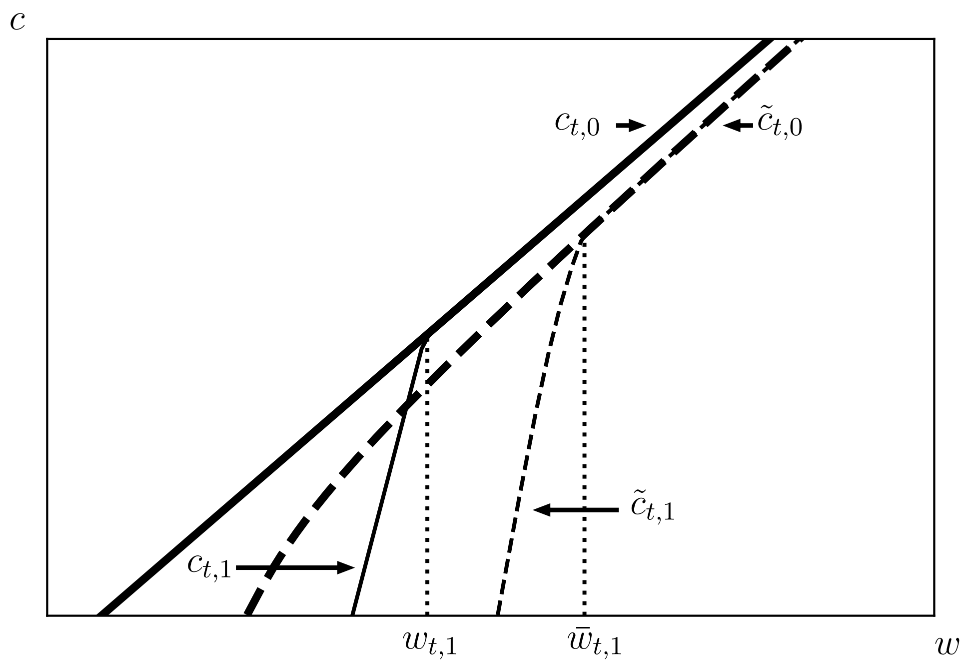

Figure 1 Consumption Functions with and without a Constraint and a Risk

Notes: $c_{t,0}$ is the consumption function with no constraint and no risk, $\tilde{c}_{t,0}$ is the consumption function with no constraint and a risk that is realized between period t and $t+1$, $c_{t,0}$ is the consumption function with one constraint in period $t+1$ and no risk, and $c_{t,0}$ is the consumption function with one constraint in period $t+1$ and a risk that is realized between period t and $t+1$. The figure illustrates that the vertical distance between $c_{t,1}$ and $\tilde{c}_{t,1}$ is always greater than the vertical distance between $c_{t,0}$ and $\tilde{c}_{t,0}$ for $w < \bar{\omega}_{t,1}$.

The darker loci reflect behavior of consumers who do not face the future constraint, and the dashed loci reflect behavior of consumers who *do* face the immediate risk. As expected, for levels of wealth above $\omega_{t,1}$ where the future constraint stops impinging on current behavior for perfect foresight consumers, behavior of the constrained and unconstrained perfect foresight consumers is the same. Similarly, $\tilde{c}_{t,1}(w_t) = \tilde{c}_{t,0}(w_t)$ for levels of wealth above $\bar{\omega}_{t,1}$ beyond which the probability of the future constraint binding is zero. And for both constrained and unconstrained consumers, the introduction of the risk reduces the level of consumption (the dashed loci are below their solid counterparts). The importance of Theorem 1 in this context is that for levels of wealth below $\bar{\omega}_{t,1}$, the vertical distance between the solid and the dashed loci is greater for the constrained (thin line) than for the unconstrained (thick line) consumers, because of the interaction between the liquidity constraint and the precautionary motive.

0.3 A More General Result?

The results in Theorem 1 is limited to the effects of an additional constraint when a household faces income risk that is realized between period t and $t+1$. We could hope to find a more general result where precautionary saving increases if we for example impose an immediate constraint or an earlier risk, or generally multiple constraints or risks. However, it turns out that the answer is "not necessary" to all these possible scenarios. In this subsection, we argue for why we cannot derive more general results.

To examine these results, we need to develop a last bit of notation. We define, $c^m_{t,n}$, as the consumption function in period t assuming that the first n constraints and the first m risks have been imposed, counting risks, like constraints, backwards from period T. Thus, relating our new notation to our previous usage, $c^0_{t,n} = c_{t,n}$ because 0 risks have been imposed. All other functions are defined correspondingly, e.g. $\Omega^m_{t,n}$ is the end-of-period-t value function assuming the first n constraints and m risks have been imposed. We will continue to use the notation $\tilde{c}_{t,n}$ to designate the effects of imposition of a single immediate risk (realized between periods t and $t+1$).

Suppose now there are m future risks that will be realized between t and T. One might hope to show that the precautionary effect of imposing all risks in the presence of all constraints would be greater than the effect of imposing all risks in the absence of any constraints:

$$c^0_{t,n}(w) - c^m_{t,n}(w) \geq c^0_{t,0}(w) - c^m_{t,0}(w). \tag{7}$$

Such a hope, however, would be in vain. In fact, we will now show that even the considerably weaker condition, involving only the single risk ζ_{t+1} and all constraints, $c^0_{t,n}(w) - c^1_{t,n}(w) \geq c^0_{t,0}(w) - c^1_{t,0}(w)$, can fail to hold for some w.

0.3.1 An Immediate Constraint

Consider a situation in which n constraints applies in between t and T. Since $c_{t,n-1}$ designates the consumption rule that will be optimal prior to imposing the period-t constraint, the consumption rule imposing all constraints will be

$$c_{t,n}(w) \;=\; \min[c_{t,n-1}(w), w]. \tag{8}$$

Now define the level of wealth below which the period t constraint binds for a consumer not facing the risk as $\omega_{t,n}$. For values of wealth $w \geq \omega_{t,n}$, analysis of the effects of the risk is identical to analysis in the previous subsection where the first $n-1$ constraints were imposed. For levels of wealth $w < \omega_{t,n}$, we have $c^1_{t,n}(w) = c_{t,n}(w) = w$ (for the simple $c \leq w$ constraint; a corresponding point applies to the more sophisticated form of constraint); that is, for consumers with wealth below $\omega_{t,n}$, the introduction of the risk in period $t+1$ has no effect on consumption in t, because for these levels of wealth the constraint at the end of t has the effect of 'hiding' the risk from view (they were constrained before the risk was imposed and remain constrained afterwards). Thus for households for whom inequality (6) in Theorem 1 holds strictly in the absence of the constraint at t, at levels of wealth below $\omega_{t,n}$, the precautionary effect of the risk is wiped out.

0.3.2 An Earlier Risk

Consider now the question of how the addition of a risk ζ_t that will be realized between periods $t-1$ and t affects the consumption function at the beginning of period $t-1$, in the absence of any constraint at the beginning of period t.

The question at hand is then whether we can say that

$$c^1_{t-1,0}(w) - c^2_{t-1,0}(w) \;\geq\; c^0_{t-1,0}(w) - c^1_{t-1,0}(w); \tag{9}$$

that is, does the introduction of the risk ζ_t have a greater precautionary effect on consumption in the presence of the subsequent risk ζ_{t+1} than in its absence?

The answer again is "not necessarily." To see why, we present an example in Appendix **??** of a CRRA utility problem in which in a certain limit the introduction of a risk produced an effect on the consumption function that is indistinguishable from the effect of a liquidity constraint. If the risk ζ_t is of this liquidity-constraint-indistinguishable form, then the logic of the previous subsection clearly applies: For some levels of wealth, the introduction of the risk at t can 'hide' the precautionary effect of any risks at $t+1$ or later.

0.4 What Can Be Said?

It might seem that the previous subsection implies that little useful can be said about the precautionary effects of introducing a new risk in the presence of preexisting constraints and risks. It turns out, however, that there is at least one useful result.

Theorem 3. *Consider an agent who has a utility function with $u' > 0$, $u'' < 0$, $u''' > 0$, and non-increasing absolute prudence $-u'''/u''$. Then the introduction of a risk ζ_{t+1} has a greater precautionary effect on level t consumption in the presence of all future risks and constraints than in the absence of any future risks and constraints, i.e.*

$$c^{m-1}_{t,n}(w) - c^{m}_{t,n}(w) \;>\; c^{0}_{t,0}(w) - c^{1}_{t,0}(w) \tag{10}$$

at levels of period-t wealth w such that in the absence of the new risk the consumer is not constrained in the current period ($c^{m-1}_{t,n}(w) > w$) and in the presence of the risk there is a positive probability that some future constraint will bind.

Appendix **??** presents the proof. It seems to us that a fair summary of this theorem is that in most circumstances the presence of future constraints and risks does increase the amount of precautionary saving induced by the introduction of a given new risk. The primary circumstance under which this should not be expected is for levels of wealth at which the consumer was constrained even in the absence of the new risk. There is no guarantee that the new risk will produce a sufficiently intense precautionary saving motive to move the initially-constrained consumer off his constraint. If it does, the effect will be precautionary, but it is possible that no effect will occur.

References

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