We now move on to the sources of consumption concavity. In our setting, there are two sources of consumption concavity: risk and constraints. The properties of consumption under risk have already been derived in Carroll and Kimball (1996). We therefore restrict our attention to showing how liquidity constraints make the consumption function concave. Once the relationship between liquidity constraints and consumption concavity is established, we use the results on consumption concavity and prudence to show under which conditions liquidity constraints increase prudence.

0.1 Notation

Throughout this paper, we are working with a finite horizon household with an horizon going from 0 to T. A liquidity constraint is a constraint on wealth that binds at a specific time $t \in (0, T]$. We first define an ordered set to keep track of the liquidity constraints.

Definition 1. *(The Set of Liquidity Constraints).*
We define $\mathcal{T}$ as an ordered set of dates at which a relevant constraint exists. We define $\mathcal{T}[1]$ as the last period in which a constraint exists, and $\mathcal{T}[2]$ is the date of the last period before $\mathcal{T}[1]$ in which a constraint exists, and so on.

For any $t \in [0, T)$, we define $c_{t,n}$ as the optimal consumption function in period t assuming that the first n constraints in $\mathcal{T}$ have been imposed. For example, $c_{t,0}(w)$ is the consumption function in period t when no constraint (aside from the intertemporal budget constraint) has been imposed, $c_{t,1}(w)$ is the consumption function in period t after the chronologically last constraint has been imposed, and so on. We define $\Omega_{t,n}, V_{t,n}$, and other functions correspondingly.

To have a distinct terminology for the effects of current-period and future-period constraints, we will restrict the use of the term 'binds' to the potential effects of a constraint in the period in which it applies ('the constraint binds if wealth is less than ...') and will use the term 'impinges' to describe the effect of a future constraint on current consumption. We can now define the concept of a kink point.

Definition 2. *(Kink Point).*
We define a kink point, $\omega_{t,n}$ as the level of wealth at which constraint n stops binding (impinging) on time t consumption.

A kink point corresponds to a transition from a level of wealth where a current constraint binds or a future constraint impinges, to a level of wealth where that constraint no longer binds or impinges.

0.2 Perfect Foresight Consumption with Liquidity Constraint

We first consider an initial situation in which a consumer is solving a perfect foresight optimization problem with a finite horizon that begins in period t and ends in period T. The consumer begins with wealth w_t and earns constant income y in each period. Wealth accumulates according to $w_{t+1} = Rs_t + y$. We are interested in how this consumer's behavior in period t changes from an initial situation with $n \geq 0$ constraints to a situation in which $n+1$ liquidity constraints has been imposed.

Theorem 1. *(Perfect Foresight Consumption with Liquidity Constraints).*

Consider an agent who has a utility function with $u' > 0$ and $u'' < 0$, faces constant income y, and is 'absolutely impatient' (by which we mean $\beta R < 1$). Assume that the agent faces a set $\mathcal{T}$ of N relevant constraints. Then $c_{t,n+1}(w)$ is a counterclockwise concavification of $c_{t,n}(w)$ around $\omega_{t,n+1}$ for all $n \leq N-1$

See Appendix **??** for the proof. Theorem 1 shows that when we have an ordered set of constraints, $\mathcal{T}$, the introduction of the next constraint in the set represents a counterclockwise concavification of the consumption function. Note that constraint $n+1$ is always at a date prior to the set of the n first constraints. From the proof of Theorem 1, we also know the shape of the perfect foresight consumption function with liquidity constraints:

Corollary 1. *(Piecewise Linear Consumption Function).*
Consider an agent who has a utility function with $u' > 0$ and $u'' < 0$, faces constant income y, and is absolutely impatient. Assume that the agent faces a set $\mathcal{T}$ of N relevant constraints. When $n \leq N$ constraints have been imposed, $c_{t,n}(w)$ is a piecewise linear increasing concave function with kink points at successively larger values of wealth at which future constraints stop impinging on current consumption.

Since the consumption function is piecewise linear, the new consumption function, $c_{t,n+1}(w)$ is not necessarily strictly more concave than $c_{t,n}(w)$ for all w. This is where the concept of counterclockwise concavification is useful. Even though $c_{t,n+1}(w)$ is not strictly more concave than $c_{t,n}(w)$ everywhere, it is a counterclockwise concavification and we apply Theorem **??** to derive the consequences of imposing one more constraint on prudence.

Theorem 2. *(Liquidity Constraints Increase Prudence).*
Consider an agent in period t who has a utility function with

$u' > 0$, $u'' < 0$, $u''' \geq 0$ and non-increasing absolute prudence $-u'''/u''$, faces constant income y, and is impatient, $\beta R < 1$. Assume that the agent faces a set $\mathcal{T}$ of N relevant constraints. When $n \leq N-1$ constraints have been imposed, the imposition of constraint $n+1$ strictly increases absolute prudence of the agent's value function if $u''' > 0$ and $w_t < \omega_{t,n+1}$ or if $u''' = 0$ and $\frac{c'_{t,n+1}}{c'_{t,n}}$ strictly declines at w.

Proof. By Theorem 1, the imposition of constraint $n+1$ constitutes a counterclockwise concavification of $c_{t,n}(w)$. By Theorem **??** and Corollary **??**, such a concavification strictly increases absolute prudence of the value function for the cases in Corollary **??**. □

Theorem 2 is the main result in the current section: the introduction of the next liquidity constraint increases absolute prudence of the value function. In the subsequent discussions, we consider cases where we relax the assumptions underlying Theorem 2. We first consider the case where we add an extra constraint to the set of relevant constraints. Next, we consider the cases with time-varying deterministic income, general constraints, and no assumption on time discounting.

0.3 Increasing the Number of Constraints

In the previous section, we analyzed a case where there was a preordained set of constraints $\mathcal{T}$ which were applied sequentially. We now examine how behavior will be modified if we add a new date to the set of dates at which the consumer is constrained.

Call the new set of dates $\hat{\mathcal{T}}$ with $N+1$ constraints (one more constraint than before), and call the consumption rules corresponding to the new set of dates $\hat{c}_{t,1}$ through $\hat{c}_{t,N+1}$. Now call m the number of constraints in $\mathcal{T}$ at dates strictly greater than $\hat{\tau}$. Then note that that $\hat{c}_{\hat{\tau},m} = c_{\hat{\tau},m}$, because at dates after the date at which the new constraint (number $m+1$) is imposed, consumption is the same as in the absence of the new constraint. Now recall that imposition of the constraint at $\hat{\tau}$ causes a counterclockwise concavification of the consumption function around a new kink point, $\omega_{\hat{\tau},m+1}$. That is, $\hat{c}_{\hat{\tau},m+1}$ is a counterclockwise concavification of $\hat{c}_{\hat{\tau},m} = c_{\hat{\tau},m}$.

The most interesting observation, however, is that behavior under constraints $\hat{\mathcal{T}}$ in periods strictly before $\hat{\tau}$ *cannot* be described as a counterclockwise concavification of behavior under $\mathcal{T}$. The reason is that the values of wealth at which the earlier constraints caused kink points in the consumption functions before period $\hat{\tau}$ will not generally correspond to kink points once the extra constraint has been added.

We present an example in Figure 1. The original $\mathcal{T}$ contains only a single constraint, at the end of period $t+1$, inducing a kink point at $\omega_{t,1}$ in the consumption rule $c_{t,1}$. The expanded set of constraints, $\hat{\mathcal{T}}$, adds one constraint at period $t+2$. $\hat{\mathcal{T}}$ induces two kink points in the updated consumption rule $\hat{c}_{t,2}$, at $\hat{\omega}_{t,1}$ and $\hat{\omega}_{t,2}$. It is true that imposition of the new constraint causes consumption to be lower than before at every level of wealth below $\hat{\omega}_{t,1}$. However, this does not imply higher prudence of the value function at every $w < \hat{\omega}_{t,1}$. In particular, note that the original consumption function is strictly concave at $w = \omega_{t,1}$, while the new consumption function is linear at $\omega_{t,1}$, so prudence can be greater before than after imposition of the new constraint at this particular level of wealth.

The intuition is simple: At levels of initial wealth below $\hat{\omega}_{t,1}$,

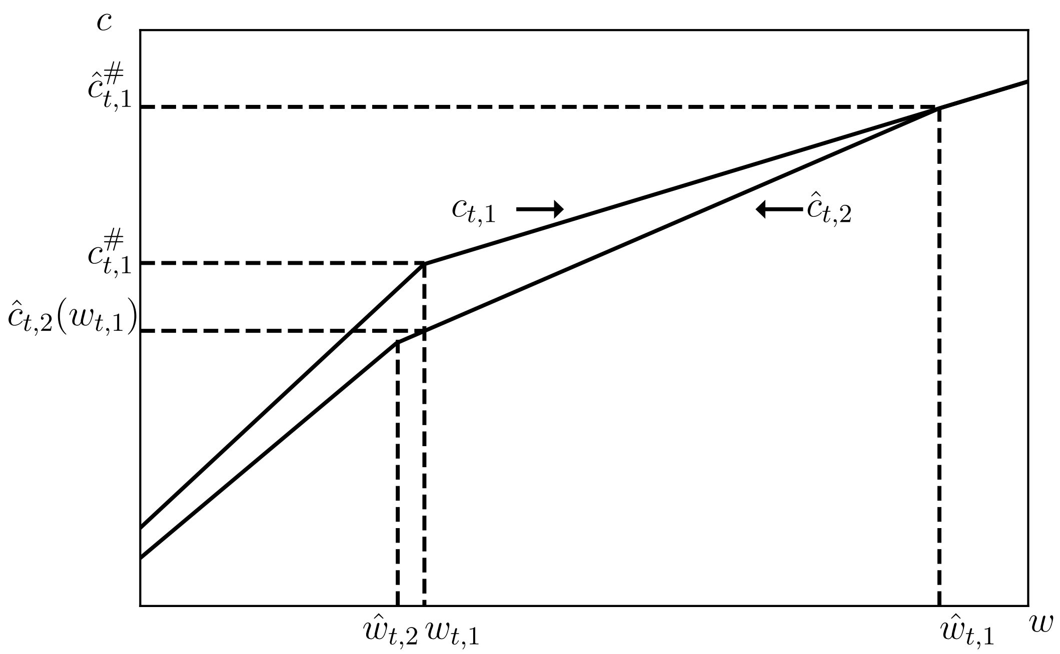

Figure 1 How a future constraint can hide a current kink

Notes: $c_{t,1}$ is the original consumption function with one constraint that induces a kink point at $\omega_{t,1}$. $\hat{c}_{t,2}$ is the modified consumption function in where we have introduced one new constraint. The two constraints affect $\hat{c}_{t,2}$ through two kink points: $\hat{\omega}_{t,1}$ and $\hat{\omega}_{t,2}$. Since we introduced the new constraint at a later point in time than the current existing constraint, the future constraint affects the position of the kink induced by the current constraint and the modified consumption function $\hat{c}_{t,2}$ is not a counterclockwise concavification of $c_{t,1}$.

the consumer had been planning to end period $t+2$ with negative wealth. With the new constraint, the old plan of ending up with negative wealth is no longer feasible and the consumer will save more for any given level of current wealth below $\hat{\omega}_{t,1}$, including $\omega_{t,1}$. But the reason $\omega_{t,1}$ was a kink point in the initial situation was that it was the level of wealth where consumption would have been equal to wealth in period $t + 1$. Now, because of the extra savings induced by the constraint in $t+2$, the larger savings induced by wealth $\omega_{t,1}$ implies that the period $t+1$ constraint will no longer bind for a consumer who begins period t with wealth $\omega_{t,1}$. In other words, at wealth $\omega_{t,1}$ the extra savings induced by the new constraint 'hides' the original constraint and prevents it from being relevant any more at $\omega_{t,1}$.

Notice, however, that all constraints that existed in $\mathcal{T}$ will remain relevant at *some* level of wealth under $\hat{\mathcal{T}}$ even after the new constraint is imposed - they just induce kink points at different levels of wealth than before, e.g. the first constraint causes a kink at $\hat{\omega}_{t,1}$ rather than at $\omega_{t,1}$.

0.4 A More General Analysis

We now want to allow time variation in the level of income, y_t, and in the location of the liquidity constraint (e.g. a constraint in period t might require the consumer to end period t with savings s_t greater than ς where ς is a negative number). We also drop the restriction that $\beta R < 1$, allowing the consumer to want to have consumption growth over time.

Under these more general circumstances, a constraint imposed in a given period can render constraints in either earlier or later periods irrelevant. For example, consider a CRRA utility consumer with $\beta R = 1$ who earns income of 1 in each period, but who is required to arrive at the end of period $T-2$ with savings of 5. Then a constraint that requires savings to be greater than zero at the end of period $T-3$ will have no effect because the consumer is required by the constraint in period $T-2$ to end period $T-3$ with savings greater than 4.

Formally, consider now imposing the first constraint, which applies in period $\tau < T$. The simplest case, analyzed before, was a constraint that requires the minimum level of end-of-period wealth to be $s_\tau \geq 0$. Here we generalize this to $s_\tau \geq \varsigma_{\tau,2}$ where in principle we can allow borrowing by choosing ς to be a negative number. Now for constraint 2 calculate the kink points for prior periods from

$$\begin{aligned} u'(c^{\#}_{\tau,2}) &= R\beta u'(c_{\tau+1,1}(R\varsigma_{\tau,1} + y_{t+1})) && (1) \\ \omega_{\tau,2} &= (V'_{\tau,1})^{-1}(u'(c^{\#}_{\tau,2})). && (2) \end{aligned}$$

In addition, for constraint 2 recursively calculate

$$\underline{\varsigma}_{\tau,2} = (\varsigma_{\tau+1,2} - y_{\tau+1,2} + \underline{c})/R \qquad (3)$$

where $\underline{c}$ is the lowest value of consumption permitted by the model (independent of constraints).[1]

Now assume that the first n constraints in $\mathcal{T}$ have been imposed, and consider imposing constraint number $n+1$, which we assume applies in period τ. The first thing to check is whether constraint number $n+1$ is relevant given the already-imposed set of constraints. This is simple: A constraint that requires $s_\tau \geq \varsigma_{\tau,n+1}$ will be irrelevant if $\min_i[\underline{\varsigma}_{\tau,i}] \leq \varsigma_{\tau,n+1}$. If the constraint is irrelevant then the analysis proceeds simply by dropping this constraint and renumbering the constraints in $\mathcal{T}$ so that the former constraint $n+2$ becomes constraint $n+1$, $n+3$ becomes $n+2$, and so on.

Now consider the other possible problem: That constraint number $n+1$ imposed in period τ will render irrelevant some of the constraints that have already been imposed. This too is simple to check: It will be true if the proposed $\varsigma_{\tau,n+1} \geq \varsigma_{\tau,i}$ for any $i \leq n$. The fix is again simple: Counting down from $i = n$, find the smallest value of i for which $\varsigma_{\tau,n+1} \geq \varsigma_{\tau,i}$. Then we know that constraint $n+1$ has rendered constraints i through n irrelevant.

[1]For example, CRRA utility is well defined only on the positive real numbers, so for a CRRA utility consumer $\underline{c} = 0$. In other cases, for example with exponential or quadratic cases, there is nothing to prevent consumption of $-\infty$, so for those models $\underline{c} = -\infty$, unless there is a desire to restrict the model to positive values of consumption, in which case the $c \geq 0$ constraint will be implemented through the use of (3).

The solution is to drop these constraints from $\mathcal{T}$ and start the analysis over again with the modified $\mathcal{T}$.

If this set of procedures is followed until the chronologically earliest relevant constraint has been imposed, the result will be a $\mathcal{T}$ that contains a set of constraints that can be analyzed as in the simpler case. In particular, proceeding from the final $\mathcal{T}[1]$ through $\mathcal{T}[N]$, the imposition of each successive constraint in $\mathcal{T}$ now causes a counterclockwise concavification of the consumption function around successively lower values of wealth as progressively earlier constraints are applied and the result is again a piecewise linear and strictly concave consumption function with the number of kink points equal to the number of constraints that are relevant at any feasible level of wealth in period t.

The preceding discussion thus establishes the following result:

Theorem 3. *(Liquidity Constraints Increase Prudence).*
Consider an agent in period t who has a utility function with $u' > 0$, $u'' < 0$, $u''' \geq 0$, and non-increasing absolute prudence $-u'''/u''$. Assume that the agent faces a set $\mathcal{T}$ of N relevant constraints. When $n \leq N-1$ constraints have been imposed, the imposition of constraint $n+1$ strictly increases absolute prudence of the agent's value function if the utility function satisfies $u''' > 0$ and $w_t < \omega_{t,n+1}$ or if $u''' = 0$ and $\frac{c'_{t,n+1}}{c'_{t,n}}$ strictly declines at w.

Theorem 3 is a generalization of Theorem 2. Even if we relax the assumptions that income is constant and the agent is impatient, the imposition of an extra constraint increases absolute prudence of the value function as long as we are careful when we select the set $\mathcal{T}$ of relevant constraints.

Finally, consider adding a new constraint to the problem and call the new set of constraints $\hat{\mathcal{T}}$. Suppose the new constraint applies in period $\hat{\tau}$. Then the analysis of the new situation will be like the analysis of an added constraint in the simpler case in section 0.3 if the new constraint is relevant given the constraints that apply after period $\hat{\tau}$ and the new constraint does not render any of those later constraints irrelevant. If the new constraint fails either of these tests, the analysis of $\hat{\mathcal{T}}$ can proceed from the ground up as described above.

References

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