Numerical solutions have supplanted analytical methods for theoretical modeling of consumption/saving choices because analytical solutions are not available for realistic descriptions of utility and uncertainty.

But a drawback to numerical solutions is that it is often difficult to know why results come out the way they do. A leading example is in the relationship between precautionary saving behavior and liquidity constraints. At least since Zeldes (1984), economists working with numerical solutions have known that liquidity constraints can strictly increase precautionary saving under very general circumstances - even for consumers with a quadratic utility function that generates no intrinsic precautionary saving motive.[1] On the other hand, simulation results have sometimes seemed to suggest that liquidity constraints and precautionary saving are substitutes rather than complements. In an early example, Samwick (1995) showed that unconstrained consumers with a precautionary saving motive in a retirement saving model behave in ways qualitatively and quantitatively similar to the behavior of liquidity constrained consumers facing no uncertainty.

This paper provides the theoretical tools needed to make sense of the interactions between liquidity constraints and precautionary saving. These tools provide a rigorous theoretical foundation that can be used to clarify the reasons for the numerical literature's apparently contrasting findings.

For example, one of the paper's simpler points is a proof that when a liquidity constraint is added to the standard consumption

[1] For the seminal numerical examination of some of the interactions between precautionary saving and liquidity constraints, see Deaton (1991), who also provides conditions under which the problem defines a contraction mapping.

problem, the resulting value function exhibits increased prudence around the level of wealth where the constraint becomes binding.[2] Constraints induce precaution because constrained agents have less flexibility in responding to shocks when the effects of the shocks cannot be spread out over time. The precautionary saving motive is heightened by the desire (in the face of risk) to make such constraints less likely to bind.

At a deeper level, we show that the effect of a constraint on prudence is an example of a general theoretical result: Prudence is induced by concavity of the consumption function. Since a constraint causes consumption concavity around the point where the constraint binds,[3] adding a constraint necessarily boosts prudence around that point. We show that this concavity-boosts-prudence result holds for any utility function with non-negative third derivative; "prudence" in the utility function 1990 is not necessary, because prudence is created by consumption concavity.

These results tie in closely with findings in Carroll and Kimball (1996) showing that, within the HARA class, the introduction of uncertainty causes the consumption function to become strictly concave (in the absence of constraints) for all but a few carefully chosen combinations of utility function and uncertainty. Taken together, that paper and this one can be seen as establishing rigorously the sense in which precautionary saving and liquidity constraints are substitutes.[4] To illustrate this point, in appendix

[2] Kimball (1990) defines prudence of the value function and shows that it is the key theoretical requirement to produce precautionary saving.

[3] Since the first version of this paper, the connection between constraints and consumption concavity has been explored in more specific settings (see e.g. Park (2006) for CRRA utility, Nishiyama and Kato (2012) for quadratic utility, and Holm (2018) for the case with infinitely-lived households with HARA utility).

[4] See Fernandez-Corugedo (2000) for a related demonstration that 'soft' liquidity constraints bear an even closer resemblance to precautionary behavior. Mendelson and Amihud (1982) provide an impressive treatment of a similar problem.

?? we provide an example of a specific kind of uncertainty that (under CRRA utility, in the limit) induces a consumption function that is point-wise identical to the consumption function that would be induced by the addition of a liquidity constraint.

We further show that, once consumption concavity is induced (either by a constraint or by uncertainty), it propagates back to periods before the period in which the concavity is first created.[5] Precautionary saving arises from the *possibility* that constraints might bind; this may help to explain why such a high percentage of households cite precautionary motives as the most important reason for saving (Kennickell and Lusardi, 2004) even though the fraction of households who report actually having been constrained in the past is relatively low (Jappelli, 1990).

Our final theoretical contribution is to show that the introduction of further liquidity constraints beyond the first one may actually *reduce* precautionary saving at some levels of wealth by 'hiding' the effects of the pre-existing constraint(s). Identical logic implies that uncertainty can hide the effects of a constraint, because the consumer may save so much for precautionary reasons that the constraint becomes irrelevant. For example, a typical perfect foresight model of retirement consumption for a consumer with Social Security income implies that the legal constraint on borrowing against Social Security benefits will cause the consumer to run assets down to zero, then set consumption equal to income for the remainder of life. Now consider adding the possibility of large medical expenses near the end of life (e.g. nursing home fees; see ?). Under reasonable assumptions, a

[5] Carroll and Kimball (1996) showed that the concavity induced by uncertainty propagated backwards, but the proofs in that paper cannot be applied to concavity created by a liquidity constraint.

consumer facing such a risk may save enough for precautionary reasons to render the constraint irrelevant.

The rest of the paper is structured as follows. To fix notation and ideas, the next section sets out our general theoretical framework. The third section shows that concavity of the consumption function heightens prudence. The fourth section shows how concavity, whether induced by constraints or uncertainty, propagates to previous periods. Section 5 shows how the introduction of a constraint creates a precautionary saving motive. The sixth section shows when the introduction of additional liquidity constraints beyond the first constraint increases the precautionary motive at any given level of wealth. The fact that this does not always occur means that the introduction of constraints or uncertainty can hide the effects of pre-existing constraints or uncertainty. The final section concludes.

References

Carroll, Christopher D., and Miles S. Kimball (1996): "On the Concavity of the Consumption Function," *Econometrica*, 64(4), 981–992, http://econ.jhu.edu/people/ccarroll/concavity.pdf.

Deaton, Angus S. (1991): "Saving and Liquidity Constraints," *Econometrica*, 59, 1221–1248, http://www.jstor.org/stable/2938366.

Fernandez-Corugedo, Emilio (2000): "Soft Liquidity Constraints and Precautionary Savings," *Manuscript, Bank of England*.

Holm, Martin Blomhoff (2018): "Consumption with liquidity constraints: An analytical characterization," *Economics Letters*, 167, 40–42.

Jappelli, Tullio (1990): "Who is Credit Constrained in the U.S. Economy?," *Quarterly Journal of Economics*, 105(1), 219–34.

Kennickell, Arthur, and Annamaria Lusardi (2004): "Disentangling the Importance of the Precautionary Saving Mode," Working Paper 10888, National Bureau of Economic Research.

Kimball, Miles S. (1990): "Precautionary Saving in the Small and in the Large," *Econometrica*, 58, 53–73.

Mendelson, Haim, and Yakov Amihud (1982): "Optimal Consumption Policy Under Uncertain Income," *Management Science*, 28(6), 683–697.

NISHIYAMA, SHIN-ICHI, AND RYO KATO (2012): "On the Concavity of the Consumption Function with a Quadratic Utility under Liquidity Constraints," *Theoretical Economics Letters*, 2(05), 566.

PARK, MYUNG-HO (2006): "An analytical solution to the inverse consumption function with liquidity constraints," *Economics Letters*, 92(3), 389–394.

SAMWICK, ANDREW A. (1995): "The Limited Offset Between Pension Wealth and Other Private Wealth: Implications of Buffer-Stock Saving," *Manuscript, Department of Economics, Dartmouth College.*

ZELDES, STEPHEN P. (1984): "Optimal Consumption with Stochastic Income," Ph.D. thesis, Massachusetts Institute of Technology.