Non Dimensionalize N-S Eqns.

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho' \\ \rho' u'_i \\ \rho' E'\end{bmatrix} + \frac{\partial}{\partial x'_j}\begin{bmatrix}\rho' u'_j \\ \rho' u'_i u'_j + P'\delta_{ij} \\ (\rho' E' + P') u'_j\end{bmatrix} = \frac{\partial}{\partial x'_j}\begin{bmatrix} 0 \\ \tau'_{ij} \\ u'_i \tau'_{ij} + \Theta'_j\end{bmatrix}$$

$$\tau'_{ij} = \mu'\left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i}\right) - \frac{2}{3}\mu' \frac{\partial u'_k}{\partial x'_k}\delta_{ij}$$

$$\Theta'_j = K' \frac{\partial T'}{\partial x'_j}$$

$$E' = C'_v T' + \frac{1}{2} u'_j \cdot u'_j$$

$$P' = \rho' R' T'$$

$$C'_p - C'_v = R' \ ; \ (\gamma - 1) = \frac{R'}{C'_v} \quad \left(1 - \frac{1}{\gamma}\right) = \frac{R'}{C'_p} \quad \gamma = \frac{C'_p}{C'_v}$$

$$u_5 = \frac{\rho' E'}{\gamma \rho^* C_v^* T^*} = \rho E \quad w/ \ \rho = \frac{\rho'}{\rho^*} \quad E = \frac{E'}{\gamma C_v^* T^*}$$

Further Define $\quad P = \frac{P'}{P^*} \ ; \ T = \frac{T'}{T^*} \ ; \ \mu = \frac{\mu'}{\mu^*} \ ; \ K = \frac{K'}{K^*}$

$$C_v = \frac{C'_v}{C_v^*} \ ; \ R = \frac{R'}{R^*} \ ; \ u_j = \frac{u'_j}{u^*}$$

$$x = \frac{x'}{L} \ ; \ t = \frac{t'}{t^*} = \frac{t'}{(L/u^*)} \ ; \ t^* = \frac{L}{u^*}$$

$$C'_p - C'_v = R'$$

$$(\gamma - 1) = \frac{R'}{C'_v} = \frac{R^*}{C_v^*} \qquad \left(1 - \frac{1}{\gamma}\right) = \frac{R'}{C'_p}$$

$$\cancel{(\gamma - 1) = \frac{R'}{R^*}\frac{R^*}{C_v^*}\frac{C_v^*}{C'_v} = \frac{R}{C_v}} \quad \frac{\gamma - 1}{\gamma} = \frac{R'}{C'_p} = \frac{R^*}{C_p^*}$$

continuity

$$\frac{L}{\rho^* u^*} \Big| \frac{\partial}{\partial t'} \rho' + \frac{\partial}{\partial x_j'} \rho' u_j' = 0$$

$$\frac{\partial}{\partial t'/(L/u^*)} (\rho'/\rho^*) + \frac{\partial}{\partial (x_j'/L)} \left(\frac{\rho'}{\rho^*} \frac{u_j'}{u^*}\right) = 0$$

$$\boxed{\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x_j}(\rho u_j) = 0}$$

$$\frac{L}{\rho^* u^* u^*} \Big| \frac{\partial}{\partial t'}(\rho' u_i') + \frac{\partial}{\partial x_j'}\left(\rho' u_j' u_i' + P' \delta_{ij}\right) = \frac{\partial}{\partial x_j'}\left[\mu'\left(\frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'}\right) - \frac{2}{3}\mu' \frac{\partial u_k'}{\partial x_k'} \delta_{ij}\right]$$

$$\frac{L}{\rho^* u^* u^*} \frac{\partial}{\partial t'}(\rho' u_i') = \frac{\partial}{\partial t'/(L/u^*)}\left(\frac{\rho'}{\rho^*} \frac{u_i'}{u^*}\right)$$

$$= \frac{\partial}{\partial t}(\rho u_i)$$

$$\frac{L}{\rho^* u^* u^*} \frac{\partial}{\partial x_j'}(\rho' u_j' u_i') = \frac{\partial}{\partial (x'/l)_j}\left(\frac{\rho'}{\rho^*} \frac{u_j'}{u^*} \frac{u_i'}{u^*}\right) = \frac{\partial}{\partial x_j}(\rho u_j u_i)$$

$$\frac{L}{\rho^* u^* u^*} \frac{\partial}{\partial x_j}(P' \delta_{ij}) = \frac{\partial}{\partial (x_j/l)} \frac{P'}{P^*} \delta_{ij} \frac{P^*}{\rho^* u^* u^*}$$

$$= \frac{\partial}{\partial x_j} P_j \delta_{ij} \frac{\gamma R^* T^*}{\gamma u^* u^*}$$

$$= \frac{\partial}{\partial x_j} \frac{1}{\gamma M^2} P_j \delta_{ij}$$

$$\frac{L}{\rho^* u^* u^*} \frac{\partial}{\partial x_j} \mu'\left(\frac{\partial u_i'}{\partial x_j'} + \frac{\partial u_j'}{\partial x_i'}\right) = \frac{\partial}{\partial (x_j'/l)} \frac{\mu'}{\mu^*} \frac{\mu^*}{\rho^* u^* l} \frac{\partial (u'/u^*)}{\partial (x_j'/l)}$$

$$- \frac{\partial}{\partial x_j} \mu' \frac{\partial u_k'}{\partial x_k'} \delta_{ij}$$

$$= \frac{1}{Re}\left\{\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k} \delta_{ij}\right]\right\}$$

Momentum Result (3)

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left[(\rho u_j u_i) + \frac{1}{\gamma M^2} P \delta_{ij}\right] = \frac{1}{Re}\frac{\partial}{\partial x_j}\mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]$$

Energy

$$\frac{1}{\gamma\rho^* c_v^* T^*}\frac{L}{u^*}\left|\frac{\partial}{\partial t'}(\rho' E') + \frac{\partial}{\partial x_j}(\rho' E' + P')u_j' = \frac{\partial}{\partial x_j}\left(u_i' \tau_{ij}' + q_j'\right)\right.$$

$$\frac{1}{\gamma\rho^* c_v^* T^*}\frac{L}{u^*}\frac{\partial}{\partial t'}\rho' E' = \frac{\partial}{\partial t'_{(L/u)}}\frac{\rho'}{\rho^*}\frac{E'}{\gamma c_v^* T^*}$$

$$\frac{E'}{\gamma c_v^* T^*} = \frac{c_v' T'}{\gamma c_v^* T^*} + \frac{\frac{1}{2}u' \cdot u'}{\gamma c_v^* T^*}$$

$$= \frac{1}{\gamma} T c_v + \frac{1}{2}\frac{u'}{u^*}\cdot\frac{u'}{u^*}\frac{u^{2*}}{\gamma R^* T^*}\frac{R^*}{c_v^*}$$

$$\boxed{E = \frac{1}{\gamma} c_v T + (\gamma - 1) M^2 \frac{1}{2} u \cdot u}$$

$$= \frac{\partial}{\partial t}(\rho E)$$

$$\frac{1}{\gamma\rho^* c_v^* T^*}\frac{L}{u^*}\frac{\partial}{\partial x_j}(\rho' E' + P')u_j' = \frac{\partial}{\partial (x/_L)}\left(\frac{\rho'}{\rho}\frac{E'}{\gamma c_v^* T} + \frac{P'}{P^*}\frac{P^*}{\rho^* \gamma c_v^* T}\right)\frac{u'}{u^*}$$

$$= \frac{\partial}{\partial x_j}\left(\rho E + P\frac{R^* T^*}{\gamma c_v^* T^*}\right)u_j$$

$$= \frac{\partial}{\partial x_j}\left(\rho E + \frac{\gamma - 1}{\gamma} P\right)u_j$$

$$\frac{1}{\gamma \rho^* c_v^* T^*}\frac{L}{u^*}\frac{\partial}{\partial x_j'} u_i' \tau_{ij}' = \frac{\partial}{\partial(x_j'/\ell)} \frac{u_i'}{u^*}\frac{\mu'}{\mu^*}\frac{\mu^*}{\rho^* u^* L}\frac{u^{*2}}{\gamma c_v^* T^*}\frac{\partial u_i'/u^*}{\partial(x_j'/\ell)}$$

$$= \frac{\partial}{\partial x_j}\mu u_i\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{u^{*2}}{\gamma R^* T}\frac{R^*}{c_v^*}\frac{1}{Re}$$

$$= \frac{\partial}{\partial x_j}\mu u_i \tau_{ij} \; M^2(\gamma-1)$$

$$= \frac{(\gamma-1)M^2}{Re}\frac{\partial}{\partial x_j}(\mu u_i\tau_{ij})$$

$$\frac{1}{\gamma\rho^* c_v^* T^*}\frac{L}{u^*}\frac{\partial}{\partial x_j}\left(K'\frac{\partial T}{\partial x_j'}\right) = \frac{\partial}{\partial(x_j/\ell)}\frac{K'}{K^*}\frac{K^*}{u\rho^* c_v^*\ell}\frac{1\,\partial(T'/T^*)}{\partial x_j'/\ell}$$

$$= \frac{\partial}{\partial x_j}K\frac{\partial T}{\partial x_j}\frac{\mu^*}{\rho^* u^*\ell}\frac{K^*}{\mu^* c_p^*}\frac{c_p^*}{c_v^*}$$

MHC $$= \frac{(1)\cancel{\gamma}}{Re\,Pr}\frac{\partial}{\partial x_j}K\frac{\partial T}{\partial x_j}$$

$$\frac{\partial(\rho E)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho E + \frac{\gamma-1}{\gamma}p\right)u_j = \frac{(\gamma-1)M^2}{Re}\frac{\partial}{\partial x_j}(\mu u_i\tau_{ij})$$

$$+\frac{\cancel{\gamma}}{Re\,Pr}\frac{\partial}{\partial x_j}K\frac{\partial T}{\partial x_j}$$ MHC

$$E = \frac{1}{\gamma}c_v T + (\gamma-1)M^2\frac{1}{2}(u\cdot u)$$

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u_i \\ \rho E\end{bmatrix} + \frac{\partial}{\partial x_j}\begin{bmatrix}\rho u \\ \rho u_j u_i + \frac{1}{\gamma M^2} P \delta_{ij} \\ \left(\rho E + \frac{\gamma-1}{\gamma} P\right) u_j\end{bmatrix} = \frac{\partial}{\partial x_j}\begin{bmatrix} 0 \\ \frac{1}{Re}\,\mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\nabla\cdot u\,\delta_{ij}\right] \\ \frac{(\gamma-1)M^2}{Re}\,\mu\left(u_i \tau_{ij}\right) + \frac{\cancel{\gamma}}{Re\,Pr}\left(K\frac{\partial T}{\partial x_j}\right)\end{bmatrix}$$

$$E = \frac{1}{\gamma} C_v T + (\gamma-1) M^2 \frac{1}{2}(\vec{u}\cdot\vec{u})$$

$$P = \rho T$$

$$\boxed{a^2 = \frac{T}{M^2}}$$