Cyl-polar coordinates (r, θ, z)

Euler eqs become (in Dimensional form)

$$(1) \quad \frac{\bar{V}_\theta^2}{\bar{r}} = \frac{1}{\bar{\rho}} \frac{\partial \bar{P}}{\partial \bar{r}} \qquad \bar{P}/\bar{\rho}^\gamma = C_1 \quad \Big\}\ \text{arbitrary Datum}$$

Non Dim.

$$\left.\begin{aligned} \bar{V}_\theta &= V_\theta\, \bar{U}_0 \\ \bar{r} &= r\, \bar{R}_0 \\ \bar{\rho} &= \rho\, \bar{\rho}_0 \end{aligned}\right\} \text{arbitrary nondim.}$$

$$(1^*) \quad \frac{V_\theta^2}{r} \frac{\bar{U}_0^2}{\bar{R}} = \frac{1}{\rho \bar{\rho}_0} \frac{1}{\bar{R}} \frac{\partial \bar{P}}{\partial r}$$

$$\frac{V_\theta^2}{r} = \frac{1}{\rho} \frac{\partial P}{\partial r} \qquad \Rightarrow \quad P = \bar{P}/\bar{\rho}_0 \bar{U}_0^2$$

$$\frac{P \bar{\rho}_0 \bar{U}_0^2}{\rho^\gamma \bar{\rho}_0^\gamma} = C_1 \quad ; \quad P/\rho^\gamma = C_2$$

$$C_2 = \frac{C_1 \bar{\rho}_0^{\gamma-1}}{\bar{U}_0^2}$$

Eqns under arbitrary nondimensionalization become

(2) $$\frac{V_\theta^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r} \qquad P = C_2 \rho^\gamma \ ; \ \frac{\partial P}{\partial r} = \gamma C_2 \rho^{\gamma-1}\frac{\partial \rho}{\partial r}$$

assume nondimensional Distribution

$$V_\theta(r) = \frac{\epsilon r}{2\pi}\exp\left(\frac{1-r^2}{2}\right)$$

(3) or $$V_\theta^2(r) = \frac{\epsilon^2 r^2}{(2\pi)^2}\exp(1-r^2)$$

(3) => (2) yields

(4) $$\frac{\epsilon^2 r}{4\pi^2}\exp(1-r^2) = \frac{1}{\rho}\gamma C_2 \rho^{\gamma-1}\frac{\partial \rho}{\partial r} = \gamma C_2 \rho^{\gamma-2}\frac{\partial \rho}{\partial r}$$

Integrating (4) yields

$$\frac{-\epsilon^2}{8\pi^2}\exp(1-r^2) = \frac{\gamma}{\gamma-1}C_2\rho^{\gamma-1} + \underbrace{C_3}_{\equiv \text{ integration const.}}$$

allow $r \to \infty$

$$C_3 = -\frac{\gamma}{\gamma-1}C_2(\rho|_\infty)^{\gamma-1}$$

Eqn becomes

$$\rho^{\gamma-1} = \rho\Big|_{\infty}^{\gamma-1} - \frac{\gamma-1}{\gamma C_2} \frac{\epsilon^2}{8\pi^2} \exp(1-r^2)$$

$$\boxed{\begin{aligned} \rho &= \left[\rho_\infty - \frac{\gamma-1}{\gamma}\frac{1}{C_2}\frac{\epsilon^2}{8\pi^2}\exp(1-r^2)\right]^{\frac{1}{\gamma-1}} \\ P &= C_2\left[\rho_\infty - \frac{\gamma-1}{\gamma}\frac{1}{C_2}\frac{\epsilon^2}{8\pi^2}\exp(1-r^2)\right]^{\frac{\gamma}{\gamma-1}} \\ \text{with}\quad V_\theta &= \frac{\epsilon r}{2\pi}\exp\left(\frac{1-r^2}{2}\right) \end{aligned}}$$

What about the nondimensionalization?

$$\text{if}\quad \left.\begin{aligned} \bar{U}_0 &= \bar{u}_\infty \\ \bar{\rho}_0 &= \bar{\rho}_\infty \\ \bar{R} &= 1 \end{aligned}\right\} \Rightarrow \quad C_1\Big|_\infty = \bar{P}_\infty / \bar{\rho}_\infty^{\gamma}$$

$$C_2 = \frac{\bar{P}_\infty}{\bar{\rho}_\infty^{\gamma}} \frac{\bar{\rho}_\infty^{\gamma-1}}{\bar{u}_\infty^2} = \frac{\bar{P}_\infty}{\bar{\rho}_\infty} \frac{1}{\bar{u}_\infty^2}$$

$$C_2 = \frac{1}{\gamma M_\infty^2}$$

$$\rho_\infty = 1$$

$$\rho = \left[1 - (\gamma-1) M^2 \frac{\epsilon^2}{8\pi^2} \exp(1-r^2)\right]^{\frac{1}{\gamma-1}}$$

$$\text{if}\quad \left.\begin{aligned} \bar{U}_0 &= \bar{a}_\infty \\ \bar{\rho}_0 &= \rho_\infty \\ \bar{R} &= 1 \end{aligned}\right\} \Rightarrow C_1\Big|_\infty = \bar{P}_\infty / \bar{\rho}_\infty^{\gamma}$$

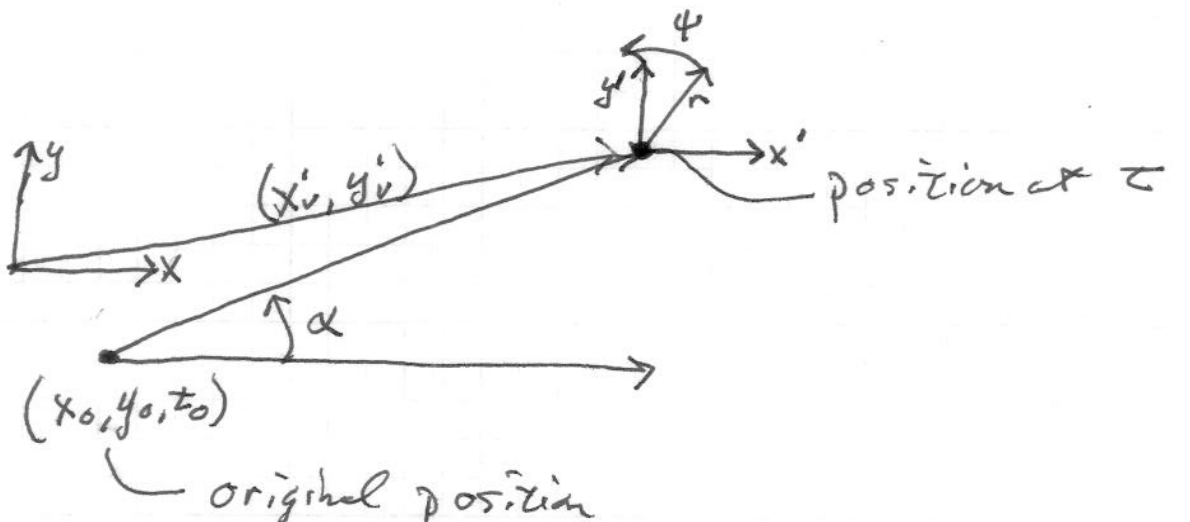
$$C_2 = \frac{\bar{P}_\infty}{\bar{\rho}_\infty^{\gamma}} \frac{\bar{\rho}_\infty^{\gamma-1}}{\bar{a}_\infty^2} = \frac{\bar{P}_\infty}{\bar{\rho}_\infty} \frac{1}{\bar{a}_\infty^2}$$

$$C_2 = \frac{1}{\gamma}$$

$$\rho_\infty = 1$$

$$\rho = \left[1 - (\gamma-1) \frac{\epsilon^2}{8\pi^2} \exp(1-r^2) \right]^{\frac{1}{\gamma-1}}$$

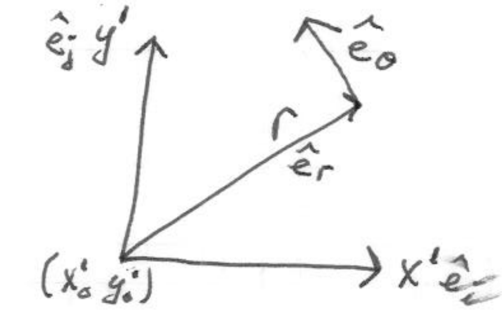
Now assume that the vortex translates with a uniform flow. The expressions relative to a fixed coordinate (x, y, t) become



The total velocity is the sum of the Translation and local

$$U \hat{e}_i = U_\infty \cos(\alpha) + U_{loc}$$
$$V \hat{e}_j = U_\infty \sin(\alpha) + V_{loc}$$

local velocity is expressed in (r, θ) local coordinates.



$$\hat{e}_r = \frac{x'}{\sqrt{x'^2+y'^2}}\hat{e}_i + \frac{y'}{\sqrt{x'^2+y'^2}}\hat{e}_j$$

Unit vectors $\begin{bmatrix}\hat{e}_r \\ \hat{e}_\theta\end{bmatrix}$, $\begin{bmatrix}\hat{e}_i \\ \hat{e}_j\end{bmatrix}$

$$\hat{e}_\theta = \frac{-y'}{\sqrt{x'^2+y'^2}}\hat{e}_i + \frac{x'}{\sqrt{x'^2+y'^2}}\hat{e}_j$$

$$\begin{bmatrix}\hat{e}_r \\ \hat{e}_\theta\end{bmatrix} = \frac{1}{\sqrt{x'^2+y'^2}}\begin{bmatrix}x' & y' \\ -y' & x'\end{bmatrix}\begin{bmatrix}\hat{e}_i \\ \hat{e}_j\end{bmatrix}$$

so represent velocity in $[x', y']$ frame

$$\vec{V}_{r,\theta} = \begin{bmatrix}0\hat{e}_r \\ V_\theta \hat{e}_\theta\end{bmatrix} \qquad V_\theta = \frac{\epsilon\sqrt{(x'^2+y'^2)}}{2\pi}\exp\left(\frac{1-(x'^2+y'^2)}{2}\right)$$

go the other direction.

$$\begin{bmatrix} \hat{e}_i \\ \hat{e}_j \end{bmatrix} = \frac{1}{\sqrt{x'^2 + y'^2}} \begin{bmatrix} x' & -y' \\ y' & x' \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \end{bmatrix}$$

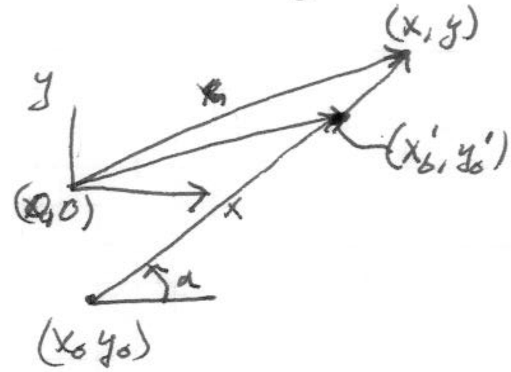
so

$$\begin{bmatrix} U_{eoc}\,\hat{e}_i \\ V_{eoc}\,\hat{e}_j \end{bmatrix} = \begin{bmatrix} -\frac{\epsilon}{2\pi} y' \exp\left[\frac{1-(x'^2+y'^2)}{2}\right] \\ \frac{\epsilon}{2\pi} x' \exp\left[\frac{1-(x'^2+y'^2)}{2}\right] \end{bmatrix}$$

now remember coordinate $[x_0', y_0']$ is moving relative to $[x, y]$

$$x_0' = x_0 + U_\infty \cos(\alpha)[t - t_0] \; ; \quad x = x_0' + x'$$
$$y_0' = y_0 + U_\infty \sin(\alpha)[t - t_0] \; ; \quad y = y_0' + x'$$

Define $f(x, y, t)$



$$x = x_0 + U_\infty \cos(\alpha)[t - t_0] + x'$$
$$y = y_0 + U_\infty \sin(\alpha)[t - t_0] + y'$$

$$(x - x_0) - U_\infty \cos(\alpha)[t] = x'$$
$$(y - y_0) - U_\infty \sin(\alpha)[t] = y'$$

Define $f(x, y, t) = \left[1 - (x'^2 + y'^2)\right]$

$$= \left\{1 - \left[(x - x_0 - u_\infty \cos(\alpha)\, t)^2 + (y - y_0 - u_\infty \sin(\alpha)\, t)^2\right]\right\}$$

$$u(x,y,t) = u_\infty \cos(\alpha) + \frac{\epsilon}{2\pi}\left(y - y_0 - u_\infty \sin(\alpha)\, t\right) \exp\left(\frac{f(x,y,t)}{2}\right)$$

$$v(x,y,t) = u_\infty \sin(\alpha) + \frac{\epsilon}{2\pi}\left(x - x_0 - u_\infty \cos(\alpha)\, t\right) \exp\left(\frac{f(x,y,t)}{2}\right)$$