

Partial-Fraction Expansion Notes

Case 1: Real and Distinct Roots.

$$F(s) = \frac{2}{(s+1)(s+2)}$$

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)}$$

To find k_1 , multiply by $s+1$, $\left| s = -1 \right.$ substitute $s = -1$.

$$\left. \frac{2}{(s+2)} \right|_{s=-1} = k_1 + \left. \frac{k_2}{(s+2)} (s+1) \right|_{s=-1}$$

$$\underline{k_1 = 2}$$

To find k_2 , multiply by $s+2$, $\left| s = -2 \right.$ substitute $s = -2$.

$$\left. \frac{2}{(s+1)} \right|_{s=-2} = \frac{k_1(s+2)}{(s+1)} + \left. k_2 \right|_{s=-2}$$

$$\underline{k_2 = -2}$$

Substitute,

inverse Laplace transform.

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$\underline{\underline{\mathcal{L}^{-1}[F(s)] = 2e^{-t} - 2e^{-2t}}}$$

Partial-Fraction Expansion Notes

Case 1: Real and Distinct Poles

$$F(s) = \frac{s}{(s+1)(s+2)}$$

$$F(s) = \frac{s}{(s+1)(s+2)} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)}$$

To find k_1 , multiply by $(s+1)$
 To find k_2 , multiply by $(s+2)$
 Substitute $s = -1$
 Substitute $s = -2$

$$\frac{s}{(s+1)(s+2)} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)}$$

$$k_1 = s$$

To find k_2 , multiply by $(s+2)$
 To find k_1 , multiply by $(s+1)$
 Substitute $s = -2$
 Substitute $s = -1$

$$\frac{s}{(s+1)(s+2)} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)}$$

$$k_2 = -2$$

Substitute, inverse Laplace transform

$$F(s) = \frac{s}{s+1} - \frac{2}{s+2}$$

$$\mathcal{L}^{-1}[F(s)] = e^{-t} - 2e^{-2t}$$

Case 2: Real and Repeated Roots.

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)}$$

k_1 , $\times s+1$, $s=-1$

$$\frac{2}{(s+2)^2} \Big|_{s=-1} = k_1 + \frac{k_2(s+1)}{(s+2)^2} + \frac{k_3(s+1)}{(s+2)} \Big|_{s=-1}$$

$$\underline{\underline{k_1 = 2.}}$$

k_2 , $\times (s+2)^2$, $s = -2$

$$\frac{2}{(s+1)} \Big|_{s=-2} = \frac{k_1(s+2)^2}{(s+1)} + k_2 + \frac{k_3(s+2)}{1}$$

$$\underline{\underline{k_2 = -2}}$$

k_3 , differentiate $F(s) \times (s+2)^2 ds$, $s = -2$.

$$\frac{-2}{(s+1)^2} \Big|_{s=-2} = \frac{(s+2)s k_1 + k_3}{(s+1)^2} \Big|_{s=-2}$$

$$\underline{\underline{k_3 = -2}}$$

$$F(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)}$$

$$\underline{\underline{L^{-1}[F(s)] = 2e^{-t} - 2e^{-2t} - 2te^{-2t}}}$$

Case 2: Real and Repeated Roots

$$\frac{F(s)}{s} = \frac{1/s^2}{(s+1)(s+2)^2}$$

$$\frac{F(s)}{s} = \frac{1/s^2}{(s+1)(s+2)^2} = \frac{k_1}{s+1} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+2)^2}$$

$$\frac{1/s^2}{(s+1)(s+2)^2} = \frac{k_1}{s+1} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+2)^2}$$

$$1/s^2 = k_1(s+2)^2 + k_2(s+1)(s+2) + k_3(s+1)$$

$$1/s^2 = k_1(s^2+4s+4) + k_2(s^2+3s+2) + k_3(s+1)$$

$$1/s^2 = (k_1+k_2)s^2 + (4k_1+3k_2+k_3)s + (4k_1+2k_2+k_3)$$

$$1/s^2 = 0s^2 + 0s + 1/s^2$$

$$k_1 = 0$$

$$\frac{1/s^2}{(s+1)(s+2)^2} = \frac{k_2}{(s+2)} + \frac{k_3}{(s+2)^2}$$

$$1/s^2 = k_2(s+2) + k_3$$

$$1/s^2 = k_2s + 2k_2 + k_3$$

$$1/s^2 = 0s + 0 + 1/s^2$$

$$k_2 = 0$$

$$k_3 = 1/s^2$$

$$\frac{1/s^2}{(s+1)(s+2)^2} = \frac{0}{s+1} + \frac{0}{(s+2)} + \frac{1/s^2}{(s+2)^2}$$

$$k_3 = 1/s^2$$

$$F(s) = \frac{1/s^2}{(s+1)(s+2)^2} = \frac{1/s^2}{(s+1)(s+2)^2}$$

Case 3. Complex or Imaginary Roots. ~~not provided~~

$$F(s) = \frac{s(3s+1) + (1+2)}{s(s^2+2s+5)} = \frac{3s^2 + s + 3}{s(s^2+2s+5)}$$

$$F(s) = \frac{3s^2 + s + 3}{s(s^2+2s+5)} = \frac{k_1}{s} + \frac{k_2s + k_3}{s^2+2s+5}$$

$$k_1, \text{ at } s=0$$

$$\frac{3}{s^2+2s+5} \Big|_{s=0} = k_1 + \frac{(k_2s + k_3)s}{s^2+2s+5} \Big|_{s=0}$$

$$\underline{k_1 = \frac{3}{5}}$$

Through cross multiplication, i.e. multiply by $s(s^2+2s+5)$, and clearing fractions.

$$3 = k_1(s^2+2s+5) + (k_2s+k_3)s \quad / k_1 = \frac{3}{5}$$

$$= \left(\frac{3}{5} + k_2\right)s^2 + \left(k_3 + \frac{6}{5}\right)s + 3$$

$$0 = \left(\frac{3}{5} + k_2\right)s^2 + \left(k_3 + \frac{6}{5}\right)s$$

$$\therefore \left(k_2 + \frac{3}{5}\right) = 0 \quad \text{and} \quad \left(k_3 + \frac{6}{5}\right) = 0 \quad \therefore k_2 = -\frac{3}{5}, \quad k_3 = -\frac{6}{5}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{5} \left(\frac{s+2}{s^2+2s+5} \right)$$

$$\mathcal{L}[Ae^{-at} \cos \omega t] = \frac{A(s+a)}{(s+a)^2 + \omega^2}$$

$$\mathcal{L}[Be^{-at} \sin \omega t] = \frac{B\omega}{(s+a)^2 + \omega^2}$$

$$\mathcal{L}[Ae^{-at} \cos \omega t + Be^{-at} \sin \omega t] = \frac{A(s+a) + B\omega}{(s+a)^2 + \omega^2}$$

completing last part of equation $F(s)$

$$F(s) = \frac{3/5}{s} - \frac{3}{5} \left(\frac{(s+1) + (1/2)s}{(s+1)^2 + 2^2} \right)$$

$$\therefore \underline{\underline{L^{-1}[F(s)] = \frac{3}{5} - \frac{3}{5} \left(e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) \right)}}$$

$$\frac{3}{s} = \frac{k_1}{s} + \frac{k_2(s+k)}{(s+1)^2 + 2^2}$$

$$\underline{\underline{\frac{3}{s} = k_1 + \frac{k_2(s+k)}{(s+1)^2 + 2^2}}}$$

Through cross multiplication, i.e. multiply by $(s+1)^2 + 2^2$ and clearing fractions

$$3 = k_1(s+1)^2 + 2^2 + (k_2 + k)s$$

$$= \left(\frac{3}{s} + k_2 \right) s + \left(k_2 + \frac{3}{2} \right) s + 3$$

$$0 = \left(\frac{3}{s} + k_2 \right) s + \left(k_2 + \frac{3}{2} \right) s + 3$$

$$\therefore \left(k_2 + \frac{3}{2} \right) = 0 \text{ and } \left(k_2 + \frac{3}{2} \right) = 0 \therefore k_2 = -\frac{3}{2}, k_3 = -\frac{3}{2}$$

$$F(s) = \frac{3/5}{s} - \frac{3/5}{s} \left(\frac{s}{s^2 + 2s + 2} \right)$$

$$L[Ae^{-at} \cos wt] = \frac{A(s+a)}{(s+a)^2 + w^2}$$

$$L[B e^{-at} \sin wt] = \frac{Bw}{(s+a)^2 + w^2}$$

$$L[Ae^{-at} \cos wt + B e^{-at} \sin wt] = \frac{A(s+a) + Bw}{(s+a)^2 + w^2}$$