Partial-Fraction Expansion Notes

Case 1: Real and Distinct Roots.

$$F(s) = \frac{2}{(s+1)(s+2)}$$

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+2)}$$

To find k_1, multiply by $s+1$, substitute $s=-1$.

k_1, $\times\ s+1$, $\Big|_{s=-1}$.

$$\left.\frac{2}{(s+2)}\right|_{s=-1} = k_1 + \left.\frac{k_2}{(s+2)}(s+1)\right|_{s=-1}$$

$$k_1 = 2$$

To find k_2, multiply by $s+2$, substitute $s=-2$

k_2, $\times\ s+2$, $\Big|_{s=-2}$

$$\left.\frac{2}{(s+1)}\right|_{s=-2} = \left.\frac{k_1(s+2)}{(s+1)} + k_2\right|_{s=-2}.$$

$$k_2 = -2$$

Substitute,

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

inverse laplace transform.

$$\mathcal{L}^{-1}[F(s)] = 2e^{-t} - 2e^{-2t}$$

Case 2: Real and Repeated Roots.

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)}$$

k_1, $\times$ $s+1$, $s=-1$

$$\left.\frac{2}{(s+2)^2}\right|_{s=-1} = \left. k_1 + \frac{k_2(s+1)}{(s+2)^2} + \frac{k_3(s+1)}{(s+2)}\right|_{s=-1}$$

$$\underline{\underline{k_1 = 2.}}$$

k_2, $\times$ $(s+2)^2$, $s = -2$

$$\left.\frac{2}{(s+1)}\right|_{s=-2} = \frac{k_1(s+2)^2}{(s+1)} + k_2 + \frac{k_3}{1}(s+2)$$

$$\underline{\underline{k_2 = -2}}$$

k_3, differentiate $F(s) \times (s+2)^2$ ds, $s = -2$.

$$\left.\frac{-2}{(s+1)^2}\right|_{s=-2} = \left.\frac{(s+2)s}{(s+1)^2}k_1 + k_3\right|_{s=-2}$$

$$\underline{\underline{k_3 = -2}}$$

$$F(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)} \qquad L^{-1}[F(s)] = \underline{\underline{2e^{-1t} - 2e^{-2t} - 2te^{-2t}}}$$

Case 2: Real and Repeated Roots

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{s+1} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

K_1, $\times\ s+1$, $s=-1$

$$\left.\frac{2}{(s+2)^2}\right|_{s=-1} = \left. K_1 + \frac{K_2(s+1)}{(s+2)^2} + \frac{K_3(s+1)}{(s+2)}\right|_{s=-1}$$

$$\underline{\underline{K_1 = 2}}$$

K_2, $\times\ (s+2)^2$, $s=-2$

$$\left.\frac{2}{(s+1)}\right|_{s=-2} = \frac{K_1(s+2)^2}{(s+1)} + K_2 + K_3(s+2)$$

$$\underline{\underline{K_2 = -2}}$$

K_3, differentiate $F(s) \times (s+2)^2\ d/ds$, $s=-2$

$$\left.\frac{-2}{(s+1)^2}\right|_{s=-2} = \left.\frac{(s+2)s\,K_1}{(s+1)^2} + K_3\right|_{s=-2}$$

$$\underline{\underline{K_3 = -2}}$$

$$F(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)^2} - \frac{2}{(s+2)} \qquad L^{-1}[F(s)] = \underline{2e^{-t} - 2e^{-2t} - 2te^{-2t}}$$

Case 3. Complex or Imaginary Roots.

$$F(s) = \frac{3}{s(s^2+2s+5)}$$

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{k_1}{s} + \frac{k_2 s + k}{s^2+2s+5}$$

k_1, $\times s$, $s=0$

$$\left.\frac{3}{s^2+2s+5}\right|_{s=0} = k_1 + \left.\frac{(k_2 s + k)s}{(s^2+2s+5)}\right|_{s=0}.$$

$$k_1 = \frac{3}{5}$$

Through cross multiplication, i.e. multiply by $s(s^2+2s+5)$, and clearing fractions.

$$3 = k_1(s^2+2s+5) + (k_2 s + k_3)s \quad \Big|_{k_1 = \frac{3}{5}}$$

$$= \left(\frac{3}{5} + k_2\right)s^2 + \left(k_3 + \frac{6}{5}\right)s + 3.$$

$$0 = \left(\frac{3}{5} + k_2\right)s^2 + \left(k_3 + \frac{6}{5}\right)s$$

$$\therefore \left(k_2 + \frac{3}{5}\right) = 0 \quad \text{and} \quad \left(k_3 + \frac{6}{5}\right) = 0 \quad \therefore \quad k_2 = \frac{-3}{5}, \; k_3 = \frac{-6}{5}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\left(\frac{s+2}{s^2+2s+5}\right)$$

$$\mathcal{L}[A e^{-at}\cos \omega t] = \frac{A(s+a)}{(s+a)^2+\omega^2}$$

$$\mathcal{L}[B e^{-at}\sin \omega t] = \frac{B\omega}{(s+a)^2+\omega^2}.$$

$$\mathcal{L}[A e^{-at}\cos\omega t + B e^{-at}\sin\omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

Converting last part of equation $F(s)$.

$$\therefore \quad F(s) = \frac{3/5}{s} - \frac{3}{5}\left(\frac{(s+1) + (1/2)2}{(s+1)^2 + 2^2}\right)$$

$$\therefore \quad \mathcal{L}^{-1}[F(s)] = 3/5 - \frac{3}{5}\left(e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right)$$