MM April Problem 2070

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12105. *Proposed by Enrique Trevino, Lake Forest College, Lake Forest, IL*
Fix a prime p. For any integer $n \geq p$, let S_n be the number of ways of coloring n points using p distinct colors, each at least once, Characterize those n such that S_n is not a multiple of p^2.

Solution
We claim that $S_n = \sum_{i=0}^{p} \binom{p}{i}(-1)^{p-i} i^n$. This follows directly from the inclusion-exclusion principle. The total number of ways to color the points using p colors is p^n. A coloring that uses only $p-1$ colors is subtracted once from the p^n total in the $-\binom{p}{p-1}n^{p-1}$ term. A coloring that uses only $p-2$ colors is subtracted twice in the $-\binom{p}{p-1}n^{p-1}$ term but is added back once in the $\binom{p}{p-2}n^{p-2}$ term. A coloring that uses only $p-3$ colors is subtracted three times in the $-\binom{p}{p-1}n^{p-1}$ term, added back three times in the $\binom{p}{p-2}n^{p-2}$ term and finally subtracted once in the $-\binom{p}{p-3}n^{p-3}$ term. Continuing the reasoning in this way, we can clearly see that all the colorings that use less that p distinct colors are subtracted once from the total p^n. Thus, we are left with only the colorings that use p distinct colors. However notice that $S(n,p) = \frac{1}{p!}\sum_{i=0}^{p} \binom{p}{i}(-1)^{p-i} i^n$, where $S(n,p)$ is a Stirling number of the second kind. This is a well-known explicit formula [2] for Stirling numbers of the second kind. Therefore, $S_n = p!S(n,p)$. Since p is prime, $p!$ only has one "p" in its prime factorization. The problem is thus reduced to finding n such that $S(n,p)$ is not a multiple of p. If p is an odd prime, according to Theorem 5.2 in [1], $S(n,p$ is not a multiple of p only when $n \equiv 1 \pmod{p-1}$. If $p=2$, then $S(n,p)$ is never even, as proved in the "Parity" section of [2]. The characterization, then, is all n congruent to 1 modulo $p-1$.

References
[1] http://www.oyeat.com/papers/stirling9.pdf
[2] https://en.wikipedia.org/wiki/Stirling$_{n}umbers_{o}f_{t}he_{s}econd_{k}ind$