AMM April Problem 12105

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12105. *Proposed by Gary Brookfield, California State University, Los Angeles, CA*
Let r be a real number, and let $f(x) = x^3 + 2rx^2 + (r^2 - 1)x - 2r$. Suppose that f has real roots a, b, and c. Prove that $a, b, c \in$ [-1,1] and |arcsin a| + |arcsin b| + |arcsin c| = π

Solution
We will first prove that $a, b, c \in$ [-1,1]. The discriminant of $f(x)$ is $-4r^4 - 44r^2 + 4$. Since $f(x)$ has 3 real roots, we know that $f(x) > 0$. It is easy to check that this is only true for $-d < r < d$, where $d \approx 0.30028$. Now, note that $f(x) = x^3 + 2rx^2 + (r^2 - 1)x - 2r = (x - a)(x - b)(x - c)$. There are three cases.
Case 1: $r = 0$
In this case, we have $f(x) = x^3 - x$, so the roots are $-1, 0, 1$. It is easy to check that they satisfy the given relationship.
Case 2 : $r > 0$
By Vieta's formulas, $abc = 2r > 0$ and $a + b + c = -2r < 0$, so we must have one positive and two negative roots. Without loss of generality, let $a > 0, b < 0, c < 0$. We know that $f(1) = (1 - a)(1 - b)(1 - c) = r^2 > 0$. Since $1 - b$ and $1 - c$ are positive, we must have that $1 - a$ is positive, so $a < 1$. Similarly, $f(-1) = (-1-a)(-1-b)(-1-c) = -r^2 < 0$. Since $-1-a$ is negative, we must have $b < -1, c < -1$ or $b > -1, c > -1$. However, the former would make $-2r = a + b + c < -1 \Rightarrow r > 0.5$, which contradicts the bounds on r. Hence, we must have $a < 1, b > -1, c > -1$.
Case 3 : $r < 0$
By the same argument as above, we have that $abc < 0$ and $a + b + c > 0$, so without loss of generality, let $a < 0, b > 0, c > 0$. Since $(-1-a)(-1-b)(-1-c) < 0$ and b and c are positive, we have $a > -1$. Since $(1-a)(1-b)(1-c) > 0$ and a is negative, we must have either $b < 1, c < 1$ or $b > 1, c > 1$. However,

the latter would make $-2r = a + b + c > 1 \Rightarrow r < -0.5$, which contradicts the bounds on r. Hence, we must have $a > -1, b < 1, c < 1$.

In all cases, $a, b, c \in [-1, 1]$. We will now prove the given arcsin relationship using complex numbers. Notice that, if $t \in [0, 1]$, then $|\arcsin(t)| = \arg(\sqrt{1-t^2} + ti)$ and, if $t \in [-1, 0]$, then $|\arcsin(t)| = \arg(\sqrt{1-t^2} - ti)$. Therefore, in Case 2 (when a is positive and b, c are negative), proving that $|\arcsin(a)| + |\arcsin(b)| + |\arcsin(c)| = \pi$ is equivalent to proving that $z := \arg(\sqrt{1-a^2} + ai)(\sqrt{1-b^2} - bi)(\sqrt{1-c^2} - ci) = \pi$ since multiplying complex numbers adds their angles. Similarly, in Case 3 (when a is negative and b, c are positive), $|\arcsin(a)| + |\arcsin(b)| + |\arcsin(c)| = \pi$ is equivalent to proving that $w := \arg(\sqrt{1-a^2} - ai)(\sqrt{1-b^2} + bi)(\sqrt{1-c^2} + ci) = \pi$. However, since z and w are nonzero (because a, b, c cannot all be 0), both cases are equivalent to proving that $\mathrm{Img}(z) = \mathrm{Img}(w) = 0$, so that they are both negative real numbers and their arguments must be π. Note that $\mathrm{Img}(w) = -\mathrm{Img}(z)$. Thus, we only need to prove that $\mathrm{Img}(w) = 0$.

Let $x = \frac{1}{a^2}, y = \frac{1}{b^2}, z = \frac{1}{c^2}$. By Newton's Sums we can prove that $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 2r^4 - 4r^2 + 2$. Also, notice that $x + y + z = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2 - 2(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}) = (\frac{r^2-1}{2r})^2 - 2(\frac{-2r}{2r}) = \frac{r^4-2r^2+1}{4r^2} + 2$ by Vieta's Formulas. Substituting these formulas, it is quite easy to prove that

$$\frac{1}{16r^4}(2r^2 - 4r^2 + 2) - \frac{1}{2r^2}(\frac{r^4 - 2r^2 + 1}{4r^2} + 2) = 0$$

$$\Rightarrow \frac{1}{16r^4}(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}) - \frac{1}{2r^2}(x + y + z) = 0$$

$$\Rightarrow x^2y^2z^2(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}) - 2xyz(x + y + z) = 0$$

$$\Rightarrow x^2y^2z^2(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}) - 2xyz(x + y + z) = 0$$

$$\Rightarrow y^2z^2 + x^2z^2 + x^2y^2 - 2x^2yz - 2xy^2z - 2xyz^2 = 0$$

$$\Rightarrow 4xyz^2 - 4xz^2 - 4yz^2 = x^2y^2 - 2x^2yz + x^2z^2 - 2xy^2z + 2xyz^2 + 4xyz - 4xz^2 + y^2z^2 - 4yz^2 + 4z^2$$

$$\Rightarrow (-2\sqrt{(x-1)(y-1)}z)^2 = (xz + yz - 2z - xy)^2$$

$$\Rightarrow -2\sqrt{(x-1)(y-1)}z = xz + yz - 2z - xy$$

$$\Rightarrow xz + yz - 2z = xy - 2\sqrt{(x-1)(y-1)}z$$

$$\Rightarrow xz - x - z + 1 + yz - y - z + 1 = xy - x - y + 1 - 2\sqrt{(x-1)(y-1)}z + 1$$

$$\Rightarrow (x-1)(y-1) + (y-1)(z-1) + 2\sqrt{(x-1)(y-1)}z = (x-1)(y-1) + 1$$

$$\Rightarrow (x-1)(y-1)+(y-1)(z-1)+2\sqrt{(x-1)(y-1)}(z-1) = (x-1)(y-1)+1-2\sqrt{(x-1)(y-1)}$$

$$\Rightarrow (\sqrt{(x-1)(z-1)} + \sqrt{(y-1)(z-1)})^2 = (\sqrt{(x-1)(y-1)} - 1)^2$$

$$\Rightarrow \sqrt{(x-1)(z-1)} + \sqrt{(y-1)(z-1)} = \sqrt{(x-1)(y-1)} - 1$$

since $\sqrt{(x-1)(z-1)} + \sqrt{(y-1)(z-1)}$ and $\sqrt{(x-1)(y-1)} - 1$ are both positive.

$$\Rightarrow -\sqrt{1-a^2}\sqrt{1-b^2}c + \sqrt{1-a^2}b\sqrt{1-c^2} + a\sqrt{1-b^2}\sqrt{1-c^2} + abc = 0$$

$$\Rightarrow Img(w) = 0$$

as desired. Thus, we have proven the given statement.