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2061. *Proposed by Florin Stanescu, Serban Cioiculescu school, Gaesti, Romania.*
Three complex numbers a, b, c satisfy $|a| = |b| = |c| = 1$ and $a^3 + b^3 + c^3 = 2abc$. Prove that a, b, c are the vertices of an isosceles triangle on the complex plane.

Solution
We will use the notation that $cis(w) = \cos(w) + i\sin(w)$, and we will denote the point represented by a complex number by its capital letter (ex. A is the point with complex number a). Since a, b, c lie on the unit circle, let $a = cis(x),\ b = cis(y),\ c = cis(z)$, where $0 \le x, y, z \le 2\pi$. Since multiplying complex numbers multiplies their magnitudes and adds their angles and dividing complex numbers divides their magnitudes and subtracts their angles, we get that $\frac{a^2}{bc} = cis(2x - y - z)$, and similarly for $\frac{b^2}{ac}$ and $\frac{c^2}{ab}$. Because a, b, c are non-zero, dividing both sides of $a^3 + b^3 + c^3 = 2abc$ by abc gives us the relation $\frac{a^2}{bc} + \frac{b^2}{ac} + \frac{c^2}{ab} = 2$. Now, substituting the polar representations of a, b, c into this equation and equating the real and imaginary parts gives:

$$\cos(2x - y - z) + \cos(2y - x - z) + \cos(2z - x - y) = 2$$
$$\sin(2x - y - z) + \sin(2y - x - z) + \sin(2z - x - y) = 0$$

However, notice that, by the product to sum identities,

$$4\sin(\frac{x}{2} + \frac{y}{2} - z)\sin(\frac{x}{2} + \frac{z}{2} - y)\sin(\frac{y}{2} + \frac{z}{2} - x)$$
$$= 4 \cdot \frac{1}{2} \cdot (\cos(\frac{3y}{2} - \frac{3z}{2}) - \cos(x - \frac{z}{2} - \frac{y}{2})) \cdot \sin(\frac{y}{2} + \frac{z}{2} - x)$$

$$= 2(\cos(\frac{3y}{2} - \frac{3z}{2}) \cdot \sin(\frac{y}{2} + \frac{z}{2} - x) - \cos(x - \frac{z}{2} - \frac{y}{2}) \cdot \sin(\frac{y}{2} + \frac{z}{2} - x))$$

$$= \sin(2x - y - z) + \sin(2y - x - z) + \sin(2z - x - y) = 0$$

Without loss of generality, let $\sin(\frac{x}{2} + \frac{z}{2} - y) = 0$ by the zero product property. It follows that

$$\frac{x}{2} + \frac{z}{2} - y = \pi n, n \in \mathbb{Z}$$

$$\Rightarrow x + z = 2y + 2\pi n$$

Therefore $arg(b^2) = 2y + 2\pi n = x + z = arg(ac)$ and $|b^2| = 1 = |ac|$, so we must have that $b^2 = ac \Rightarrow b = \frac{ac}{b}$. Now, it is well-known that, if complex numbers a, b lie on the unit circle and z is an arbitrary complex number, then the foot of the perpendicular from Z to AB is given by: [1]

$$\frac{a + b + z - ab\overline{z}}{2}$$

Thus, the perpendicular foot from B to AC is

$$\frac{a + c + b - ac\overline{b}}{2}$$

$$\Rightarrow \frac{a + c + b - \frac{ac}{b}}{2}$$

(Since b lies on the unit circle, $b = \frac{1}{b}$)

$$\Rightarrow \frac{a + c + b - b}{2}$$

$$\Rightarrow \frac{a + c}{2}$$

which is the midpoint of AC.

Therefore the altitude from B to AC passes through the midpoint of AC. This implies that AB = BC, so a, b, c form the vertices of an isosceles triangle.

References

[1] Chen, E. Bashing Geometry with Complex Numbers. http://web.evanchen.cc/handouts/cmplx/en-cmplx.pdf