PO12Q: Introduction to Quantitative Political Analysis II

Week 3 - Worksheet Solutions

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Regression

1. a. There is a positive relationship between life expectancy and per capita GDO, meaning that longer life expectancies (higher levels of health) are expected to lead to higher levels of per capita GDP (wealth)
   b. It is a sample, as there are more than those countries represented in most of the regions.
   c. The regional regression line would adapt to that particular subset of observations, so change both intercept and slope
   d. There is a curve suggested in this plot, so a quadratic regression line (second order polynomial) would be better (we will cover this in Week 9)

Matrices

1. Entries.
   a. $\boldsymbol{A}_{22} = 8$
   b. $\boldsymbol{A}_{31} = 14$
   c. $\boldsymbol{B}_{13} = 12$
   d. $\boldsymbol{B}_{24} = 3$
   e. $\boldsymbol{C}_{12} = 22$
   f. $\boldsymbol{C}_{21} = 31$

2. Dimensions.
   - $\boldsymbol{A} \in \mathbb{R}^{3\times3}$
   - $\boldsymbol{B} \in \mathbb{R}^{2\times4}$
   - $\boldsymbol{C} \in \mathbb{R}^{2\times2}$
   - $\boldsymbol{B}' \in \mathbb{R}^{4\times2}$
   - $\boldsymbol{CB} \in \mathbb{R}^{2\times4}$.

3. Transpose $\boldsymbol{B}$.

$$\boldsymbol{B} = \begin{bmatrix} 13 & 8 & 12 & 2 \\ 1 & 5 & 15 & 3 \end{bmatrix} \implies \boldsymbol{B}' = \begin{bmatrix} 13 & 1 \\ 8 & 5 \\ 12 & 15 \\ 2 & 3 \end{bmatrix}$$

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4. Product defined? (and dimensions if defined)
   a. $\boldsymbol{AB}$: not defined (3×3 with 2×4)
   b. $\boldsymbol{BA}$: not defined (2×4 with 3×3)
   c. $\boldsymbol{AC}$: not defined (3×3 with 2×2)
   d. $\boldsymbol{CA}$: not defined (2×2 with 3×3)
   e. $\boldsymbol{BC}$: not defined (2×4 with 2×2)

5. Vector–matrix products.

   With $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$:

$$\boldsymbol{Au} = \begin{bmatrix} 9 & 4 & 11 \\ 6 & 8 & 3 \\ 14 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 27 \\ 4 \\ 25 \end{bmatrix}, \qquad \boldsymbol{v}'\boldsymbol{A} = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 9 & 4 & 11 \\ 6 & 8 & 3 \\ 14 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 13 \end{bmatrix}.$$

6. Calculate $\boldsymbol{D}$ where $\boldsymbol{CB} = \boldsymbol{D}$.

$$\boldsymbol{D} = \boldsymbol{CB} = \begin{bmatrix} 25 & 22 \\ 31 & 19 \end{bmatrix} \begin{bmatrix} 13 & 8 & 12 & 2 \\ 1 & 5 & 15 & 3 \end{bmatrix} = \begin{bmatrix} 347 & 310 & 630 & 116 \\ 422 & 343 & 657 & 119 \end{bmatrix}$$

7. Linear combination with identity.

$$3\boldsymbol{A} - 2\boldsymbol{I}_3 = 3 \begin{bmatrix} 9 & 4 & 11 \\ 6 & 8 & 3 \\ 14 & 7 & 9 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & 12 & 33 \\ 18 & 22 & 9 \\ 42 & 21 & 25 \end{bmatrix}.$$

8. Determinant

   Generally we calculate the determinant with

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc,$$

   Here, we have $a = 25,\ b = 22,\ c = 31,\ d = 19$ for $\boldsymbol{C}$. Therefore

$$\det(\boldsymbol{C}) = 25 \cdot 19 - 22 \cdot 31 = 475 - 682 = -207.$$

9. Inverse of $\boldsymbol{C}$.

For $\boldsymbol{C} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\boldsymbol{C}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Here $a = 25$, $b = 22$, $c = 31$, $d = 19$, so $ad - bc = -207$ and

$$\boldsymbol{C}^{-1} = \frac{1}{-207}\begin{bmatrix} 19 & -22 \\ -31 & 25 \end{bmatrix} = \begin{bmatrix} -\frac{19}{207} & \frac{22}{207} \\ \frac{31}{207} & -\frac{25}{207} \end{bmatrix}.$$

10. Solve $\boldsymbol{Cx} = \boldsymbol{b}$.

Using $\boldsymbol{C}^{-1}$ we can solve for $\boldsymbol{x}$: $\boldsymbol{x} = \boldsymbol{C}^{-1}\boldsymbol{b}$.

- For $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$:

$$\boldsymbol{x} = \begin{bmatrix} -\frac{19}{207} & \frac{22}{207} \\ \frac{31}{207} & -\frac{25}{207} \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{85}{207} \\ \frac{106}{207} \end{bmatrix}.$$

- For $\boldsymbol{b}_2 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$:

$$\boldsymbol{x} = \begin{bmatrix} -\frac{19}{207} & \frac{22}{207} \\ \frac{31}{207} & -\frac{25}{207} \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{26}{69} \\ -\frac{17}{69} \end{bmatrix}.$$

11. Difference between $\boldsymbol{C}^{-1}\boldsymbol{b}$ and $\boldsymbol{b}\boldsymbol{C}^{-1}$.

With $\boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ (a 2×1 column) and $\boldsymbol{C} = \begin{bmatrix} 25 & 22 \\ 31 & 19 \end{bmatrix}$, we have $\boldsymbol{C}^{-1} \in \mathbb{R}^{2\times 2}$.

- $\boldsymbol{C}^{-1}\boldsymbol{b}$ is defined: (2×2 times 2×1 gives a 2×1 column vector), and equals the unique solution to $\boldsymbol{Cx} = \boldsymbol{b}$:

$$\boldsymbol{C}^{-1} = \frac{1}{-207}\begin{bmatrix} 19 & -22 \\ -31 & 25 \end{bmatrix} = \begin{bmatrix} -\frac{19}{207} & \frac{22}{207} \\ \frac{31}{207} & -\frac{25}{207} \end{bmatrix}, \qquad \boldsymbol{C}^{-1}\boldsymbol{b} = \begin{bmatrix} -\frac{85}{207} \\ \frac{106}{207} \end{bmatrix}.$$

- $\boldsymbol{b}\boldsymbol{C}^{-1}$ is not defined because the inner dimensions do not match: $(2 \times 1)(2 \times 2)$ has $1 \neq 2$.

Note: If you instead interpret $\boldsymbol{b}$ as a row vector $\boldsymbol{b}' = \begin{bmatrix} 1 & -3 \end{bmatrix}$, then $\boldsymbol{b}'\boldsymbol{C}^{-1}$ *is* defined and yields a 1×2 row:

$$\boldsymbol{b}'\boldsymbol{C}^{-1} = \begin{bmatrix} 1 & -3 \end{bmatrix} \begin{bmatrix} -\frac{19}{207} & \frac{22}{207} \\ \frac{31}{207} & -\frac{25}{207} \end{bmatrix} = \begin{bmatrix} -\frac{112}{207} & \frac{97}{207} \end{bmatrix}.$$

So the “difference” is both existence (only the left product is defined for a column $\boldsymbol{b}$) and shape (column vs row), illustrating that matrix multiplication is order-sensitive.