

PO12Q: Introduction to Quantitative Political Analysis II

Week 4: Appendix

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Appendix:

Why $\hat{\beta} = (X'X)^{-1}(X'Y)$ is not a new formula!

The following proof is taken from Li (2011) who shows in algebraic notation that the matrix formula

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

is indeed the same as our two formulae for $\hat{\beta}_0$ and $\hat{\beta}_1$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

and

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

As a first step we are replacing the rather unwieldy expressions in the nominator and denominator of $\hat{\beta}_1$ with a shorthand:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{SP_{XY}}{SS_X}$$

Then we can have a look at the new matrix notation. First up is $(X'X)^{-1}$.

$$(X'X)^{-1} = \frac{1}{n\sum x_i^2 - \sum x_i \sum x_i} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

Here we can simplify the denominator slightly, making use of the SS_X notation:

$$(X'X)^{-1} = \frac{1}{nSS_X} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix}$$

where

$$SS_X = \sum x_i^2 - n\bar{x}^2 = \sum (x_i - \bar{x})^2$$

The component of $(X'Y)$ remains unchanged for now.

$$(X'Y) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Let's put these two components together to form $(X'X)^{-1}(X'Y)$:

$$(X'X)^{-1}(X'Y) = \frac{1}{nSS_X} \begin{bmatrix} \sum x_i^2 & -\sum x_i \\ -\sum x_i & n \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Now we are going to multiply the matrices, and increasingly simplify the (admittedly at the beginning intimidating looking) result:

$$(X'X)^{-1}(X'Y) = \frac{1}{nSS_X} \begin{bmatrix} \Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i \\ -\Sigma x_i \Sigma y_i + n \Sigma x_i y_i \end{bmatrix}$$

where

$$SS_X = \Sigma x_i^2 - n\bar{x}^2 = \Sigma (x_i - \bar{x})^2$$

$$\begin{aligned}
 (X'X)^{-1}(X'Y) &= \frac{1}{nSS_X} \begin{bmatrix} \sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i \\ -\sum x_i \sum y_i + n \sum x_i y_i \end{bmatrix} \\
 &= \frac{1}{nSS_X} \begin{bmatrix} n\bar{y} \sum x_i^2 - n\bar{x} \sum x_i y_i \\ n \sum x_i y_i - n\bar{x} n\bar{y} \end{bmatrix}
 \end{aligned}$$

where

$$SS_X = \sum x_i^2 - n\bar{x}^2 = \sum (x_i - \bar{x})^2$$

$$(X'X)^{-1}(X'Y) = \frac{1}{nSS_X} \begin{bmatrix} \sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i \\ -\sum x_i \sum y_i + n \sum x_i y_i \end{bmatrix}$$

$$= \frac{1}{nSS_X} \begin{bmatrix} n\bar{y} \sum x_i^2 - n\bar{x} \sum x_i y_i \\ n \sum x_i y_i - n\bar{x} n\bar{y} \end{bmatrix}$$

$$= \frac{1}{SS_X} \begin{bmatrix} \frac{n}{n} \bar{y} \sum x_i^2 - \frac{n}{n} \bar{x} \sum x_i y_i \\ \frac{n}{n} \sum x_i y_i - \frac{n\bar{x}n\bar{y}}{n} \end{bmatrix}$$

where

$$SS_X = \sum x_i^2 - n\bar{x}^2 = \sum (x_i - \bar{x})^2$$

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 (X'X)^{-1}(X'Y) &= \frac{1}{nSS_X} \begin{bmatrix} \Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i \\ -\Sigma x_i \Sigma y_i + n \Sigma x_i y_i \end{bmatrix} \\
 &= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \Sigma x_i^2 - \bar{x} \Sigma x_i y_i \\ \Sigma x_i y_i - n \bar{x} \bar{y} \end{bmatrix}
 \end{aligned}$$

where

$$SS_X = \Sigma x_i^2 - n\bar{x}^2 = \Sigma (x_i - \bar{x})^2 \text{ and}$$

$$SP_{XY} = \Sigma x_i y_i - n\bar{x}\bar{y} = \Sigma (x_i - \bar{x})(y_i - \bar{y})$$

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 &= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \Sigma x_i^2 - \bar{x} \Sigma x_i y_i \\ SP_{XY} \end{bmatrix}
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&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \underbrace{\bar{y} n \bar{x}^2 + \bar{y} n \bar{x}^2}_{=0 \text{ SP}_{XY}} - \bar{x} \sum x_i y_i \end{bmatrix}
\end{aligned}$$

where

$SS_X = \sum x_i^2 - n \bar{x}^2 = \sum (x_i - \bar{x})^2$ and

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(X'X)^{-1}(X'Y) &= \frac{1}{nSS_X} \begin{bmatrix} \sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i \\ -\sum x_i \sum y_i + n \sum x_i y_i \end{bmatrix} \\
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&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \underbrace{\bar{y} n \bar{x}^2 + \bar{y} n \bar{x}^2}_{\substack{=0 \\ SP_{XY}}} - \bar{x} \sum x_i y_i \\ \sum x_i y_i - n \bar{x} \bar{y} \end{bmatrix} \\
&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \bar{y} n \bar{x}^2 + \bar{x} n \bar{x} \bar{y} - \bar{x} \sum x_i y_i \\ SP_{XY} \end{bmatrix}
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&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \bar{x} \sum x_i y_i \\ \sum x_i y_i - n \bar{x} \bar{y} \end{bmatrix} \\
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&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \bar{y} n \bar{x}^2 + \bar{x} n \bar{x} \bar{y} - \bar{x} \sum x_i y_i \\ \sum x_i y_i - n \bar{x} \bar{y} \end{bmatrix} \\
&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} (\sum x_i^2 - n \bar{x}^2) + \bar{x} (n \bar{x} \bar{y} - \sum x_i y_i) \\ \sum x_i y_i - n \bar{x} \bar{y} \end{bmatrix}
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&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \underbrace{\bar{y} n \bar{x}^2 + \bar{y} n \bar{x}^2}_{=0 \text{ SP}_{XY}} - \bar{x} \sum x_i y_i \end{bmatrix} \\
&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \bar{y} n \bar{x}^2 + \bar{x} n \bar{x} \bar{y} - \bar{x} \sum x_i y_i \\ \text{SP}_{XY} \end{bmatrix} \\
&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} SS_X - \text{SP}_{XY} \bar{x} \\ \text{SP}_{XY} \end{bmatrix}
\end{aligned}$$

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&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} SS_X - \text{SP}_{XY} \bar{x} \\ \text{SP}_{XY} \end{bmatrix} = \begin{bmatrix} \bar{y} - \frac{\text{SP}_{XY}}{SS_X} \bar{x} \\ \frac{\text{SP}_{XY}}{SS_X} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}
\end{aligned}$$

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&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} \sum x_i^2 - \bar{y} n \bar{x}^2 + \bar{x} n \bar{x} \bar{y} - \bar{x} \sum x_i y_i \\ \text{SP}_{XY} \end{bmatrix} \\
&= \frac{1}{SS_X} \begin{bmatrix} \bar{y} SS_X - \text{SP}_{XY} \bar{x} \\ \text{SP}_{XY} \end{bmatrix} = \begin{bmatrix} \bar{y} - \frac{\text{SP}_{XY}}{SS_X} \bar{x} \\ \frac{\text{SP}_{XY}}{SS_X} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}
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(q.e.d.)

References

List of References I

Li, B. (2011). Statistics 512: Applied Linear Models – Topic 3.

<https://www.stat.purdue.edu/~boli/stat512/lectures/topic3.pdf>