PO12Q: Introduction to Quantitative Political Analysis II

Week 4: Appendix

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Appendix:

Why $\hat{\beta} = (X^{'}X)^{-1}(X^{'}Y)$ is not a new formula!

The following proof is taken from Li (2011) who shows in algebraic notation that the matrix formula

$$\hat{\beta} = (X'X)^{-1}(X'Y)$$

is indeed the same as our two formulae for $\hat{\beta}_0$ and $\hat{\beta}_1$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

and

$$\hat{\beta}_1 = \frac{\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}$$

As a first step we are replacing the rather unwieldy expressions in the nominator and denominator of $\hat{\beta}_1$ with a shorthand:

$$\hat{\beta}_1 = \frac{\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\Sigma_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\mathrm{SP}_{\mathrm{XY}}}{\mathrm{SS}_{\mathrm{X}}}$$

Then we can have a look at the new matrix notation. First up is $(X'X)^{-1}$.

$$(X'X)^{-1} = \frac{1}{n\Sigma x_i^2 - \Sigma x_i \Sigma x_i} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

Here we can simplify the denominator slightly, making use of the SS_X notation:

$$(X'X)^{-1} = \frac{1}{nSS_X} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

where
$SS_X = \Sigma x_i^2 - n\bar{x}^2 = \Sigma(x_i - \bar{x})^2$

The component of $(X'Y)$ remains unchanged for now.

$$(X'Y) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

Let's put these two components together to form $(X'X)^{-1}(X'Y)$:

$$(X'X)^{-1}(X'Y) = \frac{1}{nSS_{\mathrm{X}}}\begin{bmatrix}\Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n\end{bmatrix}\begin{bmatrix}\Sigma y_i \\ \Sigma x_i y_i\end{bmatrix}$$

Now we are going to multiply the matrices, and increasingly simplify the (admittedly at the beginning intimidating looking) result:

$$(X'X)^{-1}(X'Y) = \frac{1}{nSS_X}\begin{bmatrix} \Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i \\ -\Sigma x_i \Sigma y_i + n\Sigma x_i y_i \end{bmatrix}$$

where
$SS_X = \Sigma x_i^2 - n\bar{x}^2 = \Sigma(x_i - \bar{x})^2$

$$(X'X)^{-1}(X'Y) = \frac{1}{nSS_X}\begin{bmatrix}\Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i \\ -\Sigma x_i \Sigma y_i + n\Sigma x_i y_i\end{bmatrix}$$

$$= \frac{1}{nSS_X}\begin{bmatrix}n\bar{y}\Sigma x_i^2 - n\bar{x}\Sigma x_i y_i \\ n\Sigma x_i y_i - n\bar{x}n\bar{y}\end{bmatrix}$$

where
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$$= \frac{1}{nSS_X}\begin{bmatrix}n\bar{y}\Sigma x_i^2 - n\bar{x}\Sigma x_i y_i \\ n\Sigma x_i y_i - n\bar{x}n\bar{y}\end{bmatrix}$$

$$= \frac{1}{SS_X}\begin{bmatrix}\frac{n}{n}\bar{y}\Sigma x_i^2 - \frac{n}{n}\bar{x}\Sigma x_i y_i \\ \frac{n}{n}\Sigma x_i y_i - \frac{n\bar{x}n\bar{y}}{n}\end{bmatrix}$$

where
$SS_X = \Sigma x_i^2 - n\bar{x}^2 = \Sigma(x_i - \bar{x})^2$

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$$= \frac{1}{SS_X}\begin{bmatrix}\bar{y}\Sigma x_i^2 - \bar{x}\Sigma x_i y_i \\ \Sigma x_i y_i - n\bar{x}\bar{y}\end{bmatrix}$$

where
$SS_X = \Sigma x_i^2 - n\bar{x}^2 = \Sigma(x_i - \bar{x})^2$ and
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where
$\mathrm{SS}_X = \Sigma x_i^2 - n\bar{x}^2 = \Sigma(x_i - \bar{x})^2$ and
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$$= \frac{1}{SS_X}\begin{bmatrix} \bar{y}(\Sigma x_i^2 - n\bar{x}^2) + \bar{x}(n\bar{x}\bar{y} - \Sigma x_i y_i) \\ SP_{XY} \end{bmatrix}$$

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$$= \frac{1}{SS_X}\begin{bmatrix} \bar{y}\mathrm{SS}_X - \mathrm{SP}_{XY}\bar{x} \\ \mathrm{SP}_{XY} \end{bmatrix} = \begin{bmatrix} \bar{y} - \frac{\mathrm{SP}_{XY}}{SS_X}\bar{x} \\ \frac{\mathrm{SP}_{XY}}{SS_X} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

where
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(q.e.d.)

References

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