

# Tabulate Equations of Common Ellipse Parameters

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May 2, 2018

## Introduction

This document tabulates the equations needed to deduce any of the seven common parameters of an ellipse given two of its parameters.

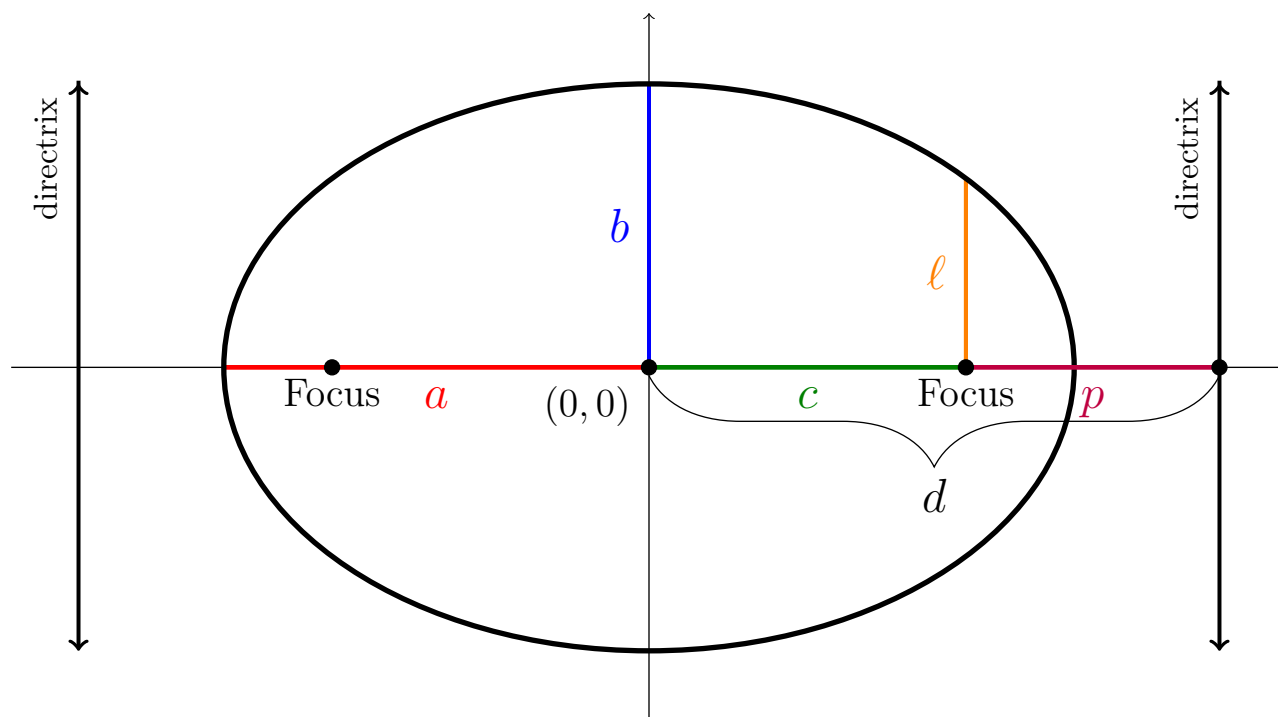


Figure 1: Common parameters labeled on an example ellipse. Eccentricity,  $e$ , is not depicted.

## Parameters

- $a$**  Semi-Major Axis. The length from the center of the ellipse to the farthest point on the curve.
- $b$**  Semi-Minor Axis. The length from the center of the ellipse to the nearest point on the curve.
- $c$**  Linear eccentricity. The length from the center of the ellipse to one of its foci.
- $d$**  Directrix (distance). The distance along the major axis from the center of the ellipse at which the directrix line lies. Sometimes denoted as  $x$  or  $y$ . In special cases it is the equation for the directrix line.
- $e$**  Eccentricity. Measurement of deviation from being circular. Sometimes denoted as  $\epsilon$ . Can be confuse with flattening that shares the symbol  $\epsilon$ .
- $\ell$**  Semi-Latus Rectum. The length of a line segment that begins at the focus and makes contact with the ellipse. It is perpendicular to the major axis.
- $p$**  Focal parameter. The length from one of the two foci to the nearest directrix.

## Other Useful Relations & Terminology

**Major Axis** Double the length of the semi-major axis ( $2a$ ). The length of the ellipse at its widest point.

**Minor Axis** Double the length of the semi-minor axis ( $2b$ ). The length of the ellipse at its thinnest point.

**Focal Length** Double the length of the linear eccentricity ( $2e$ ). The length between the ellipse's two foci.

**Flattening** A rarer type of measurement for the deviation from being circular. Flattening is given usually in terms of  $a$  and  $b$  as  $f = \frac{a-b}{a}$  or  $e$  as  $f = 1 - \sqrt{1 - e^2}$ . Sometimes denoted as  $\epsilon$ .

**Latus Rectum** Double the length of the semi-latus rectum ( $2\ell$ ). The chord that passes through a focus and is perpendicular to the major axis.

## How To Use

For the parameter of interest go to the page labeled with the parameters name. The first row and column are labeled with different variables. Select a row and column based on what information you already have. The equation that is displayed in the intersection is the function used to derived the parameter given the two variables.

## † Note On Directrix, $d$ , and Semi-Latus Rectum, $\ell$ , Equations

To solve for the parameters of an ellipse in terms of the Semi-Latus Rectum,  $\ell$ , and Directrix,  $d$  require solving a cubic function in all cases. For example in the case of linear eccentricity,  $c$ , the roots of the following equation needs to be found:

$$c^3 - 2c^2d + cd^2 - d\ell^2 = 0$$

Solving using the normal general solution to the cubic equation gives a piecewise solution that is daunting to calculate. The results would span longer than the page in some cases. Using a trigonometric cubic solution provides a much more manageable solution. However it gives a piecewise solution:

$$c = \begin{cases} \frac{2d}{3} \left( 1 + \sin \left( \frac{\arcsin \left( 1 - \frac{27\ell^2}{2d^2} \right)}{3} \right) \right) \\ \frac{4d}{3} \sin^2 \left( \frac{\arccos \left( 1 - \frac{27\ell^2}{2d^2} \right)}{6} \right) \end{cases}$$

While much more simplified than the general solution, the domain when given as a function of both  $d$  and  $\ell$  is troublesome to calculate since the switch occurs at  $\frac{a}{b} = \sqrt{\frac{2}{3}}$  which when put in terms of  $d$  and  $\ell$  is ugly.

# Semi-Major Axis

$a$	$a$	$b$	$c$	$d$	$e$	$\ell$	$p$
$a$	—	—	—	—	—	—	—
$b$	—	—	$\sqrt{b^2 + c^2}$	$\sqrt{\frac{d^2 + \sqrt{-4b^2 d^2 + d^4}}{2}}$	$\frac{b}{\sqrt{1 - e^2}}$	$\frac{b^2}{\ell}$	$b\sqrt{1 + \frac{b^2}{p^2}}$
$c$	—	$\sqrt{b^2 + c^2}$	—	$\sqrt{cd}$	$\frac{c}{e}$	$\frac{\ell + \sqrt{4c^2 + \ell^2}}{2}$	$\sqrt{cp + c^2}$
$d$	—	$\sqrt{\frac{d^2 + \sqrt{-4b^2 d^2 + d^4}}{2}}$	$\sqrt{cd}$	—	$de$	$\dagger$	$\sqrt{d^2 - dp}$
$e$	—	$\frac{b}{\sqrt{1 - e^2}}$	$\frac{c}{e}$	$de$	—	$\frac{\ell}{1 - e^2}$	$\frac{ep}{1 - e^2}$
$\ell$	—	$\frac{b^2}{\ell}$	$\frac{\ell + \sqrt{4c^2 + \ell^2}}{2}$	$\dagger$	$\frac{\ell}{1 - e^2}$	—	$\frac{\ell}{1 - \frac{\ell^2}{p^2}}$
$p$	—	$b\sqrt{1 + \frac{b^2}{p^2}}$	$\sqrt{cp + c^2}$	$\sqrt{d^2 - dp}$	$\frac{ep}{1 - e^2}$	$\frac{\ell}{1 - \frac{\ell^2}{p^2}}$	—

# Semi-Minor Axis

$b$	$a$	$b$	$c$	$d$	$e$	$\ell$	$p$
$a$	—	—	$\sqrt{a^2 - c^2}$	$a\sqrt{1 - \frac{a^2}{d^2}}$	$a\sqrt{1 - e^2}$	$\sqrt{a\ell}$	$\sqrt{\frac{\sqrt{4a^2p^2 + p^4} - p^2}{2}}$
$b$	—	—	—	—	—	—	—
$c$	$\sqrt{a^2 - c^2}$	—	—	$\sqrt{cd - c^2}$	$\frac{c}{e}\sqrt{1 - e^2}$	$\sqrt{\frac{\sqrt{4c^2\ell^2 + \ell^4} + \ell^2}{2}}$	$\sqrt{cp}$
$d$	$a\sqrt{1 - \frac{a^2}{d^2}}$	—	$\sqrt{cd - c^2}$	—	$de\sqrt{1 - e^2}$	†	$\sqrt{dp - p^2}$
$e$	$a\sqrt{1 - e^2}$	—	$\frac{c}{e}\sqrt{1 - e^2}$	$de\sqrt{1 - e^2}$	—	$\frac{\ell}{\sqrt{1 - e^2}}$	$\frac{ep}{\sqrt{1 - e^2}}$
$\ell$	$\sqrt{a\ell}$	—	$\sqrt{\frac{\sqrt{4c^2\ell^2 + \ell^4} + \ell^2}{2}}$	†	$\frac{\ell}{\sqrt{1 - e^2}}$	—	$\frac{\ell}{\sqrt{1 - \frac{\ell^2}{p^2}}}$
$p$	$\sqrt{\frac{\sqrt{4a^2p^2 + p^4} - p^2}{2}}$	—	$\sqrt{cp}$	$\sqrt{dp - p^2}$	$\frac{ep}{\sqrt{1 - e^2}}$	$\frac{\ell}{\sqrt{1 - \frac{\ell^2}{p^2}}}$	—

# Linear Eccentricity

$c$	$a$	$b$	$c$	$d$	$e$	$\ell$	$p$
$a$	—	$\sqrt{a^2 - b^2}$	—	$\frac{a^2}{d}$	$ae$	$\sqrt{a^2 - a\ell}$	$\frac{-p + \sqrt{4a^2 + p^2}}{2}$
$b$	$\sqrt{a^2 - b^2}$	—	—	$\frac{d + \sqrt{d^2 - 4b^2}}{2}$	$\frac{be}{\sqrt{1 - e^2}}$	$b\sqrt{\frac{b^2}{\ell^2} - 1}$	$\frac{b^2}{p}$
$c$	—	—	—	—	—	—	—
$d$	$\frac{a^2}{d}$	$\frac{d + \sqrt{d^2 - 4b^2}}{2}$	—	—	$de^2$	†	$d - p$
$e$	$ae$	$\frac{be}{\sqrt{1 - e^2}}$	—	$de^2$	—	$\frac{e\ell}{1 - e^2}$	$\frac{pe^2}{1 - e^2}$
$\ell$	$\sqrt{a^2 - a\ell}$	$b\sqrt{\frac{b^2}{\ell^2} - 1}$	—	†	$\frac{e\ell}{1 - e^2}$	—	$\frac{p}{\frac{p^2}{\ell^2} - 1}$
$p$	$\frac{-p + \sqrt{4a^2 + p^2}}{2}$	$\frac{b^2}{p}$	—	$d - p$	$\frac{pe^2}{1 - e^2}$	$\frac{p}{\frac{p^2}{\ell^2} - 1}$	—

# Directrix

$d$	$a$	$b$	$c$	$d$	$e$	$\ell$	$p$
$a$	—	$\frac{a^2}{\sqrt{a^2 - b^2}}$	$\frac{a^2}{c}$	—	$\frac{a}{e}$	$\sqrt{\frac{a^3}{a - \ell}}$	$\frac{p + \sqrt{4a^2 + p^2}}{2}$
$b$	$\frac{a^2}{\sqrt{a^2 - b^2}}$	—	$\frac{b^2 + c^2}{c}$	—	$\sqrt{\frac{b^2}{e^2 - e^4}}$	$\frac{b^3}{\ell\sqrt{b^2 - \ell^2}}$	$\frac{b^2 + p^2}{p}$
$c$	$\frac{a^2}{c}$	$\frac{b^2 + c^2}{c}$	—	—	$\frac{c}{e^2}$	$\frac{2c^2 + \ell^2 + \sqrt{4\ell^2 c^2 + \ell^4}}{2c}$	$c + p$
$d$	—	—	—	—	—	—	—
$e$	$\frac{a}{e}$	$\sqrt{\frac{b^2}{e^2 - e^4}}$	$\frac{c}{e^2}$	—	—	$\frac{\ell}{e - e^3}$	$\frac{p}{1 - e^2}$
$\ell$	$\sqrt{\frac{a^3}{a - \ell}}$	$\frac{b^3}{\ell\sqrt{b^2 - \ell^2}}$	$\frac{2c^2 + \ell^2 + \sqrt{4\ell^2 c^2 + \ell^4}}{2c}$	—	$\frac{\ell}{e - e^3}$	—	$\frac{p^3}{p^2 - \ell}$
$p$	$\frac{p + \sqrt{4a^2 + p^2}}{2}$	$\frac{b^2 + p^2}{p}$	$c + p$	—	$\frac{p}{1 - e^2}$	$\frac{p^3}{p^2 - \ell}$	—

# Eccentricity

$e$	$a$	$b$	$c$	$d$	$e$	$\ell$	$p$
$a$	—	$\sqrt{1 - \frac{b^2}{a^2}}$	$\frac{c}{a}$	$\frac{a}{d}$	—	$\sqrt{1 - \frac{\ell}{a}}$	$\frac{-p + \sqrt{4a^2 + p^2}}{2a}$
$b$	$\sqrt{1 - \frac{b^2}{a^2}}$	—	$\frac{c}{\sqrt{b^2 + c^2}}$	$\sqrt{\frac{1 + \sqrt{-4b^2 + d^2}}{2}}$	—	$\sqrt{1 - \frac{\ell^2}{b^2}}$	$\frac{b}{\sqrt{b^2 + p^2}}$
$c$	$\frac{c}{a}$	$\frac{c}{\sqrt{b^2 + c^2}}$	—	$\sqrt{\frac{c}{d}}$	—	$\frac{-\ell + \sqrt{4c^2 + \ell^2}}{2c}$	$\sqrt{\frac{c}{c + p}}$
$d$	$\frac{a}{d}$	$\sqrt{\frac{1 + \sqrt{-4b^2 + d^2}}{2}}$	$\sqrt{\frac{c}{d}}$	—	—	†	$\sqrt{1 - \frac{p}{d}}$
$e$	—	—	—	—	—	—	—
$\ell$	$\sqrt{1 - \frac{\ell}{a}}$	$\sqrt{1 - \frac{\ell^2}{b^2}}$	$\frac{-\ell + \sqrt{4c^2 + \ell^2}}{2c}$	†	—	—	$\frac{\ell}{p}$
$p$	$\frac{-p + \sqrt{4a^2 + p^2}}{2a}$	$\frac{b}{\sqrt{b^2 + p^2}}$	$\sqrt{\frac{c}{c + p}}$	$\sqrt{1 - \frac{p}{d}}$	—	$\frac{\ell}{p}$	—

# Semi-Latus Rectum

$\ell$	$a$	$b$	$c$	$d$	$e$	$\ell$	$p$
$a$	—	$\frac{b^2}{a}$	$\frac{a^2 - c^2}{a}$	$a - \frac{a^3}{d^2}$	$a(1 - e^2)$	—	$\frac{-p^2 + \sqrt{4a^2p^2 + p^4}}{2a}$
$b$	$\frac{b^2}{a}$	—	$\frac{b^2}{\sqrt{b^2 + c^2}}$	$\sqrt{\frac{b^2 - \sqrt{b^4 - \frac{4b^6}{d^2}}}{2}}$	$b\sqrt{1 - e^2}$	—	$\frac{bp}{\sqrt{b^2 + p^2}}$
$c$	$\frac{a^2 - c^2}{a}$	$\frac{b^2}{\sqrt{b^2 + c^2}}$	—	$\sqrt{\frac{c}{d}}(d - c)$	$\frac{c}{e}(1 - e^2)$	—	$\sqrt{\frac{cp^2}{c + p}}$
$d$	$a - \frac{a^3}{d^2}$	$\sqrt{\frac{b^2 - \sqrt{b^4 - \frac{4b^6}{d^2}}}{2}}$	$\sqrt{\frac{c}{d}}(d - c)$	—	$d(e - e^3)$	—	$\sqrt{p - \frac{p^3}{d}}$
$e$	$a(1 - e^2)$	$b\sqrt{1 - e^2}$	$\frac{c}{e}(1 - e^2)$	$d(e - e^3)$	—	—	$ep$
$\ell$	—	—	—	—	—	—	—
$p$	$\frac{-p^2 + \sqrt{4a^2p^2 + p^4}}{2a}$	$\frac{bp}{\sqrt{b^2 + p^2}}$	$\sqrt{\frac{cp^2}{c + p}}$	$\sqrt{p - \frac{p^3}{d}}$	$ep$	—	—



# Focal Parameter

$p$	$a$	$b$	$c$	$d$	$e$	$\ell$	$p$
$a$	—	$\frac{b^2}{\sqrt{a^2 - b^2}}$	$\frac{a^2 - c^2}{c}$	$\frac{d^2 - a^2}{d}$	$\frac{a}{e}(1 - e^2)$	$\sqrt{\frac{a\ell^2}{a - \ell}}$	—
$b$	$\frac{b^2}{\sqrt{a^2 - b^2}}$	—	$\frac{b^2}{c}$	$\frac{d - \sqrt{d^2 - 4b^2}}{2}$	$\frac{b}{e}\sqrt{1 - e^2}$	$\frac{b\ell}{\sqrt{b^2 - \ell^2}}$	—
$c$	$\frac{a^2 - c^2}{c}$	$\frac{b^2}{c}$	—	$d - c$	$\frac{c}{e^2}(1 - e^2)$	$\frac{\ell^2 + \sqrt{4c^2\ell^2 + \ell^4}}{2c}$	—
$d$	$\frac{d^2 - a^2}{d}$	$\frac{d - \sqrt{d^2 - 4b^2}}{2}$	$d - c$	—	$d(1 - e^2)$	†	—
$e$	$\frac{a}{e}(1 - e^2)$	$\frac{b}{e}\sqrt{1 - e^2}$	$\frac{c}{e^2}(1 - e^2)$	$d(1 - e^2)$	—	$\frac{\ell}{e}$	—
$\ell$	$\sqrt{\frac{a\ell^2}{a - \ell}}$	$\frac{b\ell}{\sqrt{b^2 - \ell^2}}$	$\frac{\ell^2 + \sqrt{4c^2\ell^2 + \ell^4}}{2c}$	†	$\frac{\ell}{e}$	—	—
$p$	—	—	—	—	—	—	—