Math 146 - Calculus II Name______________________

Slope Fields and Euler's Method

In this activity we will look at how we can approximate solutions to differential equations *without* actually solving them. We will consider a model for a tumor growth known as the *Gompertz growth function*. Let $M(t) > 0$ be the mass of a tumor at time $t \geq 0$. The relevant differential equation is

$$\frac{dM}{dt} = -rM \ln\left(\frac{M}{K}\right),$$

where r and K are positive constants.

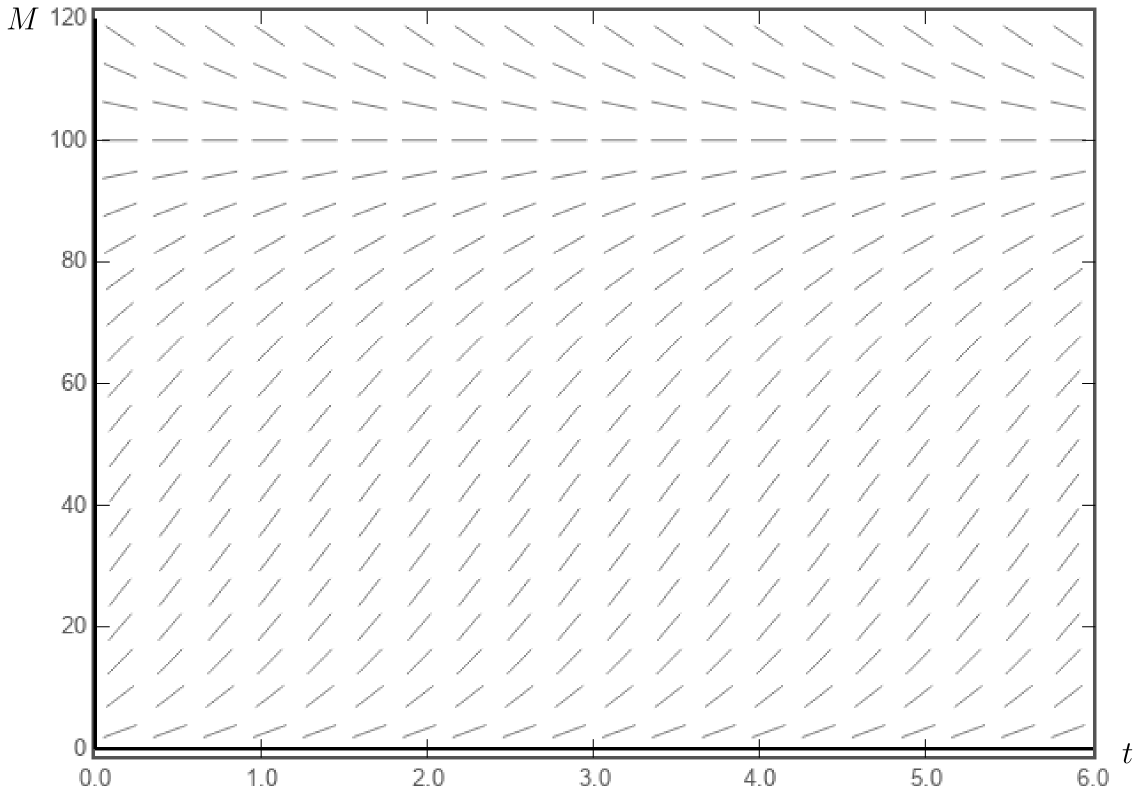
1. Verify that the differential equation has a constant solution when $M = K$. (You want to show that $\frac{dM}{dt}$ is equal to 0. This is called an equilibrium solution.)

2. Suppose that at some time t the mass of the tumor, $M(t)$, is between 0 and K. What is the sign of the rate of change of M? Is M increasing or decreasing at that point?

3. What if $M(t) > K$ at some time t?

4. What does the derivative $\dfrac{dM}{dt}$ tell you in terms of the graph of $M(t)$?

5. Hopefully your answer to the previous question included the word "slope." A nice way to visualize solutions to differential equations is with a *slope field*. For any value of t and M, we plug into right-hand side of the differential equation to get the slope, dM/dt, and indicate the slope at that point in the t-M plane by a short line segment as shown in the figure below. In this plot, $r = 1$ and $K = 100$, and the horizontal axis is t and vertical is M.



Sketch the equilibrium solution $M(t) = K = 100$ from part 1. into the plot. Does it agree with the plotted slopes/directions at every point it passes through?

6. Do the slopes indicated in the plot agree with your answers to parts 2. and 3.? Explain.

7. Suppose the initial mass of the tumor is $M(0) = 10$. Mark this point with a small filled circle.

8. The plotted line segments indicate the slope/direction of the solution to our differential equation that passes through the corresponding point. Sketch carefully the solution that corresponds to $M(0) = 10$ into the plot. Follow the slopes as closely as you can.

9. What is the long-term behavior of your hand drawn solution; that is, what is $\lim_{t\to\infty} M(t)$ equal to?

10. Suppose $M(0)$ is any value between 0 and $K = 100$. Can we say anything about the long-term behavior of the corresponding solutions?

11. Finally, we would like to get some quantitative information about the solution at $t = 1$.

    (a) We know that $M(0) = 10$. What is $M'(0)$? (Use the differential equation with $r = 1$ and $K = 100$.) Round your answer to the closest whole number.

    (b) This answer tells us the slope of the solution curve at that starting point. If we move in that direction from $t = 0$ to $t = 1$, by what amount will M change?

    (c) Use this answer to come up with an estimate for $M(1)$, the mass of the tumor at time 1.

    (d) Sketch the resulting point from part (b) into the plot. Where does the point lie in relation to the curve you sketched in part 8.? Where should it lie if the approximation was in fact exact?

    Comment: What you have just done is one step of a numerical method called Euler's method. *Its accuracy improves with more steps/shorter steps.*