Math 145 - Calculus I Name________________________

Population Growth - Day 26

Today we are going back to our study of differential equations (DEs). Remember that these are just equations that have derivatives in them, like $\frac{dy}{dx} = x + y$. Today we'll be learning about initial value problems and solutions to differential equations.

One model for how a population grows in a confined space, like bacteria in a test tube or fish in a pond, is called a *saturation model*. The equation has the form

$$\frac{dP}{dt} = k(L - P),$$

where k and L are positive constants, and $P(t)$ is the population size at time t measured in weeks.

1. Suppose we have a population of fish in a pond, and that $k = 0.1$, $L = 800$, and t is time. What is $\frac{dP}{dt}$ when $P = 500$? *(As usual, include units throughout the activity.)*

2. What is $\frac{dP}{dt}$ when $P = 1000$? What does that mean about the fish population?

3. For what population size P is $\frac{dP}{dt}$ equal to zero? What happens then?

We'd like to find a formula for the solution to this differential equation (DE). You can check whether you have a solution by plugging the **proposed solution** into the differential equation to see if you get a true result. This is done by evaluating the left- and the right-hand sides of the differential equation separately and comparing them to each other.

4. Let's show that $P = Ce^{0.1t}$ is **not** a solution to $\frac{dP}{dt} = k(L - P)$. We will do this in steps. Remember that $L = 800$ and $k = 0.1$.

(Left-hand side of the DE)

Compute: $\frac{dP}{dt} = \frac{d}{dt}\left(Ce^{0.1t}\right) =$

(Right-hand side of the DE)

Compute: $k\left(L - P\right) = 0.1\left(800 - Ce^{0.1t}\right) =$

Are the left- and right-hand sides of the equation the same?

5. Repeat this process to show that $P = 800 + Ce^{-0.1t}$ **is** a solution to this differential equation.

(Left hand side of the DE)

Compute: $\frac{dP}{dt} = \frac{d}{dt}\left(800 + Ce^{-0.1t}\right) =$

(Right hand side of the DE)

Compute: $k\left(L - P\right) = 0.1\left(800 - (800 + Ce^{-0.1t})\right) =$

Are the left and right hand sides of the equation the same?

6. An *initial value problem* is a differential equation together with some fixed conditions, which can be used to find the values of any constants in the solution. E.g., you could be asked:

   Solve the initial value problem $\frac{dP}{dt} = 0.1(800 - P)$ with the initial condition $P(0) = 500$.

   (Start with the solution $P = 800 + Ce^{-0.1t}$, determine the value of C that fits the initial condition, and then write down the final formula for $P(t)$.)

7. What would C and $P(t)$ be with the initial condition $P(0) = 1200$?

8. Finally, graph both solutions $P(t)$ from parts 6. and 7. for $0 \leq t \leq 25$ into one set of axes and re-sketch below. What can you say in both cases about $\lim_{t\to\infty} P(t)$?