Intro to Social Science Data Analysis

Week 11 Lecture: Simple Linear Regression

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1. Recap
2. Correlation
3. Best Fit Lines & Least Squares Regression
4. Hypothesis Testing
5. Some Special Issues in Simple Linear Regression

Assignment 3

Like all data assignments in this course, your response to Assignment 3 must be **reproducible**.

Quick Quiz 1

Find the sample proportions of the following party's supporters:

Saenuri	DUP	Other	Total
1064	891	520	2475

Quick Quiz 1

Saenuri	DUP	Other	Total
1064	891	520	2475
(0.43)	(0.36)	(0.21)	(1)

Quick Quiz 2

If we wanted to make inferences about **population proportions** from sampling proportions, what **distribution** do we often assume the sampling proportions follow?

What are it's **parameters**?

Quick Quix 3

Imagine we have a two-way contingency table.

	Attend University	No University
Married		
Not Married		

If we conducted a χ^2 test with this data and found a p-value of < 0.001 what would we conclude?

Quick Quiz 4

Write the simple linear regression equation for how a person's height is related to their income.

Quick Quiz 5

Describe how a linear regression line would look if the relationship between two variables was negative.

How would it look if the relationship was positive?

What about no relationship?

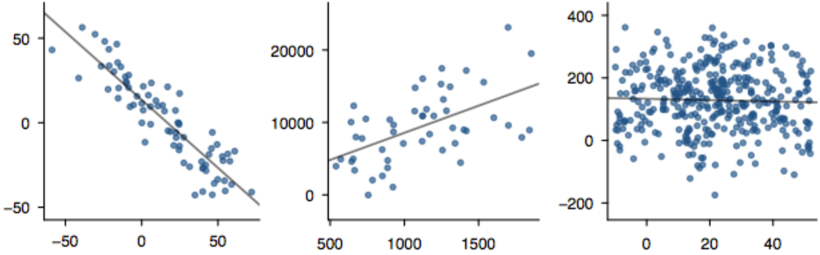
Quick Quiz 6

Describe the variables in the simple linear regression equation.

$$y = \alpha + \beta x$$

Motivation Since almost no interesting relationship is perfectly linear, how do we find the **best fit line** that describes the relationship between some x and some y?

How?



Source: Diaz et. al. (2011, 216)

In **simple linear regression** we are trying to find the straight line that is **as close to all of the data points as possible**.

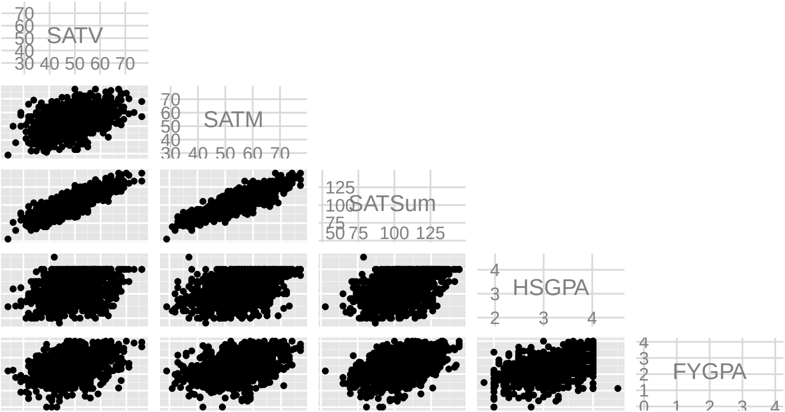
How do we find this line?

Let's use the SAT/GPA data from the openintro package:

```
# Load library
library(openintro)

# Load data
data(satGPA)

# Show variables
names(satGPA)

## [1] "sex"    "SATV"   "SATM"   "SATSum" "HSGPA"
## [6] "FYGPA"


# Subset to remove the sex variable
satGPASlim <- satGPA[, 2:6]
```

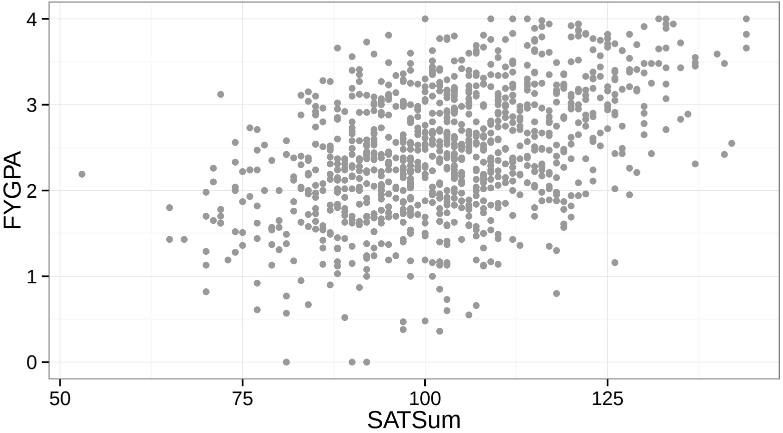
Plot the SAT Scores & GPAs

```r
library(GGally)

ggpairs(satGPASlim, upper = "blank")
```



First Year GPA

Universities want to know how well student's total SAT scores (`SATSum`) relate to their academic performance in the first year of university (`FYGPA`).



Correlation

One way to describe the overall relationship between `SATSum` and `FYGPA` is to find the **correlation** between the two variables.

Correlation (R):

Describes the **strength** of a linear relationship.

It ranges from -1 to 1.

-1 indicates a **perfect negative relationship**.

1 indicates a **perfect positive relationship**.

0 indicates **no correlation/relationship**.

Correlation

To find the correlation for observations $(x_1, y_1),\ (x_2, y_2) \ldots (x_n, y_n)$

$$R = \frac{1}{n-1} \sum_{i=1}^{n} \frac{x_i - \bar{x}}{s_x} \frac{y_i - \bar{y}}{s_y}$$

Or. . .

Or we can have R do the maths for us.

```
cor(satGPA$SATSum, satGPA$FYGPA)

## [1] 0.4603
```

Statistical Significance & Correlation

If we wanted to test to see if the correlation is statistically significant, what would the null hypothesis be?

Statistical Significance & Correlation

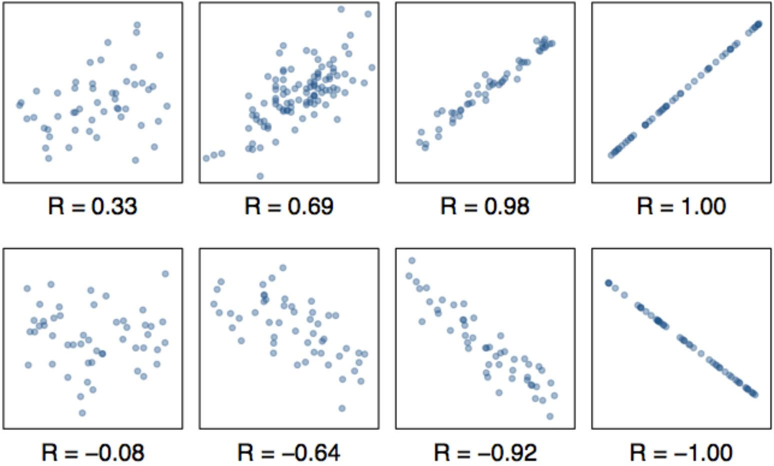
H_0: $R = 0$

H_a: R $\neq 0$

Hypothesis Testing Correlation Coefficients in R

```
cor.test(satGPA$SATSum, satGPA$FYGPA)

##
##  Pearson's product-moment correlation
##
## data:  satGPA$SATSum and satGPA$FYGPA
## t = 16.38, df = 998, p-value < 2.2e-16
## alternative hypothesis: true correlation is not equal to
## 95 percent confidence interval:
##  0.4100 0.5078
## sample estimates:
##    cor
## 0.4603
```

More Correlation Examples



R = 0.33

R = 0.69

R = 0.98

R = 1.00

R = –0.08

R = –0.64

R = –0.92

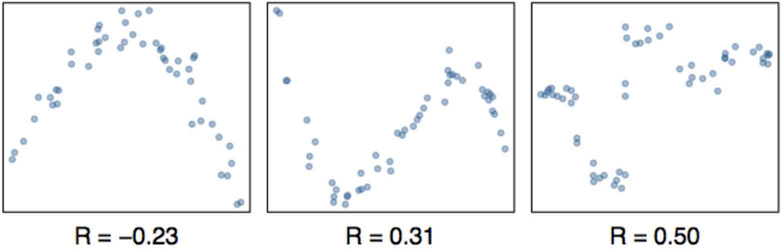
R = –1.00

Source: Diaz et al. (2011, 282)

Caution

A low linear correlation **does not necessarily** mean a weak relationship.

It means a weak **linear** relationship.



Source: Diaz et al. (2011, 282)

Best Fit Lines & Least Squares Regression

Ok, linear correlations are useful for finding:

- the **direction** of a linear relationship,
- the **strength** of a linear relationship.

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More specific

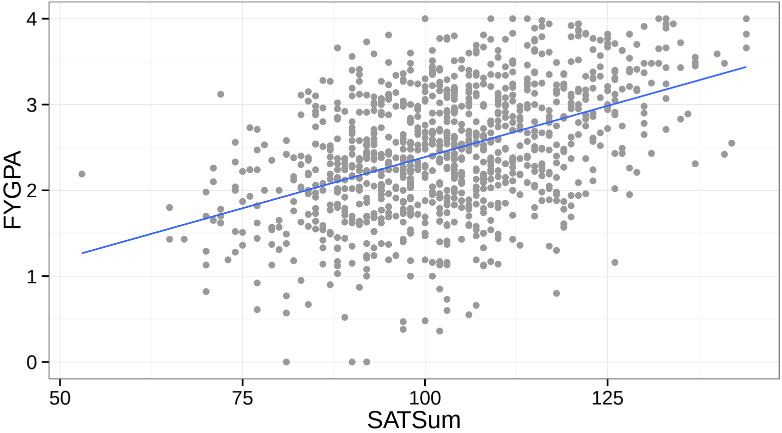
What if we want to be more specific?

For example, using a student's total SAT score to predict their first year university GPA.

Note: the estimated value of the dependent variable (y) is often written $\hat{y}$ (*"y hat"*).

The Linear Best Fit Line

The blue line is the closest straight line ("best fit") to all of the data points.



How?

How do we find the best fit line?

Residuals

Well, the best fit line would do something like have the smallest **residuals** possible.

What is a residual?

Residual:

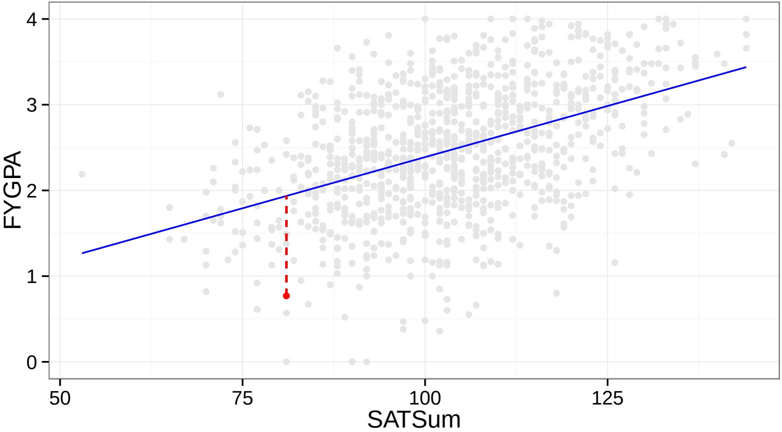
the difference between the observed and expected values based on the best fit model.

More formally: the residual (e_i) of the observation (x_i, y_i) is the difference between the observed value of y_i and the expected value $\hat{y}_i$:

$$e_i = y_i - \hat{y}_i$$

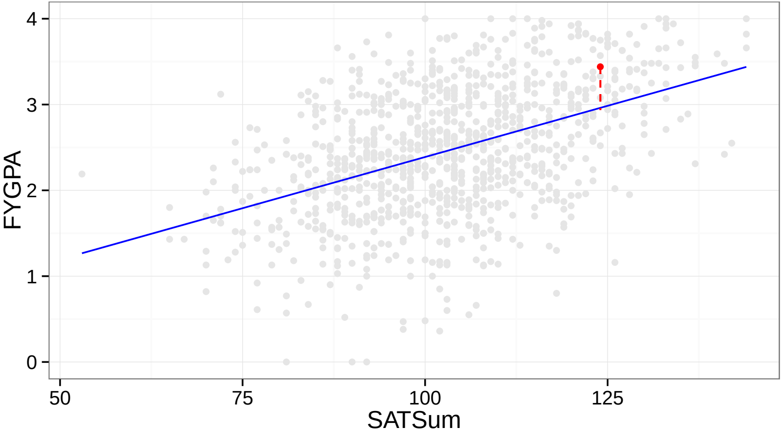
Residuals

The red point is at (81, 0.77). Given that SATSum is 81, it is expected to be at 1.935. So, it's residual is $0.77 - 1.935 = -1.65$.



Residuals

The red point is at (124, 3.44). Given that `SATSum` is 124, it is expected to be at 2.94. So, it's residual is $3.44 - 2.94 = 0.5$.



Aim

We want to find the line that gets us the smallest **sum of squared residuals** (SSR) where the SSR is:

$$SSR = e_1^2 + e_2^2 + \ldots + e_n^2$$

Um. . .

So where do the expected values ($\hat{y}$) come from?

$\hat{\beta}$

Remember the simple linear regression equation:

$$y = \alpha + \beta x$$

We want to find the $\hat{\beta}$ that **minimizes** the SSR.

Formally:

$$\hat{\beta} = \underset{b \in \mathbb{R}p}{\arg\min}\, \mathrm{SSR}$$

This is called **ordinary least squares simple linear regression.**

How?

How do we find the $\hat{\beta}$ that **minimizes** the SSR?

By Hand

One way to estimate the correlation coefficient parameter β for x is with the following equation:

$$\hat{\beta} = \frac{s_y}{s_x} R$$

Calculate $\hat{\beta}$

```
# Find standard errors
SDy <- sd(satGPA$FYGPA)
SDx <- sd(satGPA$SATSum)

# Find correlation
CorXY <- cor(satGPA$SATSum, satGPA$FYGPA)

# Estiamte correlation coefficient
BetaHat <- (SDy/SDx) * CorXY

BetaHat

## [1] 0.02387
```

Finding the intercept.

How can we find the intercept?

Finding the intercept ($\hat{\alpha}$).

You might remember from maths that we can find the whole equation for a line if we know:

- a point on the line.
- the slope.

We know the slope and we know that the point at the mean of x and y $(\bar{x}, \bar{y})$ will be on the line so we can use the equation for the **point-slope** form of a line:

$$y - \bar{y} = \hat{b}(x - \bar{x})$$

Finding the intercept ($\hat{\alpha}$).

If $\bar{y} = 2.468$, $\bar{x} = 103.329$, and $\hat{\beta} = 0.02387$, then:

$$y - \bar{y} = \hat{\beta}(x - \bar{x})$$

$$y - 2.468 = 0.02387(x - 103.329)$$

$$y - 2.468 = 0.02387x - 2.466463$$

$$y = 0.2387x + 0.001537$$

$$\widehat{FYGPA} = 0.001537 + 0.2387SATSum$$

This class

In this class we will let the computer find the $\hat{\beta}$ and $\hat{\alpha}$.

Linear Model

In R you can use the `lm` (linear model) command. For example,

```
M1 <- lm(FYGPA ~ SATSum, data = satGPA)

M1

##
## Call:
## lm(formula = FYGPA ~ SATSum, data = satGPA)
##
## Coefficients:
## (Intercept)       SATSum
##     0.00193      0.02387
```

The Regression Equation

So again, our estimated regression equation is:

$$\widehat{FYGPA} = 0.00193 + 0.02387 SATSum$$

Linear Regression Assumptions:

- The data follow a **linear trend**,
- Nearly **normally distributed residuals**,
- There is **constant variability**.

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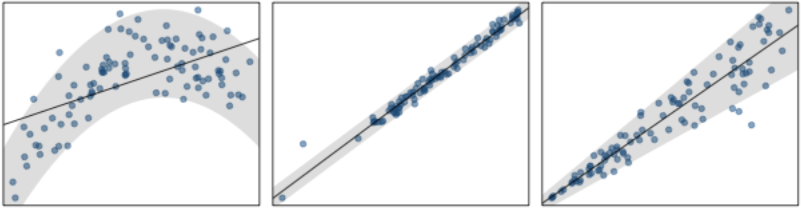
Example Assumption Violations

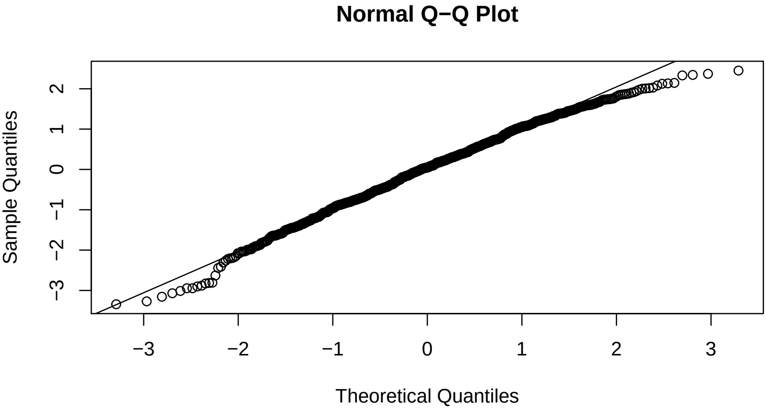
Which assumptions do these data violate?



Source: Diaz et al. (2011, 285)

To determine if the residuals in the model `M1` are normally distributed:

```
# Find standardized residuals
M1.residuals <- rstandard(M1)
# Create Quantile-Quantile Plot
qqnorm(M1.residuals)
qqline(M1.residuals)
```



Remember that $\hat{\beta}$ is a **point estimate** of the **population parameter** β.

How can we make **inferences** about β from $\hat{\beta}$?

Note: some people use b to refer to $\hat{\beta}$.

Hypotheses

What would our null and alternative hypotheses be?

Hypotheses

For our example, with a positive slope of 0.2387:

H_0: $\beta = 0$

H_a: $\beta > 0$

P-values for $\hat{\beta}$

We usually assume that the sampling distribution of $\hat{\beta}$ follows a t distribution.

Remember the equation for the $t\,test$ statistic:

$$T = \frac{\text{point estimate} - \text{null value}}{\text{SE}}$$

with $n - 2$ degrees of freedom.

The procedure is the same as before to find the p-value.

Model Summary

```
# Summarize M1 model output
summary(M1)
```

```
##
## Call:
## lm(formula = FYGPA ~ SATSum, data = satGPA)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.1976 -0.4495  0.0315  0.4557  1.6115
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.00193    0.15199    0.01     0.99
## SATSum       0.02387    0.00146   16.38   <2e-16
##
## (Intercept)
## SATSum      ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.658 on 998 degrees of freedom
## Multiple R-squared: 0.212, Adjusted R-squared: 0.211
## F-statistic:  268 on 1 and 998 DF,  p-value: <2e-16
```

Inference & p-values

So, we find evidence against the null hypothesis that the slope of the line summarizing the relationship between first year university grades and SAT total scores is 0 in the population.

Note: the p-vale given by `summary` is based on a **two-sided** hypothesis. If we have a one sided hypothesis we can **halve** the p-value.

In our example this would be impractical, since the p-value is already so small.

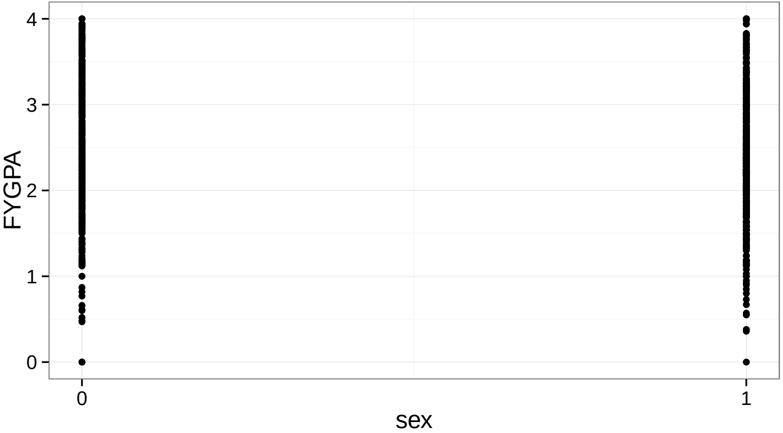
Dummy Variables

So far we have only looked at creating simple linear regression models with **continuous numeric** dependent and independent variables.

What if we have a **continuous dependent** variable and a **dichotomous (dummy) independent** variable?

Dichotomous Independent Variables

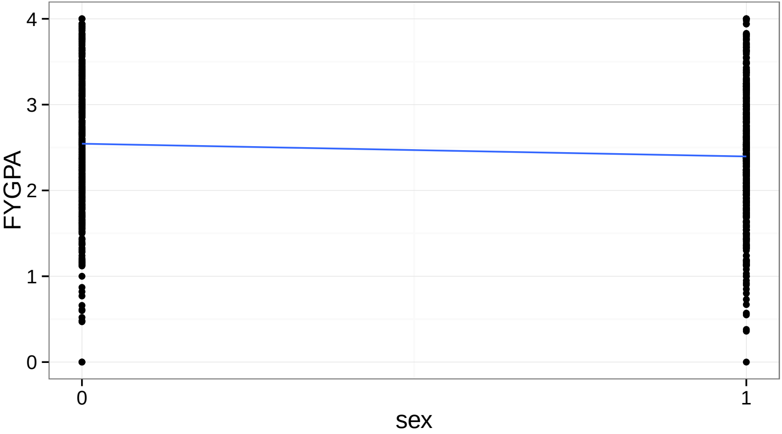
For example:



Note, I recoded the values of the original variable.

Dichotomous Independent Variables

β is pretty similar. It is still the slope of the line for a one unit change in x. The only difference is that the variable only goes between 0 and 1.



Note, I recoded the values of the original variable.

Categorical Dependent

What if our **dependent variable** is categorical, for example, the party someone voted for?

For these situations you need to use a different type of regression, for example **logistic regression**.

We do not cover this type of regression in this course.

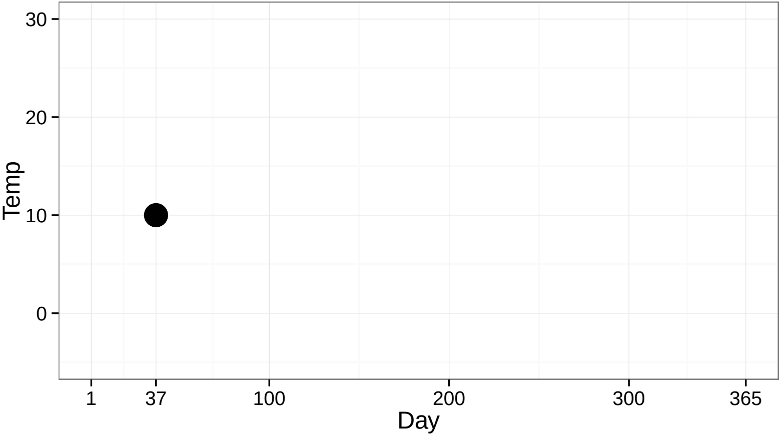
Extrapolation

Be careful about **extrapolating** beyond your data.

We don't know how the data beyond what we observe will behave.

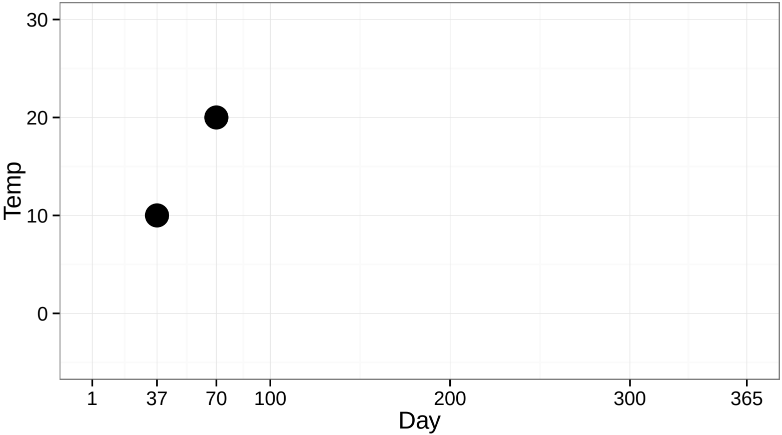
Extrapolation

On the 37th day of the year it was 10 degrees.



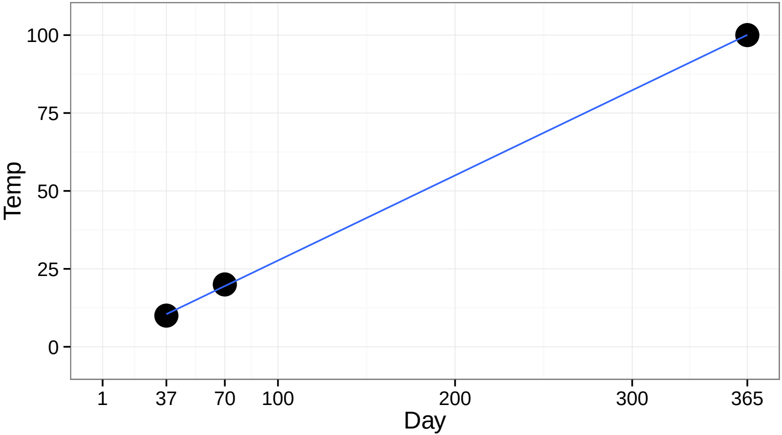
Extrapolation

On the 70th day of the year it was 20 degrees.



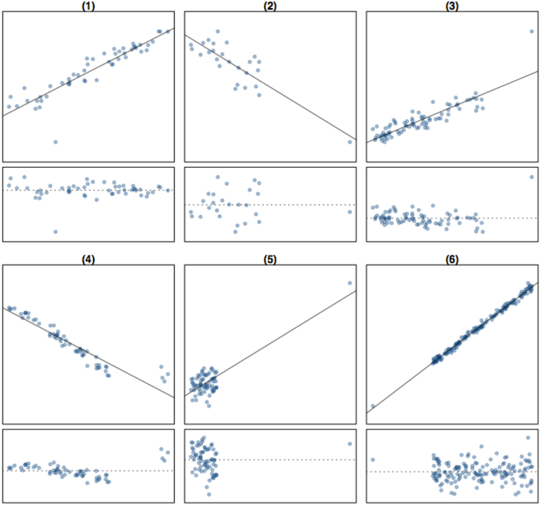
Extrapolation

So on the last day of the year it will be about 100 degrees?.



Outliers

Be careful about outliers.



Source: Diaz et al. (2011, 290)

Removing Outliers

Only remove outliers if you have a **good reason** to.

Try to find out **substantively** why they are outliers.

References I

Crawley, Michael J. 2005. Statistics: An Introduction Using R. Chichester: John Wiley Sons. Ltd.

Diaz, David M., Christopher D. Barr, and Mine Çetinkaya-Rundel. 2011. OpenIntro Statistics. 1st ed. `http://www.openintro.org/stat/downloads.php`.