V.V.V.V Imp (10m)

* Solving recurrence relations by Characteristic roots:

Def: Let $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k} = 0$, $n \geq k$, $c_k \neq 0$ be a linear recurrence relation of degree 'k'. Then the equation

$$r^k + c_1 r^{k-1} + \cdots + c_k = 0$$

is said to be the characteristic equation of the given recurrence relation

* If the characteristic equation of a linear homogenous recurrence relation of degree 'k' has 'k' distinct roots $r_1, r_2 \ldots r_k$ then

$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots c_k r_k^n, \ c_1, c_2 \ldots c_k \text{ are constants}$$

* If the characteristics eqn of a linear homogenous recurrence relation of degree 'k' has a root 'k' repeated k-times then,

$$a_n = (D_1 + D_2 n + D_3 n^2 + \cdots D_k n^{k-1}) r^n,$$

where $D_1, D_2 \ldots D_k$ are constants.

Q. Solve $a_n - 7a_{n-1} + 12a_{n-2} = 0$ for $n \geq 2$

Sol: Characteristic eqn.

$$r^2 + (-7)r + (12)(r^0) = 0$$

$$r^2 - 7r + 12 = 0$$

$$r^2 - 3r - 4r + 12 = 0$$

$$r = 3, 4$$

$$a_n = c_1 r_1^n + c_2 r_2^n$$

$$= c_1 3^n + c_2 4^n$$

Q. [Solve $a_n - 5a_{n-1} + 6a_{n-2} = 0$ where $a_0 = 2, a_1 = 5$

Sol: Characteristic eqn

$$r^2 - 5r + 6 = 0$$

$$r = 2, 3$$

$$\boxed{a_n = c_1 2^n + c_2 3^n}$$

$$a_0 = c_1 2^0 + c_2 3^0 \qquad a_1 = c_1 2^1 + c_2 3^1$$

$$2 = c_1 + c_2 \quad \text{--- (1)} \times 2 \qquad 5 = 2c_1 + 3c_2 \quad \text{--- (1)}$$

Solve (1) & (2)

$$\cancel{2c_1} + 2c_2 = 4$$
$$\cancel{2c_1} + 3c_2 = 5$$
$$-c_2 = -1$$
$$\boxed{c_2 = 1}$$

$$c_1 + c_2 = 2$$
$$c_1 + 1 = 2$$
$$\boxed{c_1 = 1}$$

$$a_n = 1.2^n + 1.3^n$$
$$a_n = 2^n + 3^n \;]$$

Q. Solve $a_n - 6a_{n-1} + 9a_{n-2} = 0$

Sol: characteristics eqn

$$r^2 - 6r + 9 = 0$$

$$r = 3, 3$$

$$a_n = (D_1 + D_2 n) r^n$$

$$(D_1 + D_2 n) 3^n$$

$a_0 = 2, a_1 = 5$

$C_2 3^1$

$3C_2$ — 1

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Q. [Solve $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$, $a_0 = 1$, $a_1 = 4$, $a_2 = 8$

Sol: characteristic eqn

$$r^3 - 7r^2 + 16r - 12 = 0$$

$$r = 2, 2, 3$$

$$a_n = (c_1 + c_2 n)2^n + c_3 3^n \,]$$

$$a_0 = (c_1 + c_2(0))2^0 + c_3 3^0$$

$$1 = c_1 + c_3 \quad \text{—— (1)} \times 4$$

$$a_1 = (c_1 + c_2)2^1 + c_3 3^1$$

$$4 = 2c_1 + 2c_2 + 3c_3 \quad \text{—— (2)} \times 4$$

$$a_2 = (c_1 + 2c_2)2^2 + c_3 3^2$$

$$8 = 4c_1 + 8c_2 + 9c_3 \quad \text{—— (3)}$$

Solve (2) & (3)

$$8c_1 + \cancel{8c_2} + 12c_3 = 16$$
$$\underline{-\,4c_1 \cancel{+8c_2} + 9c_3 = 8}$$
$$4c_1 + 3c_3 = 8 \quad \text{—— (4)}$$

Solve (1) & (4)

$$\cancel{4c_1} + 4c_3 = 4$$
$$\underline{-\,\cancel{4c_1} + 3c_3 = 8}$$
$$\boxed{c_3 = -4}$$

$$c_1 + c_3 = 1 \qquad\qquad 2c_1 + 2c_2 + 3c_3 = 4$$

$$c_1 - 4 = 1 \qquad\qquad 10 + 2c_2 - 12 = 4$$

$$\boxed{c_1 = 5} \qquad\qquad 2c_2 = 6$$

$\therefore a_n = [5 + \quad)2^n + (-4)3^n$

Q. Q. Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $n \geq 2$, $a_0 = 10$, $a_1 = 41$

Sol: Sol:

$$r^2 - 7r + 10 = 0$$

$$r = 5, 2$$

$$a_n = c_1 5^n + c_2 2^n$$

$$a_0 = c_1 5^0 + c_2 2^0$$

$$10 = c_1 + c_2 \quad \text{--- (1)} \times 2$$

$$a_1 = c_1 5^1 + c_2 2^1$$

$$41 = 5c_1 + 2c_2 \quad \text{--- (2)}$$

$\frac{4}{2}$
2

Solve

$$2c_1 + \cancel{2c_2} = 20$$

$$-5c_1 \mp \cancel{2c_2} = -41$$

$$-3c_1 = -21$$

$$\boxed{c_1 = 7}$$

$$c_1 + c_2 = 10$$

$$7 + c_2 = 10$$

$$\boxed{c_2 = 3}$$

$$\therefore a_n = 7(5^n) + 3(2^n)$$

Q. Q. Solve $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$, $n \geq 3$

Sol: $a_0 = 0$, $a_1 = 1$, $a_2 = 10$

Sol:

$$r^3 - 9r^2 + 26r - 24 = 0$$

$$r = 2, 3, 4$$

$$a_n = c_1 2^n + c_2 3^n + c_3 4^n$$

$$a_0 = c_1 2^0 + c_2 3^0 + c_3 4^0$$

$0 = c_1 + c_2 + c_3$ — (1) $\times 2$

$a_1 = c_1 2^1 + c_2 3^1 + c_3 4^1$

$1 = 2c_1 + 3c_2 + 4c_3$ — (2) $\times 2$

$a_2 = c_1 2^2 + c_2 3^2 + c_3 4^2$

$10 = 4c_1 + 9c_2 + 16c_3$ — (3)

$$\begin{array}{l} \cancel{2c_1} + 2c_2 + 2c_3 = 0 \\ \cancel{2c_1} + 3c_2 + 4c_3 = 1 \\ \hline -c_2 - 2c_3 = -1 \end{array}$$ — (4) $\times 3$

$$\begin{array}{l} \cancel{4c_1} + 6c_2 + 8c_3 = 2 \\ \cancel{4c_1} + 9c_2 + 16c_3 = 10 \\ \hline -3c_2 - 8c_3 = -8 \end{array}$$ — (5)

$$\begin{array}{l} \cancel{-3c_2} - 6c_3 = -3 \\ \cancel{-3c_2} - 8c_3 = -8 \\ \hline 2c_3 = 5 \end{array}$$

$\boxed{c_3 = 5/2}$

$-c_2 - 2(5/2) = -1$

$-c_2 = 4$

$\boxed{c_2 = -4}$

$c_1 + c_2 + c_3 = 0$

$c_1 - 4 + 5/2 = 0$

$c_1 = \frac{8-5}{2}$

$\boxed{c_1 = 3/2}$

$a_n = 3/2 \cdot 2^n - 4 \cdot 3^n + 5/2 \cdot 4^n$

. Solve $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$

l:

$r^3 - 8r^2 + 21r - 18 = 0$

$r = 2, 3, 3$

$\cancel{a_n = c_1 2^n + (c_2 3^n}$

$a_n = c_1 2^n + (c_2 + c_3 n) 3^n.$

Q. Q. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0, n \geq 3$

Sol Sol:

$$r^3 - 6r^2 + 12r - 8 = 0$$

$$r = 2, 2, 2$$

$$a_n = (c_1 + c_2 n + c_3 n^2) 2^n$$

* Solving Recurrence relation by generating functio

Equivalent expressions used for this are

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ then

1. $\sum_{n=k}^{\infty} a_n x^n = A(x) - a_0 - a_1 x - \cdots - a_{k-1} \cdot x^{k-1}$

2. $\sum_{n=k}^{\infty} a_{n-1} x^n = x\left[A(x) - a_0 - a_1 x \cdots a_{k-2} x^{k-2}\right]$

3. $\sum_{n=k}^{\infty} a_{n-2} x^n = x^2\left[A(x) - a_0 - a_1 x \cdots a_{k-3} x^{k-3}\right]$

$\vdots$

$$\sum_{n=k}^{\infty} a_{n-k} \cdot x^n = x^k\left[A(x)\right]$$

where $A(x)$ is called a generating function for a given recurrence relation.

Q.

Sol Solve $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ by using generating functions

Sol: Step 1: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

Step 2: Multiply by x^n and sum from 2 to ∞ to the given expression

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Step 3: Replace each infinite sum by an expression from the equivalent expression

$$[A(x) - a_0 - a_1 x] - 7[x[A(x) - a_0]] + 10x^2[A(x)] = 0$$

$$A(x) - a_0 - a_1 x - 7x A(x) + 7x a_0 + 10x^2 A(x) = 0$$

$$A(x)[1 - 7x + 10x^2] - a_0 - a_1 x + 7a_0 x = 0$$

$$A(x) = \frac{a_0 + a_1 x - 7a_0 x}{(1 - 7x + 10x^2)}$$

$$= \frac{a_0(1-7x) + a_1 x}{(1-2x)(1-5x)}$$

$$A(x) = \frac{c_1}{1-2x} + \frac{c_2}{1-5x}$$ 9986181466

where c_1 & c_2 are constants

$$A(x) = \frac{c_1}{1-2x} + \frac{c_2}{1-5x}$$

$$= c_1 \sum_{n=0}^{\infty} 2^n x^n + c_2 \sum_{n=0}^{\infty} 5^n x^n$$

$$a_n = c_1 2^n + c_2 5^n$$

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Q. Solve $a_n - 7a_{n-1} + 12a_{n-2} = 0$, $n \geq 2$

Sol:

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

Multiply with x^n & $\sum_{n=2}^{\infty}$

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 12 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\left[A(x)-a_0-a_1x\right]-7\left[x\left[A(x)-a_0\right]\right]+12x^2A(x)=0$$

$$A(x)-a_0-a_1x-7xA(x)+7xa_0+12x^2A(x)=0$$

$$A(x)\left[1-7x+12x^2\right]-a_0-a_1x+7xa_0=0$$

$$A(x)=\frac{a_0+a_1x-7xa_0}{1-7x+12x^2}$$

$$A(x)=\frac{a_0+a_1x-7xa_0}{(1-3x)(1-4x)}$$

$$A(x)=\frac{c_1}{1-3x}+\frac{c_2}{1-4x}$$

$$a_n=c_1\sum_{n=0}^{\infty}3^nx^n+c_2\sum_{n=0}^{\infty}4^nx^n$$

$$a_n=c_13^n+c_24^n \,]$$

Q. Solve $a_n-4a_{n-1}+3a_{n-2}=0$, $n\geq 2$, $a_0=2, a_1=4$

Sol: $$A(x)=\sum_{n=0}^{\infty}a_nx^n$$

$$\sum_{n=2}^{\infty}a_nx^n-4\sum_{n=2}^{\infty}a_{n-1}x^n+3\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$\left[A(x)-a_0-a_1x\right]-4\left[x\left[A(x)-a_0\right]\right]+3\left[x^2A(x)\right]=0$$

$$A(x)-a_0-a_1x-4xA(x)+4xa_0+3x^2A(x)=0$$

$$A(x)\left[1-4x+3x^2\right]-a_0-a_1x+4xa_0=0$$

$$A(x) = \frac{a_0 + a_1 x - 4x a_0}{(1-1x)(1-3x)}$$

$$A(x) = \frac{2+4x-8x}{(1-x)(1-3x)} = \cancel{\frac{2-4x}{1-4x+3x^2}}$$

$$= \frac{2-4x}{(1-x)(1-3x)}$$

$$= \frac{1}{1-x} + \frac{1}{1-3x}$$

$$\cancel{c_1 \sum_{n=0}^{\infty} x^n + c_2 \sum_{n=0}^{\infty} (3)^n x^n}$$

$$\cancel{a_n} = c_1 + c_2 (3)^n$$

$$\cancel{= \frac{c_1 \cdot 3^n + c_2}{3^n}}$$

$$A(x) = \sum_{n=0}^{\infty} 1^n + \sum_{n=0}^{\infty} 3^n$$

$$a_n = 1^n + 3^n$$

questions may

$$a_n = 1 + 3^n$$

$a_n = c_1 1^n + c_2 \cdot 3^n$

$a_0 = 2 = c_1 + c_2$

$a_1 = 4 = c_1 + 3c_2$

$c_1 + c_2 = 2$

$c_1 + 3c_2 = 4$

$-2c_2 = -2$

$c_2 = 1$

$c_1 = 1$

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* Solution of inhomogenous linear recurrence relations:

$a_0 + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n)$, for $n \geq k$, where $c_k \neq 0$ and $f(n)$ is some specified function of n. It is called Inhomogenous recurrence relation. It can be solved by using characteristic roots.

* A solution which satisfies the recurrence relation when the right hand side of the eqn

is set to '0', is called homogeous solution.

* A solution which satisfies the recurrence relatio with $f(n)$ on the right hand side is called particular solution.

* The homogeneous sol^n is denoted by $a_n^{(h)}$ and the particular sol^n is denoted by $a_n^{(P)}$

$\therefore$ General sol^n = Homogeneous sol^n + particular sol^n

$$a_n = a_n^{(h)} + a_n^{(P)}$$

Rule 1: If $f(n)$ is of the form of a polynomial of degree m in n (i.e.)

$(b_0+b_1 n + b_2 n^2 + \ldots \ldots b_m n^m)$, then the particula sol^n will be

$(Q_0 + Q_1 n + Q_2 n^2 + \ldots \ldots + Q_m n^m)$, provided 1 (one) is not the root

Rule 2: If $f(n)$ is of the form $(b_0+b_1 n + b_2 n^2 \ldots b_m n^m)$ then the particular sol will be

$(Q_0 + Q_1 n + Q_2 n^2 + \ldots \ldots Q_m b^m) a^n$, when a is n the root

Rule 3: If a is the characteristic root of the multiplic r, when $f(n)$ is of the form $(b_0 + b_1 n + \ldots\ldots b_m n^m) a^n$, then

$n(Q_0 + Q_1 n + \ldots \ldots Q_m n^m) a^n$

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Q. Solve $a_n - 7a_{n-1} + 10 a_{n-2} = 0$ by generating function $a_0 = 10$, $a_1 = 41$

Q. Sol: $A(x) - a_0 - a_1 x - 7x[A(x) - a_0] + 10x^2 A(x) = 0$

$A(x) - a_0 - a_1 x - 7x A(x) + 7x a_0 + 10x^2 A(x) = 0$

$A(x)[1 - 7x + 10x^2] - a_0 - a_1 x + 7x a_0 = 0$

$$A(x) = \frac{a_0 + a_1 x - 7x a_0}{[1 - 7x + 10x^2]}$$

$$A(x) = \frac{10 + 41x - 70x}{(1-2x)(1-5x)}$$

$$A(x) = \frac{10 - 29x}{(1-2x)(1-5x)}$$

$10 - 29x = A(1-5x) + B(1-2x)$

$A = 3 \quad B = 7$

$a_n = c_1 2^n + c_2 5^n$

$a_0 = 10 \Rightarrow c_1 2^0 + c_2 5^0$

$\Rightarrow c_1 + c_2 = 10$ — (1)

$a_1 = 41 \Rightarrow 2c_1 + 5c_2$

$\Rightarrow 2c_1 + 5c_2 = 41$ — (2)

Solve (1) x (2)

$c_1 = 3, \; c_2 = 7$

$a_n = 3 \cdot 2^n + 7 \cdot 5^n$

Q. $a_n + 3a_{n-1} - 10a_{n-2} = 0, \; n \geq 2, \; a_0 = 1, \; a_1 = 4$

Sol: $A(x) - a_0 - a_1 x + 3x[A(x) - a_0] - 10x^2 A(x) = 0$

$A(x)[1 + 3x - 10x^2] = a_0 + a_1 x + 3x a_0$

$$A(x) = \frac{1 + 4x + 3x}{1 + 3x - 10x^2} = \frac{1 + 7x}{1 + 3x - 10x^2}$$

$$A(x) = \frac{1 + 7x}{(1+5x)(1-2x)}$$

$a_n = c_1(-5)^n + c_2 2^n$

$a_0 = 1 = c_1 + c_2$

$a_1 = 4 = -5c_1 + 2c_2$

$c_1 = -2/7 \quad c_2 = 9/7$

$a_n = -2/7 \cdot (-5)^n + 9/7 \cdot 2^n$

Q. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

Sol: $A(x) - a_0 - a_1x - a_2x^2 - 6x[A(x) - a_0 - a_1x] + 12x^2[A(x) - a_0] - 8x^3A(x) = 0$

$A(x)[1 - 6x + 12x^2 - 8x^3] - a_0 - a_1x - a_2x^2 + 6xa_0 + 6x^2a_1 - 12x^2a_0$

$$A(x) = \frac{a_0 + a_1x + a_2x^2 - 6xa_0 - 6x^2a_1 + 12x^2a_0}{1 - 6x + 12x^2 - 8x^3}$$

$$\begin{array}{c|cccc} 2 & 1 & -6 & 12 & -8 \\ & 0 & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

$$A(x) = \frac{a_0 + a_1x + a_2x^2 - 6xa_0 - 6x^2a_1 + 12x^2a_0}{(1-2x)^3}$$

$x = 2, 2, 2$

$a_n = (c_1 + c_2n + c_3n^2)2^n$

Q. Solve $a_n - 9a_{n-1} + 20a_{n-2} = 1$

Sol: Homogeneous sol^n $a_n^{(h)}$

$a_n - 9a_{n-1} + 20a_{n-2} = 0$

$r = 4, 5$

$a_n^{(h)} = c_1 4^n + c_2 5^n$

1 is not the root of the eqn

Apply rule 1

$f(n) = 1$

$$Q_0 = Q$$

$$Q - 9Q + 20Q = 1$$

$$12Q = 1$$

$$Q = 1/12$$

$$a_n^{(p)} = \frac{1}{12}$$

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1 4^n + c_2 5^n + 1/12$$

Q. Solve $a_n - a_{n-1} - 6a_{n-2} = 30$, $a_0 = 20$, $a_1 = 5$

Sol: Homogeneous solⁿ $a_n^{(h)}$

$$a_n - a_{n-1} - 6a_{n-2} = 0$$

$$r = -2, 3$$

$$a_n^{(h)} = c_1(-2)^n + c_2 3^n$$

1 is not the root of the eqn

Apply Rule I

$$f(n) = 1$$

$$Q_0 = Q$$

$$Q - \cancel{1}Q - 6Q = 30$$

$$Q = -5$$

$$a_n^{(p)} = -5$$

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1(-2)^n + c_2 3^n - 5$$

$$a_0 = 20 = c_1 + c_2 - 5$$

$$c_1 + c_2 = 25 \quad \text{——} \ (1) \times 3$$

$$a_1 = 5 = -2c_1 + 3c_2 - 5$$

$$-2c_1 + 3c_2 = 10 \quad \text{——} \ (2)$$

Solve (1) & (2)

$$3c_1 + \cancel{3c_2} = 75$$

$$\underline{-2c_1 \pm \cancel{3c_2} = \pm 10}$$

$$5c_1 = 65$$

$$c_1 = 13$$

$$c_2 = 25 - 13$$

$$c_2 = 12$$

$$a_n = 13(-2)^n + 12(3)^n - 5$$

Q. $a_n - 7a_{n-1} + 10a_{n-2} = 6+8n$, $a_0 = 1$, $a_1 = 2$

Sol:

$$r = 2, 5$$

$$a_n^{(h)} = c_1 2^n + c_2 5^n$$

1 is not the root of the eqn

Apply Rule I

$$f(n) = 6 + 8n$$

$$a_n = Q_0 + Q_1 n$$

$$a_{n-1} = Q_0 + Q_1(n-1)$$

$$a_{n-2} = Q_0 + Q_1(n-2)$$

$$Q_0 + Q_1 n - 7(Q_0 + Q_1(n-1)) + 10(Q_0 + Q_1(n-2)) = 6+8n$$

$$Q_0 + Q_1 n - 7Q_0 - 7Q_1 n + 7Q_1 + 10Q_0 + 10Q_1 n - 20Q_1 = 6+8n$$

$$4Q_0 + n(Q_1 - 7Q_1 + 10Q_1) - 13Q_1 = 6+8n$$

$$4Q_0 - 13Q_1 + 4Q_1 n = 6 + 8n$$

$$4Q_0 - 13Q_1 = 6 \qquad 4Q_1 = 8$$

$$4Q_0 - 26 = 6 \qquad Q_1 = 2$$

$$4Q_0 = 32$$

$$Q_0 = 8$$

$$a_n^{(p)} = Q_0 + Q_1 n$$

$$a_n^{(p)} = 8 + 2n$$

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1 2^n + c_2 5^n + 8 + 2n$$

$$a_0 = 1 = c_1 + c_2 + 8 \qquad a_1 = \cancel{2} = 2c_1 + 5c_2 + 8 + \cancel{2}$$

$$c_1 + c_2 = -7 \quad \text{—(1)} \qquad 2c_1 + 5c_2 = -8 \quad \text{—(2)}$$

Solve ① × ②

$\cancel{2c_1} + 2c_2 = 14$

$\underline{-\cancel{2c_1} \pm 5c_2 = \mp 8}$

$-3c_2 = -6$

$\boxed{c_2 = 2}$

$c_1 = -7 - 2$

$\boxed{c_1 = -9}$

$a_n = (-9)2^n + (2)5^n + 8 + 2n$

Q. $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

Sol:

$r = -2, -3$

$a_n^{(h)} = c_1(-2)^n + c_2(-3)^n$

1 is not the root

$f(n) = Q_0 + Q_1 n + Q_2 n^2$

$a_n = Q_0 + Q_1 n + Q_2 n^2$

$a_{n-1} = Q_0 + Q_1(n-1) + Q_2(n-1)^2$

$a_{n-2} = Q_0 + Q_1(n-2) + Q_2(n-2)^2$

$Q_0 + Q_1 n + Q_2 n^2 + 5\left(Q_0 + Q_1(n-1) + Q_2 \overbrace{(n-1)^2}^{n^2+1-2n}\right) +$

$6\left(Q_0 + Q_1(n-2) + Q_2 \underbrace{(n-2)^2}_{n^2+4-4n}\right) = 3n^2 - 2n + 1$

$Q_0 + Q_1 n + Q_2 n^2 + 5Q_0 + 5Q_1 n \boxed{- 5Q_1} + 5Q_2 n^2 + 5Q_2$

$-10Q_2 n + 6Q_0 + 6Q_1 n \boxed{-12Q_1} + 6Q_2 n^2 +$

$24Q_2 - 24Q_2 n = 3n^2 - 2n + 1$

$12Q_0 - 17Q_1 + 29Q_2 + n(Q_1 + 5Q_1 - 10Q_2 + 6Q_1 - 24Q_2)$

$+ n^2(Q_2 + 5Q_2 + 6Q_2) = 3n^2 - 2n + 1$

$12Q_0 - 17Q_1 + 29Q_2 = 1$

$12Q_0 - 17\left(\frac{13}{24}\right) + \frac{29}{4} = 1$

$Q_0 = \frac{17}{288}$

$12Q_1 - 34Q_2 = -2$

$12Q_1 - \frac{34}{4} = -2$

$12Q_1 = -2 + \frac{34}{4}$

$= \frac{-8+34}{4}$

$Q_1 = \frac{26^{13}}{4 \times 12_{2}}$

$Q_1 = \frac{13}{24}$

$12Q_2 = 3$

$Q_2 = \frac{1}{4}$

$$a_n^{(p)} = Q_0 + Q_1 n + Q_2 n^2$$

$$= \frac{17}{288} + \frac{13n}{24} + \frac{n^2}{4}$$

$$a_n = a_n^{(h)} + a_n^{(p)} = C_1(-2)^n + C_2(-3)^n + \frac{17}{288} + \frac{13n}{24} + \frac{n^2}{4}$$

Q. Solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$

Sol: $a_n - 7a_{n-1} + 10a_{n-2} = 0$

$r = 2, 5$

$a_n^{(h)} = C_1 2^n + C_2 5^n$

4 is not root, apply rule 2

$f(n) = Q \cdot a^n = Q \cdot 4^n$

$a_n = Q \cdot 4^n$

$a_{n-1} = Q \cdot 4^{n-1}$

$a_{n-2} = Q \cdot 4^{n-2}$

$$Q\cdot 4^n - 7(Q\cdot 4^{n-1}) + 10(Q\cdot 4^{n-2}) = 4^n$$

$$\cancel{4^{n-2}}\, Q[16-28+10] = 4^2\cdot \cancel{4^{n-2}}$$

$$-2Q = 16$$

$$Q = -8$$

$$a_n = -8(4^n)$$

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1 2^n + c_2 5^n - 8(4^n)$$

Q. $a_n + 5a_{n-1} + 6a_{n-2} = 42(4^n)$

Sol: $r = -2, -3$

$$a_n^{(h)} = c_1(-2)^n + c_2(-3)^n$$

$$f(n) = Q\cdot(4^n)$$

$$a_n = Q\cdot(4^n)$$

$$a_{n-1} = Q\cdot(4^{n-1})$$

$$a_{n-2} = Q\cdot(4^{n-2})$$

20
16
6
42

$$Q\cdot(4^n) + 5(Q\cdot(4^{n-1})) + 6(Q\cdot(4^{n-2})) = 42\cdot 4^n$$

$$Q\cdot 4^{n-2}[16 + 5(4) + 6] = 42\cdot 4^n$$

$$Q\cdot \cancel{4^{n-2}}[\cancel{42}] = \cancel{42}\cdot 4^2\cdot \cancel{4^{n-2}}$$

$$Q = 16$$

$$a_n = 16\cdot 4^n$$

$$a_n = a_n^{(h)} + a_n^{(p)} = c_1(-2)^n + c_2(-3)^n + 16\cdot 4^n$$

Q. Solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$

Sol:

$$r = 2, 5$$

$$a_n^{(h)} = c_1 2^n + c_2 \cdot 5^n$$

4 is not root

Apply rule 2

$$F(n) = Q \cdot 4^n$$

$$a_n = Q \cdot 4^n$$

$$a_{n-1} = Q \cdot 4^{n-1}$$

$$a_{n-2} = Q \cdot 4^{n-2}$$

$$Q \cdot 4^n - 7 \cdot Q 4^{n-1} + 10 \cdot Q \cdot 4^{n-2} = 4^n$$

Q. Solve $a_n + a_{n-1} = 3n \cdot 2^n$

Sol:

$$r + 1 = 0$$

$$r = -1$$

$$a_n^{(h)} = c_1(-1)^n$$

2 is not root

Apply rule 2

$$a_n = (Q_0 + Q_1 n) a^n$$

$$a_n = (Q_0 + Q_1 n) 2^n$$

$$a_{n-1} = (Q_0 + Q_1(n-1)) 2^{n-1}$$

$$\times \; a_{n-2} = (Q_0 + Q_1(n-2)) 2^{n-2}$$

$$(Q_0 + Q_1 n) 2^n + (Q_0 + Q_1(n-1)) 2^{n-1} = 0 + 3n \cdot 2^n$$

$$\cancel{2^{n-1}} \left[[2Q_0 + 2Q_1 n] + [Q_0 + Q_1 n - Q_1] \right] = 3n \cdot \cancel{2^{n-1}} \cdot 2$$

$$3Q_0 + \underline{3Q_1 n} - Q_1 = \underline{6n}$$

$\boxed{Q_0 = 2/3}$ $3Q_0 - Q_1 = 0$ $3Q_1 = 6$

$3Q_0 - 2 = 0$ $\boxed{Q_1 = 2}$

$$a_n^{(P)} = \left(2/3 + 2n\right) 2^n$$

$$\therefore a_n = a_n^{(h)} + a_n^{(P)} = (-1)^n c_1 + \left(\frac{2}{3} + 2n\right) 2^n$$

Q. Solve $a_n - 3a_{n-1} - 4a_{n-2} = 4^n$

Sol:

$$-4 + 1$$

$$r = 4, -1$$

$$a_n^{(h)} = c_1 (4)^n + c_2 (-1)^n$$

4 is the root

Apply rule 3

$$F(n) = n \cdot Q (4^n)$$

$$a_n = n Q 4^n$$

$$a_{n-1} = (n-1) Q 4^{n-1}$$

$$a_{n-2} = (n-2) Q 4^{n-2}$$

$$nQ4^n - 3(n-1)Q4^{n-1} - 4(n-2)Q4^{n-2} = 4^n$$

$$nQ4^n - 3nQ4^{n-1} + 3Q4^{n-1} - 4nQ4^{n-2} + 8Q4^{n-2} = 4^n$$

$$\cancel{3Q4^{n-1} + 8Q4^{n-2}}$$

$$\cancel{4^{n-2}} \left[16\cancel{n}Q - 12\cancel{n}Q + 12Q - \cancel{4nQ} + 8Q\right] = \cancel{4^{n-2}} \cdot 4^2$$

$$20Q = 16$$

$$\boxed{Q = 4/5}$$

$$a_n^{(P)} = n \cdot (4/5) \cdot 4^n$$

$$a_n = a_n^{(h)} + a_n^{(P)} = c_1 (4)^n + c_2 (-1)^n + n \cdot \frac{4}{5} \cdot 4^n$$

Q. Solve $a_n - 6a_{n-1} + 8a_{n-2} = n \cdot 4^n$, $a_0 = 8$, $a_1 = 22$

<u>Sol:</u>

$r = 2, 4$

$a_n^{(h)} = c_1 2^n + c_2 4^n$

4 is root

Apply rule 3

$a_n = (Q_0 + Q_1 n) 4^n \cdot n$

$a_{n-1} = (Q_0 + Q_1(n-1)) 4^{n-1} (n-1)$

$a_{n-2} = (Q_0 + Q_1(n-2)) 4^{n-2} \cdot (n-2)$

$$\begin{array}{r} 32 \\ 24 \\ \hline 56 \end{array}$$

$n(Q_0 + Q_1 n) 4^n - 6(Q_0 + Q_1(n-1)) 4^{n-1}(n-1)) + 8(Q_0 + Q_1(n-2)) 4^{n-2} (n-2)$

$= n4^n$

$\cancel{4^{n-2}} [\cancel{16Q_0 n} + \cancel{16n^2 Q_1} - \cancel{24Q_0 n} - \cancel{24Q_1 n^2} + \cancel{24}^{8} Q_1 n +$
$24Q_0 + \cancel{24}^{8} Q_1 n - 24Q_1 +$
$\cancel{8Q_0 n} + \cancel{8Q_1 n^2} - \cancel{16}^{8} Q_1 n - 16Q_0 -$
$-\cancel{16}^{8} Q_1 n + 32Q_1] = n \cdot \cancel{4^{n-2}} \cdot 4^2$

$16Q_1 n + 8Q_1 - 8Q_0 = 16n$

$16Q_1 = 16$

$\boxed{Q_1 = 1}$

$8Q_1 + 8Q_0 = 0$

$\cancel{56 - 16Q_0 = 0}$

$\cancel{16Q_0 = 56}$

$\cancel{Q_0 = \frac{56}{16}}$

$\cancel{8 + 8Q_0 = 0}$

$\cancel{8 = -8Q_0}$

$\cancel{Q_0 = -1}$

$\boxed{Q_0 = -1}$

$a_n^{(p)} = (-1 + n) 4^n \cdot n$

$a_n = c_1 2^n + c_2 4^n + (-1 + n) 4^n \cdot n$

$a_0 = 8 = c_1 + c_2$ ——— (1) $\times 2$

$a_1 = 22 = 2c_1 + 4c_2$ ——— (2)

$$\frac{22}{\frac{16}{6}}$$

$$\cancel{2c_1} + 2c_2 = 16$$
$$\cancel{2c_1} + 4c_2 = 22$$
$$-2c_2 = -6$$
$$\boxed{c_2 = 3}$$

$$c_1 + 3 = 8$$
$$\boxed{c_1 = 5}$$

$$a_n = 5.2^n + 3.4^n + n(n-1)4^n$$

★ **Binomial Theorem:**

Any sum of 2 unlike symbols such as $(x+y)$ is called a binomial or a binomial expression.

Binomial theorem $(x+y)^n = \sum_{r=0}^{\infty} nC_r x^{n-r} y^r$

$$= nC_0 x^n + nC_1 x^{n-1} y^1 + \dots + nC_n y^n$$

★ **Note:**

1. The expansion $(x+y)^n$ contains $n+1$ terms
2. The sum of the powers of the x and y in each term is equal to n
3. The $r+1^{th}$ term of the expansion $(x+y)^n$ is $nC_r x^{n-r} y^r$ is called the general term. It is denoted as T_{r+1}

$$\therefore T_{r+1} = nC_r x^{n-r} y^r$$

4. The integers in $C_0, C_1, C_2 \dots C_n$ are called the binomial coeff. of the expansion $(x+y)^n$

Q. Expand $(x+y)^7$

<u>Sol:</u> $nC_r\, x^{n-r} y^{r}$

$7C_r\, x^{n-r} y^{r}$

$= 7C_0\, x^{7-0} y^{0} + 7C_1\, x^{6} y^{1} + \dots\dots 7C_r\, x^{r} y^{7-r} + \dots 7C_7\, x^{0} y^{7-0}$

Q. Expand $(2a+5b)^6$

<u>Sol:</u> $6C_r (2a)^{6-r} (5b)^{r}$

$6C_0 (2a)^6 (5b)^0 + 6C_1 (2a)^5 (5b)^1 + \dots\dots + 6C_6 (2a)^0 (5b)^6$

Q. Find middle terms of

i) $\left(2x - \frac{1}{3x}\right)^{10}$

<u>Sol:</u> $n = 10$

Total terms $= 10+1 = 11$

Middle term $= \cancel{\frac{n+1}{2}}\ \cancel{\frac{11+1}{2}} = 6 = \frac{11+1}{2}$

$T_{r+1} = T_{5+1} = nC_r\, x^{n-r} y^{r}$

$= 10C_5 (2x)^{10-5} \left(\frac{-1}{3x}\right)^5$

$= 10C_5 \cdot \frac{2^5 \cancel{x^5} (-1)^5}{3^5 \cancel{x^5}}$

$= 10C_5 \left(\frac{2}{3}\right)^5 (-1)^5$

ii) $\left(x - \frac{3}{y}\right)^9$

$n = 9$

$n+1 = 10$

$5, 6 \leftarrow \frac{10+1}{2}$

T_5, T_6

$T_{4+1} : 9C_4 \cdot x^5 \cdot \frac{3^4}{y^4}$

$T_{5+1} : 9C_5 \cdot x^4 \frac{(-3)^5}{y^5}$

Q. Find the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$

Sol: $12C_r (2x)^{12-r} (-3y)^r$

$12 - r = 9 \quad | \quad r = 3$

$\boxed{r = 3}$

$12C_3 (2x)^9 (-3y)^3$

$\underbrace{12C_3 \cdot 2^9 (-3)^3}. x^9 y^3$

Q. Find the term independent of x in the expansion $\left(x^2 + \frac{1}{x}\right)^{12}$

Sol: $T_{r+1} : 12C_r (x^2)^{12-r} \left(\frac{1}{x}\right)^r$

$= 12C_r \cdot x^{24-2r} \cdot x^{-r}$

$= 12C_r \cdot x^{\boxed{24-3r}}$

$24 - 3r = 0$

$r = 8$

$= 12C_8$

Q. Find the two successive terms in the expansion of $(1+x)^{24}$ whose coefficients are in the ratio 4:1.

Q. Sol: $n = 24$

$(1+x)^{24}$

$nC_r x^{n-r} y^r$

$\Rightarrow 24C_r x^{24-r} y^r$

$\Rightarrow 24C_r (1)^{24-r} (x)^{24-r}$

$\Rightarrow 24C_r x^r$

$$\frac{24C_r}{24C_{r+1}} = \frac{4}{1}$$

$$\frac{\dfrac{\cancel{24!}}{r!(24-r)!}}{\dfrac{\cancel{24!}}{(r+1)!(24-r-1)!}} = \frac{4}{1}$$

$$\frac{(r+1)\cancel{r!}\cancel{(24-r-1)!}}{\cancel{r!}\cancel{(24-r-1)!}(24-r)} = \frac{4}{1}$$

$$\frac{r+1}{24-r} = \frac{4}{1}$$

$r+1 = 96-4r$

$5r = 95$

$\boxed{r = 19}$

$24C_{19} \cdot x^{19}, \quad 24C_{20} \cdot x^{20}$

$24C_{19} = 24C_5 \longrightarrow 24C_5, \; 24C_6$