(1)

Separation principle

Given a plant

$\dot{x}(t) = A x(t) + B u(t)$

$y(t) = C x(t)$

and an observer based controller

$\dot{x}_k(t) = A_k x_k(t) + B_k y(t)$

$r(t) = C_k x_k(t) + D_k y(t)$

with order n

(2)

and an $\tilde{x}(t) = \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix}$

Then the closed loop system is

$\dot{\tilde{x}}(t) = \left(\tilde{A} + \tilde{B} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \tilde{C}\right) \tilde{x}(t)$

where

$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ $\tilde{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}$

$\hat{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$.

(3)

Let

$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = \begin{bmatrix} A - LC - BK & L \\ -K & 0 \end{bmatrix}$

Theorem

Then the closed loop system is stable iff the eigenvalues of $A + BK$ and $A - LC$ are negative

(roots of $\det(sI - (A + BK)) = 0$ and $\det(sI - (A + LC)) = 0$

(4)

are in LHP.

proof find eigenvalues of CL syst.

$\dot{\tilde{x}}(t) = \left(\tilde{A} + \tilde{B} \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \tilde{C}\right) \tilde{x}(t)$

$= \begin{bmatrix} A + BD_kC & BC_k \\ B_kC & A_k \end{bmatrix}$

$= \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix}$

(5)

let $\hat{x} = \underbrace{\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}}_{Q^{-1}} \tilde{x}$

$= \begin{bmatrix} x(t) \\ x(t) - x_k(t) \end{bmatrix}$

$Q^{-1} \dot{\tilde{x}} = Q^{-1} A Q Q^{-1} \tilde{x}$

(similarity transform)

(6)

New closed loop system is

$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$

$= \begin{bmatrix} A & -BK \\ A - LC & LC + \cancel{BK} - A - \cancel{BK} \end{bmatrix} \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$

$= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}$

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(7)

$$\dot{\hat{x}} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}}_{\hat{A}} \hat{x}$$

o The eigenvalues of $\hat{A}$ are the union of the eigenvalues of $A-BK$ and $A-KC$, because the matrix is in block diagonal form and similarity transforms maintain the

(8)

eigenvalues of the original system.

o As a result, all system eigenvalues/poles are located where they are placed by K, L designs

o Remember, a system is stable iff all poles/eigenvalues are in LHP.

(9)

o since my poles/eigenvalues come from $A-BK$, $A-LC$, then the system is stable iff poles/eigenvalues of $A-BK$, $A-LC$ are in LHP.

QED

(10)

o A necessary and sufficient condition for the existence of a pair (K, L) that can make the system stable is that

$\exists$ (K, L) s.t. system stable iff (A, B, C) is controllable and observable.

there exists

if and only if

proof next time + example!

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(1)

Theorem

$\exists$ (K, L) so closed loop system with combined control is stable iff (A, B, C) is controllable and observable.

Proof:

$\exists K$ so $A-BK$ is stable iff (A, B) controllable

$\exists L$ so $A-LC$ is stable iff (A, C) is observable.

(2)

- As a result (A, B, C) should be both controllable and observable

QED.

(3)

Recall

$$\begin{bmatrix} \dot{x} \\ \dot{x} - \dot{x}_k \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ x - x_k \end{bmatrix}$$

let $e = x - x_k$ (error)

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

(4)

- The characteristic eq. for the combined system is found by solving

$$\det(sI-(A-BK))\det(sI-(A-LC))=0$$

- roots are found from

$$\det(sI-(A-BK))=0$$

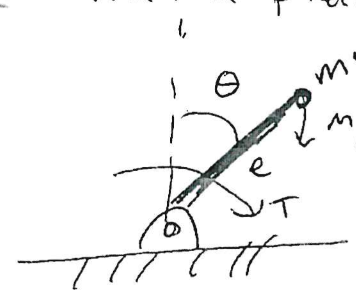
$$\det(sI-(A-LC))=0$$

(5)

- This result is important! This means the design of observer and controller can be done separately! * !! :)
- Typically, design observer to have faster dynamics than controller then design controller assuming we know all states

(6)

EX: Inverted pendulum



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$$-ml^2\ddot{\theta} = mgl\sin\theta + T$$

nonlinear → linear approximation

when θ small

$\sin\theta = \theta$

$$ml^2\ddot{\theta} - mgl\theta = T$$

$$\ddot{\theta} - \frac{g}{l}\theta = \frac{1}{ml^2}T$$

In our example,

$$\ddot{\theta} - 16\theta = r$$

$y = \theta$

set

$x_1 = \theta$

$x_2 = \dot{\theta}$

$e_1 = \theta - \hat{\theta}$

$e_2 = \dot{\theta} - \dot{\hat{\theta}}$

$\dot{x} = Ax + Br \quad y = Cx$

$A = \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$C = [1 \; 0]$

see handout

CL system

∘ from previous lecture examples!

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -4 & 21 & 4 \\ 0 & 0 & -3.2 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}$$