Methods Qualifying Exam

January 2000

Instructions:

1) Do not put your name on the exam. Place the number assigned to you on the upper left hand corner of each page of your exam.

2) Please start your answer to each question on a separate sheet of paper.

3) Answer all the questions.

4) Be sure to attempt all parts of every question. It may be possible to answer a later part of a question without having solved the earlier parts.

5) Be sure to hand in all of your exam. No additional material will be accepted once the exam has ended and you have left the exam room.

PROBLEM #1:

The tensile strength of a material is the ability that the material possesses to resist deformation when a force or a load is applied to it. A metallurgist conducts a study to evaluate the tensile strength of ductile iron strengthened at two different temperatures. She thinks that the lower temperature will yield the higher mean tensile strength. At each of the two temperatures, $800°C$ and $1000°C$, 300 specimens of ductile iron were heat treated. The data consists of the tensile strengths from 300 specimens heated to $800°C$: $X_1, \ldots, X_{300}$ which are iid with mean μ_1 and standard deviation σ_1 and the tensile strengths from 300 specimens heated to $1000°C$: $Y_1, \ldots, Y_{300}$ which are iid with mean μ_2 and standard deviation σ_2. Furthermore, the X's and Y's are independent.

a. The metallurgist is interested in the null hypothesis $\mathrm{H}_0 : \mu_1 \leq \mu_2$ versus the alternative hypothesis $\mathrm{H}_1 : \mu_1 > \mu_2$ Use the following steps to present the customary t-test of this null hypothesis based on $X_1, \ldots, X_{300}$ and $Y_1, \ldots, Y_{300}$.

   1. Write down a general formula for the t test statistic commonly used for this hypothesis test.
   2. Write down the decision rule for this hypothesis test. Use $\alpha = 0.05$.
   3. State the necessary conditions needed for your procedure to be valid and how you would verify whether the conditions are satisfied in this experimental setting.

b. For parts (b) and (c) of this question, you may assume that $\sigma_1 = \sigma_2 = 1$ and that the sample sizes are large enough to invoke the central limit theorem if necessary.

   1. Calculate the power of your test for the following six values of the parameter:

$$\Delta = \frac{\mu_1 - \mu_2}{\sqrt{1/300 + 1/300}} = \quad .5, \quad 1.0, \quad 1.5, \quad 2.0, \quad 2.5, \quad 3$$

   2. Use your results from (b.1) to sketch a power curve for your test. Be sure to label your axes clearly.

c. The metallurgist in discussing your results from (a) and (b) states, "The power of the test when $\Delta = 2.0$ is not large enough to meet industry standards. What needs to be done to increase it?" Answer the metallurgist's question, paying careful attention to: (i) your specific recommendation on how to increase the power; and (ii) explanation (based on the ideas from parts (a) and (b)) of **why** your recommendation will result in an increase in power.

d. The 600 observations considered above represent the tensile strength obtained from the two levels of heat treatment. However, after the experiments were conducted, the metallurgist informs you that the heat treatment for the 300 specimens for each heat level were conducted in the following manner. The furnace used to heat treat the specimens could hold only 5 specimens at a time. Thus, a tray containing 5 randomly selected specimens was heated to the specified temperature for the prescribed length of time and then the tensile strength measurements were taken on the 5 specimens. The metallurgist states that there is some variation in the conditions within the furnace from one experimental run to the next. Thus, there may be a strong *positive* correlation between tensile strength readings for specimens on the same tray. Given this additional information, answer the following questions without carrying out any additional calculations.

   1. How will this positive correlation within specimens affect the expectation of the variance estimator you used in part (a.1)?
   2. Suppose you did not adjust for the positive correlation within specimens and proceeded to use the ordinary t-test you proposed in part (a). Will the positive correlation in the data increase or decrease the numerical values of power you calculated for the test statistic in part (b)? Explain.

e. In light of your answer to (d), the metallurgist states, "Using the t-test from (a) to test the research hypothesis is obviously flawed. What is an alternative approach to testing the research hypothesis?" Answer the metallurgist's question by presenting a standard testing method that will account appropriately for the sampling design described in (d). Be sure to give clear, explicit statements of both your test statistic formula and your decision rule.

PROBLEM #2:

Consider the usual (full rank) linear regression model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\mathbf{Y}(n\text{x}1)$ is a vector of observed response variables, $\mathbf{X}(n\text{x}p)$ is a matrix of fixed and observed explanatory variables and whose first column consists of ones, $\boldsymbol{\beta}(p\text{x}1)$ is a vector of unknown parameters and $\boldsymbol{\epsilon}(n\text{x}1)$ is a vector of unobservable random variables assumed to have mean $\mathbf{0}$ and variance-covariance matrix $\sigma^{\mathbf{2}}\mathbf{I}$. Let $\hat{\boldsymbol{\beta}}$ and s^2 be the usual least squares estimators of $\boldsymbol{\beta}$ and σ^2, respectively.

Suppose that $\mathbf{x_{n+1}^T} = (1 \;\; x_{n+1,1} \;\; \ldots \;\; x_{n+1,p-1})$ is a 1 x p row vector of explanatory variables for which no response variable has been observed. Define augmented matrices $\mathbf{Y_a}((n+1)\text{x}1)$ and $\mathbf{X_a}((n+1)\text{x}(p+1))$ as follows:

$$\mathbf{Y_a} = \begin{bmatrix} \mathbf{Y} \\ 0 \end{bmatrix} \qquad \text{and} \qquad X_a = \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{x_{n+1}^T} & -1 \end{bmatrix}.$$

(a) Prove that the $(p+1)^{st}$ element of $(\mathbf{X_a^T X_a})^{\mathbf{-1}}\mathbf{X_a^T Y_a}$ is equal to $\hat{Y}_{x_{n+1}} \equiv \mathbf{x_{n+1}^T}\hat{\boldsymbol{\beta}}$.

[**Hint:** The least squares criterion can be used to prove this result.]

(b) Show the following result for partioned matrices:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B^T} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A^{-1}} + \mathbf{FE^{-1}F^T}) & (-\mathbf{FE^{-1}}) \\ (-\mathbf{E^{-1}F^T}) & (\mathbf{E^{-1}}) \end{bmatrix},$$

where $\mathbf{E} = \mathbf{D} - \mathbf{B^T A^{-1} B}$ and $\mathbf{F} = \mathbf{A^{-1}B}$.

c) Use the matrix result of part b) to prove that the $(p+1)^{st}$ diagonal element of the matrix $s^2\left(\mathbf{X_a^T X_a}\right)^{\mathbf{-1}}$ is equal to the least squares estimate of the variance of the *prediction* error of $\hat{Y}_{x_{n+1}}$.

PROBLEM #3:

A company that provides data input for billing operations is considering replacing the usual desktop computer/monitor/keyboard setup with notebook computers. Before doing so, they want to conduct a study to determine if the change will affect production (measured in number of entries per unit of time and in error rate=number of errors/total number of entries). They have asked you to help design a study to compare six possible configurations:

T1. Usual desktop configuration with external keyboard, mouse, and monitor

T2. A notebook computer with external keyboard and mouse

T3. A notebook computer on raised blocks with external keyboard and mouse

T4. A notebook computer with external cordless keyboard and mouse

T5. A notebook computer on raised blocks with external cordless keyboard and mouse

T6. A notebook computer alone

a. There are a large number of employees that are available for the study. However, the company would like to use as few as possible because while the employees are involved in the study they will be unavailable for their usual duties.

   1. What information do you need in order to provide an estimate of the number of employees that should be used for the study?
   2. What type of design do you suggest? Describe briefly the randomization procedure that you would use. Outline the ANOVA for this design showing sources of variation and degrees of freedom. (Assuming the number of employees used in the study=r).
   3. Would you suggest inference procedures other than the usual ANOVA? If so, which procedures?

b. Suppose it is not possible for each employee to be tested for more than four different configurations (treatments). What would you suggest? Does this place any restrictions on the number of employees used for the study? If so, what are the restrictions?

c. The employees have a wide range of experience, ranging from those who have been performing the duties for seven years to those who have been on the job for only one year. It is thought there may be a relationship between the measures of production and the time an individual has been performing the duty. In addition, the typing skills (a measure of ability to use a keyboard) differs among the employees. It is possible to give each employee a standardized typing test to obtain a measure of his or her typing skill.

   Does this information change your suggested design in Part a.2? If so, how?

PROBLEM #4:

Let Y have a double exponential distribution, that is, Y has pdf and cdf as follows:

$$f(y) = \frac{1}{2\sigma} e^{-|y-\mu|/\sigma}; \qquad F(y) = \begin{cases} \frac{1}{2} e^{(y-\mu)/\sigma}, & \text{for } y < \mu; \\ 1 - \frac{1}{2} e^{-(y-\mu)/\sigma}, & \text{for } y \geq \mu. \end{cases}$$

a. Show that μ and σ are location-scale parameters for the above cdf.

b. If $Y_1, \ldots, Y_n$ are iid with cdf G(.), describe in detail a graphical procedure for evaluating whether G(.) is a double exponential cdf.

c. If $Y_1, \ldots, Y_n$ are iid with cdf G(.), describe in detail a statistical test of the hypothesis that G(.) has a double exponential cdf.