STAT 638: Solution for Homework #1

2.1 a)
$P(Y_1 = \text{Farm}) = 0.11$
$P(Y_1 = \text{Operatives}) = 0.279$
$P(Y_1 = \text{Craftsmen}) = 0.277$
$P(Y_1 = \text{Sales}) = 0.099$
$P(Y_1 = \text{Professional}) = 0.235$

b)
$P(Y_2 = \text{Farm}) = 0.023$
$P(Y_2 = \text{Operatives}) = 0.260$
$P(Y_2 = \text{Craftsmen}) = 0.240$
$P(Y_2 = \text{Sales}) = 0.125$
$P(Y_2 = \text{Professional}) = 0.352$

c)
$P(Y_2 = \text{Farm}|Y_1 = \text{Farm}) = 0.164$
$P(Y_2 = \text{Operatives}|Y_1 = \text{Farm}) = 0.318$
$P(Y_2 = \text{Craftsmen}|Y_1 = \text{Farm}) = 0.282$
$P(Y_2 = \text{Sales}|Y_1 = \text{Farm}) = 0.073$
$P(Y_2 = \text{Professional}|Y_1 = \text{Farm}) = 0.164$

d)
$P(Y_1 = \text{Farm}|Y_2 = \text{Farm}) = 0.783$
$P(Y_1 = \text{Operatives}|Y_2 = \text{Farm}) = 0.087$
$P(Y_1 = \text{Craftsmen}|Y_2 = \text{Farm}) = 0.043$
$P(Y_1 = \text{Sales}|Y_2 = \text{Farm}) = 0.043$
$P(Y_1 = \text{Professional}|Y_2 = \text{Farm}) = 0.043$

2.2 a)
$E(a_1Y_1 + a_2Y_2) = a_1\mu_1 + a_2\mu_2$
$\text{Var}(a_1Y_1 + a_2Y_2) = a_1^2\text{Var}(Y_1) + a_2^2\text{Var}(Y_2) + 2a_1a_2\text{Cov}(Y_1, Y_2) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$, since independence of Y_1 and Y_2 implies that $\text{Cov}(Y_1, Y_2) = 0$.

b)
$E(a_1Y_1 - a_2Y_2) = a_1\mu_1 - a_2\mu_2$
$\text{Var}(a_1Y_1 - a_2Y_2) = a_1^2\text{Var}(Y_1) + a_2^2\text{Var}(Y_2) - 2a_1a_2\text{Cov}(Y_1, Y_2) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$

2.3 a)

$$p(x|y,z) = \frac{p(x,y,z)}{p(y,z)} \propto f(x,z)g(y,z)h(z) \propto f(x,z)$$

b)

$$p(y|x,z) = \frac{p(x,y,z)}{p(x,z)} \propto f(x,z)g(y,z)h(z) \propto g(y,z)$$

c)

$$p(x,y|z) = \frac{p(x,y,z)}{p(z)} \propto f(x,z)g(y,z)h(z) \propto f(x,z)g(y,z)$$

As a function of x and y, the last expression is a product of a function of x alone and a function of y alone, and therefore X and Y are conditionally independent given Z.

2.5 a)
$P(X=0,Y=0) = P(Y=0|X=0)P(X=0) = 0.4(0.5) = 0.2$
$P(X=0,Y=1) = P(Y=1|X=0)P(X=0) = 0.6(0.5) = 0.3$
$P(X=1,Y=0) = P(Y=0|X=1)P(X=1) = 0.6(0.5) = 0.3$
$P(X=1,Y=1) = P(Y=1|X=1)P(X=1) = 0.4(0.5) = 0.2$

b)
$E(Y) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1) = 0.5$

c)
The distribution of Y given $X=0$ is Bernoulli with success probability 0.60, and hence $\mathrm{Var}(Y|X=0) = 0.6(0.4) = 0.24$. The distribution of Y given $X=1$ is Bernoulli with success probability 0.40, and hence $\mathrm{Var}(Y|X=1) = 0.4(0.6) = 0.24$. We have

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = 1^2 \cdot P(Y=1) - (0.5)^2 = 0.5 - 0.25 = 0.25.$$

It makes sense that $\mathrm{Var}(Y)$ is larger than the two conditional variances since there is less uncertainty about the value of Y when we know what the value of X is.

d)

$$P(X=0|Y=1) = \frac{P(X=0,Y=1)}{P(Y=1)} = \frac{0.3}{0.5} = 0.6$$

2.6

$$\begin{gathered}
P(A^c \cap B|C) + P(A \cap B|C) = P(B|C) \Longrightarrow \\
P(A^c \cap B|C) = P(B|C) - P(A \cap B|C) \Longrightarrow \\
P(A^c \cap B|C) = P(B|C) - P(A|C)P(B|C) \Longrightarrow \\
P(A^c \cap B|C) = P(B|C)(1 - P(A|C)) \Longrightarrow \\
P(A^c \cap B|C) = P(B|C)P(A^c|C)
\end{gathered}$$

The other two proofs are done in a similar way.

Define the following probabilities:

$$P(A) = 0.5 \quad P(B) = 0.6 \quad P(C) = 0.6$$

$$P(A \cap C) = 0.2 \quad P(B \cap C) = 0.3 \quad P(A \cap B) = 0.3$$

$$P(A \cap B \cap C) = 0.1$$

We have

$$P(A \cap B|C) = \frac{0.1}{0.6} = \frac{1}{6},$$

$$P(A|C) = \frac{0.2}{0.6} = \frac{1}{3} \quad \text{and} \quad P(B|C) = \frac{0.3}{0.6} = \frac{1}{2},$$

and so A and B are conditionally independent given C. Now,

$$P(A|C^c) = \frac{P(A \cap C^c)}{P(C^c)} = \frac{0.5 - 0.2}{1 - 0.6} = \frac{3}{4},$$

$$P(B|C^c) = \frac{P(B \cap C^c)}{P(C^c)} = \frac{0.6 - 0.3}{1 - 0.6} = \frac{3}{4},$$

and

$$P(A \cap B|C^c) = \frac{P(A \cap B \cap C^c)}{P(C^c)} = \frac{0.3 - 0.1}{1 - 0.6} = \frac{1}{2}.$$

Since $(3/4)^2 = 9/16 \neq 1/2$, A and B are not conditionally independent given C^c.