STAT 626: Outline of Lecture 9

Review of ARMA Models; Difference Equations (§3.2, §3.3)

1. **Evaluation of Project Presentations:**

   Your group's presentation/project is evaluated based on the level of interest/question/enthusiasm it generates;

   Local students are evaluated based on asking questions, interest, involvement,,,,,.

2. **Review of One-Sided MA(∞) or Causal Process:** Is a time series involving only **the past and present values** of a white noise (shocks, inputs):

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

   with absolutely summable coefficients. Its *autocovariance function* is given by

$$\gamma(h) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_{j+h}\psi_j.$$

3. **Autoregressive and Moving Average (ARMA (p,q)) Models:**

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}.$$

   OR

$$\phi(B)x_t = \theta(B)w_t,$$

   where $\phi(z), \theta(z)$ are the AR and MA polynomials, respectively.
   Focus on ARMA(1,1) Models: $x_t = \phi x_{t-1} + w_t + \theta w_{t-1}$.

4. **Causal Solution:** When x_t can be written as a one-sided MA or linear process, i.e. in terms of the **past and present values** of the WN:

$$w_t, w_{t-1}, \ldots.$$

   This is important for computing the ACF of various ARMA models.

5. **Invertible ARMA:** w_t can be written in terms of **the past and present values**, i.e.

$$x_t, x_{t-1}, \ldots,$$

OR

$$w_t = x_t + \pi_1 x_{t-1} + \pi_2 x_{t-2} + \ldots = \sum_{j=0}^{\infty} \pi_j x_{t-j}.$$

This is important for parameter estimation and computing predictors.

6. **Difference Equations:** $u_n - \phi_1 u_{n-1} - \phi_2 u_{n-2} = f_n$.

   Homogeneous DE: $f_n = 0$, for all n.

   How does one solve a DE?

7. The AR, MA and ARMA models are good examples of difference equations. Their ACF, Ψ- and π-weights, predictors satisfies similar difference equations.

8. **HOW to compute ACF of Causal ARMA Models?**
   Ans: Use the moving average (MA) representation of x_t.

Autoregressive Models of order p or AR(p) Models:

$$x_t = \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t, \quad \phi_p \neq 0.$$

QUESTION: Is a time series $\{x_t\}$ defined via an AR(p) model always stationary? If so, what is its autocovariance function?

What happens when $p = 1$ and $\phi = 1$?

AR(1) with $|\phi| < 1$

Consider the general AR(1):

$$x_t = \phi x_{t-1} + w_t,$$

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}.$$

$$\gamma(h) = \frac{\sigma_w^2}{1-\phi^2}\phi^h, \quad h = 0, 1, 2 \ldots.$$

Autocorrelation Function (ACF) of AR(1): $\rho(h) = \phi^h, \quad h = 1, 2, \ldots.$

What is the shape of the population (theoretical) correlogram of an AR(1)?

3.3 Difference Equations

The study of the behavior of ARMA processes and their ACFs is greatly enhanced by a basic knowledge of difference equations, simply because they are difference equations. This topic is also useful in the study of time domain models and stochastic processes in general. We will give a brief and heuristic account of the topic along with some examples of the usefulness of the theory. For details, the reader is referred to Mickens (1990).

Suppose we have a sequence of numbers $u_0, u_1, u_2, \ldots$ such that

$$u_n - \alpha u_{n-1} = 0, \quad \alpha \neq 0, \quad n = 1, 2, \ldots \,. \tag{3.29}$$

For example, recall (3.9) in which we showed that the ACF of an AR(1) process is a sequence, $\rho(h)$, satisfying

$$\rho(h) - \phi\rho(h-1) = 0, \quad h = 1, 2, \ldots \,.$$

Equation (3.29) represents a homogeneous difference equation of order 1. To solve the equation, we write:

$$\begin{aligned} u_1 &= \alpha u_0 \\ u_2 &= \alpha u_1 = \alpha^2 u_0 \\ &\;\vdots \\ u_n &= \alpha u_{n-1} = \alpha^n u_0. \end{aligned}$$

Given an initial condition $u_0 = c$, we may solve (3.29), namely, $u_n = \alpha^n c$.

In operator notation, (3.29) can be written as $(1 - \alpha B)u_n = 0$. The polynomial associated with (3.29) is $\alpha(z) = 1 - \alpha z$, and the root, say, z_0, of this polynomial is $z_0 = 1/\alpha$; that is $\alpha(z_0) = 0$. We know a solution (in fact, *the* solution) to (3.29), with initial condition $u_0 = c$, is

$$u_n = \alpha^n c = \left(z_0^{-1}\right)^n c. \tag{3.30}$$

That is, the solution to the difference equation (3.29) depends only on the initial condition and the inverse of the root to the associated polynomial $\alpha(z)$.

Now suppose that the sequence satisfies

$$u_n - \alpha_1 u_{n-1} - \alpha_2 u_{n-2} = 0, \quad \alpha_2 \neq 0, \quad n = 2, 3, \ldots \tag{3.31}$$

This equation is a homogeneous difference equation of order 2. The corresponding polynomial is

$$\alpha(z) = 1 - \alpha_1 z - \alpha_2 z^2,$$

which has two roots, say, z_1 and z_2; that is, $\alpha(z_1) = \alpha(z_2) = 0$. We will consider two cases. First suppose $z_1 \neq z_2$. Then the general solution to (3.31) is

$$u_n = c_1 z_1^{-n} + c_2 z_2^{-n}, \tag{3.32}$$

where c_1 and c_2 depend on the initial conditions. The claim that is *a* solution can be verified by direct substitution of (3.32) into (3.31):

$$\begin{aligned}
\left(c_1 z_1^{-n} + c_2 z_2^{-n}\right) - \ & \alpha_1 \left(c_1 z_1^{-(n-1)} + c_2 z_2^{-(n-1)}\right) - \alpha_2 \left(c_1 z_1^{-(n-2)} + c_2 z_2^{-(n-2)}\right) \\
&= c_1 z_1^{-n} \left(1 - \alpha_1 z_1 - \alpha_2 z_1^2\right) + c_2 z_2^{-n} \left(1 - \alpha_1 z_2 - \alpha_2 z_2^2\right) \\
&= c_1 z_1^{-n} \alpha(z_1) + c_2 z_2^{-n} \alpha(z_2) = 0.
\end{aligned}$$

Given two initial conditions u_0 and u_1, we may solve for c_1 and c_2:

$$u_0 = c_1 + c_2 \quad \text{and} \quad u_1 = c_1 z_1^{-1} + c_2 z_2^{-1},$$

where z_1 and z_2 can be solved for in terms of α_1 and α_2 using the quadratic formula, for example.

When the roots are equal, $z_1 = z_2$ $(= z_0)$, a general solution to (3.31) is

$$u_n = z_0^{-n}(c_1 + c_2 n). \tag{3.33}$$

This claim can also be verified by direct substitution of (3.33) into (3.31):

$$\begin{aligned}
z_0^{-n}(c_1 + c_2 n) - \ & \alpha_1 \left(z_0^{-(n-1)}[c_1 + c_2(n-1)]\right) - \alpha_2 \left(z_0^{-(n-2)}[c_1 + c_2(n-2)]\right) \\
&= z_0^{-n}(c_1 + c_2 n) \left(1 - \alpha_1 z_0 - \alpha_2 z_0^2\right) + c_2 z_0^{-n+1} \left(\alpha_1 + 2\alpha_2 z_0\right) \\
&= c_2 z_0^{-n+1} \left(\alpha_1 + 2\alpha_2 z_0\right).
\end{aligned}$$

To show that $(\alpha_1 + 2\alpha_2 z_0) = 0$, write $1 - \alpha_1 z - \alpha_2 z^2 = (1 - z_0^{-1} z)^2$, and take derivatives with respect to z on both sides of the equation to obtain $(\alpha_1 + 2\alpha_2 z) = 2z_0^{-1}(1 - z_0^{-1} z)$. Thus, $(\alpha_1 + 2\alpha_2 z_0) = 2z_0^{-1}(1 - z_0^{-1} z_0) = 0$, as was to be shown. Finally, given two initial conditions, u_0 and u_1, we can solve for c_1 and c_2:

$$u_0 = c_1 \quad \text{and} \quad u_1 = (c_1 + c_2) z_0^{-1}.$$

It can also be shown that these solutions are unique.

To summarize these results, in the case of distinct roots, the solution to the homogeneous difference equation of degree two was

$$\begin{aligned}
u_n = z_1^{-n} &\times (\text{a polynomial in } n \text{ of degree } m_1 - 1) \\
+ \ z_2^{-n} &\times (\text{a polynomial in } n \text{ of degree } m_2 - 1),
\end{aligned} \tag{3.34}$$

where m_1 is the multiplicity of the root z_1 and m_2 is the multiplicity of the root z_2. In this example, of course, $m_1 = m_2 = 1$, and we called the polynomials of degree zero c_1 and c_2, respectively. In the case of the repeated root, the solution was

$$u_n = z_0^{-n} \times (\text{a polynomial in } n \text{ of degree } m_0 - 1), \tag{3.35}$$

where m_0 is the multiplicity of the root z_0; that is, $m_0 = 2$. In this case, we wrote the polynomial of degree one as $c_1 + c_2 n$. In both cases, we solved for c_1 and c_2 given two initial conditions, u_0 and u_1.

Example 3.9 The ACF of an AR(2) Process

Suppose $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$ is a causal AR(2) process. Multiply each side of the model by x_{t-h} for $h > 0$, and take expectation:

$$E(x_t x_{t-h}) = \phi_1 E(x_{t-1} x_{t-h}) + \phi_2 E(x_{t-2} x_{t-h}) + E(w_t x_{t-h}).$$

The result is

$$\gamma(h) = \phi_1 \gamma(h-1) + \phi_2 \gamma(h-2), \quad h = 1, 2, \ldots . \tag{3.36}$$

In (3.36), we used the fact that $E(x_t) = 0$ and for $h > 0$,

$$E(w_t x_{t-h}) = E\Big(w_t \sum_{j=0}^{\infty} \psi_j w_{t-h-j}\Big) = 0.$$

Divide (3.36) through by $\gamma(0)$ to obtain the difference equation for the ACF of the process:

$$\rho(h) - \phi_1 \rho(h-1) - \phi_2 \rho(h-2) = 0, \quad h = 1, 2, \ldots . \tag{3.37}$$

The initial conditions are $\rho(0) = 1$ and $\rho(-1) = \phi_1/(1-\phi_2)$, which is obtained by evaluating (3.37) for $h = 1$ and noting that $\rho(1) = \rho(-1)$.

Using the results for the homogeneous difference equation of order two, let z_1 and z_2 be the roots of the associated polynomial, $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Because the model is causal, we know the roots are outside the unit circle: $|z_1| > 1$ and $|z_2| > 1$. Now, consider the solution for three cases:

(i) When z_1 and z_2 are real and distinct, then

$$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h},$$

so $\rho(h) \to 0$ exponentially fast as $h \to \infty$.

(ii) When $z_1 = z_2$ $(= z_0)$ are real and equal, then

$$\rho(h) = z_0^{-h}(c_1 + c_2 h),$$

so $\rho(h) \to 0$ exponentially fast as $h \to \infty$.

(iii) When $z_1 = \bar{z}_2$ are a complex conjugate pair, then $c_2 = \bar{c}_1$ (because $\rho(h)$ is real), and

$$\rho(h) = c_1 z_1^{-h} + \bar{c}_1 \bar{z}_1^{-h}.$$

Write c_1 and z_1 in polar coordinates, for example, $z_1 = |z_1| e^{i\theta}$, where θ is the angle whose tangent is the ratio of the imaginary part and the real part of z_1 (sometimes called $\arg(z_1)$; the range of θ is $[-\pi, \pi]$). Then, using the fact that $e^{i\alpha} + e^{-i\alpha} = 2\cos(\alpha)$, the solution has the form

$$\rho(h) = a|z_1|^{-h} \cos(h\theta + b),$$

where a and b are determined by the initial conditions. Again, $\rho(h)$ dampens to zero exponentially fast as $h \to \infty$, but it does so in a sinusoidal fashion. The implication of this result is shown in the next example.

Example 3.10 An AR(2) with Complex Roots

Figure 3.3 shows $n = 144$ observations from the AR(2) model

$$x_t = 1.5x_{t-1} - .75x_{t-2} + w_t,$$

with $\sigma_w^2 = 1$, and with complex roots chosen so the process exhibits pseudo-cyclic behavior at the rate of one cycle every 12 time points. The autoregressive polynomial for this model is $\phi(z) = 1 - 1.5z + .75z^2$. The roots of $\phi(z)$ are $1 \pm i/\sqrt{3}$, and $\theta = \tan^{-1}(1/\sqrt{3}) = 2\pi/12$ radians per unit time. To convert the angle to cycles per unit time, divide by 2π to get 1/12 cycles per unit time. The ACF for this model is shown in §3.4, Figure 3.4.

To calculate the roots of the polynomial and solve for arg in R:

```
z = c(1,-1.5,.75)      # coefficients of the polynomial
(a = polyroot(z)[1])  # print one root: 1+0.57735i = 1 + i/sqrt(3)
arg = Arg(a)/(2*pi)    # arg in cycles/pt
1/arg                  # = 12,  the pseudo period
```

To reproduce Figure 3.3:

```
set.seed(90210)
ar2 = arima.sim(list(order=c(2,0,0), ar=c(1.5,-.75)), n = 144)
plot(1:144/12, ar2, type="l", xlab="Time (one unit = 12 points)")
abline(v=0:12, lty="dotted", lwd=2)
```

To calculate and display the ACF for this model:

```
ACF = ARMAacf(ar=c(1.5,-.75), ma=0, 50)
plot(ACF, type="h", xlab="lag")
abline(h=0)
```

We now exhibit the solution for the general homogeneous difference equation of order p:

$$u_n - \alpha_1 u_{n-1} - \cdots - \alpha_p u_{n-p} = 0, \quad \alpha_p \neq 0, \quad n = p, p+1, \ldots . \tag{3.38}$$

The associated polynomial is

$$\alpha(z) = 1 - \alpha_1 z - \cdots - \alpha_p z^p.$$

Suppose $\alpha(z)$ has r distinct roots, z_1 with multiplicity m_1, z_2 with multiplicity m_2, ..., and z_r with multiplicity m_r, such that $m_1 + m_2 + \cdots + m_r = p$. The general solution to the difference equation (3.38) is

$$u_n = z_1^{-n} P_1(n) + z_2^{-n} P_2(n) + \cdots + z_r^{-n} P_r(n), \tag{3.39}$$

where $P_j(n)$, for $j = 1, 2, \ldots, r$, is a polynomial in n, of degree $m_j - 1$. Given p initial conditions $u_0, \ldots, u_{p-1}$, we can solve for the $P_j(n)$ explicitly.

Example 3.11 The ψ-weights for an ARMA Model

For a causal ARMA(p, q) model, $\phi(B)x_t = \theta(B)w_t$, where the zeros of $\phi(z)$ are outside the unit circle, recall that we may write

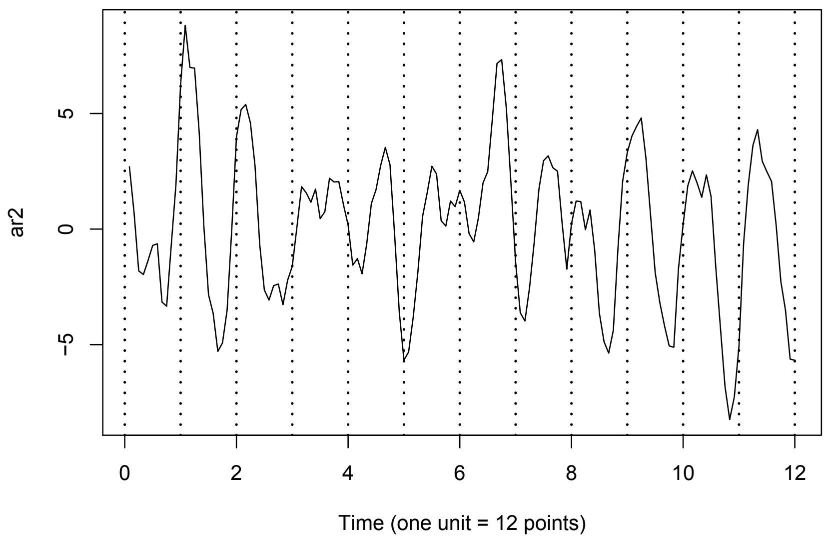

Fig. 3.3. Simulated AR(2) model, $n = 144$ with $\phi_1 = 1.5$ and $\phi_2 = -.75$.

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j},$$

where the ψ-weights are determined using Property 3.1.

For the pure MA(q) model, $\psi_0 = 1$, $\psi_j = \theta_j$, for $j = 1, \ldots, q$, and $\psi_j = 0$, otherwise. For the general case of ARMA(p, q) models, the task of solving for the ψ-weights is much more complicated, as was demonstrated in Example 3.7. The use of the theory of homogeneous difference equations can help here. To solve for the ψ-weights in general, we must match the coefficients in $\phi(z)\psi(z) = \theta(z)$:

$$(1 - \phi_1 z - \phi_2 z^2 - \cdots)(\psi_0 + \psi_1 z + \psi_2 z^2 + \cdots) = (1 + \theta_1 z + \theta_2 z^2 + \cdots).$$

The first few values are

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 - \phi_1 \psi_0 &= \theta_1 \\ \psi_2 - \phi_1 \psi_1 - \phi_2 \psi_0 &= \theta_2 \\ \psi_3 - \phi_1 \psi_2 - \phi_2 \psi_1 - \phi_3 \psi_0 &= \theta_3 \\ &\vdots \end{aligned}$$

where we would take $\phi_j = 0$ for $j > p$, and $\theta_j = 0$ for $j > q$. The ψ-weights satisfy the homogeneous difference equation given by

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = 0, \quad j \geq \max(p, q+1), \tag{3.40}$$

with initial conditions

$$\psi_j - \sum_{k=1}^{j} \phi_k \psi_{j-k} = \theta_j, \quad 0 \le j < \max(p, q+1). \tag{3.41}$$

The general solution depends on the roots of the AR polynomial $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$, as seen from (3.40). The specific solution will, of course, depend on the initial conditions.

Consider the ARMA process given in (3.27), $x_t = .9x_{t-1} + .5w_{t-1} + w_t$. Because $\max(p, q+1) = 2$, using (3.41), we have $\psi_0 = 1$ and $\psi_1 = .9 + .5 = 1.4$. By (3.40), for $j = 2, 3, \ldots$, the ψ-weights satisfy $\psi_j - .9\psi_{j-1} = 0$. The general solution is $\psi_j = c\,.9^j$. To find the specific solution, use the initial condition $\psi_1 = 1.4$, so $1.4 = .9c$ or $c = 1.4/.9$. Finally, $\psi_j = 1.4(.9)^{j-1}$, for $j \ge 1$, as we saw in Example 3.7.

To view, for example, the first 50 ψ-weights in R, use:

```
ARMAtoMA(ar=.9, ma=.5, 50)         # for a list
plot(ARMAtoMA(ar=.9, ma=.5, 50))   # for a graph
```

3.4 Autocorrelation and Partial Autocorrelation

We begin by exhibiting the ACF of an MA(q) process, $x_t = \theta(B)w_t$, where $\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q$. Because x_t is a finite linear combination of white noise terms, the process is stationary with mean

$$E(x_t) = \sum_{j=0}^{q} \theta_j E(w_{t-j}) = 0,$$

where we have written $\theta_0 = 1$, and with autocovariance function

$$\gamma(h) = \operatorname{cov}(x_{t+h}, x_t) = \operatorname{cov}\Big(\sum_{j=0}^{q} \theta_j w_{t+h-j}, \sum_{k=0}^{q} \theta_k w_{t-k}\Big)$$

$$= \begin{cases} \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}, & 0 \le h \le q \\ 0 & h > q. \end{cases} \tag{3.42}$$

Recall that $\gamma(h) = \gamma(-h)$, so we will only display the values for $h \ge 0$. The cutting off of $\gamma(h)$ after q lags is the signature of the MA(q) model. Dividing (3.42) by $\gamma(0)$ yields the ACF of an MA(q):

$$\rho(h) = \begin{cases} \dfrac{\sum_{j=0}^{q-h} \theta_j \theta_{j+h}}{1 + \theta_1^2 + \cdots + \theta_q^2} & 1 \le h \le q \\ 0 & h > q. \end{cases} \tag{3.43}$$

For a causal ARMA(p, q) model, $\phi(B)x_t = \theta(B)w_t$, where the zeros of $\phi(z)$ are outside the unit circle, write

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}. \tag{3.44}$$

It follows immediately that $E(x_t) = 0$. Also, the autocovariance function of x_t can be written as

$$\gamma(h) = \text{cov}(x_{t+h}, x_t) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}, \quad h \geq 0. \tag{3.45}$$

We could then use (3.40) and (3.41) to solve for the ψ-weights. In turn, we could solve for $\gamma(h)$, and the ACF $\rho(h) = \gamma(h)/\gamma(0)$. As in Example 3.9, it is also possible to obtain a homogeneous difference equation directly in terms of $\gamma(h)$. First, we write

$$\begin{aligned} \gamma(h) &= \text{cov}(x_{t+h}, x_t) = \text{cov}\Big(\sum_{j=1}^{p} \phi_j x_{t+h-j} + \sum_{j=0}^{q} \theta_j w_{t+h-j},\, x_t\Big) \\ &= \sum_{j=1}^{p} \phi_j \gamma(h-j) + \sigma_w^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}, \quad h \geq 0, \end{aligned} \tag{3.46}$$

where we have used the fact that, for $h \geq 0$,

$$\text{cov}(w_{t+h-j},\, x_t) = \text{cov}\Big(w_{t+h-j}, \sum_{k=0}^{\infty} \psi_k w_{t-k}\Big) = \psi_{j-h}\sigma_w^2.$$

From (3.46), we can write a general homogeneous equation for the ACF of a causal ARMA process:

$$\gamma(h) - \phi_1\gamma(h-1) - \cdots - \phi_p\gamma(h-p) = 0, \quad h \geq \max(p, q+1), \tag{3.47}$$

with initial conditions

$$\gamma(h) - \sum_{j=1}^{p} \phi_j \gamma(h-j) = \sigma_w^2 \sum_{j=h}^{q} \theta_j \psi_{j-h}, \quad 0 \leq h < \max(p, q+1). \tag{3.48}$$

Dividing (3.47) and (3.48) through by $\gamma(0)$ will allow us to solve for the ACF, $\rho(h) = \gamma(h)/\gamma(0)$.

Example 3.12 The ACF of an AR(p)

In Example 3.9 we considered the case where $p = 2$. For the general case, it follows immediately from (3.47) that

$$\rho(h) - \phi_1\rho(h-1) - \cdots - \phi_p\rho(h-p) = 0, \quad h \geq p. \tag{3.49}$$

Let $z_1, \dots, z_r$ denote the roots of $\phi(z)$, each with multiplicity $m_1, \dots, m_r$, respectively, where $m_1 + \cdots + m_r = p$. Then, from (3.39), the general solution is