STAT 630 Fall 2014
Homework 3 Solution

2.4.2

(a) Since W is defined on interval $[1, 4]$, thus $P(W \geq 5) = 0$.

(b) $P(W \geq 2) = \frac{4-2}{4-1} = \frac{2}{3}$.

(c) $P(W^2 \leq 9) = P(W \leq 3) = \frac{3-1}{4-1} = \frac{2}{3}$.

2.4.4

In this question, we apply the fact that the integral of density function over its definition interval equals to 1.

(a) Let $\int_0^1 cx = \frac{c}{2}x^2|_0^1 = \frac{c}{2}$ equal to one, thus $c = 2$. So $f(x) = 2x\mathrm{I}_{\{0 \leq x \leq 1\}}$

(b) Let $\int_0^1 cx^n = \frac{c}{n+1}x^{n+1}|_0^1 = \frac{c}{n+1}$ equal to one, thus $c = n+1$. So $f(x) = (n+1)x^n\mathrm{I}_{\{0 \leq x \leq 1\}}$

(c) Let $\int_0^2 cx^{\frac{1}{2}} = \frac{2}{3}cx^{\frac{3}{2}}|_0^2 = \frac{2}{3}c \cdot 2\sqrt{2}$ equal to one, thus $c = \frac{3}{4\sqrt{2}}$. So $f(x) = \frac{3}{4\sqrt{2}}x^{1/2}\mathrm{I}_{\{0 \leq x \leq 2\}}$

2.4.6

(b)$P(0 < X < 3) = \int_0^3 3exp(-3x) = -exp(-3x)|_0^3 = 1 - e^{-9}$.
(e)$P(2 < X < 10) = \int_2^{10} 3exp(-3x) = -exp(-3x)|_2^{10} = e^{-6} - e^{-30}$.

2.4.19

First we can see $f(x) \geq 0$, then we need to prove its integral over definition interval equals to 1.

$$\int_0^\infty \alpha x^{\alpha-1}e^{-x^\alpha}dx = -\int_0^\infty d(e^{-x^\alpha}) = -e^{-x^\alpha}|_0^\infty$$

Because $\alpha > 0$, thus this integral equals to 1. Here f is a density.

2.4.22

First, it is easy to see $f(x) \geq 0$. Then

$$\int_{-\infty}^{\infty} \frac{1}{2} e^{-|x|} dx = \int_{-\infty}^{0} \frac{1}{2} e^{x} dx + \int_{0}^{\infty} \frac{1}{2} e^{-x} dx = \frac{1}{2}(e^{x}|_{-\infty}^{0}) + \frac{1}{2} = 1$$

Hence, f is a valid density function.

2.5.3

(a). We can see $F(x)$ can be negative or greater than 1, thus it is not a valid cumulative distribution function.
(c). First $0 \leq F(x) \leq 1$ and also it is nondecreasing over its definition interval, thus it is a valid cdf.
(d). $F(x)$ can be greater than 1 when $1 < x \leq 3$, thus it is not a valid cdf.
(f). You can easily verify $0 \leq F(x) \leq 1$ and it is nondecreasing. Thus it is a valid cdf.
(g). It does not satisfy the property (b). For example, let $x_1 = -0.5, x_2 = 0.3$, we can see $F(-0.5) > F(0.3)$, therefore it is not nondecreasing.

2.5.5

In this question, we should transform value of Y into z-score in order to find the value in z-table.

(a) $P(Y \leq -5) = P(z \leq \frac{-5-(-8)}{2}) = P(z \leq \frac{3}{2}) = \Phi(\frac{3}{2}) = 0.9331928$.

(b) $P(-2 \leq Y \leq 7) = P(Y \leq 7) - P(Y \leq -2) = P(z \leq \frac{15}{2}) - P(z \leq 3) = 0.001349898$.

(c) $P(Y \geq 3) = 1 - P(Y \leq 3) = 1 - \Phi(\frac{11}{2}) = 1.898956 \times 10^{-8}$.

(d) Use function *qnorm* in R to find the percentiles. $35th$ percentile is -8.7706 and $84th$ percentile is -6.0111.

2.5.7

Since $F(x)$ is a continous function, then $P(X = x) = 0$ and $P(X < x) = P(X \leq x) - P(X = x) = F(x) - 0 = F(x)$
(a) $P(X < 1/3) = F(1/3) = (1/3)^2 = 1/9$
(b) $P(1/4 < X < 1/2) = P(X < 1/2) - P(X \leq 1/4) = F(1/2) - F(1/4) = (1/2)^2 - (1/4)^2 = 3/16 = 0.1875$
(f) Since $0 \leq P(X < -1) \leq P(X \leq 0) = F(0) = 0$, then $P(X < -1) = 0$
(g) Since $1 \geq P(X < 3) \geq P(X \leq 1) = F(1) = 1$, then $P(X < 3) = 1$
(h) $P(X = 3/7) = P(X \leq 3/7) - P(X = 3/7) = F(3/7) - F(3/7) = 0$
(i) Suppose q is the sth percentile of the distribution of X, then $q^2 = s/100$, i.e $q = \sqrt{\frac{s}{100}}$. Hence, the $40th$ percentile is $\sqrt{.40} = 0.6324555$ and the $72th$ percentile is $\sqrt{.72} = 0.8485$.

2.5.8

(a) $P(1/3 < Y < 3/4) = F_Y(3/4) - F_Y(1/3) = 1 - (1-3/4)^3 - (1/3)^3 = \frac{1637}{1728} = 0.947338.$

(b) $P(Y = 1/3) = 0$ since F_Y is continuous at 1/3.

(c) $P(Y = 1/2) = P(Y \leq 2) - P(Y < 1/2) = F_Y(1/2) - F_Y(1/2-) = 1 - (1-1/2)^3 - (1/2)^3 = \frac{3}{4}.$

2.5.21

(a) First the density function of Weibull distribution is $\alpha t^{\alpha-1} e^{-t^\alpha}$, then if $x > 0$

$$F(x) = \int_0^x \alpha t^{\alpha-1} e^{-t^\alpha} dt = -de^{-t^\alpha}|_0^x = 1 - e^{-x^\alpha}$$

When $x \to \infty$, this integral intends to be 1.

(b) Let q be a certain percentile. Then let the cumulative distribution function of Weibull distribution $F(x) = q$, that is:

$$1 - e^{-x^\alpha} = q$$

we can solve it for x, which is $\{-\log(1-q)\}^{\frac{1}{\alpha}}$.

2.5.24

We will discuss the cdf for two cases. If $x \leq 0$, then $F(x) = \int_{-\infty}^x \frac{1}{2} e^t dt = \frac{1}{2} e^t|_{-\infty}^x = \frac{1}{2} e^x$. If $x > 0$,$F(x) = \int_{-\infty}^0 \frac{1}{2} e^t dt + \int_0^x \frac{1}{2} e^{-t} dt = \frac{1}{2} - \frac{1}{2} e^{-t}|_0^x = 1 - \frac{1}{2} e^{-x}$. So,
$F(x) = \begin{cases} \frac{1}{2} e^x & x \leq 0 \\ 1 - \frac{1}{2} e^{-x} & x > 0 \end{cases}$

2.6.1

Since $x \in [L, R]$, thus $Y \in [cL + d, cR + d]$. Apply theorem 2.6.2, we know $f_Y(y) = \frac{1}{R-L} \cdot \frac{1}{c} = \frac{1}{c(R-L)}$. This is just the density for the uniform distribution on $[cL + d, cR + d]$.

2.6.5

Let $h(x) = x^3$. Then $Y = h(X)$ and h os strictly increasing, and $h^{-1}(y) = y^{\frac{1}{3}}$. Hence, $f_Y(y) = f_X(h^{-1}(y))/|h'(h^{-1}(y))| = f_X(y^{\frac{1}{3}})/3(y^{\frac{1}{3}})^2$, which equals $\lambda e^{-\lambda y^{\frac{1}{3}}}/3y^{\frac{2}{3}} = \frac{\lambda}{3} y^{-\frac{2}{3}} e^{-\lambda y^{\frac{1}{3}}}$ for $y > 0$, otherwise equals 0.

2.6.9

(a) Since $X = \sqrt{Y}, Y = h(X) = X^2, h^{'}(X) = 2X$, then the density of Y is $(\sqrt{y})^3/(4 \cdot 2 \cdot \sqrt{y}) = \frac{y}{8}$ for $0 < Y < 4$, $f_y(y) = 0$ for otherwise.

(b) Since $X = Z^2, Z = h(X) = \sqrt{X}, h^{'}(X) = \frac{1}{2} X^{-1/2}$. Thus the density of Z is $\frac{(z^2)^3}{4}/(\frac{1}{2} \cdot (z^2)^{-1/2}) = \frac{z^7}{2}$, $Z \in (0, \sqrt{2})$.

2.6.12

Since $Y(x) = x^{\frac{1}{3}}$ is increasing, Y is also 1-1. The inverse is $Y^{-1}(y) = y^3$ and the derivative is $Y'(Y^{-1}(y)) = \frac{1}{3}(y^3)^{-\frac{2}{3}}$. By applying theorem 2.6.2, we get $f_Y(y) = f_X(y^3)/|\frac{1}{3}(y^3)^{-\frac{2}{3}}| = 3y^{-4}$.

2.6.18

For X, its density is $\alpha x^{\alpha-1}e^{-x^{\alpha}}$ and we also know $X = Y^{\frac{1}{\beta}}, h^{'}(X) = \beta X^{\beta-1}$. Then Apply theorem 2.6.2, we can obtain:
$f_Y(y) = \alpha \cdot y^{\frac{\alpha-1}{\beta}} e^{-y^{\frac{\alpha}{\beta}}}/(\beta(y^{\frac{1}{\beta}})^{\beta-1}) = \frac{\alpha}{\beta}y^{\frac{\alpha}{\beta}-1} \cdot e^{-y^{\frac{\alpha}{\beta}}}$ for $y > 0$.
You can see it is the density function of Weibull distribution with parameter $\frac{\alpha}{\beta}$.