STAT 630 Fall 2014
Homework 11 Solution

6.3.1

The sample mean $\bar{x} = 4.88$. If H_0 is true , then $P(\mu < 4.88) = \Phi((4.88 - 5)/(\sqrt{0.5/10})) = \Phi(-0.54) = 0.2958$. So the two-sided p-value is 0.5916.

6.3.2

In this case, the true variance is unknown and we have to estimate it by sample variance. Since observations are assumed to have normal distributions, thus $\frac{\sqrt{n}(\bar{x}-5)}{sd} \sim t_{n-1}$, where sd is the sample standard deviation. The test statistic is $\frac{\sqrt{n}(\bar{x}-5)}{sd} = -0.5454$, p-value$= 2P(t_9 < -0.5454) = 0.5987$.

6.3.8

For likelihood ratio statistic, we have $2\{\sum y_i \log\frac{\hat{p}}{p} + (n - \sum y_i)\log\frac{1-\hat{p}}{1-p}\} \xrightarrow{d} \chi^2(1)$. The p-value equals $1 - \mathrm{pchisq}(0.9768, 1) = 0.3230$. For Wald statistic, we have $\frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \xrightarrow{d} \mathrm{N}(0,1)$, then p-value equals $2 * \Phi\left(\frac{0.62-0.65}{\sqrt{\frac{0.62(1-0.62)}{250}}}\right) = 0.3284$. We conclude that we do not have enough evidence against H_0. For score statistic, we have $\frac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} \mathrm{N}(0,1)$, then then p-value equals $2 * \Phi\left(\frac{0.62-0.65}{\sqrt{\frac{0.65(1-0.65)}{250}}}\right) = 0.32$.

8.2.16

Without loss of generality. assume $\mu_0 = 0$. For $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 = \sigma_1^2$ with $\sigma_0^2 < \sigma_1^2$, the corresponding UMP size α test rejects H_0 whenever

$$\frac{L(\sigma_1^2|x_1, \cdots, x_n)}{L(\sigma_0^2|x_1, \cdots, x_n)} = \frac{(\sigma_1^2)^{-n/2}\exp\{-\sum\limits_{i=1}^{n} x_i^2/(2\sigma_1^2)\}}{(\sigma_0^2)^{-n/2}\exp\{-\sum\limits_{i=1}^{n} x_i^2/(2\sigma_0^2)\}} = \left(\frac{\sigma_0^2}{\sigma_1^2}\right)^{n/2} \exp(\frac{1}{2}\sum_{i=1}^{n} x_i^2(1/\sigma_0^2 - 1/\sigma_1^2)) > c_0,$$

or equivalently,

$$\frac{n}{2}(\log\sigma_0^2 - \log\sigma_1^2) + \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_{i=1}^n x_i^2 > \log c_0,$$

or equivalently,

$$\sum_{i=1}^n x_i^2/\sigma_0^2 > \{2\log c_0 - n(\log\sigma_0^2 - \log\sigma_1^2)\}\left(1 - \frac{\sigma_0^2}{\sigma_1^2}\right)^{-1} = c_0'$$

Since $\sum_{i=1}^n x_i^2/\sigma_0^2$ has a χ^2 distribution with degree of freedom $n-1$ under the null hypothesis, thus we reject the null hypothesis if this ratio is greater than $\chi^2_{1-\alpha}$. c_0 is chosen so that $c_0' = \chi^2_{1-\alpha}$.
We notice that this test does not involve σ_1^2, so it is the UMP test with size α for the hypothesis $H_0 : \sigma^2 \le \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_1^2$.
(c). The maximum likelihood estimator for σ^2 is $\frac{1}{n}\sum_{i=1}^n (x_i - \mu_0)^2$, denoted by $\hat{\sigma^2}_{MLE}$. Known that the log likelihood function of σ^2 is

$$\begin{aligned}\mathcal{L}(\sigma^2|x_1,\cdots,x_n) &= -\sum_{i=1}^n (0.5\log 2\pi + 0.5\log\sigma^2 + (x_i-\mu_0)^2/(2\sigma^2)) \\ &= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{\sum_{i=1}^n (x_i-\mu_0)^2}{2\sigma^2}\end{aligned}$$

we can obtain the log-likelihood ratio test statistic: $2(\mathcal{L}(\hat{\sigma}^2_{MLE}|x_1,\cdots,x_n) - \mathcal{L}(\hat{\sigma}_0^2|x_1,\cdots,x_n)) = n\{\hat{\sigma}^2_{MLE}/\sigma_0^2 - \log(\hat{\sigma}^2_{MLE}/\sigma_0^2) - 1\}$ which has a χ^2_1 distribution.

8.2.20

The likelihood ratio of λ_1 and λ_0 is

$$\frac{L(\lambda_1|x_1,\cdots,x_n)}{L(\lambda_0|x_1,cdots,x_n)} = \prod_{i=1}^n \exp(\lambda_0-\lambda_1)\left(\frac{\lambda_1}{\lambda_0}\right)^{x_i} = \exp(n(\lambda_0-\lambda_1))\left(\frac{\lambda_1}{\lambda_0}\right)^{n\bar{x}}$$

Then let the ratio is greater than c_0 and take the logarithm on both sides of the inequality, we can obtain:

$$n(\lambda_0-\lambda_1) + n\bar{x}(\log\lambda_1 - \log\lambda_0) > \log c_0$$

It is equivalent to

$$n\bar{x} > \frac{\log c_0 - n(\lambda_0-\lambda_1)}{\log\lambda_1 - \log\lambda_0}$$

Since $n\bar{x}$ has poisson distribution with parameter $n\lambda_0$, we need make use of this distribution and find a c_1 so that $P(n\bar{x} > c_1) = \alpha$. Then we can choose c_0 so that $\frac{\log c_0 - n(\lambda_0-\lambda_1)}{\log\lambda_1-\log\lambda_0} = c_1$. To obtain $P(n\bar{x} > c_1)$, we can use the result of 8.2.19, which gives that $P(n\bar{x} > c_1) = 1 - \frac{1}{c_1!}\int_{n\lambda_0}^{\infty} y^{c_1}e^{-y}dy$. For this test, we can see it does not include λ_1, so it is UMP test with size α.

Addtional Problem: A

The log likelihood function is $\mathcal{L}(\lambda|x_1, \cdots, x_n) = -n\lambda + n\bar{x}\log\lambda - \sum_{i=1}^{n}\log(x_i!)$. So $\frac{\partial \mathcal{L}}{\partial \lambda} = -n + \frac{n\bar{x}}{\lambda}$ and the MLE is $\hat{\lambda}_{MLE} = \bar{x}$. Since $\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = -\frac{n\bar{x}}{\lambda^2}$, the asymptotic variance of $\hat{\lambda}_{MLE}$ is the inverse of the expected fisher information, which is $-E[\frac{\partial^2 \mathcal{L}}{\partial \lambda^2}]^{-1} = \frac{\lambda}{n}$. For the Wald test, we use $\bar{x}/n$ as the estimate of variance of $\hat{\lambda}_{MLE}$ for the test statistic, which is $\frac{\bar{x}-\lambda_0}{\sqrt{\bar{x}/n}} \sim \mathrm{N}(0,1)$;for the score test, we use λ_0/n as the variance of $\hat{\lambda}_{MLE}$. The test statistic is $\frac{\bar{x}-\lambda_0}{\sqrt{\lambda_0/n}} \sim \mathrm{N}(0,1)$; The log likelihood ratio test statistic is $2(\mathcal{L}(\hat{\lambda}_{MLE}|x_1, \cdots, x_n) - \mathcal{L}(\lambda_0|x_1, \cdots, x_n)) \sim \chi^2_1$.

Addtional Problem: B

We plug in the maximum likelihood estimator for the variance $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2$ for σ^2 in the LR. Thus, $LR = \left(\frac{\sigma_0}{\hat{\sigma}}\right)^n \exp\left(-\frac{n}{2}\left(1 - \frac{\hat{\sigma}^2}{\sigma_0^2}\right)\right)$
Since that we know $2n\log\left(\frac{\sigma_0}{\hat{\sigma}}\right) - n\left(1 - \frac{\hat{\sigma_0}}{\sigma_0^2}\right)$ is converging to χ^2_1 in distribution, then likelihood ration test rejects $H_0 : \sigma^2 = \sigma_0^2$ in favor of $H_a : \sigma^2 \neq \sigma_0^2$ when $2n\ln\left(\frac{\sigma_0}{\hat{\sigma}}\right) - n\left(1 - \frac{\hat{\sigma_0}}{\sigma_0^2}\right) > \chi^2_{1,1-\alpha}$

Addtional Problem: C

i The MLE of θ is $\hat{\theta} = \frac{2x_1+x_2}{2(x_1+x_2+x_3)} = \frac{2\cdot 10+68}{2(10+68+112)} = 0.2316$.

ii The likelihood function is $L(\theta|s_1, \cdots, s_n) = 2^{x_2}\theta^{2x_1+x_2}(1-\theta)^{x_2+2x_3}$. By plugging in the MLE, we get that $LR = \frac{L(\hat{\theta}|s_1,\cdots,s_n)}{L(\theta|s_1,\cdots,s_n)} = (\frac{\hat{\theta}}{\theta})^{2x_1+x_2}(\frac{1-\hat{\theta}}{1-\theta})^{x_2+2x_3}$ and $2\log(LR) = 2(2x_1 + x_2)\log(\frac{\hat{\theta}}{\theta}) + 2(x_2 + 2x_3)\log(\frac{1-\hat{\theta}}{1-\theta}) \to \chi^2_1$.

iii $2\log(LR) = 115.497 > \chi^2_{0.05,1} = 3.84$. There is significant evidence to reject the H_0.

Addtional Problem: D

We learned from slides 79 and 80 of Chapter 6 that a level $1-\alpha$ confidence interval for the variance σ^2 of a normal distribution is $(\frac{\sum_{i=1}^{n}(x_i-\bar{x_n})^2}{c_2}, \frac{\sum_{i=1}^{n}(x_i-\bar{x_n})^2}{c_1})$. where c_1 and c_2 are constant such that $P(c_1 < W < c_2) = 1-\alpha$, and $W \sim \chi^2(n-1)$.Thus, according to Theorem B, an acceptance region for a level α test of $H_0 : \sigma^2 = \sigma_0^2$ is $A = \{X : \sigma_0^2 \in (\frac{\sum_{i=1}^{n}(x_i-\bar{x_n})^2}{c_2}, \frac{\sum_{i=1}^{n}(x_i-\bar{x_n})^2}{c_1})\}$, where c_1 and c_2 are constant such that $P(c_1 < W < c_2) = 1-\alpha$, and $W \sim \chi^2(n-1)$. That is, $A = \{X : c_1 \cdot \sigma_0^2 < \sum_{i=1}^{n}(x_i - \bar{x_n})^2 < c_2 \cdot \sigma_0^2\} = \{X : c_1 < \frac{(n-1)S^2}{\sigma_0^2} < c_2\}$, where S^2 is the sample variance of $x_1, ..., x_n$. When $\sigma_0 = 2, n = 16, \alpha = 0.05$, let's we choose $c_1 = \chi^2_{\alpha/2,n-1} = 6.262$ and $c_2 = \chi^2_{1-\alpha/2,n-1} = 27.488$. Then the acceptance region is $A = \{X : 1.670 < S^2 < 7.330\}$, where S^2 is the sample variance of $x_1, ..., x_n$. So the rejection region is $R = \{X : 1.670 > S^2 \ or \ S^2 > 7.330\}$.