STATISTICS 630 - Solution to Test I

June 21, 2013

1. Suppose that we independently toss five fair coins. Let $X =$ the number of heads face up on the five coins.

   (a) Find the probability distribution of X.

$$\begin{aligned} P(X = x) &= \binom{5}{x}\left(\frac{1}{2}\right)^5, \quad & x = 0, 1, 2, 3, 4, 5 \\ &= 0, & \text{otherwise.} \end{aligned}$$

   (b) Find the conditional probability of tossing exactly three heads given that you toss an odd number of heads.

$$\begin{aligned} P(X = 3 | X \in \{1, 3, 5\}) &= \frac{P(X = 3)}{P(X = 1) + P(X = 3) + P(X = 5)} \\ &= \frac{\binom{5}{3}}{\binom{5}{1} + \binom{5}{3} + \binom{5}{5}} = \frac{10}{5 + 10 + 1} = \frac{5}{8}. \end{aligned}$$

2. The joint probability mass function (pmf) for (X, Y) is given in the following table:

		y			
		0	1	2	3
	1	1/6	1/6	0	0
x	2	1/12	2/12	1/12	0
	3	1/24	3/24	3/24	1/24

   (a) Compute $P(X = Y)$ and $P(X > Y)$.

$$\begin{aligned} P(X = Y) &= p(1, 1) + p(2, 2) + p(3, 3) = 1/6 + 1/12 + 1/24 = 7/24 \\ P(X > Y) &= p(1, 0) + p(2, 0) + p(3, 0) + p(2, 1) + P(3, 1) + p(3, 2) \\ &= 1/6 + 1/12 + 1/24 + 2/12 + 3/24 + 3/24 = 17/24. \end{aligned}$$

   (b) Obtain the marginal pmf of X and the conditional pmf of X given $Y = 2$. Based on these pmfs, what can you say about the independence of X and Y?

   $p_X(1) = 1/6 + 1/6 = 1/3$, $p_X(2) = 1/12 + 2/12 + 1/12 = 1/3$, $p_X(3) = 1/24 + 3/24 + 3/24 + 1/24 = 1/3$, $p_X(x) = 0$ for all other x.
   $P(X = 1|Y = 2) = 0$, $P(X = 2|Y = 2) = (1/12)/(5/24) = 2/5$, $P(X = 3|Y = 2) = (3/24)/(5/24) = 3/5$, $P(X = x|Y = 2) = 0$ for all other x.
   Since these two pmfs differ, X and Y are not independent.

3. A professor carries 12 bills of various denominations in her wallet. Suppose that she has 5 one-dollar bills, 3 five-dollar bills, and 4 ten-dollar bills. She selects three of the bills at random from her wallet.

   (a) Find the probability that the bills are all of the same denomination.

$$P(\text{All 3 bills are of the same denomination}) = \frac{\binom{5}{3}+\binom{3}{3}+\binom{4}{3}}{\binom{12}{3}} = \frac{10+1+4}{220} = \frac{15}{220} = \frac{3}{44}$$

   (b) Find the probability that the bills are all of different denominations.

$$P(\text{All 3 bills are of the different denominations}) = \frac{\binom{5}{1}\binom{3}{1}\binom{4}{1}}{\binom{12}{3}} = \frac{5\times 3\times 4}{220} = \frac{60}{220} = \frac{3}{11}$$

4. Suppose that X is a random variable with the probability density function

$$f_X(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the cumulative distribution function of X.

   Let $0 < x < 2$. Then

$$F_X(x) = \int_0^x \frac{3}{8}t^2 dt = \left(\frac{3}{8}\right)\left(\frac{t^3}{3}\right)\Big|_0^x = \frac{x^3}{8}.$$

   Thus,

$$F_X(x) = \begin{cases} 1 & \text{for } x \geq 2 \\ \frac{x^3}{8} & \text{for } 0 < x < 2 \\ 0 & \text{for } x \leq 0. \end{cases}$$

   (b) Let $Y = X^2$. Derive the cumulative distribution function of Y.

   Let $0 < y < 4$. Then

$$P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y}) = \frac{y^{3/2}}{8}.$$

   Thus,

$$F_Y(y) = \begin{cases} 1 & \text{for } y \geq 4 \\ \frac{y^{3/2}}{8} & \text{for } 0 < y < 4 \\ 0 & \text{for } y \leq 0. \end{cases}$$

5. A professor at a major university uses a plagiarism checker website to check for the originality of the papers turned in for his class. Of papers that have been copied from the web, $\frac{5}{6}$ will test positive (labelled "copied"). Of papers that are original (not copied from the web), $\frac{2}{3}$ will test negative (labelled "original"). Suppose that $\frac{1}{4}$ of all papers in this class have been copied from the web.

   (a) Obtain the probability that a random selected paper from this class will test negative (be identified as "original.")

   Let $A =$ the event that the paper is copied from the internet, and let $B =$ the event that the paper is classified as copied from the internet. Then

$$P(B^c) = P(B^C|A)P(A)+P(B^C|A^C)P(A^C) = \left(1 - \frac{5}{6}\right)\left(\frac{1}{4}\right)+\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = \frac{1}{24}+\frac{1}{2} = \frac{13}{24}.$$

   (b) Given that a paper tests positive (is identified as "copied"), obtain the probability that the paper was actually copied from the web.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{(\frac{5}{6})(\frac{1}{4})}{(\frac{5}{6})(\frac{1}{4}) + (\frac{1}{3})(\frac{3}{4})} = \frac{5}{11}.$$

   Notice that the denominator is $P(B) = \frac{11}{24} = 1 - \frac{13}{24}$.