STAT 636, Fall 2015 - Assignment 2

SOLUTIONS

1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

Without using a computer:

(a) Find the eigenvalues and normalized eigenvectors of $\mathbf{A}$.

WE SOLVE THE CHARACTERISTIC EQUATION

$$|\mathbf{A} - \lambda \mathbf{I}| = \lambda^2 + \lambda - 6 = 0$$

WITH SOLUTIONS $\lambda_1 = 2$ AND $\lambda_2 = -3$. THE FIRST EIGENVALUE $\mathbf{x}_1$ THEN SATISFIES

$$\mathbf{A}\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$$

THIS CORRESPONDS TO THE TWO EQUATIONS

$$\begin{aligned} x_{11} + 2x_{12} &= 2x_{11} \\ 2x_{11} - 2x_{12} &= 2x_{12} \end{aligned}$$

WHICH CAN BE SOLVED WITH ANY $\mathbf{x}_1$ SUCH THAT $x_{11} = 2x_{12}$. ARBITRARILY TAKING $x_{12} = 1$, WE HAVE $\mathbf{x}_1' = [2, 1]$ WITH LENGTH $\sqrt{5}$. THUS

$$\mathbf{e}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

SIMILARLY,

$$\mathbf{e}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

(b) Write the spectral decomposition of $\mathbf{A}$.

THE SPECTRAL DECOMPOSITION IS

$$\mathbf{A} = \lambda_1 \mathbf{e}_1 \mathbf{e}_1' + \lambda_2 \mathbf{e}_2 \mathbf{e}_2'$$

WE HAVE

$$\mathbf{e}_1 \mathbf{e}_1' = \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{AND} \quad \mathbf{e}_2 \mathbf{e}_2' = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

THEN WE HAVE

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \frac{1}{5} \left(2 \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \right)$$

(c) Verify that the determinant of $\mathbf{A}$ equals the product of its eigenvalues.

THE DETERMINANT IS

$$|\mathbf{A}| = (1)(-2) - (2)(2) = -6 = \lambda_1 \times \lambda_2$$

(d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of $\mathbf{A}$ equals the sum of its eigenvalues.

WE HAVE

$$\mathrm{tr}\mathbf{A} = 1 - 2 = \lambda_1 + \lambda_2$$

(e) Is $\mathbf{A}$ orthogonal? Why or why not?

FOR $\mathbf{A}$ TO BE ORTHOGONAL, ITS COLUMNS MUST BE UNIT LENGTH AND MUTUALLY PERPENDICULAR (WITH INNER PRODUCTS EQUAL TO ZERO). NEITHER OF THESE CONDITIONS HOLD FOR $\mathbf{A}$.

(f) Is $\mathbf{A}$ positive definite? Why or why not?

SINCE ONE OF THE EIGENVALUES IS NEGATIVE, $\mathbf{A}$ CAN NOT BE POSITIVE DEFINITE. FOR EXAMPLE, LET $\mathbf{x}' = [1, 3]$. THEN $\mathbf{x}'\mathbf{A}\mathbf{x} = -5$.

(g) Find $\mathbf{A}^{-1}$ and determine its eigenvalues and normalized eigenvectors.

WE CAN USE THE RELATION

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{\lambda_1}\mathbf{e}_1\mathbf{e}_1' + \frac{1}{\lambda_2}\mathbf{e}_2\mathbf{e}_2' \\ &= \frac{1}{10}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} - \frac{1}{15}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \\ &= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\end{aligned}$$

THE EIGENVALUES OF $\mathbf{A}^{-1}$ ARE THE INVERSE OF THOSE FOR $\mathbf{A}$. THE EIGENVECTORS ARE THE SAME.

2. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$

These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of $\mathbf{A}$ and $\mathbf{B}$ are nearly linearly dependent. Show that $\mathbf{A}^{-1} \approx (-3)\mathbf{B}^{-1}$. So, small changes - perhaps due to rounding - can result in substantially different inverses.

3. Derive expressions for the means and variances of the following linear combinations in terms of the means and covariances of the random variables X_1, X_2, and X_3.

(a) $X_1 - 2X_2$.

THIS IS $\mathbf{c}'\mathbf{X}$, WITH $\mathbf{c}' = [1, -2, 0]$. WE HAVE $E(X_1 - 2X_2) = \mathbf{c}'\boldsymbol{\mu} = \mu_1 - 2\mu_2$ AND

$$\begin{aligned}\text{VAR}(X_1 - 2X_2) = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} = [1, -2, 0] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \\ = \sigma_{11} - 4\sigma_{12} + 4\sigma_{22}\end{aligned}$$

YOU MAY RECALL THAT, FOR A LINEAR COMBINATION OF TWO RANDOM VARIABLES X_1 AND X_2:

$$\text{VAR}(aX_1 + bX_2) = a^2\text{VAR}(X_1) + b^2\text{VAR}(X_2) + 2ab\text{COV}(X_1, X_2)$$

WE HAVE JUST LEARNED HOW TO COMPUTE SUCH EXPRESSIONS MORE GENERALLY.

(b) $X_1 + 2X_2 - X_3$.
THIS IS $\mathbf{c}'\mathbf{X}$, WITH $\mathbf{c}' = [1, 2, -1]$. WE HAVE $E(X_1 + 2X_2 - X_3) = \mathbf{c}'\boldsymbol{\mu} = \mu_1 + 2\mu_2 - \mu_3$ AND

$$\begin{aligned}\text{VAR}(X_1 + 2X_2 - X_3) = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} = [1, 2, -1] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \\ = \sigma_{11} + 4\sigma_{22} + \sigma_{33} + 4\sigma_{12} - 2\sigma_{13} - 4\sigma_{23}\end{aligned}$$

FOR A LINEAR COMBINATION OF THREE RANDOM VARIABLES, WE HAVE

$$\begin{aligned}\text{VAR}(aX_1 + bX_2 + cX_3) = a^2\text{VAR}(X_1) + b^2\text{VAR}(X_2) + c^2\text{VAR}(X_3) + \\ 2ab\text{COV}(X_1, X_2) + 2ac\text{COV}(X_1, X_3) + 2bc\text{COV}(X_2, X_3)\end{aligned}$$

(c) $3X_1 - 4X_2$ if X_1 and X_2 are independent (so, $\sigma_{12} = 0$).
THIS IS $\mathbf{c}'\mathbf{X}$, WITH $\mathbf{c}' = [3, -4, 0]$. WE HAVE $E(3X_1 - 4X_2) = \mathbf{c}'\boldsymbol{\mu} = 3\mu_1 - 4\mu_2$ AND

$$\begin{aligned}\text{VAR}(3X_1 - 4X_2) = \mathbf{c}'\boldsymbol{\Sigma}\mathbf{c} = [3, -4, 0] \begin{bmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} \\ = 9\sigma_{11} + 16\sigma_{22}\end{aligned}$$

FOR TWO INDEPENDENT VARIABLES, $\text{VAR}(aX_1 + bX_2) = a^2\text{VAR}(X_1) + b^2\text{VAR}(X_2)$.

4. Let $\boldsymbol{\mu}' = [1, 1]$, and consider the following covariance matrices

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 1.00 & 0.80 \\ 0.80 & 1.00 \end{bmatrix} \quad \boldsymbol{\Sigma}_2 = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \quad \boldsymbol{\Sigma}_3 = \begin{bmatrix} 1.00 & -0.80 \\ -0.80 & 1.00 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_4 = \begin{bmatrix} 1.00 & 0.40 \\ 0.40 & 0.25 \end{bmatrix} \quad \boldsymbol{\Sigma}_5 = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 0.25 \end{bmatrix} \quad \boldsymbol{\Sigma}_6 = \begin{bmatrix} 1.00 & -0.40 \\ -0.40 & 0.25 \end{bmatrix}$$

$$\boldsymbol{\Sigma}_7 = \begin{bmatrix} 0.25 & 0.40 \\ 0.40 & 1.00 \end{bmatrix} \quad \boldsymbol{\Sigma}_8 = \begin{bmatrix} 0.25 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \quad \boldsymbol{\Sigma}_9 = \begin{bmatrix} 0.25 & -0.40 \\ -0.40 & 1.00 \end{bmatrix}$$

For each covariance matrix:

(a) Draw the ellipse consisting of all points $\mathbf{x}' = [x_1, x_2]$ for which

$$(\mathbf{x} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\mathbf{x} - \boldsymbol{\mu}) = \chi_2^2(0.05)$$

where $\chi_2^2(0.05)$ is the 95th percentile of the chi square distribution with $p = 2$ degrees of freedom. You can draw it by hand if you want, as long as you label the axis tick marks carefully. Alternatively, you can use the `draw.ellipse` function from the `plotrix` package.

SEE THE FIGURES AT THE END OF THE DOCUMENT. IN EACH CASE, LET λ_i AND $\mathbf{e}_i$ BE THE iTH EIGENVALUE, EIGENVECTOR PAIR FOR $\boldsymbol{\Sigma}$, $i = 1, 2$. THE ELLIPSE HAS AXES EQUAL TO THE $\mathbf{e}_i$, WITH THE HALF-LENGTH IN THE DIRECTION OF $\mathbf{e}_i$ EQUAL TO $\sqrt{\chi_2^2(0.05)\lambda_i}$. NOTICE THAT THE ORIENTATION OF THE ELLIPSE DEPENDS ON THE CORRELATION BETWEEN X_1 AND X_2, AND ITS SCALING DEPENDS ON THE RELATIVE MAGNITUDES OF THE VARIANCES OF X_1 AND X_2.

(b) Simulate 5000 realizations from the corresponding bivariate normal distribution using `rmvnorm` function from the `mvtnorm` package and compute the proportion that are inside the ellipse you just drew.

SEE THE CODE BELOW. IN EACH CASE, ABOUT 95% OF ALL SIMULATED REALIZATIONS FELL WITHIN THE ELLIPSE.

For an arbitrary multivariate normal random vector $\mathbf{X} = [X_1, X_2, \ldots, X_p]$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, what would you guess $P\left((\mathbf{X} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\mathbf{X} - \boldsymbol{\mu}) \leq \chi_p^2(\alpha)\right)$ equals?

BASED ON WHAT WE SAW FOR $p = 2$, A SENSIBLE GUESS WOULD BE THAT

$$P\left((\mathbf{X} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\mathbf{X} - \boldsymbol{\mu}) \leq \chi_p^2(\alpha)\right) = 1 - \alpha$$

IT DOES INDEED, AS WE WILL SEE IN TOPIC 4.

5. Consider the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}' = [4, 3, 2, 1]$ and covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

and consider the linear combinations $\mathbf{AX}^{(1)}$ and $\mathbf{BX}^{(2)}$. Find the following:

(a) $E\left(\mathbf{X}^{(1)}\right)$.

THIS IS JUST $(\boldsymbol{\mu}^{(1)})' = [\mu_1, \mu_2]$.

(b) $E\left(\mathbf{B}\mathbf{X}^{(2)}\right)$.
THIS IS

$$\mathbf{B}\boldsymbol{\mu}^{(2)} = \begin{bmatrix} \mu_3 - 2\mu_4 \\ 2\mu_3 - \mu_4 \end{bmatrix}$$

(c) $\text{Cov}\left(\mathbf{A}\mathbf{X}^{(1)}\right)$.
THIS IS

$$\mathbf{A}\boldsymbol{\Sigma}_{11}\mathbf{A}' = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 7$$

(d) $\text{Cov}\left(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}\right)$.
THIS IS

$$\boldsymbol{\Sigma}_{12} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$$

(e) $\text{Cov}\left(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)}\right)$.
WE CAN WRITE

$$\begin{aligned} \text{Cov}\left(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)}\right) &= E\left(\mathbf{A}\mathbf{X}^{(1)} - \mathbf{A}\boldsymbol{\mu}^{(1)}\right)\left(\mathbf{B}\mathbf{X}^{(2)} - \mathbf{B}\boldsymbol{\mu}^{(2)}\right)' \\ &= \mathbf{A}E\left(\mathbf{X}^{(1)} - \boldsymbol{\mu}^{(1)}\right)\left(\mathbf{X}^{(2)} - \boldsymbol{\mu}^{(2)}\right)'\mathbf{B}' = \mathbf{A}\boldsymbol{\Sigma}_{12}\mathbf{B}' \\ &= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 6 \end{bmatrix} \end{aligned}$$

6. Generate a random sample of $n = 100$ observations from the bivariate normal distribution with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1.0 & 0.8 \\ 0.8 & 1.0 \end{bmatrix}$$

So that we all end up with the same numbers, first set your random seed to 101: `set.seed(101)`.
Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means of the two components and

$$s_{11} = \frac{1}{n}\sum_{j=1}^{n}\left(x_{1j} - \bar{x}_1\right)^2, \quad s_{22} = \frac{1}{n}\sum_{j=1}^{n}\left(x_{2j} - \bar{x}_2\right)^2, \quad \text{and} \quad s_{12} = \frac{1}{n}\sum_{j=1}^{n}\left(x_{1j} - \bar{x}_1\right)\left(x_{2j} - \bar{x}_2\right)$$

be the sample variances and sample covariance, computed by dividing by n instead of $n-1$. Thus,

$$\mathbf{S}_n = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Also, let

$$r_{12} = \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}}$$

be the sample correlation between the two variables. Finally, with $\mathbf{y}_i$ the vector of n observations on variable i, let $\mathbf{d}_i = \mathbf{y}_i - \bar{x}_i\mathbf{1}$ be the ith deviation vector, and $\mathbf{D}$ be the $n \times 2$ matrix with columns equal to the $\mathbf{d}_i$, $i = 1, 2$. Verify the following relations:

(a) $s_{11} = \frac{1}{n}\mathbf{d}_1'\mathbf{d}_1$.
SEE THE CODE BELOW.

(b) $s_{22} = \frac{1}{n}\mathbf{d}_2'\mathbf{d}_2$.
SEE THE CODE BELOW.

(c) $s_{12} = \frac{1}{n}\mathbf{d}_1'\mathbf{d}_2$.
SEE THE CODE BELOW.

(d) $S_n = \frac{1}{n}\sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'$.
SEE THE CODE BELOW.

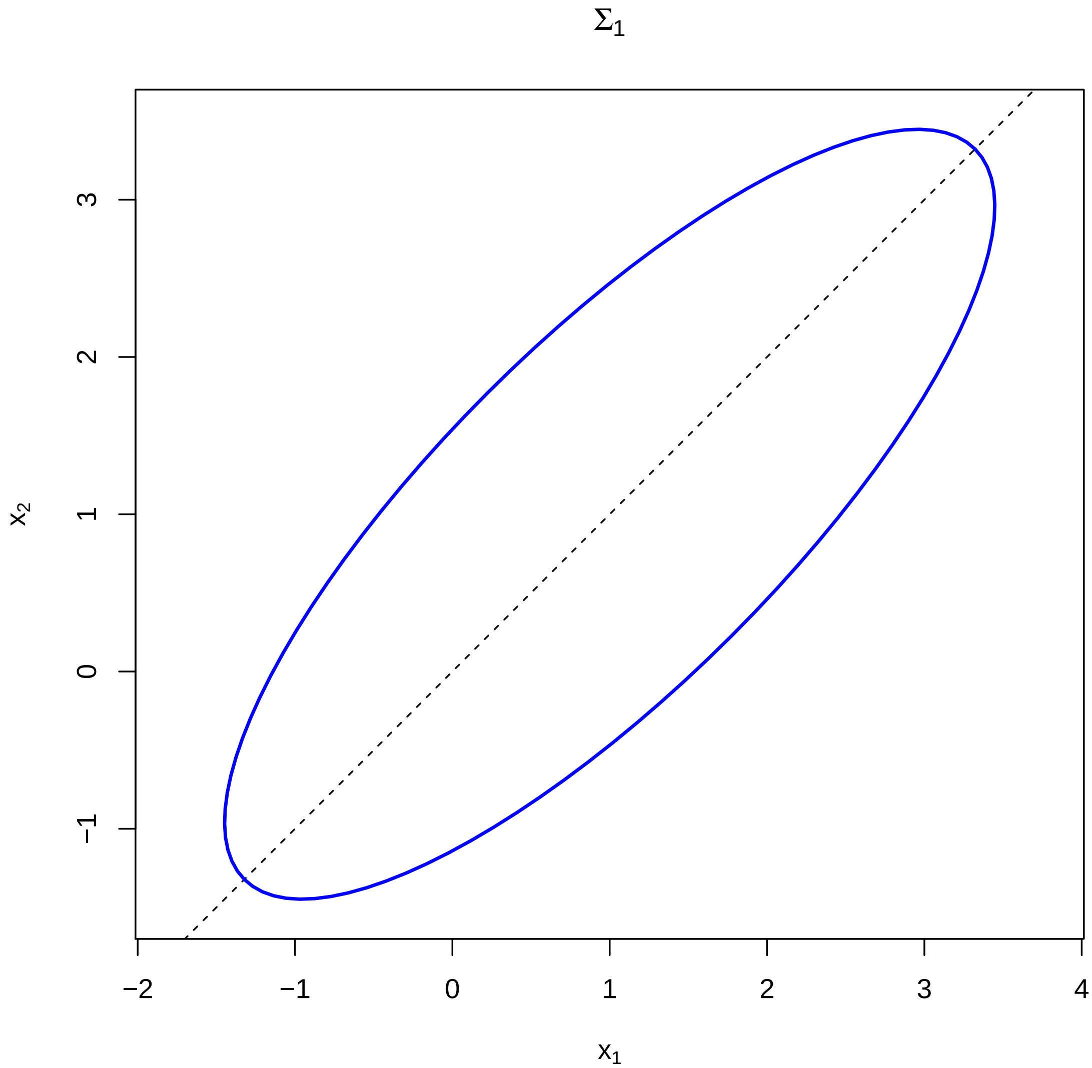
(e) $S_n = \frac{1}{n}\mathbf{D}'\mathbf{D}$.
SEE THE CODE BELOW.

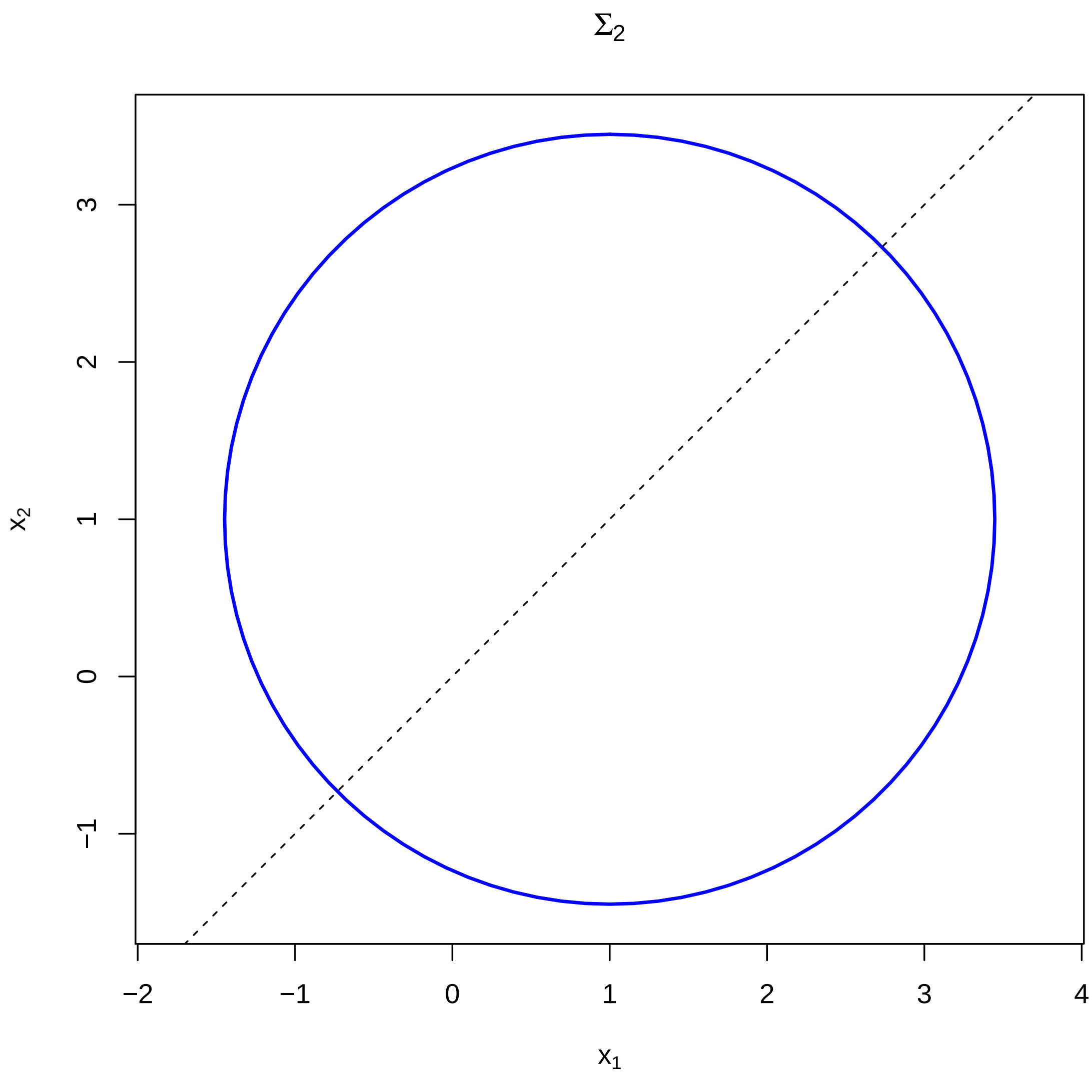
(f) $r_{12} = \cos(\theta)$, where θ is the angle between $\mathbf{d}_1$ and $\mathbf{d}_2$.
SEE THE CODE BELOW.

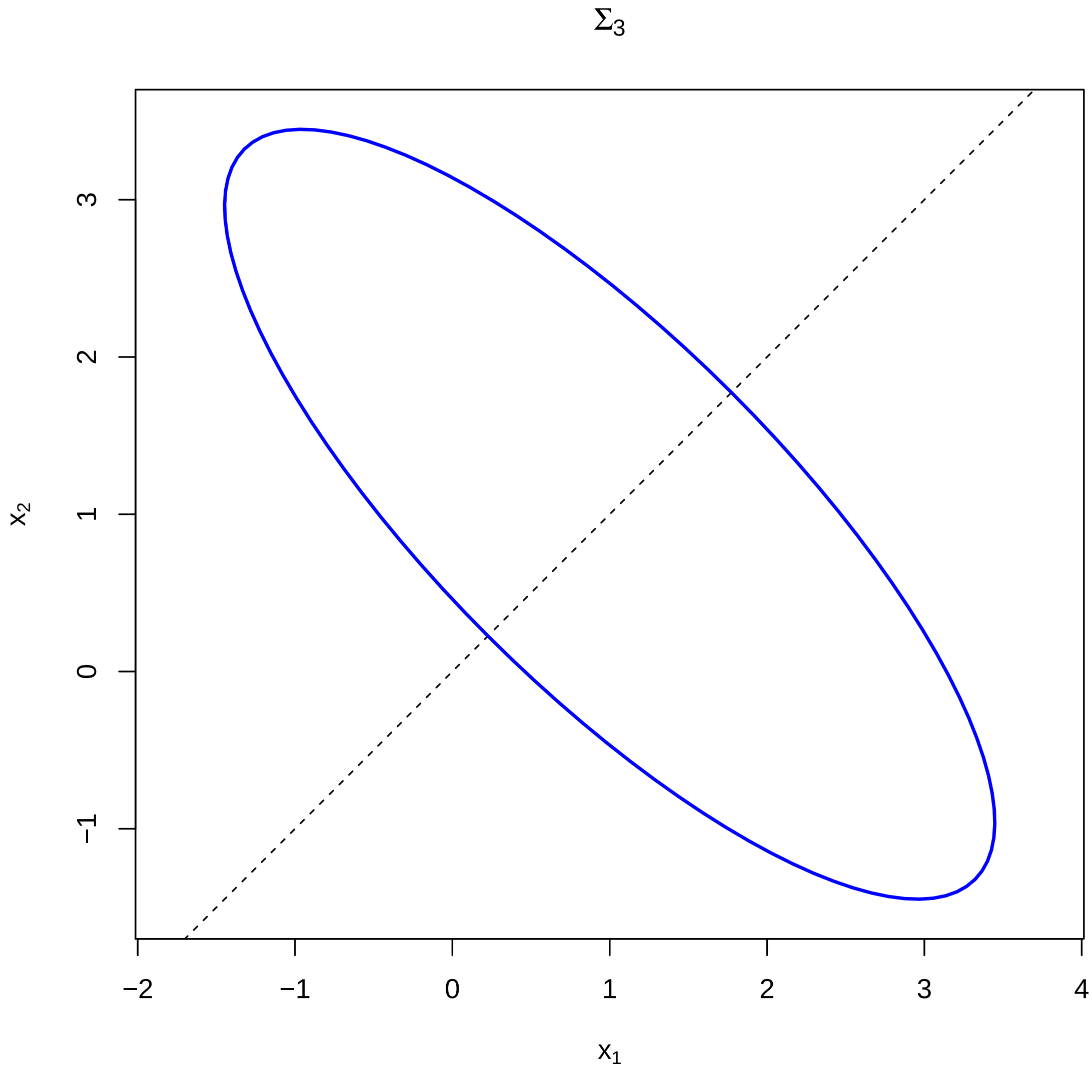
```
####
#### (1)
####

A <- matrix(c(1, 2, 2, -2), nrow = 2)

## Eigenvalues and eigenvectors of A.
ee <- eigen(A)
lambda <- ee$values
ee <- ee$vectors

## Spectral decomposition.
lambda[1] * ee[, 1] %*% t(ee[, 1]) + lambda[2] * ee[, 2] %*% t(ee[, 2])

## Determinant.
det(A)
prod(lambda)

## Trace.
sum(diag(A))
sum(lambda)

## Orthogonal?
t(A[, 1]) %*% A[, 1]
t(A[, 1]) %*% A[, 2]

## Positive definite?
any(lambda < 0)
x <- c(1, 3)
t(x) %*% A %*% x

## Inverse.
solve(A)
A_inv <- (1 / lambda[1]) * ee[, 1] %*% t(ee[, 1]) +
  (1 / lambda[2]) * ee[, 2] %*% t(ee[, 2])
eigen(A_inv)

####
#### (2)
####

A <- B <- matrix(c(4, 4.001, 4.001, 4.002), nrow = 2)
B[2, 2] <- B[2, 2] + 0.000001

solve(A)
```

```
solve(B)

####
#### (4)
####

library(mvtnorm)
library(plotrix)

mu <- c(1, 1)
rho <- 0.8

## Function to draw ellipse of constant distance from mu.
ellipse_f <- function(mu, Sigma, alpha) {
  c2 <- qchisq(1 - alpha, 2)

  ee <- eigen(Sigma)
  lambda <- ee$values
  theta <- acos(ee$vec[1, 1]) * 360 / (2 * pi) * sign(ee$vec[2, 1])

  plot(c(-1.5, 3.5), c(-1.5, 3.5), xlab = expression(x[1]), ylab = expression(x[2]),
    asp = 1, type = "n")
  abline(0, 1, lty = 2)
  draw.ellipse(mu[1], mu[2], sqrt(c2 * lambda[1]), sqrt(c2 * lambda[2]), angle = theta,
    border = "blue", lwd = 2)

}

## Function to construct Sigma, given variances and correlation.
Sigma_f <- function(sg, rho) {
  matrix(c(sg[1] ^ 2, rho * sg[1] * sg[2], rho * sg[1] * sg[2], sg[2] ^ 2), nrow = 2)
}

## The Sigma matrices to consider.
Sigma <- list(Sigma_f(c(1, 1), rho), Sigma_f(c(1, 1), 0), Sigma_f(c(1, 1), -rho),
  Sigma_f(c(1, 0.5), rho), Sigma_f(c(1, 0.5), 0), Sigma_f(c(1, 0.5), -rho),
  Sigma_f(c(0.5, 1), rho), Sigma_f(c(0.5, 1), 0), Sigma_f(c(0.5, 1), -rho))

for(i in 1:9) {
  ## Draw ellipse.
  pdf(paste("figures/Sigma_", i, ".pdf", sep = ""))
  ellipse_f(mu, Sigma[[i]], alpha = 0.05)
  title(main = bquote(Sigma[.(i)]))
  dev.off()
```

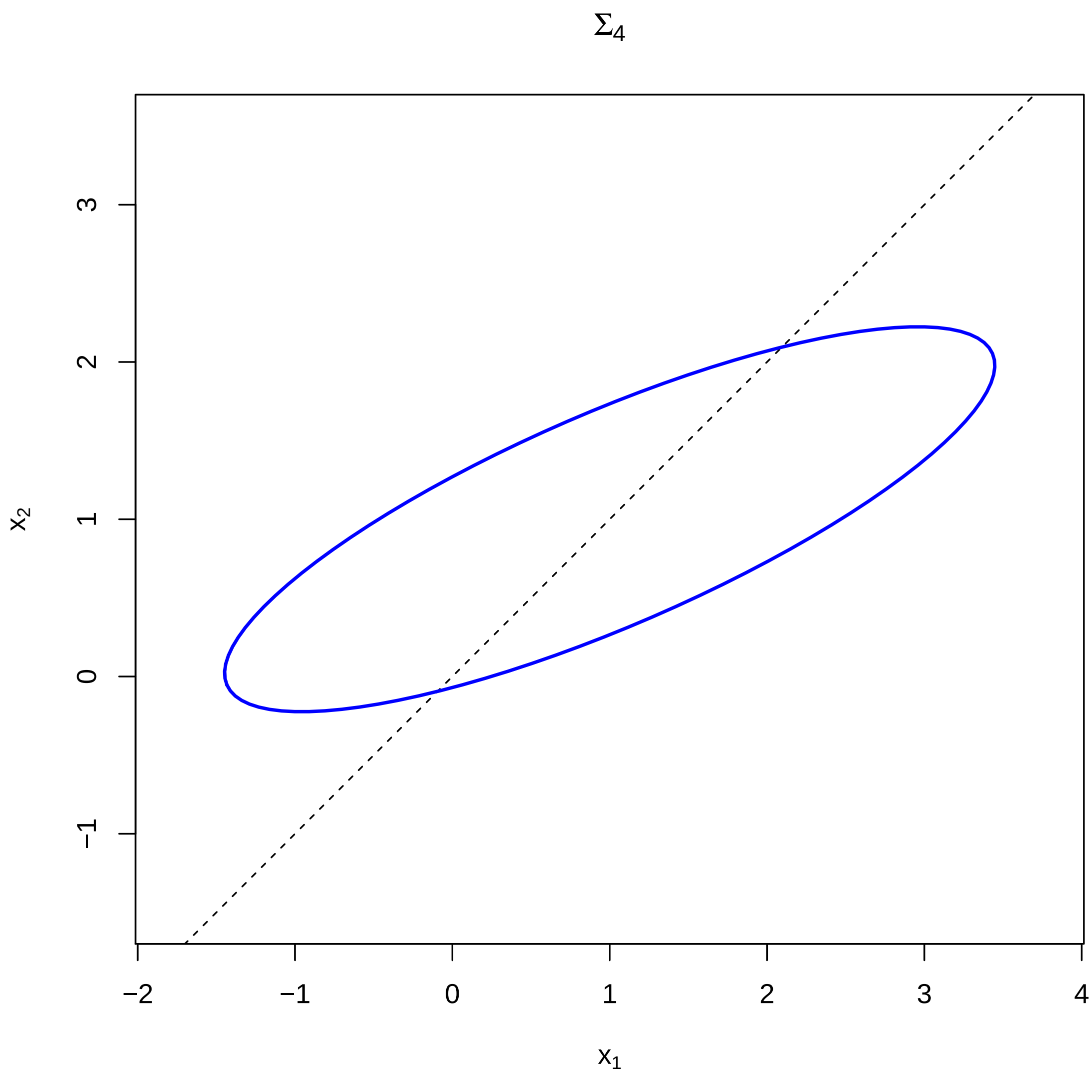
```
  ## Simulated realizations and proportions inside the ellipse.
  X <- rmvnorm(5000, mean = mu, sigma = Sigma[[i]])
  Sigma_inv <- solve(Sigma[[i]])
  dd <- apply(X, 1, function(x) { t(x - mu) %*% Sigma_inv %*% (x - mu) })
  print(round(mean(dd <= qchisq(0.95, 2)), 3))
}

####
#### (6)
####

set.seed(101)

n <- 100
mu <- c(1, -1)
Sigma <- matrix(c(1, 0.8, 0.8, 1), nrow = 2)

## Simulate data.
X <- rmvnorm(n, mean = mu, sigma = Sigma)
x_bar <- colMeans(X)

## Sample variances and covariance.
S <- var(X) * (n - 1) / n
S_0 <- matrix(0, nrow = 2, ncol = 2)
for(j in 1:n)
  S_0 <- S_0 + (X[j, ] - x_bar) %*% t(X[j, ] - x_bar)
S_0 / n

## Sample correlation.
cor(X)
S[1, 2] / (sqrt(S[1, 1]) * sqrt(S[2, 2]))

## Construct deviation vectors, resulting in the mean-centered version of X.
D <- X - matrix(1, nrow = n, ncol = 1) %*% x_bar
colMeans(D)
colMeans(scale(X, center = TRUE, scale = FALSE))

## Reproduce sample variances and covariance using the deviation vectors.
t(D[, 1]) %*% D[, 1] / n
t(D[, 2]) %*% D[, 2] / n
t(D[, 1]) %*% D[, 2] / n
t(D) %*% D / n

## Reproduce sample correlation using the deviation vectors.
(t(D[, 1]) %*% D[, 2]) / (sqrt(t(D[, 1]) %*% D[, 1]) * sqrt(t(D[, 2]) %*% D[, 2]))
```

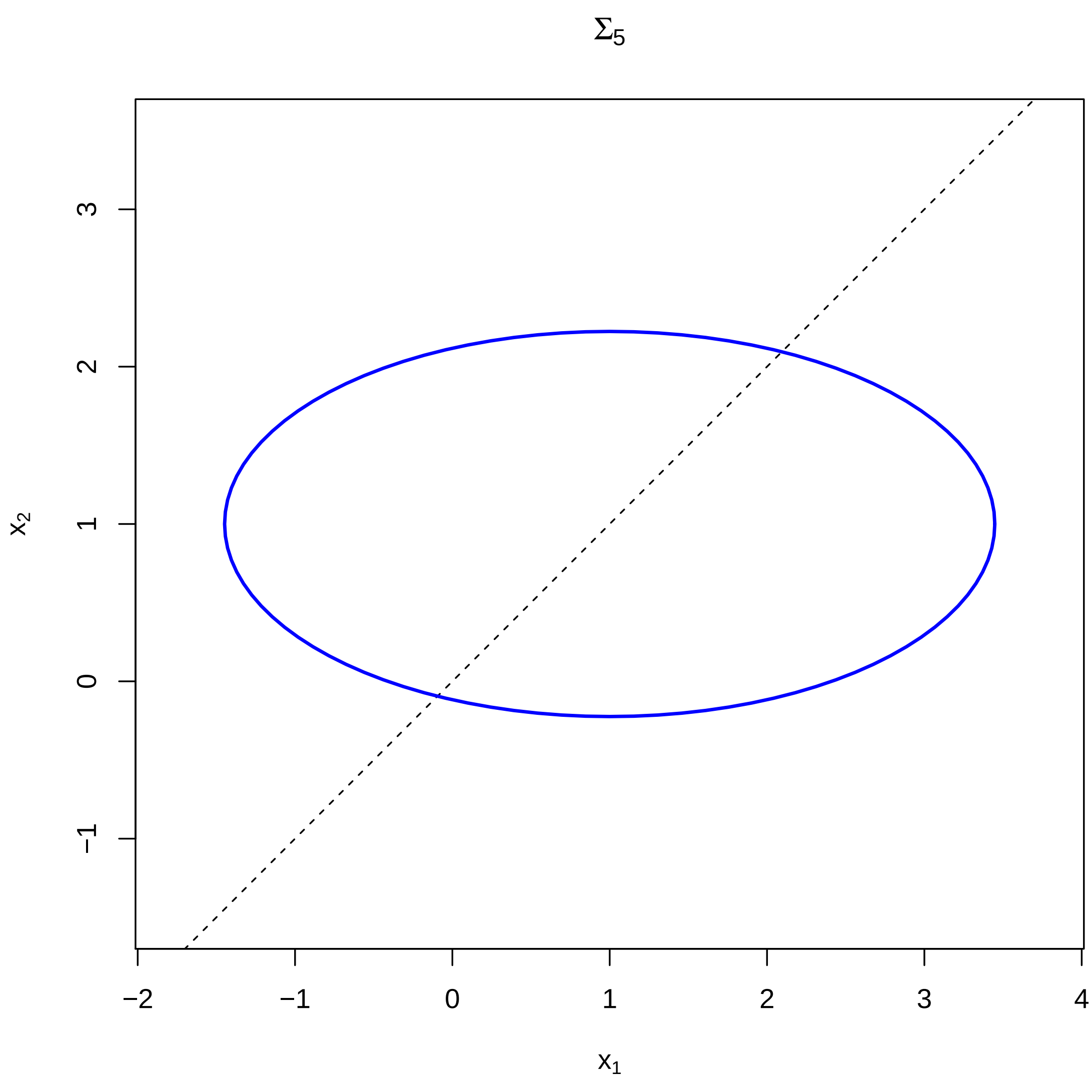

Σ1
3
2
1
0
−1
x2
−2
−1
0
1
2
3
4
x1


Σ2
x1
x2
−2
−1
0
1
2
3
4
−1
0
1
2
3



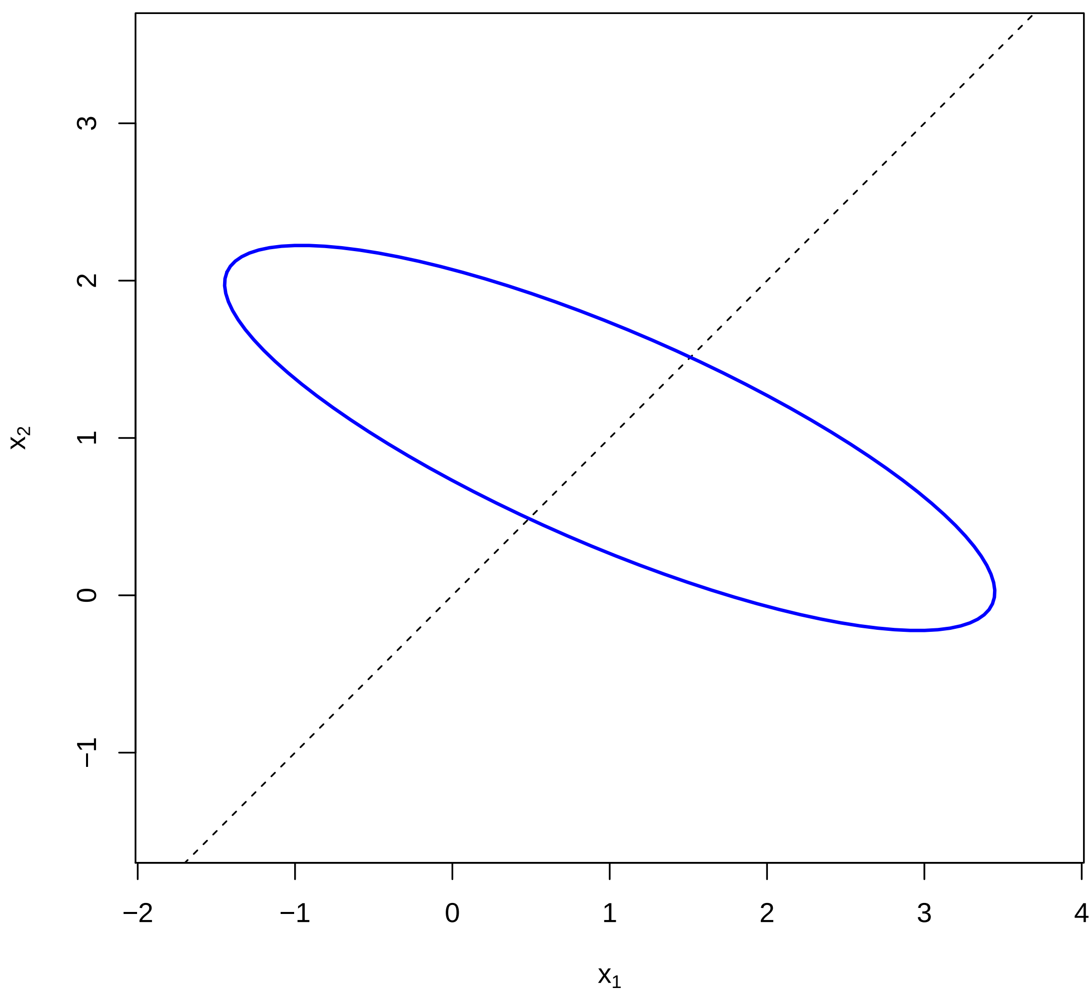
Σ3
x2
x1
3
2
1
0
−1
−2
−1
0
1
2
3
4

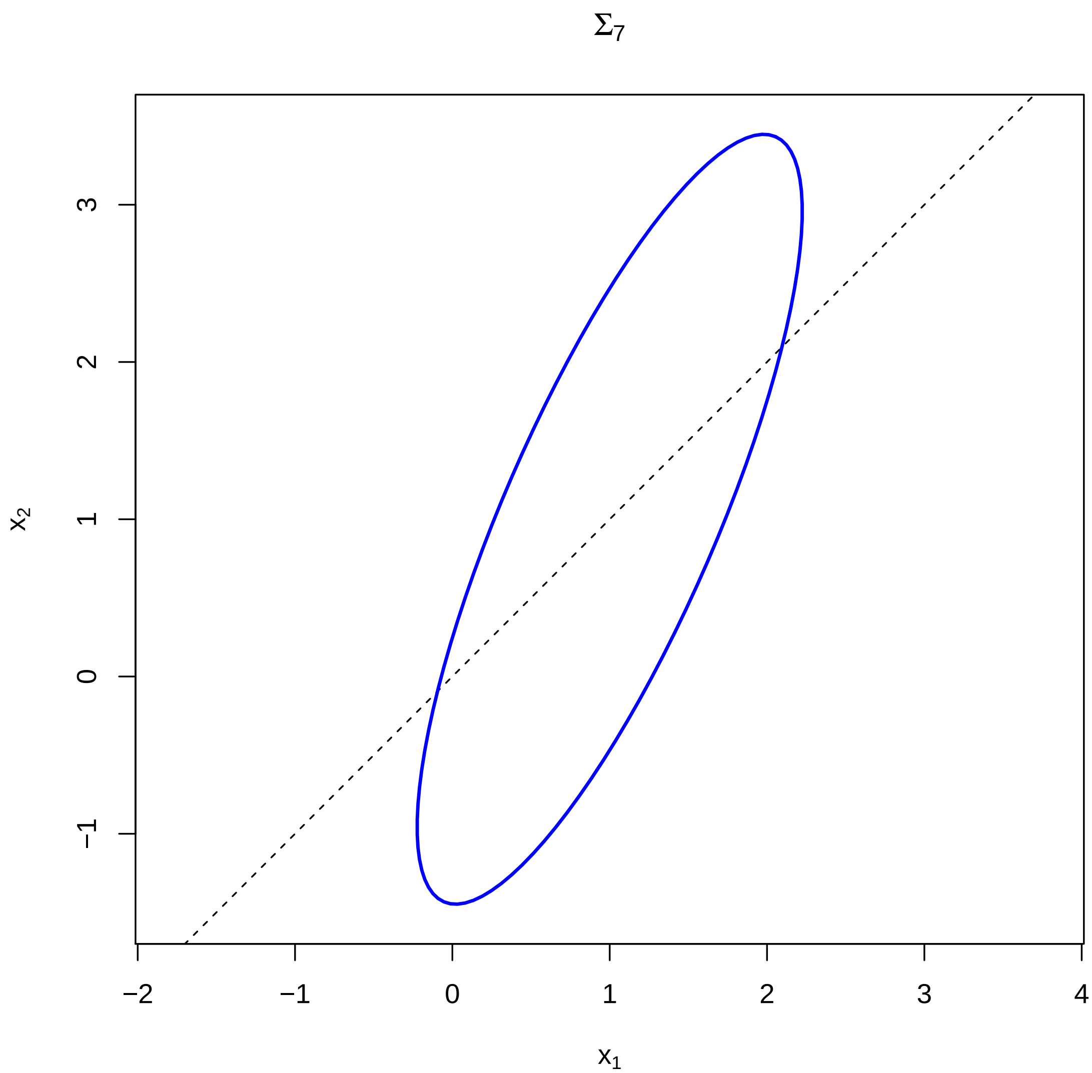

Σ4
x2
x1
3
2
1
0
−1
−2
−1
0
1
2
3
4



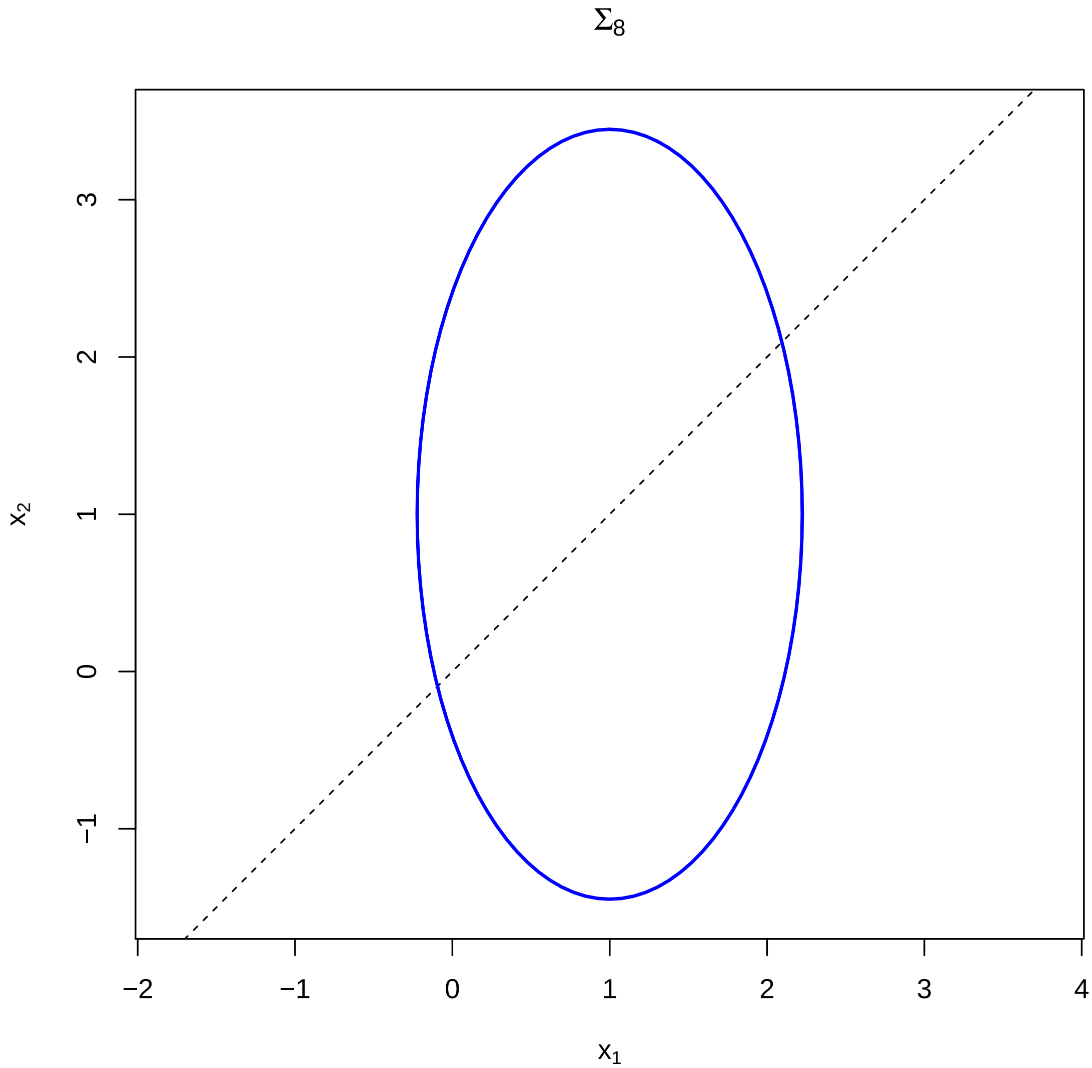
Σ5
x2
x1
3
2
1
0
−1
−2
−1
0
1
2
3
4

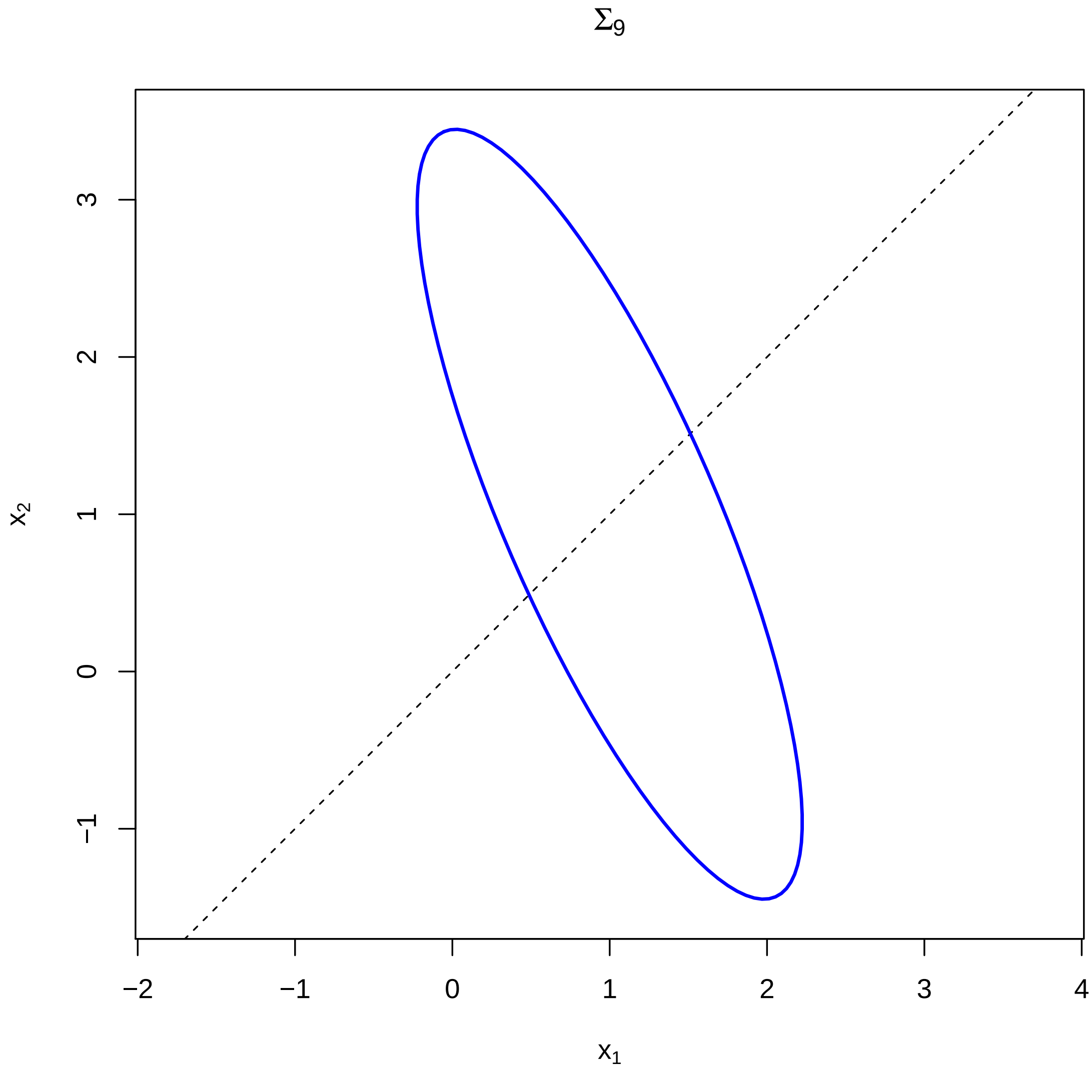
Σ_6





Σ7
x2
3
2
1
0
−1
x1
−2
−1
0
1
2
3
4


Σ8
x2
x1
3
2
1
0
−1
−2
−1
0
1
2
3
4


Σ9
x1
x2
−2
−1
0
1
2
3
4
−1
0
1
2
3