STAT 641 - ASSIGNMENT 8 - SOLUTIONS

1. (10 points) Test $H_o: \mu \leq 10000$ vs $H_1: \mu > 10000$. Decision Rule: Reject H_o if $\bar{Y} \geq 10500$.

   P[Type I error at $\mu = 10000$] = P[$\bar{Y} \geq 10500$ when $\mu = 10000$] = $P[Z \geq \sqrt{10}(10500 - 10000)/1000] = 1 - pnorm(1.58) = .057$

   P[Type II error at $\mu = 10700$] = P[$\bar{Y} \leq 10500$ when $\mu = 10700$] = $P[Z \leq \sqrt{10}(10500 - 10700)/1000] = pnorm(-.632) = .264$

2. (10 points) Refer to the previous question. Suppose the company wants the probability of Type I error to be at most .01.

   (a.) Decision Rule: Reject H_o if $\bar{Y} \geq 10000 + Z_{.01}1000/\sqrt{10} = 10735.7$

   (b.) The probability of a Type II error at μ_1 is computed by noting $\bar{Y}$ has a $N(\mu_1, (1000)^2/10)$ distribution P[$\bar{Y} \leq 10735.7$ when $\mu = \mu_1$] = $P[Z \leq \sqrt{10}(10735.7 - \mu_1)/1000] = pnorm(\sqrt{10}(10735.7 - \mu_1)/1000)$

μ_1	10600	10800	11000	11500
P[Type II error at μ_1]	0.666	0.419	0.202	0.00783

3. (10 points) Let μ be the reaction time in a chemical process using the new additive. Test the hypotheses: $H_o: \mu \geq 10$ vs $H_1: \mu < 10$

   (a.) From the n=15 batches: $\bar{Y} = 8.7$ and $S = 2$. Using $\alpha = .01$, Reject H_o if $\bar{Y} < 10 - t_{.01}S/\sqrt{n} = 10 - (2.624)(2)/\sqrt{15} = 8.645$

   $p - value = P[t_{14} < \sqrt{15}(8.7 - 10)/2] = pt(-2.517, 14) = .0123 > .01 = \alpha \;\Rightarrow$ Fail to reject H_o and conclude there is not sufficient evidence that the average reaction time has been reduced using the new additive.

   (b.) Using $\sigma \approx 2$, compute the power at $\mu = 8.5$:

   $\gamma(8.5)$ = P[reject H_o at $\mu = 8.5$] = $P[t < -t_{.01,14}] = P[t < -2.624]$

   where t has a non-central t-distribution with df = 14 and non-centrality parameter,

   $\Delta = \sqrt{15}(8.5 - 10)/2 = -2.9047$. Therefore, $\gamma(8.5) = pt(-2.624, 14, -2.9047) = .6165$

   (c.) Using the table on page 27 of Handout 12 with a one-sided test having $\alpha = .05$; $\beta = 1 - .80 = .2$; $\phi = |9 - 10|/2 = .5$; we have n=27.

4. (10 points) The following normal reference distribution plot along with a p-value of .993 from the Shapiro-Wilk test indicates that a normal distribution provides an excellent fit to the data with $\bar{X} = 118.48$, $S = 6.1922$.

(a.) Test the hypotheses: $H_o : \sigma \geq 10$ versus $H_1 : \sigma < 10$ with rejection region:

Reject H_o if $(n-1)S^2/(10)^2 \leq \chi^2_{.10,24} = 15.659$

From the data, $(n-1)S^2/(10)^2 = (25-1)(6.192)^2/(10)^2 = 9.20 < 15.659 \Rightarrow Reject H_o$ and conclude there is significant evidence that the new device produces readings which have a standard deviation less than 10.

p-value $= P[\chi^2_{24} \leq 9.20] = pchisq(9.20, 24) = .003$ which is less than $\alpha. = 01$

(b.) $\beta(\sigma_1) = P[\text{Type II error at } \sigma_1] = P[\chi^2_{24} \geq \frac{(10)^2}{\sigma_1^2} 15.659] = 1 - pchisq\left(\frac{(10)^2}{\sigma_1^2}\ 15.659\right)$

σ_1	5	6	7	8	9	10
$\beta(\sigma_1)$	.0000269	.00872	.0128	.435	.734	0

$\beta(10) = 0$ because 10 is in the null space and hence a Type II error cannot occur.

(c.) From $P\left[\frac{(n-1)S^2}{\sigma^2} \geq \chi^2_{n-1,1-.9}\right] = .9$, we have that an upper 90% confidence bound on the standard deviation of the new device is given by
$\frac{\sqrt{n-1}S}{\sqrt{\chi^2_{n-1,.1}}} = \frac{\sqrt{25-1}(6.1922)}{\sqrt{15.659}} = 7.666$. Thus, we are 90% confident that σ is less than 7.666 which would be consistent with our conclusion that the data indicated that σ was less than 10.

5. (10 points) Let $\tilde{\mu}$ be the median reading for the distribution of blood sugar device readings. Test $H_o : \tilde{\mu} \geq 120$ versus $H_1 : \tilde{\mu} < 120$

(a.) **Sign Test**: Because 120 was one of the data values, we delete it and the sample size is now n=25-1=24.

Let S_+ be the number of readings in the data less than 120: The decision rule is

Reject H_o if $S_+ \leq 7$, because $P[B \geq 7] = .032 < .05$ and $P[B \leq 8] = .076 > .05$, where B has a Binomial(n=24,p=.5) distribution.

From the data, $S_+ = 14 > 7$ therefore, conclude there is not significant evidence that the median is less than 120.

p-value $= P[B \leq S_+] = P[B \leq 14] = pbinom(14, 24, .5) = 0.846 > 0.05 = \alpha$

(b.) **Wilcoxon signed rank test**: Let W_+ be the sum of the ranks associated with the positive values of $X_i = Y_i - 120$:

One of the values of X was 0, so we delete that observation and the sample size is now n=25-1=24.

Reject $H_o : \tilde{\mu} \geq 120$ if $W_+ \leq qsignrank(.05, 24, TRUE) = 92$

From the data, we have the sum of the ranks of $|X_i|$ associated with the positive values of X_i is $W_+ = 112.5 > 92$, therefore, fail to reject H_o and conclude there is not significant evidence that the median is less than 120.

p-value $= P[W_+ \leq 112.5] = psignrank(112.5, 24, TRUE) = .145 > .05 = \alpha$

Using the R function wilcox.test(x,c,alternative="less",paired=TRUE) we obtain the following:

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        Wilcoxon signed rank test with continuity correction

data:  x and c
V = 112.5, p-value = 0.1447
alternative hypothesis: true location shift is less than 0
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(c.) There is strong evidence that the population distribution is a normal distribution, therefore a 95% lower bound on the median blood sugar reading is given by $\bar{Y} - t_{.05,24}S_Y/\sqrt{25} = 118.48 - (1.711)(6.1922)/5 = 116.361$

A distribution-free lower bound would be $(X_{(r)},\ \infty)$ where r is the largest integer such that $.95 = P[Y_{(r)} \leq Q(.5)] = 1 - pbinom(r, 25, .5)$ which yields r=8. Therefore, the distribution-free lower bound would be $(X_{(8)},\ \infty) = (115,\ \infty)$ which is nearly the same as the normal based lower bound.

6. (10 points) Let p be the probability of identifying patients at risk of sudden cardiac death using the new method. From the data, $\hat{p} = y/n = 46/50 = .92$

(a.) Because $min(n\hat{p}, n(1-\hat{p}) = 4 < 5$, the Agresti-Coull C.I. may not be appropriate. The Clopper-Pearson C.I. is given by $(C_L,\ C_U)$ where

$$C_L = \frac{1}{1+\frac{5}{46}F_{10,92,.025}} = .808; \qquad C_U = \frac{\frac{47}{4}F_{94,8,.025}}{1+\frac{47}{4}F_{94,8,.025}} = .978$$

The 95% Agresti-Coull C.I. for p is given by $\tilde{p} \pm Z_{.025}\frac{\tilde{p}(1-\tilde{p})}{\tilde{n}}$ where

$\tilde{n} = n + Z^2_{.025} = 50 + (1.96)^2 = 53.8416$, $\tilde{p} = (Y + .5Z^2_{.025})/\tilde{n} = (46 + .5(1.96)^2)/(53.8416) = .89$

The 95% C.I. on p is $.89 \pm 1.96\sqrt{(.89)(1-.89)}/\sqrt{53.8416} = .89 \pm .084 = (.806,\ .974)$

Even though, $min(n\hat{p}, n(1-\hat{p}) < 5$, the Agresti-Coull yields nearly the same interval as Clopper-Pearson.

(b.) Test the hypotheses $H_o : p \leq .8$ versus $H_1 : p > .8$ at the $\alpha = .05$ level.

Let Y be the number of patients that were identified as being at risk out of the 50 patients.

Reject H_o if $Y \geq B_{.05,.8} = qbinom(1-.05, 50, .8) = 44$.

From the data, $Y = 46 > 44$. Thus, reject H_o and conclude that there is significant evidence that the new method has increased the accuracy relative to the old method.

Let B have a Binomial(n=50,p=.8) distribution, then $p-value = P[B \geq 46] = 1-pbinom(45, 50, .80 = .0185 < .05 = \alpha$

(c.) The power of the test in part (b.) is given by $\gamma(p) = 1 - binom(44, 50, p)$ for p= 75%, 80%, 85%, 90%, 95%:

p	.75	.8	.85	.9	.95
$\gamma(p)$	0.007046	0.04803	0.2194	0.6161	0.9622

(d.) The required sample size n to achieve $\beta(.9) = 1 - \gamma(.9) = 1 - .8 = .2$ using an $\alpha = .05$ test is given by

$$n = \left[\frac{Z_\alpha\sqrt{p_o(1-p_o)} + Z_\beta\sqrt{p_1(1-p_1)}}{\delta}\right]^2 = \left[\frac{1.645\sqrt{.8(1-.8)} + .84\sqrt{.9(1-.9)}}{(.8-.9}\right]^2 = 82.8$$

Therefore, n=83 is required to achieve the stated goals.

Multiple Choice (40 points) SELECT ONE of the following letters **(A, B, C, D, or E)** corresponding to the **BEST** answer. Show details for partial credit.

(MC1.) **C.** The P.I. will be too narrow and hence will have a level of confidence less than 95%.

(MC2.) **A.** $n = \left(\frac{(9)(1.96)}{1.5}\right)^2 = 138.3$

(MC3.) **A.** $n = \frac{\sigma^2(1.645+1.28)^2}{(.5\sigma)^2} = 34.2$

(MC4.) **D.** The power, $\gamma(\mu)$, is a function of μ

(MC5.) **C.** $\beta(47.9) = P\left[\chi_9^2 \le \frac{(23.8)^2}{47.9)^2} 16.919\right] = pchisq(4.177, 9) = .101$

(MC6.) **C.** Test the hypotheses $H_o : p \le .2$ versus $H_1 : p > .2$ at the $\alpha = .05$ level.

Reject H_o if $Y > 7 = qbinom(.95, 20.2)$

$\beta(.4) = P[B \le 7] = pbinom(7, 20.4) = .416$

(MC7.) **D.** The test statistic would be $t = \sqrt{15}(\bar{Y} - 20)/S$ which has a non-central t-distribution with non-centrality parameter $\Delta = \frac{\sqrt{n}(\mu_1 - 20)}{\sigma} = \frac{\sqrt{15}(20+.8\sigma-20)}{\sigma} = .8\sqrt{15}$

$\beta(20 + .8\sigma) = P[t_{14,\Delta} < t_{.01,14}] = pt(2.6245, 14, .8\sqrt{15}) = .32$

Alternatively, **C.** If you use the a Z-test: Reject H_o if $\bar{Y} \ge 20 + 2.33\sigma/\sqrt{15}$

$\beta(20 + .8\sigma) = P[\bar{Y} < 20 + 2.33\sigma/\sqrt{15}$ when $\mu = 20 + .8\sigma] = P[Z < 2.33 + .8\sqrt{15}] = pnorm(-.768) = .22$

(MC8.) **A.** See the discussion on page 51 in Handout 12

(MC9.) **B.** See the discussion on page 33 in Handout 12

(MC10.) **B.** See the discussion on page 33 in Handout 12