STAT 630 Fall 2014
Homework 2 Solution

1.5.7

(b) $P(curve\ ball|hit) = \frac{P(hit \cap curve\ ball)}{P(hit)} = \frac{P(hit|curve\ ball)P(curve\ ball)}{P(hit)} = \frac{0.05*0.2}{0.074} = 0.135$
(c) $P(curve\ ball|not\ hit) = \frac{P(not\ hit \cap curve\ ball)}{P(not\ hit)} = \frac{P(not\ hit|curve\ ball)P(curve\ ball)}{1-P(hit)} = \frac{0.95*0.2}{1-0.074} = 0.205$

1.5.9

First we obtain the probabilities for each event.
$P(A) = \frac{6}{36} = \frac{1}{6}$; Since sum of the two dices equals to 12 means both dices show a number 6, thus $P(B) = \frac{1}{36}$; $P(C) = P(D) = \frac{6}{36} = \frac{1}{6}$.

(a) The event $A \cap B$ means both dices show the same numbers and the number is 6, thus $P(A \cap B) = \frac{1}{36} \neq P(A)P(B)$. So A and B are not independent.

(b) The event $A \cap C$ means both dices show the same numbers and the number is 4, thus $P(A \cap C) = \frac{1}{36} = P(A)P(C)$. So A and C are independent.

(c) The event $A \cap D$ means both dices show the same numbers and the number is 4, thus $P(A \cap D) = \frac{1}{36} = P(A)P(D)$. So A and D are independent.

(d) The event $C \cap D$ means both the red die and blue die show number 4. Thus $P(C \cap D) = \frac{1}{36} = P(C)P(D)$. So C and D are independent.

(e) The event $A \cap C \cap D$ means both the red die and blue die show number 4. Thus $P(A \cap C \cap D) = \frac{1}{36} \neq P(A)P(C)P(D)$. So A, C and D are not all independent.

1.5.13

(a) $P(red) = P(card1)P(red|card1) + P(card2)P(red|card2) + P(card3)P(red|card3) = \frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$

(b) $P(card1|red) = \frac{P(card1,red)}{P(red)} = \frac{P(card1)P(red|card1)}{P(red)} = \frac{\frac{1}{3}\times 1}{\frac{1}{2}} = \frac{2}{3}$

(c) Make three cards as specified and run the experiment repeatedly. Discard all experiments where the one side showing on the desk is black. Of the experiments where the one side showing is red, count the fraction that the other side is also red. If sufficient many experiments are done, this fraction should be close to $\frac{2}{3}$.

1.5.14

($\Longrightarrow$) If A and B are independent, then $P(A \cap B) = P(A)P(B)$. $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(B)(1 - P(A)) = P(A^c)P(B)$. Thus, A^c and B are independent
($\Longleftarrow$) If A^c and B are independent, then $P(A^c \cap B) = P(A^c)P(B)$. $P(A \cap B) = P(B) - P(A^c \cap B) = P(B) - P(A^c)P(B) = P(B)(1 - P(A^c)) = P(A)P(B)$. Thus, A and B are independent

1.5.18

(a) First, we try to obtain some information from the question. Since a car is randomly selected to be behind door A, B or C, then $P(car\ behind\ A) = P(car\ behind\ B) = P(car\ behind\ C) = \frac{1}{3}$. If you choose door A, then $P(open\ A) = P(open\ B) = \frac{1}{2}$. Also $P(open\ B|car\ behind\ A) = P(open\ C|car\ behind\ A) = \frac{1}{2}$. If you choose door A and do not change your choice when host opens door B, then the probability to win is

$$\begin{aligned} P(car\ behind\ A|open\ B) &= \frac{P(car\ behind\ A \cap\ open\ B)}{P(open\ B)} \\ &= \frac{P(open\ B|car\ behind\ A)P(car\ behind\ A)}{P(open\ B)} \\ &= \frac{1/2 * 1/3}{1/2} = \frac{1}{3} \end{aligned}$$

(b) Since conditional on car is behind C the probability of opening B is 1, thus if you change your choice to C when the host opens B, the probability to win is:

$$\begin{aligned} P(car\ behind\ C|open\ B) &= \frac{P(car\ behind\ C \cap\ open\ B)}{P(open\ B)} \\ &= \frac{P(open\ B|car\ behind\ C)P(car\ behind\ C)}{P(open\ B)} \\ &= \frac{1 * 1/3}{1/2} = \frac{2}{3} \end{aligned}$$

(c) To do the experiment, you can hide a key in one of three small cups. Then let one person to point out one of them in which he believes the key is hidden. Then you reveal one of the cups without the key and ask the person whether he wants to change his mind. Repeat the above steps for several times and count the fraction of the time they win with or without changing their first choice.

Problem A

Since the gene that a parent transmits to one offspring is independent of the one he transmits to another, thus event B,C and D have the sample probability: $\frac{2}{4} = \frac{1}{2}$. Then it is easy to show B,C and D are pair-wise independent. But $P(B \cap C \cap D) = \frac{2}{2^3} = \frac{1}{4} \neq P(B)P(C)P(D)$, thus they are not mutually independent.

Problem B

This parallel system fails means that all three sub-series fail. The top series fails when either one of its two components fails. Thus $P(\text{series1 fails}) = 1 - P(\text{both 1 and 2 work}) = 1 - (1-p)^2$. So does the bottom series. The mid series only have one component, thus the probability that it fails is simply p. Therefore, $P(\text{parallel system fails}) = (1-(1-p)^2)^2 * p = p^3(2-p)^2 = p^5 - 4p^4 + 4p^3$.

2.1.5

(b) If $x \in A \cap B^c$, then $I_{A\cup B} = 1, I_A = 1, I_B = 0$, we have $I_{A\cup B} = max\{I_A, I_B\}$. If $x \in A^c \cap B$, then $I_{A\cup B} = 1, I_A = 0, I_B = 1$. If $x \in A \cap B$, then $I_{A\cup B} = I_A = I_B = 1$. Thus $I_{A\cup B} = max\{I_A, I_B\}$.
(c) If $x \in A^c$, then $x \notin A$ and $I_A = 0$. Thus $I_{A^c} = 1 = 1 - I_A$. If $x \notin A^c$, then $x \in A$ and $I_A = 1$. Thus $I_{A^c} = 0 = 1 - I_A$.

2.1.8

(a) By definition, $W(1) = X(1)+Y(1)+Z(1) = I_{\{1,2,3\}}(1) - I_{\{2,3\}}(1) + I_{\{3,4,5\}}(1) = 1-0+0 = 1$

(b) $W(2) = X(2) + Y(2) + Z(2) = I_{\{1,2,3\}}(2) - I_{\{2,3\}}(2) + I_{\{3,4,5\}}(2) = 1 - 1 + 0 = 0$

(c) $W(5) = X(5) + Y(5) + Z(5) = I_{\{1,2,3\}}(5) - I_{\{2,3\}}(5) + I_{\{3,4,5\}}(5) = 0 - 0 + 1 = 1$

(d) Note that for $B \subset A$ we have$I_A - I_B = I_{A\cap B} \geq 0$. Then, $W(s) = X(s) - Y(s) + Z(s) = I_{\{1,2,3\}}(s) - I_{\{2,3\}}(s) + Z(s) = I_{\{1\}}(s) + Z(s) \geq Z(s)$ for all $s \in S$. Therefore, $W \geq Z$.

2.2.4

First, $P(Z = z) = \frac{1}{6}$ if $z = 1, 2, 3, 4, 5, 6$ and $P(Z = z) = 0$ otherwise.

(a) $P(W = w) = \frac{1}{6}$, if $w = 5, 12, 31, 68, 129, 220$ and $P(W = w) = 0$ otherwise.

(b) $P(V = v) = \frac{1}{6}$, if $v = 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}$ and $P(V = v) = 0$ otherwise.

(c) $P(ZW = x) = \frac{1}{6}$, if $x = 5, 24, 93, 272, 645, 1320$ and $P(ZW = x) = 0$ otherwise.

(d) $P(VW = y) = \frac{1}{6}$, if $y = 5, 12\sqrt{2}, 31\sqrt{3}, 136, 129\sqrt{5}, 220\sqrt{6}$ and $P(VW = y) = 0$ otherwise.

(e) $P(V + W = r) = \frac{1}{6}$, if $r = 6, 12 + \sqrt{2}, 31 + \sqrt{3}, 70, 129 + \sqrt{5}, 220 + \sqrt{6}$ and $P(V + W = r) = 0$ otherwise.

2.3.4

(a) $P_Z(0) = P(Z = 0) = \frac{1}{4}$; $P_Z(1) = P(Z = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$; $P_Z(2) = P(Z = 2) = \frac{1}{4}$;$P_Z(z) = 0$ otherwise.

(b) $P_W(0) = P(W = 0) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$; $P_W(1) = P(W = 1) = \frac{1}{4}$; $P_W(w) = 0$ otherwise.

2.3.8

$P(W = 11) = \exp(-\lambda)\frac{\lambda^{11}}{11!}$ is a function of λ.
Let $f(\lambda) = \exp(-\lambda)\frac{\lambda^{11}}{11!}$. Let $f'(\lambda) = \exp(-\lambda)\frac{\lambda^{10}}{11!}(-\lambda + 11)$. Let $f'(\lambda)$ equal to zero, then we can get $\lambda = 11$.

2.3.10

First, the geometric distribution is a discrete distribution of nonnegative integers, thus we have

$$\begin{aligned}
P(X^2 \le 15) &= P(X < 4) \\
&= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\
&= p + (1 - p)p + (1 - p)^2 p + (1 - p)^3 p \\
&= 1/5 + 4/5 * 1/5 + 16/25 * 1/5 + 1/5 * (4/5)^3 \\
&= 369/625 = 0.5904
\end{aligned}$$

2.3.13

$P(X = 3) = \binom{7}{3} \cdot \binom{13}{5} / \binom{20}{8} = 0.3575851$. If $X = 8$, since the maximum value of X is only 7, thus $P(X = 8) = 0$.

2.3.14

(a) Binomial(20,$\frac{2}{3}$).

(b) $P(\text{This event occurs five times}) = \binom{20}{5}(\frac{2}{3})^5(1-\frac{2}{3})^{20-5} = 1.4229 \times 10^{-4}$.

2.3.15

(a) The number of baskets the player sinks for independent throws has a binomial distribution with $p = 0.35$. Therefore, $P(X = 3) = \begin{pmatrix} 10 \\ 3 \end{pmatrix} * 0.35^3 * (1 - 0.35)^7 = 0.2522196$.

(b) This probability obeys a geometric distribution with $p = 0.35$. So $P(X = 9) = (1 - 0.35)^9 * 0.35 = 7.2492 * 10^{-3}$.

(c) The certain number of failures before obtaining a certain number of successes has a negative binomial distribution and in this case, number of failures is 8 and number of successes $r = 2$. Thus the probability is $\binom{9}{1}0.35^2 0.65^8 = 3.513 * 10^{-2}$.

2.3.18

(a) The number of calls arrive in the next two minutes has a poisson distribution with parameter $\lambda t = 4$. Thus $P(X = 5) = \frac{4^5 * e^{-4}}{5!} = 0.15629$.

(b) Because the number of arrivals is non-overlapping periods are independent, thus $P(X = 5, Y = 5) = P(X = 5)P(Y = 5) = 2.4428 \times 10^{-2}$

(c) The number of calls arrive in the next ten minutes has a poisson distribution with parameter $\lambda t = 20$. Thus $P(X = 0) = \frac{20^0 * e^{-20}}{0!} = e^{-20} = 2.061 \times 10^{-9}$.