# STAT 408/608 Homework 10 Solutions: Written Section

May 1, 2015

1. (a)

$$
\begin{aligned}
e &= Y - X\hat{\beta} \\
e'e &= (Y - X\hat{\beta})'(Y - X\hat{\beta}) = Y'Y - 2\hat{\beta}X'Y + (X\hat{\beta})'(X\hat{\beta})
\end{aligned}
$$

Take derivative with respect to $\hat{\beta}$, then we have,

$$
\begin{aligned}
&-2X'Y + 2X'X\hat{\beta} = 0 \\
&\hat{\beta} = (X'X)^{-1}X'Y
\end{aligned}
$$

(b)

$$
\begin{aligned}
&Y = X\beta + e, \text{ where } \ e \sim \mathrm{N}(0, \sigma^2) \\
&E(Y) = E(X\beta + e) = X\beta \\
&E(\hat{\beta}) = E((X'X)^{-1}X'Y) = (X'X)^{-1}X'E(Y) = (X'X)^{-1}X'X\beta = \beta
\end{aligned}
$$

(c)

$$
Var(\hat{\beta}) = Var((X'X)^{-1}X'Y) = (X'X)^{-1}X'Var(Y)X(X'X)^{-1} = \sigma^2(X'X)^{-1}
$$

2. (a)

$$
L(\beta) = -\frac{n}{2}log2\pi - \frac{1}{2}log|\Sigma| - \frac{1}{2}(Y - X\beta)'\Sigma^{-1}(Y - X\beta)
$$

because only the last term depend on $\beta$. We expand this term and take derivative with respect to $\beta$,

$$
\begin{aligned}
&RSS = Y'\Sigma^{-1}Y - 2\beta'X'\Sigma^{-1}Y + \beta'X'\Sigma^{-1}X\beta \\
&\frac{RSS}{\beta} \Rightarrow -2X'\Sigma^{-1}Y + 2X'\Sigma^{-1}X\beta = 0 \\
&\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y
\end{aligned}
$$

(b) The covariance matrix filled with ones multiplied by the scalar $\sigma_e^2$ which is not a full rank. Therefore, $\Sigma$ is not invertible, and the expression for $\beta_{GLS}$ cannot be evaluated.

(c)

$$\begin{aligned}
e_t &= \rho e_{t-1} + v_t \\
&= \rho(\rho e_{t-2} + v_{t-1}) + v_t \\
&= \rho^2 e_{t-2} + \rho v_{t-1} + v_t \\
&= \rho^2(e_{t-3} + v_{t-2}) + \rho v_{t-1} + v_t \\
&= \rho^3 e_{t-3} + \rho^2 v_{t-2} + \rho v_{t-1} + v_t \\
E[e_t e_{t-3}] &= E[(\rho^3 e_{t-3} + \rho^2 v_{t-2} + \rho v_{t-1} + v_t)e_{t-3}] \\
&= \rho^3 E[e_{t-3}^2] + 0 + 0 + 0 \\
&= \rho^3 \sigma_3^2 \\
Corr(e_t e_{t-3}) &= \frac{\rho^3 \sigma_e^2}{\sqrt{\sigma_e^2 \sigma_e^2}} \\
&= \rho^3
\end{aligned}$$

(d)

$$\begin{aligned}
E(\hat{\beta}) &= E((X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y) = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}E(Y) \\
&= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}X\beta = \beta
\end{aligned}$$

(e) Let $Var(e_t) = Var(\rho e_{t-1} + v_t) = \Sigma$

$$\begin{aligned}
Var((X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y) &= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Var(Y)\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1} \\
&= (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}\Sigma\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1} \\
&= (X'\Sigma^{-1}X)^{-1}
\end{aligned}$$

(f) Both $\hat{\beta}$ and $\hat{\beta_{GLS}}$ are unbiased estimators of $\beta$.
When we do the transformation: $X \rightarrow L^{-1}X$, $Y \rightarrow L^{-1}Y$, where $\Sigma = L'L$, we have:
$Var(\hat{\beta}) = \sigma^2((L^{-1}X)'(L^{-1}X))^{-1} = \sigma^2(X'L^{-1\prime}L^{-1}X)^{-1} = (X'\Sigma^{-1}X)^{-1} = Var(\hat{\beta}_{GLS})$

3. In this question, we can first look at the data graphically to decide which structure we are going to deal with. In Fig.1 we see there is a time dependent, therefore, I decided to go with a model that included quarters.

   The initial model will look like:

   Sale $\sim$ Temp + Sun + Temp:Sun + Q2 + Q3 + Q4

   To examine the data autocorrelated or not, we can check residuals via ACF plot. Fig.2 shows there exists an autocorrelation AR(1) structure.

   We fit GLS model with correlation equal to 0.8 (from AR(1) coefficient). The final model becomes,

   See corresponding R code to get the outputs and better looking graphs :)

   The p-value $=1.118e-13$ for the model is smaller than 0.05, which means at least one parameter is differ from 0. It is a significant model. In order to examine the valid model, we need the diagnostic plot and MM plot.

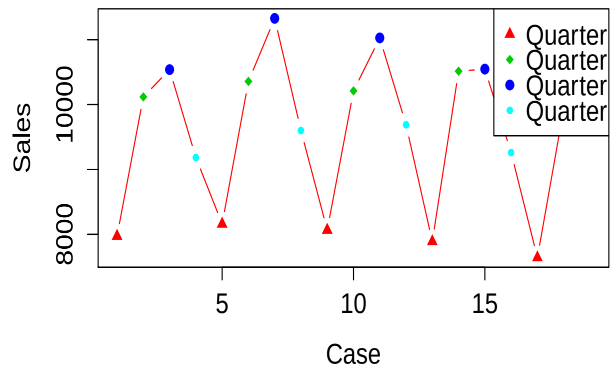


Figure 1: Time Series Plot

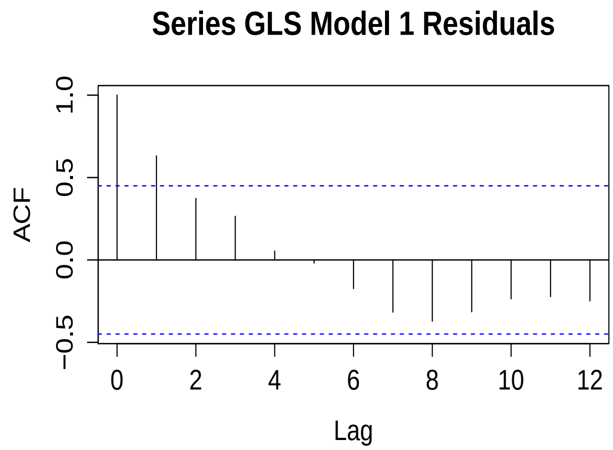


Figure 2: Autocorrelation Function to Determine Correlation Function.

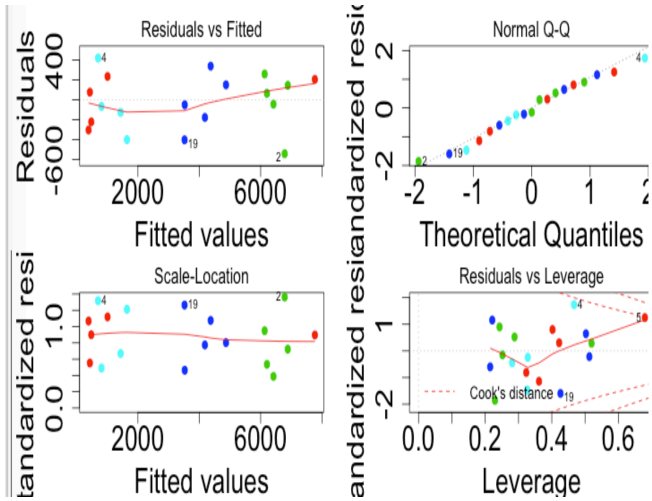


Figure 3: Marginal Model Plot for Overall Model.

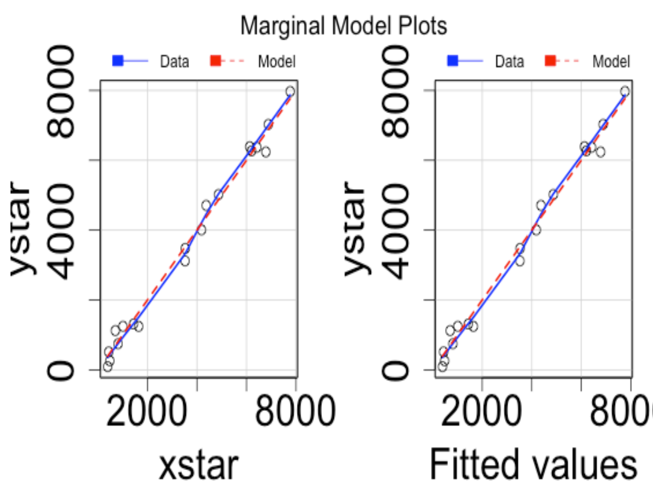


Figure 4: Model Diagnoistics.

Fig.3 is marginal model plot for the whole model. It indicates that model is valid. Fig.4 is diagnostic plot. It appears we have a valid model with constant variance, normally distributed residuals and no major influence points.