Homework 05
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6.1

a) θ_A and θ_B are not independent under this prior distribution because $\theta_A = \theta, \theta_B = \theta\gamma$ both depend on θ

b) $\theta, \gamma = \theta_b/\theta_a, Y_a = (y_1...y_n), Y_b = (y_1..y_m)$

$$\begin{aligned} p(y_a, y_b|\theta, \gamma) &\propto e^{-n\theta - m\theta\gamma}\theta^{n\bar{y}}(\theta\gamma)^{m\bar{z}} \\ p(\theta, \gamma) &= \theta^{a-1}e^{-b\theta}\gamma^{c-1}e^{-d\gamma} \end{aligned}$$

Full Conditional:

$$\begin{aligned} p(\theta|\gamma, y_a, y_b) &\propto \theta^{n\bar{y}}e^{-n\theta}e^{-m\theta\gamma}(\theta\gamma)^{m\bar{z}}\theta^{a-1}e^{-b\theta} \\ &= e^{-\theta(n+m\gamma+b)}\theta^{n\bar{y}+a-1}\theta^{m\bar{z}}\gamma^{m\bar{z}} \\ &= \theta^{n\bar{y}+m\bar{z}+a-1}e^{-\theta(n+m\gamma+b)} \\ &\propto gamma(n\bar{y} + m\bar{z} + a, n + m\gamma + b) \end{aligned}$$

Using estimates from the data:

$$\propto gamma(359 + a, 385.59 + b)$$

c)

Full Conditional:

$$\begin{aligned} p(\gamma|\theta, y_a, y_b) &\propto p(\theta, \gamma|y_a, y_b) \\ &\propto e^{-n\theta - m\theta\gamma}(\theta\gamma)^{m\bar{z}}\gamma^{c-1}e^{-d\gamma} \\ &\propto e^{-n\theta}e^{-m\theta\gamma}e^{-d\gamma}\theta^{m\bar{z}}\gamma^{m\bar{z}}\gamma^{c-1} \\ &\propto \gamma^{m\bar{z}+c-1}e^{-\gamma(m\theta+d)} \\ &\propto gamma(m\bar{z} + c, m\theta + d) \end{aligned}$$

Using estimates from the data:

$$\propto gamma(305 + c, 202.96 + d)$$

d)

As the values of the prior for gamma get larger, the expectation of the difference between the two parameters increases

	Sim	Param	Mu.Diff
1	1	a=2, b=1, c=d=8	0.1817122
2	2	a=2, b=1, c=d=16	-0.1036834
3	3	a=2, b=1, c=d=32	-0.4976019
4	4	a=2, b=1, c=d=64	-0.9638915
5	5	a=2, b=1, c=d=128	-1.5307103

7.1

a)

$|\sum|^{-(p+2)/2}$ is impropper and does not depend on θ.

b)

$$\begin{aligned}
pj(\theta,\Sigma|y_1...y_n) &\propto |\Sigma|^{-(p+2)/2}|\Sigma|^{-n/2}exp[-1/2\sum_{i=1}^{n}(y_i-\theta)^T\Sigma^{-1}(y_i-\theta)]|\Sigma|^{-(p+2)/2} \\
&\propto |\Sigma|^{-(n+p+2)/2}exp[-1/2\sum_{i=1}^{n}(y_i-\theta)^T\Sigma^{-1}(y_i-\theta)]|\Sigma|^{-(p+2)/2}
\end{aligned}$$

$$\begin{aligned}
pj(\theta|\Sigma,y_1...y_n) &\propto |\Sigma|^{-(n+p+2)/2}exp[-.5tr[\sum_{i=1}^{n}(y_i-\theta)(y_i-\theta)^T\Sigma^{-1}]]|\Sigma|^{-(p+2)/2} \\
&\propto |\Sigma|^{-(n+p+2)/2}exp[-.5tr[\sum_{i=1}^{n}(y_i-\bar{y})(y_i-\bar{y})^T\Sigma^{-1}]]|\Sigma|^{-(p+2)/2} \\
&\propto |\Sigma|^{-1/2}exp[-.5(y_i-\theta)^T\Sigma^{-1}(y_i-\theta)] \\
&\propto N(\bar{y},\Sigma/n)
\end{aligned}$$

$$\begin{aligned}
pj(\Sigma|y_1...y_n) &\propto |\Sigma|^{-(p+2)/2}exp[-1/2tr[S_0\Sigma^{-1}]] \\
&\propto |\Sigma|^{(-v_0+p+1)/2}exp[-1/2(tr(S_0\Sigma^{-1}))] \\
&\propto InverseWishart(n-1,(nS^2)^{-1})
\end{aligned}$$

7.2

a)

$$
\begin{aligned}
p(y|\theta,\psi) =& (2\pi)^{-p/2}|\Sigma|^{-1/2}exp[-1/2(y-\theta)^T\Sigma^{-1}(y-\theta)] \\
lp(y|\theta,\psi) =& \frac{-np}{2}log[2\pi] + \frac{n}{2}log[|\psi|] - \frac{1}{2}(y-\theta)^T\psi(y-\theta)
\end{aligned}
$$

The likelihood is proportional to:

$$
\begin{aligned}
\sqrt{|\psi|}exp[-1/2n\sum_{i=1}^{n}(y_i-\theta)^T\psi(y_i-\theta)] =& \sqrt{|\psi|}exp[-1/2tr[s^2\psi + (\theta-\bar{y})^2]] \\
p_u(\psi) =& InverseWishart(1, 1/\psi) \\
p_u(\theta,\psi|y) =& MVN(\bar{y}, 1/\psi)
\end{aligned}
$$

b)

Yes because it is constructed from the product of the prior of Σ, the prior of $\theta|\Sigma$, and the likelihood.

$$
\begin{aligned}
pu(\theta|\Sigma)pu(\Sigma)pu(y|\theta,\Sigma) =& |\Sigma|^{-(v_0+p+2)/2}exp[-1/2tr[(S_o\psi)]]exp[-\rho_o/2(\theta-\mu_0)\psi(\theta-\mu_o)] \\
& \sqrt{|\psi|}exp[-1/2n\sum_{i=1}^{n}(y_i-\theta)^T\psi(y_i-\theta)] \\
& \propto MVN(\mu,\psi)
\end{aligned}
$$