1. Suppose a time series $\{x_t\}$ follows the model $\beta_0 + \beta_1 t + w_t$ where w_t is a white noise with the mean zero and variance 4.0.

(a) (5 points) Compute the mean and autocovariance function of $\{x_t\}$. Is $\{x_t\}$ a stationary time series?

(b) (7 points) Consider the new time series $y_t = x_t - x_{t-1}$. Compute its mean and autocovariance function, and determine whether $\{y_t\}$ is stationary.

(c) (3 points) Plot the ACF of $\{y_t\}$. What is the name of such a plot of an ACF?

(a) $E(x_t) = E(\beta_0 + \beta_1 t + w_t) = \beta_0 + \beta_1 t + \overbrace{E(w_t)}^{0} = \beta_0 + \beta_1 t.$

$$\begin{aligned} Cov(x_{t+h}, x_t) &= Cov[\beta_0 + \beta_1(t+h) + w_{t+h}, \ \beta_0 + \beta_1 t + w_t] \\ &= Cov(w_{t+h}, w_t) = \text{cov. of a white noise} \\ &= \begin{cases} \sigma_w^2 = 4.0, & h = 0 \\ 0, & h > 0. \end{cases} \end{aligned}$$

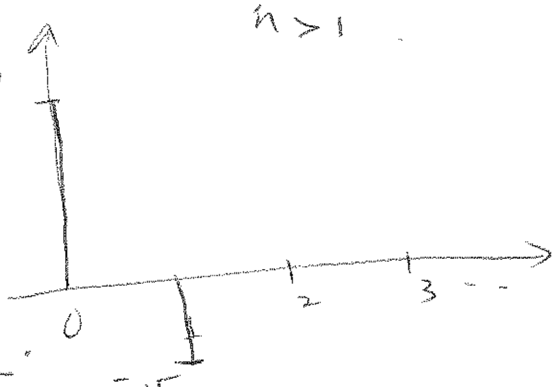
$\{x_t\}$ is nonstationary due to $E(x_t)$ varying with time.

(b) $y_t = x_t - x_{t-1} = \beta_1 + w_t - w_{t-1}$, is a moving average of order 1 with $\theta = -1$. Its ACF is

$$\gamma(h) = \begin{cases} 2\sigma_w^2 = 8, & h = 0 \\ -\sigma_w^2 = -4, & h = 1, \\ 0 & h > 1 \end{cases}$$

$\{x_t\}$ is stationary.

(c) $\rho(h) = \begin{cases} 1 & h = 0 \\ -1/2 & h = 1 \\ 0 & h > 1 \end{cases}$



Plot of ACF is called the correlogram.

2. (10) A time series is defined as

$$x_t = \text{Price of a galon of milk for day } t, \quad t = 1, \ldots, 365.$$

You know that price of a commodity increases slowly over time (which may simply be the result of inflation) with increasing variability.

Describe a strategy for transforming the above time series data to stationarity, i.e. to a series with (nearly) constant mean and variance from day to day.

Take log of the data, to reduce variability.

Difference of the log of data, to remove the trend.

3. Let $\{w_t\}$ be a WN$(0, \sigma_w^2)$ and define a time series by

$$x_t = w_t + 0.9 w_{t-2}.$$

(a). (5) Compute var(x_t).

(b). (5) Compute cov(x_5, x_6).

(c). (5) Find the ACF $\rho(h)$ of the time series and plot it against the lag h.

(d). (5) Define the notion of invertibility of an ARMA model. Is the above model invertible? Why?

(a) $Var(x_t) = Var(w_t + 0.9 w_{t-2}) = \sigma_w^2 (1 + 0.81) = 1.81 \sigma_w^2$.

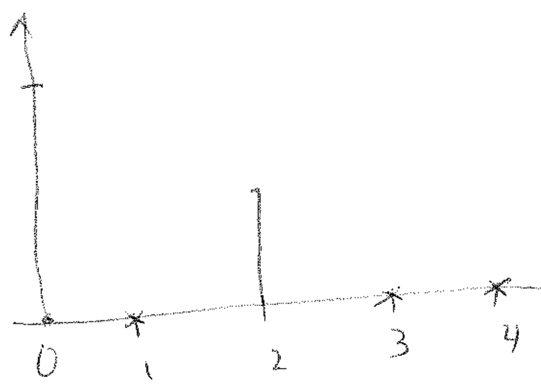
(b) $Cov(x_5, x_6) = Cov(w_5 + 0.9 w_3, w_6 + 0.9 w_4) = 0 = \gamma(1)$

(c) $\gamma(2) = Cov(x_{t+2}, x_t) = Cov(w_{t+2} + 0.9 w_t, w_t + 0.9 w_{t-2})$

$= 0.9\, Cov(w_t, w_t) = 0.9 \sigma_w^2$.

$\gamma(h) = 0, h > 2$.

$$\rho(h) = \begin{cases} 1 & h = 0, \\ 0 & h = 1, \\ \cancel{1.81}\ \frac{0.9}{1.81}, & h = 2 \\ 0 & h > 2. \end{cases}$$



(d) ARMA model is invertible if w_t can be written in terms of $x_t, x_{t-1}, \ldots$. This happens if the roots of $\theta(z) = 1 + 0.9 z^2 = 0$ are outside the unit circle. Here

$z^2 = -\frac{1}{0.9}$, $z = \pm i \sqrt{10/9}$, $|z| > 1$.

So $\{x_t\}$ is <u>invertible</u>.

4. Let w_t be a white noise process with mean zero and variance 1.0, and consider the series

$$x_t = w_t + (2t + 1)w_{t-1}.$$

(a) (10) Determine the mean and autocovariance function of x_t.

(b) (5) Is the series stationary? Why?

$E(x_t) = 0$.

(a) $Var(x_t) = Var\left[w_t + (2t+1)w_{t-1}\right] = \sigma_w^2 + (2t+1)^2 \sigma_w^2$.

$Cov(x_{t+1}, x_t) = Cov\left[w_{t+1} + (2t+2+1)w_t,\ w_t + (2t+1)w_{t-1}\right]$

$= Cov(w_t, w_t) \cdot (2t+3) = (2t+3)\sigma_w^2$.

$Cov(x_{t+h}, x_t) = 0, \quad h \geq 2$.

(b) The series is nonstationary, since the cov. depends on time and not on the lag h.

5. (5) Suppose the sample ACF for a series x_t is equal to 0.8 and 0.4 at lags 1 and 2, respectively, and is quite close to 0 at lags 3 and higher. Which of the following is true? (Circle the correct statement.)

(a) series x_t is definitely long memory.

(b) the sample partial correlogram for x_t will be small at lags 3 and higher.

(c) after adjusting for the linear dependence of $x(t)$ and $x(t+2)$ on $x(t+1)$, $x(t)$ and $x(t+2)$ are almost uncorrelated.

(d) moving average model is a good candidate for series x_t.

6. Consider a time series defined recursively by

$$x_t = 3 + x_{t-1} + w_t,$$

for $t = 1, 2, \ldots$, with $x_0 = 0$ where w_t is a WN with mean zero and variance one.

(a) (5) Express x_t's in terms of present and past values of the white noise.

(b) (5) Find the mean function and the autocovariance function of x_t (show details of your computation). Is this time series stationary? Why?

(c) (5) Compute $\rho_x(t, t-1)$, and find its limit as t goes to infinity.

This is a HW problem,
Problem 1.8 from the text.

7. Let a time series x_t be defined by

$$x_t = -0.9x_{t-2} + w_t,$$

where w_t is a white noise.
(a) (10) Define the notion of causality for ARMA models. Is the above model causal? If so write x_t as a one-sided moving average of the WN w_t.

(b) (5) Compute $\mathrm{Var}(x_t)$.

(c) (5) Compute the ACF of x_t and plot it.

(a) ARMA model is causal if x_t can be written in terms $w_t, w_{t-1}, \ldots$. This happens when the roots of $\phi(z) = 1 + 0.9z^2 = 0$ are outside the unit circle. Here the roots are as Problem 3:

$z = \pm i\sqrt{\frac{10}{9}}$, $|z| > 1$, so the model is <u>causal</u>.

(b). $Var(x_t) = Var(-0.9x_{t-2} + w_t) = 0.81\, Var(x_t) + \sigma_w^2,$

$(1 - 0.81)\, Var(x_t) = \sigma_w^2, \quad Var(x_t) = \frac{\sigma_w^2}{0.19} = 5.26\,\sigma_w^2.$

$$\boxed{\begin{aligned} (1 + 0.9B^2)x_t &= w_t, \quad x_t = \frac{1}{1 + 0.9B^2} w_t = \\ &= [1 - 0.9B^2 + 0.81B^4 - \ldots]\, w_t \\ &= w_t - 0.9w_{t-2} + 0.81w_{t-4} - \ldots \end{aligned}}$$

$$\psi_j = \begin{cases} (-0.9)^j, & j \text{ even}, \\ 0 & j \text{ odd}. \end{cases}$$

(c) $\gamma(h) = \mathrm{cov}(x_{t+h}, x_t) = \mathrm{Cov}\left(-0.9\, x_{t+h-2} + w_{t+h}, x_t\right)$

$= \mathrm{cov}(-0.9\, x_{t+h-2}, x_t)$

$= -0.9\, \mathrm{cov}(x_{t+h-2}, x_t) = -0.9\, \gamma(h-2).$

$h=1, \quad \gamma(1) = -0.9\,\gamma(-1) = -0.9\,\gamma(1) \Rightarrow \gamma(1) = 0.$

$h=2, \quad \gamma(2) = -0.9\,\gamma(0) = -0.9\,(5.26\,\sigma_w^2)$

$\vdots$

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & h=0 \\ (-0.9)^{h-1} & h = \text{even}, \\ 0 & h = \text{odd}. \end{cases}$$