# STAT 630 Fall 2014
# Homework 9 Solution

### 6.3.1

If $H_0$ is true, then $\frac{\sqrt{n}(\bar{x}-5)}{\sigma_0} \sim N(0,1)$, where $\bar{x} = 4.88$. So the 0.95-confidence interval for $\mu$ is $\bar{x} \pm 1.96 \cdot \frac{\sigma_0}{\sqrt{n}}$, that is $(4.4417, 5.3183)$. Since this confidence interval contains 5, we can not reject $H_0$.

### 6.3.2

In this case, the true variance is unknown and we have to estimate it by sample variance. Since observations are assumed to have normal distributions, thus $\frac{\sqrt{n}(\bar{x}-5)}{sd} \sim t_{n-1}$, where sd is the sample standard deviation. Since $n = 10$ and $t_{0.975,9} = 2.2622$, we can obtain the 0.95 confidence interval: $(4.382, 5.378)$. We can see 5 is inside this confidence interval, therefore we can not reject $H_0$.

### 6.3.8

From the question we know $\hat{p} = 0.62$ and sample size $n = 250$. Since if $\frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})/n}} \sim N(0,1)$, then the Wald interval is $\hat{p} \pm z_{0.95}\sqrt{\hat{p}(1-\hat{p})/n}$. After plugging in the values of $\hat{p}$ and $n$, we can obtain the Wald interval: $(0.5695, 0.6705)$. For the score interval, we have $\frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \sim N(0,1)$. So $-z_{0.95} \le \frac{\hat{p}-p}{\sqrt{p(1-p)/n}} \le z_{0.95}$. Through solving this inequality for $p$, we can obtain the score interval: $\frac{2n\hat{p}+z_{0.95}^2 \pm z_{0.95}\sqrt{z_{0.95}^2+4n\hat{p}-4n\hat{p}^2}}{2(z_{0.95}^2+n)}$, that is $(0.56848, 0.66896)$. For both intervals, 0.65 is inside. thus we can not reject $H_0$.

### 6.4.18

In my simulation, nonparametric bootstrap percentile confidence interval is (1.471087,4.288050); nonparametric bootstrap t confidence interval is (1.456253,4.343747); parametric bootstrap percentile confidence interval is (1.530721,4.264086 ) parametric bootstrap t confidence interval is (1.487014,4.312986).
R code:

```
x=c(3.27,-1.24,3.97,2.25,3.47,-0.09,7.45,6.20,3.74,4.12,
    1.42,2.75,-1.48,4.97,8.00,3.26,0.15,-3.64,4.88,4.55)
m=1000
n=length(x)
#nonparametric bootstrap quantitle
temp=rep(0,m)
for (i in 1:m)temp[i]=mean(sample(x,replace=TRUE))
quantile(temp,c(0.025,0.975))
#nonparametric bootstrap t
mean(x)-qt(0.975,n-1)*sd(temp)
mean(x)+qt(0.975,n-1)*sd(temp)
#parametric bootstrap quantitle
temp=rep(0,m)
for (i in 1:m)temp[i]=mean(rnorm(n,mean=mean(x),sd=sd(x)))
quantile(temp,c(0.025,0.975))
#parametric bootstrap t
mean(x)-qt(0.975,n-1)*sd(temp)
mean(x)+qt(0.975,n-1)*sd(temp)
```

### 6.5.1

First we can write down the log likelihood function:

$$\mathcal{L}(\sigma^2) = \sum_{i=1}^{n} \left( -0.5 * \log(2\pi) - 0.5 * \log(\sigma^2) - 0.5 * \frac{(x_i - \mu_0)^2}{\sigma^2} \right)$$

Then take second order derivative with respect to $\sigma^2$ for $\mathcal{L}(\sigma^2)$, we can obtain: $\frac{\partial^2 \mathcal{L}(\sigma^2)}{\partial^2 \sigma^2} = \frac{n}{2\sigma^4} - \sum\limits_{i=1}^{n} \frac{(x_i-\mu_0)^2}{\sigma^6}$. Therefore,

$$E[-\frac{\partial^2 \mathcal{L}(\sigma^2)}{\partial^2 \sigma^2}] = -\frac{n}{2\sigma^4} + \sum_{i=1}^{n} E[\frac{(x_i - \mu_0)^2}{\sigma^6}] = -\frac{n}{2\sigma^4} + n * \frac{\sigma^2}{\sigma^6} = \frac{n}{2\sigma^4}.$$

Hence the fisher information is $\frac{n}{2\sigma^4}$.

### 6.5.3

The score function for $Gamma\,(\alpha_0, \theta)$, where $alpha_0$ is known, is given by $S\,(\theta | x_1, \ldots, x_n) = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln(1 + x_i)$. The Fisher information is then given by

$$nI\,(\alpha) = -E_\theta \left( \frac{\partial}{\partial \alpha} S\,(\alpha | X_1, \cdots, X_n) \right) = -E_\alpha \left( -\frac{n}{\alpha^2} \right) = \frac{n}{\alpha^2}.$$

**6.5.4**

(a) For poisson distribution, the log likelihood function is $\mathcal{L}\lambda) = \sum_{i=1}^{n}(-\lambda + x_i \log(\lambda) - \log(x_i!))$. Then we take second order derivative of it with respect to $\lambda$ and obtain: $\frac{\partial^2 \mathcal{L}}{\partial \lambda^2} = -\frac{\sum_i x_i}{\lambda^2}$. Therefore, the fisher information is $E[-\frac{\partial^2 \mathcal{L}}{\partial \lambda^2}] == \frac{n*\lambda}{\lambda^2} = \frac{n}{\lambda}$. Then the Wald interval is $\hat{\lambda} \pm z_{0.975}\sqrt{\frac{\hat{\lambda}}{n}}$ and we can use $\bar{x}$ as $\hat{\lambda}$. After calculations, the Wald interval is $(8.28854, 11.01146)$. Since $\lambda = 10$ is inside the interval, we can not reject $H_0 : \lambda = 10$.

(b) If we use true $\lambda$ for the standard deviation, then we have $\frac{\hat{\lambda}-\lambda}{\sqrt{\frac{\lambda}{n}}} \sim N(0,1)$. Thus $\left(\frac{\hat{\lambda}-\lambda}{\sqrt{\frac{\lambda}{n}}}\right)^2 \leq z_{0.975}^2$. Then solve this inequality for $\lambda$, we have

$$0.5\left(2\hat{\lambda} + z_{0.975}^2/n - \sqrt{(2\hat{\lambda} + z_{0.975}^2/n)^2 - 4\hat{\lambda}^2}\right) \leq \lambda \leq 0.5\left(2\hat{\lambda} + z_{0.975}^2/n + \sqrt{(2\hat{\lambda} + z_{0.975}^2/n)^2 - 4\hat{\lambda}^2}\right)$$

After some calculations, the score interval is $(8.381198, 11.11088)$.

(c) We can see the score interval shifts right compared with the Wald interval. To obtain the coverage of the two intervals, you can directly obtain the samples of Poisson(11) distribution for many times, then calculate two intervals and find the percentage of 11 inside each interval. If the sample size $n$ is large, the Wald interval and Score interval should have similar coverage; If the sample size $n$ is small, we expect score interval has better coverage.

R code:

```
N=100000
s=20
lambda = 10
sample = matrix (rpois(N*s, lambda), nrow=N)
msample = apply (sample, 1, mean)
z = qnorm (1.95/2)
wald_u = msample + z*sqrt (msample/s)
wald_d = msample - z*sqrt (msample/s)
score_u = msample + z*z/s/2 + sqrt ( (2*msample+ z*z/s) ^2 - 4*msample*msample)/2
score_d = msample + z*z/s/2 - sqrt ( (2*msample+ z*z/s)^2 - 4*msample*msample)/2
w_in = (lambda - wald_u)*(lambda - wald_d)<0
s_in = (lambda - score_u)*(lambda - score_d)<0
r_w = sum (w\_in)/N
r_s = sum (s\_in)/N
```

### 6.5.5

First we need to find the fish information. For this gamma distribution, the density function is $f(x) = \theta^2 x e^{-\theta x}$, so the log-likelihood function is $\mathcal{L}(\theta) = \sum_{i=1}^{n}(2\log(\theta) + \log(x_i) - \theta x_i)$. Then we take second order derivative of it with respect to $\theta$, we can obtain $\frac{\partial^2 \mathcal{L}}{\partial \theta^2} = \frac{-2n}{\theta^2}$. Therefore, the fisher information is $\frac{2n}{\theta^2}$. Since MLE of $\theta$ is $\frac{2}{\bar{x}}$, we can use it as an estimate $\hat{\theta}$. Then the Wald interval can be constructed: $\hat{\theta} \pm z_{0.95}\hat{\theta}\sqrt{\frac{1}{2n}}$. After plugging in the data, the Wald interval is $(9.538 \times 10^{-4}, 1.504 \times 10^{-3})$.

### 6.5.6

In this case, the density function is $f(x) = \theta e^{-\theta x}$. Similar to 6.5.5, we can obtain the fisher information which is $\frac{n}{\theta^2}$. So the Wald interval is $\hat{\theta} \pm z_{0.95}\hat{\theta}/\sqrt{n}$, where $\hat{\theta} = \frac{1}{\bar{x}}$. After plugging in the data, the result is $(4.199 \times 10^{-4}, 8.0896 \times 10^{-4})$. We can see this interval is narrower than the interval in 6.5.5 and it is shifted to the left.

### Additional Problem A

With LLN, $\bar{X} \xrightarrow{P} E(X) = \frac{\theta}{1-\theta}$. Then, with continuous mapping theorem, $\hat{\theta} = \frac{1}{1+\bar{X}} \xrightarrow{P}= \frac{1}{1+\frac{\theta}{1-\theta}} = \theta$.

### Additional Problem B

We know that $MSE\left(\hat{\sigma}^2\right) = \frac{2\sigma^n}{n}$. Since $\lim_{n\to\infty} MSE\left(\hat{\sigma}^2\right) = \lim_{n\to\infty} \frac{2\sigma^n}{n} = 0$, the mle of $\sigma^2$ is consistent.

### Additional Problem C

Due to previous problem, we know all the regularity conditions are satisfied for the normal model. Thus according to theorem 6.5.3, $(n(I(\sigma^2)))^{1/2}(\hat{\sigma^2}_{mle} - \sigma^2) \to N(0,1)$ as $n$ goes to infinity, where $n(I(\sigma^2))$ is the fisher information which is $\frac{n}{2\sigma^4}$ obtained in Question 6.5.1.