STAT 626: Outline of Lecture 14

The ARIMA (p, d, q) Model Building Process (§3.8)

1. **Plot the Data, Transform to Stationarity if Necessary,**
   Select the Differencing Order d.
2. **Model Formulation: Use the ACF and PACF to Select p, q:**
   ARIMA(p, d, q).
3. **Model Estimation: Find the MLE of the $p + q + 1$ Parameters**
4. **Model Diagnostic: Check the Residuals for Independence**
5. **If Not Happy, Go to Step 2 and Repeat the PROCESS**
6. **Choose from the Competing Models Using AIC/BIC**

Example 3.38: Analysis of GNP Data

Example 3.40 Diagnostics for the Glacial Varve Series

ALL Models Are Wrong, But SOME Are Useful.

Who said the above?

STAT 626: Outline of Future Lectures

1. Forecating: Begins when a good model is identified for the time series,

2. Given the time series data $x_1, \ldots, x_n$: **What are the principles for model-based forecasting ?**

$$x_t = f(\beta, \text{Past of the Series}) + w_t.$$

Example:

$$x_t = \phi x_{t-1} + w_t.$$

Principle: Replace the unknowns by the best ESTIMATES.

Example:

$$x_t = w_t + \theta w_{t-1}.$$

3. **Forecasting ARMA Models**

Recall that causal ARMA models can be written as One-Sided MA(∞) of a white noise:

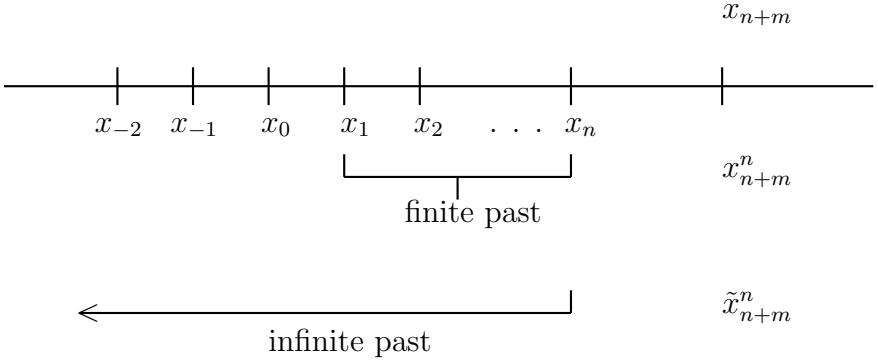
$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j},$$

and **invertible** ARMA models can be written as One-Sided AR(∞);

$$x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} + w_t.$$

Forecasting

4. A pictorial setup for forecasting the future values $x_{n+m}, m = 1, 2, \ldots$:



5. What are their forecasts, forecast error, and forecast error variances?

   Forecast error: $x_{n+m} - x^n_{n+m}$

   Error variance: $P^n_{n+m} = \mathrm{Var}(x_{n+m} - x^n_{n+m})$

6. Their 95% forecast intervals?

$$x^n_{n+m} \pm 1.96\sqrt{P^n_{n+m}}.$$

From (3.149), we see that the new forecast is a linear combination of the old forecast and the new observation. Based on (3.149) and the fact that we only observe $x_1, \ldots, x_n$, and consequently $y_1, \ldots, y_n$ (because $y_t = x_t - x_{t-1}$; $x_0 = 0$), the truncated forecasts are

$$\tilde{x}^n_{n+1} = (1-\lambda)x_n + \lambda\tilde{x}^{n-1}_n, \quad n \geq 1, \tag{3.150}$$

with $\tilde{x}^0_1 = x_1$ as an initial value. The mean-square prediction error can be approximated using (3.144) by noting that $\psi^*(z) = (1-\lambda z)/(1-z) = 1 + (1-\lambda)\sum_{j=1}^{\infty} z^j$ for $|z| < 1$; consequently, for large n, (3.144) leads to

$$P^n_{n+m} \approx \sigma_w^2[1 + (m-1)(1-\lambda)^2].$$

In EWMA, the parameter $1-\lambda$ is often called the smoothing parameter and is restricted to be between zero and one. Larger values of λ lead to smoother forecasts. This method of forecasting is popular because it is easy to use; we need only retain the previous forecast value and the current observation to forecast the next time period. Unfortunately, as previously suggested, the method is often abused because some forecasters do not verify that the observations follow an IMA(1, 1) process, and often arbitrarily pick values of λ. In the following, we show how to generate 100 observations from an IMA(1,1) model with $\lambda = -\theta = .8$ and then calculate and display the fitted EWMA superimposed on the data. This is accomplished using the Holt-Winters command in R (see the help file `?HoltWinters` for details; no output is shown):

```
set.seed(666)
x = arima.sim(list(order = c(0,1,1), ma = -0.8), n = 100)
(x.ima = HoltWinters(x, beta=FALSE, gamma=FALSE))  # alpha below is 1-lambda
  Smoothing parameter: alpha: 0.1663072
plot(x.ima)
```

3.8 Building ARIMA Models

There are a few basic steps to fitting ARIMA models to time series data. These steps involve plotting the data, possibly transforming the data, identifying the dependence orders of the model, parameter estimation, diagnostics, and model choice. First, as with any data analysis, we should construct a time plot of the data, and inspect the graph for any anomalies. If, for example, the variability in the data grows with time, it will be necessary to transform the data to stabilize the variance. In such cases, the Box–Cox class of power transformations, equation (2.37), could be employed. Also, the particular application might suggest an appropriate transformation. For example, suppose a process evolves as a fairly small and stable percent-change, such as an investment. For example, we might have

$$x_t = (1 + p_t)x_{t-1},$$

where x_t is the value of the investment at time t and p_t is the percentage-change from period $t-1$ to t, which may be negative. Taking logs we have

$$\log(x_t) = \log(1+p_t) + \log(x_{t-1}),$$

or

$$\nabla \log(x_t) = \log(1+p_t).$$

If the percent change p_t stays relatively small in magnitude, then $\log(1+p_t) \approx p_t$[8] and, thus,

$$\nabla \log(x_t) \approx p_t,$$

will be a relatively stable process. Frequently, $\nabla \log(x_t)$ is called the return or growth rate. This general idea was used in Example 3.32, and we will use it again in Example 3.38.

After suitably transforming the data, the next step is to identify preliminary values of the autoregressive order, p, the order of differencing, d, and the moving average order, q. We have already addressed, in part, the problem of selecting d. A time plot of the data will typically suggest whether any differencing is needed. If differencing is called for, then difference the data once, $d=1$, and inspect the time plot of ∇x_t. If additional differencing is necessary, then try differencing again and inspect a time plot of $\nabla^2 x_t$. Be careful not to overdifference because this may introduce dependence where none exists. For example, $x_t = w_t$ is serially uncorrelated, but $\nabla x_t = w_t - w_{t-1}$ is MA(1). In addition to time plots, the sample ACF can help in indicating whether differencing is needed. Because the polynomial $\phi(z)(1-z)^d$ has a unit root, the sample ACF, $\widehat{\rho}(h)$, will not decay to zero fast as h increases. Thus, a slow decay in $\widehat{\rho}(h)$ is an indication that differencing may be needed.

When preliminary values of d have been settled, the next step is to look at the sample ACF and PACF of $\nabla^d x_t$ for whatever values of d have been chosen. Using Table 3.1 as a guide, preliminary values of p and q are chosen. Recall that, if $p=0$ and $q>0$, the ACF cuts off after lag q, and the PACF tails off. If $q=0$ and $p>0$, the PACF cuts off after lag p, and the ACF tails off. If $p>0$ and $q>0$, both the ACF and PACF will tail off. Because we are dealing with estimates, it will not always be clear whether the sample ACF or PACF is tailing off or cutting off. Also, two models that are seemingly different can actually be very similar. With this in mind, we should not worry about being so precise at this stage of the model fitting. At this stage, a few preliminary values of p, d, and q should be at hand, and we can start estimating the parameters.

Example 3.38 Analysis of GNP Data

In this example, we consider the analysis of quarterly U.S. GNP from 1947(1) to 2002(3), $n = 223$ observations. The data are real U.S. gross

[8] $\log(1+p) = p - \frac{p^2}{2} + \frac{p^3}{3} - \cdots$ for $-1 < p \leq 1$. If p is a small percent-change, then the higher-order terms in the expansion are negligible.

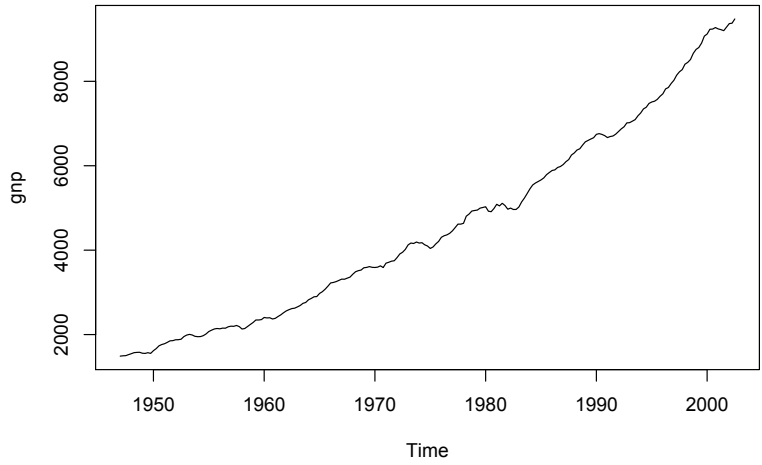


Fig. 3.12. Quarterly U.S. GNP from 1947(1) to 2002(3).

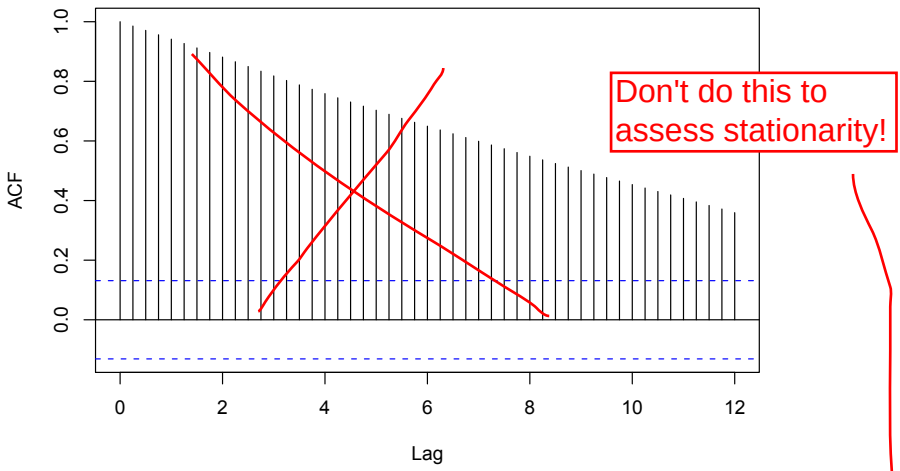


Fig. 3.13. Sample ACF of the GNP data. Lag is in terms of years.

national product in billions of chained 1996 dollars and have been seasonally adjusted. The data were obtained from the Federal Reserve Bank of St. Louis (`http://research.stlouisfed.org/`). Figure 3.12 shows a plot of the data, say, y_t. Because strong trend hides any other effect, it is not clear from Figure 3.12 that the variance is increasing with time. For the purpose of demonstration, the sample ACF of the data is displayed in Figure 3.13. Figure 3.14 shows the first difference of the data, ∇y_t, and now that the trend has been removed we are able to notice that the variability in the second half of the data is larger than in the first half of the data. Also, it appears as though a trend is still present after differencing. The growth

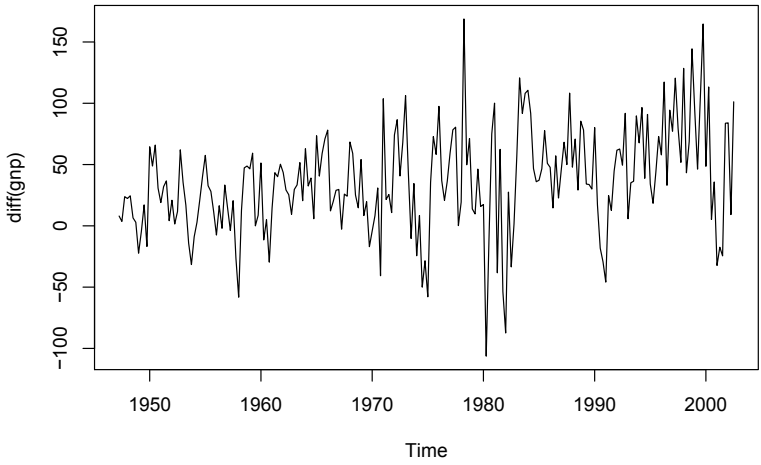


Fig. 3.14. First difference of the U.S. GNP data.

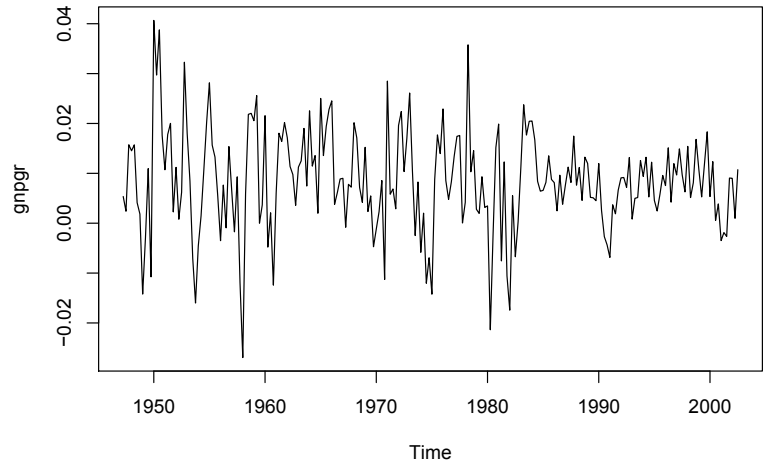


Fig. 3.15. U.S. GNP quarterly growth rate.

rate, say, $x_t = \nabla \log(y_t)$, is plotted in Figure 3.15, and, appears to be a stable process. Moreover, we may interpret the values of x_t as the percentage quarterly growth of U.S. GNP.

The sample ACF and PACF of the quarterly growth rate are plotted in Figure 3.16. Inspecting the sample ACF and PACF, we might feel that the ACF is cutting off at lag 2 and the PACF is tailing off. This would suggest the GNP growth rate follows an MA(2) process, or log GNP follows an ARIMA$(0, 1, 2)$ model. Rather than focus on one model, we will also suggest that it appears that the ACF is tailing off and the PACF is cutting off at

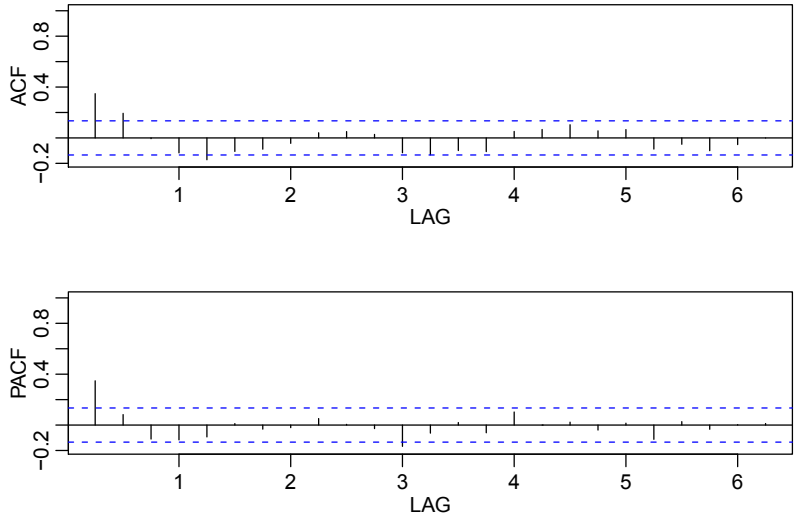


Fig. 3.16. Sample ACF and PACF of the GNP quarterly growth rate. Lag is in terms of years.

lag 1. This suggests an AR(1) model for the growth rate, or ARIMA$(1,1,0)$ for log GNP. As a preliminary analysis, we will fit both models.

Using MLE to fit the MA(2) model for the growth rate, x_t, the estimated model is

$$x_t = .008_{(.001)} + .303_{(.065)}\widehat{w}_{t-1} + .204_{(.064)}\widehat{w}_{t-2} + \widehat{w}_t, \tag{3.151}$$

where $\widehat{\sigma}_w = .0094$ is based on 219 degrees of freedom. The values in parentheses are the corresponding estimated standard errors. All of the regression coefficients are significant, including the constant. *We make a special note of this because, as a default, some computer packages do not fit a constant in a differenced model.* That is, these packages assume, by default, that there is no drift. In this example, not including a constant leads to the wrong conclusions about the nature of the U.S. economy. Not including a constant assumes the average quarterly growth rate is zero, whereas the U.S. GNP average quarterly growth rate is about 1% (which can be seen easily in Figure 3.15). We leave it to the reader to investigate what happens when the constant is not included.

This is important.

The estimated AR(1) model is

$$x_t = .008_{(.001)}\,(1 - .347) + .347_{(.063)}x_{t-1} + \widehat{w}_t, \tag{3.152}$$

where $\widehat{\sigma}_w = .0095$ on 220 degrees of freedom; note that the constant in (3.152) is $.008\,(1 - .347) = .005$.

We will discuss diagnostics next, but assuming both of these models fit well, how are we to reconcile the apparent differences of the estimated models

(3.151) and (3.152)? In fact, the fitted models are nearly the same. To show this, consider an AR(1) model of the form in (3.152) without a constant term; that is,

$$x_t = .35x_{t-1} + w_t,$$

and write it in its causal form, $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$, where we recall $\psi_j = .35^j$. Thus, $\psi_0 = 1, \psi_1 = .350, \psi_2 = .123, \psi_3 = .043, \psi_4 = .015, \psi_5 = .005, \psi_6 = .002, \psi_7 = .001, \psi_8 = 0, \psi_9 = 0, \psi_{10} = 0$, and so forth. Thus,

$$x_t \approx .35w_{t-1} + .12w_{t-2} + w_t,$$

which is similar to the fitted MA(2) model in (3.152).

The analysis can be performed in R as follows.

```
plot(gnp)
acf2(gnp, 50)
gnpgr = diff(log(gnp))  # growth rate
plot(gnpgr)
acf2(gnpgr, 24)
sarima(gnpgr, 1, 0, 0)  # AR(1)
sarima(gnpgr, 0, 0, 2)  # MA(2)
ARMAtoMA(ar=.35, ma=0, 10)  # prints psi-weights
```

The next step in model fitting is diagnostics. This investigation includes the analysis of the residuals as well as model comparisons. Again, the first step involves a time plot of the innovations (or residuals), $x_t - \widehat{x}_t^{t-1}$, or of the standardized innovations

$$e_t = \left(x_t - \widehat{x}_t^{t-1}\right) / \sqrt{\widehat{P}_t^{t-1}}, \tag{3.153}$$

where $\widehat{x}_t^{t-1}$ is the one-step-ahead prediction of x_t based on the fitted model and $\widehat{P}_t^{t-1}$ is the estimated one-step-ahead error variance. If the model fits well, the standardized residuals should behave as an iid sequence with mean zero and variance one. The time plot should be inspected for any obvious departures from this assumption. Unless the time series is Gaussian, it is not enough that the residuals are uncorrelated. For example, it is possible in the non-Gaussian case to have an uncorrelated process for which values contiguous in time are highly dependent. As an example, we mention the family of GARCH models that are discussed in Chapter 5.

Investigation of marginal normality can be accomplished visually by looking at a histogram of the residuals. In addition to this, a normal probability plot or a Q-Q plot can help in identifying departures from normality. See Johnson and Wichern (1992, Chapter 4) for details of this test as well as additional tests for multivariate normality.

There are several tests of randomness, for example the runs test, that could be applied to the residuals. We could also inspect the sample autocorrelations of the residuals, say, $\widehat{\rho}_e(h)$, for any patterns or large values. Recall that, for a white noise sequence, the sample autocorrelations are approximately independently and normally distributed with zero means and variances $1/n$. Hence, a

good check on the correlation structure of the residuals is to plot $\widehat{\rho}_e(h)$ versus h along with the error bounds of $\pm 2/\sqrt{n}$. The residuals from a model fit, however, will not quite have the properties of a white noise sequence and the variance of $\widehat{\rho}_e(h)$ can be much less than $1/n$. Details can be found in Box and Pierce (1970) and McLeod (1978). This part of the diagnostics can be viewed as a visual inspection of $\widehat{\rho}_e(h)$ with the main concern being the detection of obvious departures from the independence assumption.

In addition to plotting $\widehat{\rho}_e(h)$, we can perform a general test that takes into consideration the magnitudes of $\widehat{\rho}_e(h)$ as a group. For example, it may be the case that, individually, each $\widehat{\rho}_e(h)$ is small in magnitude, say, each one is just slightly less that $2/\sqrt{n}$ in magnitude, but, collectively, the values are large. The Ljung–Box–Pierce Q-statistic given by

$$Q = n(n+2)\sum_{h=1}^{H} \frac{\widehat{\rho}_e^2(h)}{n-h} \tag{3.154}$$

can be used to perform such a test. The value H in (3.154) is chosen somewhat arbitrarily, typically, $H = 20$. Under the null hypothesis of model adequacy, asymptotically $(n \to \infty)$, $Q \sim \chi^2_{H-p-q}$. Thus, we would reject the null hypothesis at level α if the value of Q exceeds the $(1-\alpha)$-quantile of the χ^2_{H-p-q} distribution. Details can be found in Box and Pierce (1970), Ljung and Box (1978), and Davies et al. (1977). The basic idea is that if w_t is white noise, then by Property 1.1, $n\widehat{\rho}_w^2(h)$, for $h = 1, \ldots, H$, are asymptotically independent χ^2_1 random variables. This means that $n\sum_{h=1}^{H}\widehat{\rho}_w^2(h)$ is approximately a χ^2_H random variable. Because the test involves the ACF of residuals from a model fit, there is a loss of $p+q$ degrees of freedom; the other values in (3.154) are used to adjust the statistic to better match the asymptotic chi-squared distribution.

Example 3.39 Diagnostics for GNP Growth Rate Example

We will focus on the MA(2) fit from Example 3.38; the analysis of the AR(1) residuals is similar. Figure 3.17 displays a plot of the standardized residuals, the ACF of the residuals, a ~~boxplot of the standardized residuals~~, and the p-values associated with the Q-statistic, (3.154), at lags $H = 3$ through $H = 20$ (with corresponding degrees of freedom $H - 2$).

Inspection of the time plot of the standardized residuals in Figure 3.17 shows no obvious patterns. Notice that there are outliers, however, with a few values exceeding 3 standard deviations in magnitude. The ACF of the standardized residuals shows no apparent departure from the model assumptions, and the Q-statistic is never significant at the lags shown. The normal Q-Q plot of the residuals shows departure from normality at the tails due to the outliers that occurred primarily in the 1950s and the early 1980s.

The model appears to fit well except for the fact that a distribution with heavier tails than the normal distribution should be employed. We discuss

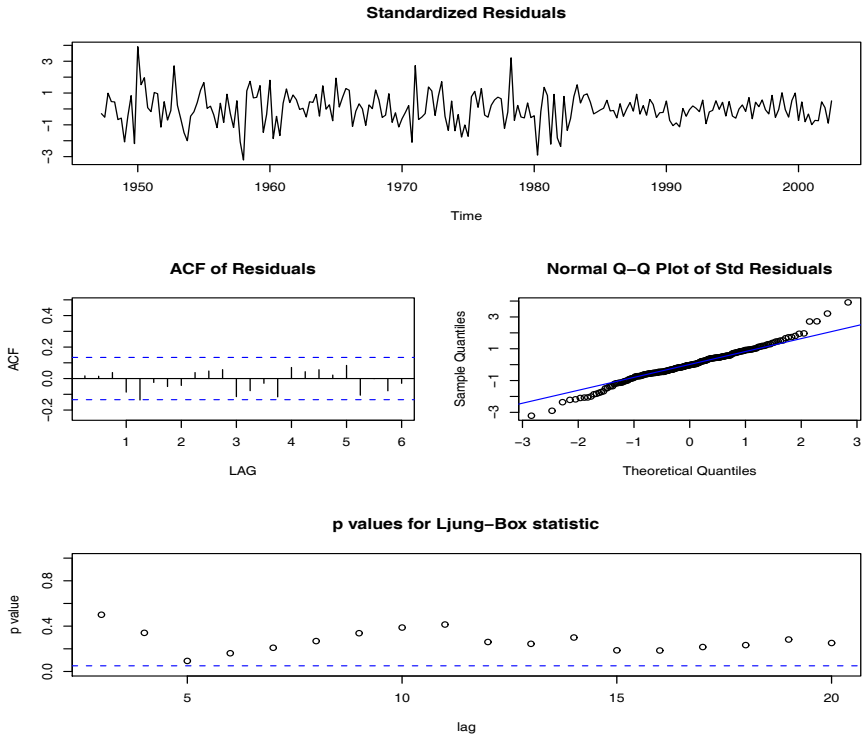


Fig. 3.17. Diagnostics of the residuals from MA(2) fit on GNP growth rate.

some possibilities in Chapters 5 and 6. The diagnostics shown in Figure 3.17 are a by-product of the `sarima` command from the previous example.[9]

Example 3.40 Diagnostics for the Glacial Varve Series

In Example 3.32, we fit an ARIMA$(0, 1, 1)$ model to the logarithms of the glacial varve data and there appears to be a small amount of autocorrelation left in the residuals and the Q-tests are all significant; see Figure 3.18.

To adjust for this problem, we fit an ARIMA$(1, 1, 1)$ to the logged varve data and obtained the estimates

$$\widehat{\phi} = .23_{(.05)},\ \widehat{\theta} = -.89_{(.03)},\ \widehat{\sigma}_w^2 = .23.$$

Hence the AR term is significant. The Q-statistic p-values for this model are also displayed in Figure 3.18, and it appears this model fits the data well.

As previously stated, the diagnostics are byproducts of the individual `sarima` runs. We note that we did not fit a constant in either model because

[9] The script `tsdiag` is available in R to run diagnostics for an ARIMA object, however, the script has errors and we do not recommend using it.

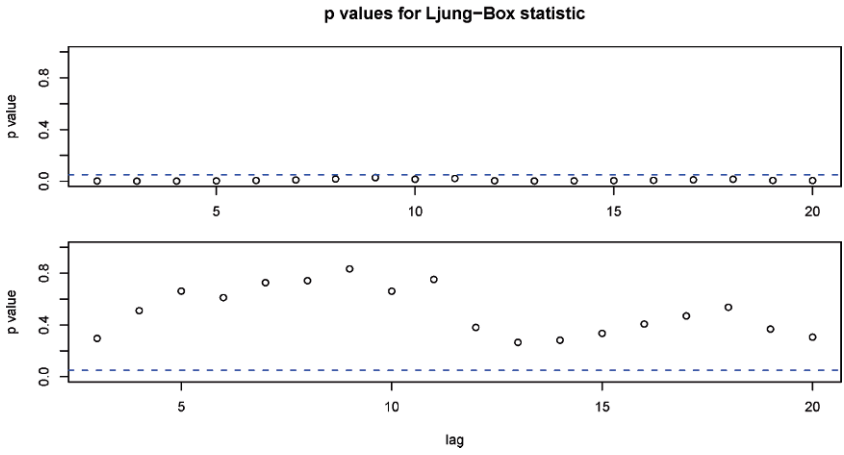


Fig. 3.18. Q-statistic p-values for the ARIMA$(0,1,1)$ fit [top] and the ARIMA$(1,1,1)$ fit [bottom] to the logged varve data.

there is no apparent drift in the differenced, logged varve series. This fact can be verified by noting the constant is not significant when the command `no.constant=TRUE` is removed in the code:

```
sarima(log(varve), 0, 1, 1, no.constant=TRUE)    # ARIMA(0,1,1)
sarima(log(varve), 1, 1, 1, no.constant=TRUE)    # ARIMA(1,1,1)
```

In Example 3.38, we have two competing models, an AR(1) and an MA(2) on the GNP growth rate, that each appear to fit the data well. In addition, we might also consider that an AR(2) or an MA(3) might do better for forecasting. Perhaps combining both models, that is, fitting an ARMA$(1,2)$ to the GNP growth rate, would be the best. As previously mentioned, we have to be concerned with overfitting the model; it is not always the case that more is better. Overfitting leads to less-precise estimators, and adding more parameters may fit the data better but may also lead to bad forecasts. This result is illustrated in the following example.

Example 3.41 A Problem with Overfitting

Figure 3.19 shows the U.S. population by official census, every ten years from 1910 to 1990, as points. If we use these nine observations to predict the future population, we can use an eight-degree polynomial so the fit to the nine observations is perfect. The model in this case is

$$x_t = \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots + \beta_8 t^8 + w_t.$$

The fitted line, which is plotted in the figure, passes through the nine observations. The model predicts that the population of the United States will be close to zero in the year 2000, and will cross zero sometime in the year 2002!

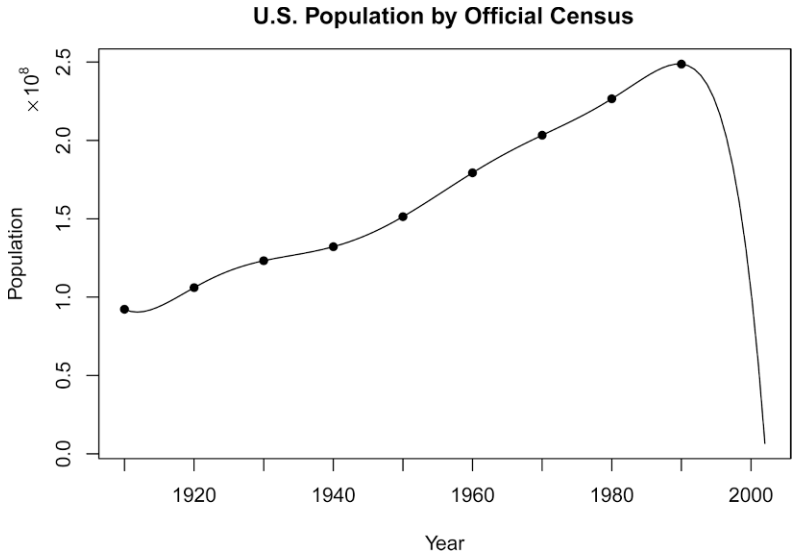


Fig. 3.19. A perfect fit and a terrible forecast.

The final step of model fitting is model choice or model selection. That is, we must decide which model we will retain for forecasting. The most popular techniques, AIC, AICc, and BIC, were described in §2.2 in the context of regression models.

Example 3.42 Model Choice for the U.S. GNP Series

Returning to the analysis of the U.S. GNP data presented in Examples 3.38 and 3.39, recall that two models, an AR(1) and an MA(2), fit the GNP growth rate well. To choose the final model, we compare the AIC, the ~~AICc,~~ and the BIC for both models. These values are a byproduct of the `sarima` runs displayed at the end of Example 3.38, but for convenience, we display them again here (recall the growth rate data are in `gnpgr`):

```
1 sarima(gnpgr, 1, 0, 0)  # AR(1)
     $AIC: -8.294403   $AICc: -8.284898   $BIC: -9.263748
2 sarima(gnpgr, 0, 0, 2)  # MA(2)
     $AIC: -8.297693   $AICc: -8.287854   $BIC: -9.251711
```

The AIC and ~~AICc~~ both prefer the MA(2) fit, whereas the BIC prefers the simpler AR(1) model. It is often the case that the BIC will select a model of smaller order than the AIC or AICc. It would not be unreasonable in this case to retain the AR(1) because pure autoregressive models are easier to work with.

Parsimony Principle