STATISTICS 630 - Solution to Final Exam **December 10, 2013**

1. Let X and Y be jointly distributed random variables with means $\mu_X = 0$ and $\mu_Y = -2$, variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$, and covariance $\text{Cov}(X, Y) = -1$. Let $U = X - Y - 3$ and $V = 2X + 3Y + 5$. Find $E(U)$, $\text{Var}(U)$, $E(V)$, $\text{Var}(V)$ and $\text{Cov}(U, V)$.

$$E(U) = E(X) - E(Y) - 3 = 0 - (-2) - 3 = -1,$$
$$\text{Var}(U) = \text{Var}(X) + \text{Var}(Y) - 2\text{Cov}(X, Y) = 2 + 4 - (2)(-1) = 8,$$
$$E(V) = 2E(X) + 3E(Y) + 5 = 2(0) + 3(-2) + 5 = -1,$$
$$\text{Var}(V) = 4\text{Var}(X) + 9\text{Var}(Y) + 2(2)(3)\text{Cov}(X, Y) = 4(2) + 9(4) - 12 = 32,$$
$$\text{Cov}(U, V) = 2\text{Var}(X) + 3\text{Cov}(X, Y) - 2\text{Cov}(X, Y) - 3\text{Var}(Y) = 2(2) + (-1) - 3(4) = -9.$$

2. Let $X_1, \ldots, X_n$ be a random sample from the normal$(0, \sigma_1^2)$ distribution for $\sigma_1 > 0$ and $Y_1, \ldots, Y_m$ be a random sample from the normal$(0, \sigma_2^2)$ distribution for $\sigma_2 > 0$. Assume that $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are mutually independent.

   (a) Let
   $$W = \frac{\sum_{i=1}^n X_i^2/(n\sigma_1^2)}{\sum_{j=1}^m Y_j^2/(m\sigma_2^2)}.$$
   Explain why W has an $F(n, m)$ distribution.

   Since $X_i \sim N(0, \sigma_1^2)$, by the properties of the normal distribution $X_i/\sigma_1 \sim N(0, 1)$ and $X_i^2/\sigma_1^2 \sim \chi^2(1)$. By independence of $X_1, \ldots, X_n$, $U = \sum_{i=1}^n X_i^2/\sigma_1^2 \sim \chi^2(n)$. Similarly, $V = \sum_{j=1}^m Y_j^2/m\sigma_2^2 \sim \chi^2(m)$. Thus, $W = (U/n)/(V/m) \sim F(n, m)$ by the representation of the F distribution since U and V are independent chi-square random variables with n and m degrees of freedom, respectively.

   (b) Using W as a pivot, derive a level γ confidence interval for σ_1^2/σ_2^2.

   Let $c_1 > 0$ and c_2 be values such that $G(c_2) - G(c_1) = \gamma$ where $G(\cdot)$ is the cumulative distribution function of the $F(n, m)$ distribution. Then

   $$\begin{aligned} \gamma &= P[c_1 < W < c_2] = P\left[c_1 < \frac{\sum_{i=1}^n X_i^2/(n\sigma_1^2)}{\sum_{j=1}^m Y_j^2/(m\sigma_2^2)} < c_2\right] \\ &= P\left[\frac{1}{c_2}\frac{\sum_{i=1}^n X_i^2/n}{\sum_{j=1}^m Y_j^2/m} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{1}{c_1}\frac{\sum_{i=1}^n X_i^2/n}{\sum_{j=1}^m Y_j^2/m}\right]. \end{aligned}$$

   Thus,
   $$\left[\frac{1}{c_2}\frac{\sum_{i=1}^n X_i^2/n}{\sum_{j=1}^m Y_j^2/m}, \frac{1}{c_1}\frac{\sum_{i=1}^n X_i^2/n}{\sum_{j=1}^m Y_j^2/m}\right]$$
   is a level γ confidence interval for σ_1^2/σ_2^2.

3. Suppose that T_1 and T_2 are independent random variables such that $E(T_1) = \theta$, $E(T_2) = 2\theta$, $\mathrm{Var}(T_1) = 2\theta^2$ and $\mathrm{Var}(T_2) = 4\theta^2$. Consider the following estimators of θ:

$$\hat{\theta}_1 = \frac{T_1 + T_2}{3} \qquad \text{and} \qquad \hat{\theta}_2 = \frac{T_1 + T_2}{4}.$$

Find the bias, variance, and mean squared error of each of these estimators. Then determine which estimator is preferable.

$$E(\hat{\theta}_1) = \frac{E(T_1) + E(T_2)}{3} = \frac{\theta + 2\theta}{3} = \theta, \text{ bias}(\hat{\theta}_1) = \theta - \theta = 0.$$

$$E(\hat{\theta}_2) = \frac{E(T_1) + E(T_2)}{4} = \frac{\theta + 2\theta}{4} = \frac{3\theta}{4}, \text{ bias}(\hat{\theta}_2) = \frac{3\theta}{4} - \theta = -\frac{\theta}{4}.$$

$$\mathrm{Var}(\hat{\theta}_1) = \frac{\mathrm{Var}(T_1) + \mathrm{Var}(T_2)}{9} = \frac{2\theta^2 + 4\theta^2}{9} = \frac{2\theta^2}{3}.$$

$$\mathrm{Var}(\hat{\theta}_2) = \frac{\mathrm{Var}(T_1) + \mathrm{Var}(T_2)}{16} = \frac{2\theta^2 + 4\theta^2}{16} = \frac{3\theta^2}{8}.$$

$$MSE(\hat{\theta}_1) = \frac{2\theta^2}{3} + 0^2 = \frac{2\theta^2}{3}.$$

$$MSE(\hat{\theta}_2) = \frac{3\theta^2}{8} + \left(\frac{-\theta}{4}\right)^2 = \frac{7\theta^2}{16}.$$

Since $\hat{\theta}_2$ has smaller MSE, it is the preferred estimator.

4. Let $X_1, \ldots, X_n$ be a random sample from the normal$(0, 1/\theta)$ distribution for $\theta > 0$ with probability density function

$$f_\theta(x) = \frac{\theta^{1/2}}{\sqrt{2\pi}} e^{-\theta x^2/2}, \quad -\infty < x < \infty.$$

Suppose that θ has the prior density

$$\pi(\theta) = \begin{cases} 4\theta^2 e^{-2\theta}, & \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Obtain the posterior distribution of θ given $X_1 = x_1, \ldots, X_n = x_n$. Then obtain the mean and variance of the posterior distribution.

The posterior pdf

$$\pi(\theta|x) \propto L(\theta|x_1, ..., x_n) \times \pi(\theta) \propto \theta^2 e^{-2\theta} \times \theta^{n/2} e^{-\theta \sum x_i^2/2} = \theta^{2+n/2} e^{-(2+\sum x_i^2/2)\theta}.$$

We recognize this as the kernel of the gamma$(3+n/2, 2+\sum_{i=1}^n x_i^2/2)$ distribution. Thus, the posterior distribution of θ given $X_1 = x_1, \ldots, X_n = x_n$ is the gamma$(3 + n/2, 2 + \sum_{i=1}^n x_i^2/2)$ distribution. The posterior mean and variance are

$$E[\theta|x_1, ..., x_n] = \frac{3 + n/2}{2 + \sum_{i=1}^n x_i^2/2} \quad \text{and} \quad \mathrm{Var}[\theta|x_1, ..., x_n] = \frac{3 + n/2}{(2 + \sum_{i=1}^n x_i^2/2)^2}.$$

5. A statistics professor enjoys playing tennis and needs to practice his serves. Suppose that he attempts three serves and the number of good serves is a random variable Y with moment generating function

$$M_Y(s) = \frac{1}{27} + \frac{6}{27}e^s + \frac{12}{27}e^{2s} + \frac{8}{27}e^{3s}.$$

(a) Use the moment generating function to show that

$$E(Y) = 2 \quad \text{and} \quad \mathrm{Var}(Y) = \frac{2}{3}.$$

$$E(Y) = m_Y'(0) = \frac{6}{27}e^s + \frac{12(2)}{27}e^{2s} + \frac{8(3)}{27}e^{3s}\bigg|_{s=0} = \frac{6}{27} + \frac{12(2)}{27} + \frac{8(3)}{27} = 2.$$

$$E(Y) = m_Y''(0) = \frac{6}{27}e^s + \frac{12(2^2)}{27}e^{2s} + \frac{8(3^2)}{27}e^{3s}\bigg|_{s=0} = \frac{6}{27} + \frac{12(2^2)}{27} + \frac{8(3^2)}{27} = \frac{126}{27} = \frac{14}{3}.$$

$$\mathrm{Var}(Y) = \frac{14}{3} - 2^2 = \frac{2}{3}.$$

(b) Suppose $Z_n = Y_1 + \cdots + Y_n$ where $Y_1, ..., Y_n$ are independent random variables with the above mean and variance. Find a number m (with proof) such that $\frac{1}{n}Z_n \xrightarrow{P} m$.

Since $E(Y_i) = 2$ and $\mathrm{Var}(Y_i) = 2/3$, we can apply the Weak Law of Large Numbers to obtain

$$\frac{1}{n}Z_n \xrightarrow{P} E(Y_i) = 2.$$

6. Suppose that $X_1, \ldots, X_n$ are a random sample from a distribution with probability density function

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

(a) Obtain the maximum likelihood estimator and Fisher's information for θ.

The log likelihood is

$$\log L(\theta) = \log\left(\Pi_{i=1}^n \theta x_i^{\theta-1}\right) = n\log(\theta) + \sum_{i=1}^n (\theta-1)\log(x_i).$$

Thus, the score function is

$$\frac{\partial \log L(\theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n \log(x_i).$$

Set the score function equal to zero and solve for θ to obtain the mle:

$$\hat{\theta} = \frac{n}{-\sum_{i=1}^n \log(x_i)}.$$

Since $\frac{\partial^2 \log L(\theta)}{\partial \theta^2} = -n/\theta^2 < 0$, we have found a maximum.
Fisher's information for θ in the sample $X_1, \ldots, X_n$ is

$$I_n(\theta) = -E\left[\frac{\partial^2 \log L(\theta)}{\partial \theta^2}\right] = \frac{n}{\theta^2}.$$

(b) Write out expressions for the Wald statistic and the score statistic for testing $H_0 : \theta = \theta_0$ versus $H_0 : \theta \neq \theta_0$. The Wald statistic is

$$W = I_n(\hat{\theta})(\hat{\theta} - \theta_0)^2 = \frac{n(\hat{\theta} - \theta_0)^2}{\hat{\theta}^2}.$$

The score statistic is

$$S = \frac{U(\theta_0)^2}{I_n(\theta_0)} = \frac{\left(\frac{n}{\theta_0} + \sum_{i=1}^n \log(x_i)\right)^2}{(n/\theta_0^2)} = \frac{\theta_0^2\left(\frac{n}{\theta_0} - \frac{n}{\hat{\theta}}\right)^2}{n}.$$

After a little algebra, one can show that $W = S$ in this problem.