STAT 626: Outline of Lecture 3

Stationary Time Series and Its ACF (§1.5)

$\boxed{\text{Computing Covar.}}$

1. Stationary Time Series

2. Autocovariance Function

3. Autocorrelation Function (ACF)

4. The Correlogram

~~**Example 1.15 Mean Function of Signal Plus Noise**~~

~~A great many practical applications depend on assuming the observed data have been generated by a fixed signal waveform superimposed on a zero-mean noise process, leading to an additive signal model of the form (1.5). It is clear, because the signal in (1.5) is a fixed function of time, we will have~~

$$\cancel{\begin{aligned}\mu_{xt} &= E(x_t) = E\big[2\cos(2\pi t/50 + .6\pi) + w_t\big] \\ &= 2\cos(2\pi t/50 + .6\pi) + E(w_t) \\ &= 2\cos(2\pi t/50 + .6\pi),\end{aligned}}$$

~~and the mean function is just the cosine wave.~~

~~The lack of independence between two adjacent values x_s and x_t can be assessed numerically, as in classical statistics, using the notions of covariance and correlation. Assuming the variance of x_t is finite, we have the following definition.~~

Definition 1.2 *The* **autocovariance function** *is defined as the second moment product*

$$\gamma_x(s,t) = \mathrm{cov}(x_s, x_t) = E[(x_s - \mu_s)(x_t - \mu_t)], \tag{1.10}$$

for all s and t. When no possible confusion exists about which time series we are referring to, we will drop the subscript and write $\gamma_x(s,t)$ as $\gamma(s,t)$.

Note that $\gamma_x(s,t) = \gamma_x(t,s)$ for all time points s and t. The autocovariance measures the *linear* dependence between two points on the same series observed at different times. Very smooth series exhibit autocovariance functions that stay large even when the t and s are far apart, whereas choppy series tend to have autocovariance functions that are nearly zero for large separations. The autocovariance (1.10) is the average cross-product relative to the joint distribution $F(x_s, x_t)$. Recall from classical statistics that if $\gamma_x(s,t) = 0$, x_s and x_t are not linearly related, but there still may be some dependence structure between them. If, however, x_s and x_t are bivariate normal, $\gamma_x(s,t) = 0$ ensures their independence. It is clear that, for $s = t$, the autocovariance reduces to the (assumed finite) variance, because

$$\gamma_x(t,t) = E[(x_t - \mu_t)^2] = \mathrm{var}(x_t). \tag{1.11}$$

Example 1.16 Autocovariance of White Noise

The white noise series w_t has $E(w_t) = 0$ and

$$\gamma_w(s,t) = \mathrm{cov}(w_s, w_t) = \begin{cases} \sigma_w^2 & s = t, \\ 0 & s \neq t. \end{cases} \tag{1.12}$$

A realization of white noise with $\sigma_w^2 = 1$ is shown in the top panel of Figure 1.8.

Example 1.17 Autocovariance of a Moving Average

Consider applying a three-point moving average to the white noise series w_t of the previous example as in Example 1.9. In this case,

$$\gamma_v(s,t) = \mathrm{cov}(v_s, v_t) = \mathrm{cov}\left\{\tfrac{1}{3}(w_{s-1}+w_s+w_{s+1}), \tfrac{1}{3}(w_{t-1}+w_t+w_{t+1})\right\}.$$

When $s = t$ we have[3]

$$\begin{aligned}\gamma_v(t,t) &= \tfrac{1}{9}\mathrm{cov}\{(w_{t-1}+w_t+w_{t+1}),(w_{t-1}+w_t+w_{t+1})\}\\ &= \tfrac{1}{9}[\mathrm{cov}(w_{t-1},w_{t-1})+\mathrm{cov}(w_t,w_t)+\mathrm{cov}(w_{t+1},w_{t+1})]\\ &= \tfrac{3}{9}\sigma_w^2.\end{aligned}$$

When $s = t+1$,

$$\begin{aligned}\gamma_v(t+1,t) &= \tfrac{1}{9}\mathrm{cov}\{(w_t+w_{t+1}+w_{t+2}),(w_{t-1}+w_t+w_{t+1})\}\\ &= \tfrac{1}{9}[\mathrm{cov}(w_t,w_t)+\mathrm{cov}(w_{t+1},w_{t+1})]\\ &= \tfrac{2}{9}\sigma_w^2,\end{aligned}$$

using (1.12). Similar computations give $\gamma_v(t-1,t) = 2\sigma_w^2/9$, $\gamma_v(t+2,t) = \gamma_v(t-2,t) = \sigma_w^2/9$, and 0 when $|t-s| > 2$. We summarize the values for all s and t as

$$\gamma_v(s,t) = \begin{cases}\tfrac{3}{9}\sigma_w^2 & s=t,\\ \tfrac{2}{9}\sigma_w^2 & |s-t|=1,\\ \tfrac{1}{9}\sigma_w^2 & |s-t|=2,\\ 0 & |s-t|>2.\end{cases} \tag{1.13}$$

Example 1.17 shows clearly that the smoothing operation introduces a covariance function that decreases as the separation between the two time points increases and disappears completely when the time points are separated by three or more time points. This particular autocovariance is interesting because it only depends on the time separation or lag and not on the absolute location of the points along the series. We shall see later that this dependence suggests a mathematical model for the concept of weak stationarity.

Example 1.18 Autocovariance of a Random Walk

For the random walk model, $x_t = \sum_{j=1}^t w_j$, we have

$$\gamma_x(s,t) = \mathrm{cov}(x_s,x_t) = \mathrm{cov}\left(\sum_{j=1}^s w_j, \sum_{k=1}^t w_k\right) = \min\{s,t\}\,\sigma_w^2,$$

because the w_t are uncorrelated random variables. Note that, as opposed to the previous examples, the autocovariance function of a random walk

[3] If the random variables $U = \sum_{j=1}^m a_j X_j$ and $V = \sum_{k=1}^r b_k Y_k$ are linear combinations of random variables $\{X_j\}$ and $\{Y_k\}$, respectively, then $\mathrm{cov}(U,V) = \sum_{j=1}^m \sum_{k=1}^r a_j b_k \mathrm{cov}(X_j, Y_k)$. Furthermore, $\mathrm{var}(U) = \mathrm{cov}(U,U)$.

depends on the particular time values s and t, and not on the time separation or lag. Also, notice that the variance of the random walk, $\text{var}(x_t) = \gamma_x(t,t) = t\,\sigma_w^2$, increases without bound as time t increases. The effect of this variance increase can be seen in Figure 1.10 where the processes start to move away from their mean functions $\delta\,t$ (note that $\delta = 0$ and .2 in that example).

As in classical statistics, it is more convenient to deal with a measure of association between -1 and 1, and this leads to the following definition.

Definition 1.3 *The* **autocorrelation function (ACF)** *is defined as*

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}}. \tag{1.14}$$

The ACF measures the linear predictability of the series at time t, say x_t, using only the value x_s. We can show easily that $-1 \le \rho(s,t) \le 1$ using the Cauchy–Schwarz inequality.[4] If we can predict x_t *perfectly* from x_s through a linear relationship, $x_t = \beta_0 + \beta_1 x_s$, then the correlation will be $+1$ when $\beta_1 > 0$, and -1 when $\beta_1 < 0$. Hence, we have a rough measure of the ability to forecast the series at time t from the value at time s.

Often, we would like to measure the predictability of another series y_t from the series x_s. Assuming both series have finite variances, we have the following definition.

Definition 1.4 *The* **cross-covariance function** *between two series, x_t and y_t, is*

$$\gamma_{xy}(s,t) = \text{cov}(x_s, y_t) = E[(x_s - \mu_{xs})(y_t - \mu_{yt})]. \tag{1.15}$$

There is also a scaled version of the cross-covariance function.

Definition 1.5 *The* **cross-correlation function (CCF)** *is given by*

$$\rho_{xy}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}}. \tag{1.16}$$

We may easily extend the above ideas to the case of more than two series, say, $x_{t1}, x_{t2}, \ldots, x_{tr}$; that is, multivariate time series with r components. For example, the extension of (1.10) in this case is

$$\gamma_{jk}(s,t) = E[(x_{sj} - \mu_{sj})(x_{tk} - \mu_{tk})] \qquad j,k = 1,2,\ldots,r. \tag{1.17}$$

In the definitions above, the autocovariance and cross-covariance functions may change as one moves along the series because the values depend on both s

[4] The Cauchy–Schwarz inequality implies $|\gamma(s,t)|^2 \le \gamma(s,s)\gamma(t,t)$.

DEFINITIONS

1. A Time Series $\{x_t\}$ is **stationary** if

   (a) the mean function $E(x_t)$ does not depend on the time t,

   (b) the covariance function $\mathrm{cov}(x_s, x_t)$ depends on the times s, t only through the distance $|s-t|$.

2. **Autocovariance Function** of a Stationary Time Series:

$$\gamma(h) = \mathrm{cov}(x_{t+h}, x_t), \quad h = 0, 1, \ldots.$$

   NOTE:

$$\gamma(0) = \mathrm{cov}(x_t, x_t) = \mathrm{var}(x_t).$$

3. **The Autocorrelation Function (ACF)**

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)}, \quad h = 0, 1, \ldots.$$

4. **Correlogram** is the plot of $\rho(h)$ vs h.

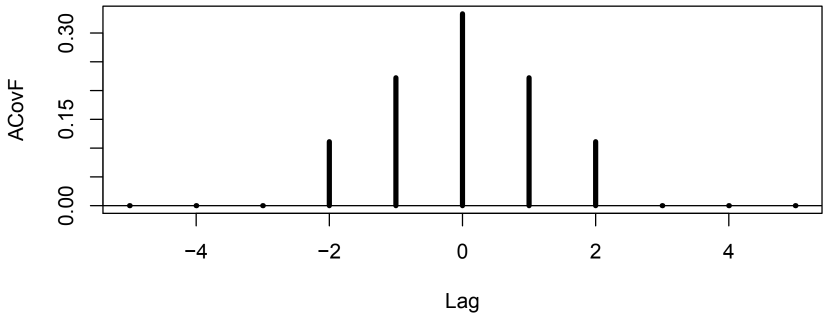

Fig. 1.12. Autocovariance function of a three-point moving average.

Definition 1.8 *The* **autocovariance function of a stationary time series** *will be written as*

$$\gamma(h) = \text{cov}(x_{t+h}, x_t) = E[(x_{t+h} - \mu)(x_t - \mu)]. \tag{1.22}$$

Definition 1.9 *The* **autocorrelation function (ACF) of a stationary time series** *will be written using (1.14) as*

$$\rho(h) = \frac{\gamma(t+h, t)}{\sqrt{\gamma(t+h, t+h)\gamma(t, t)}} = \frac{\gamma(h)}{\gamma(0)}. \tag{1.23}$$

The Cauchy–Schwarz inequality shows again that $-1 \leq \rho(h) \leq 1$ for all h, enabling one to assess the relative importance of a given autocorrelation value by comparing with the extreme values -1 and 1.

Example 1.19 Stationarity of White Noise

The mean and autocovariance functions of the white noise series discussed in Examples 1.8 and 1.16 are easily evaluated as $\mu_{wt} = 0$ and

$$\gamma_w(h) = \text{cov}(w_{t+h}, w_t) = \begin{cases} \sigma_w^2 & h = 0, \\ 0 & h \neq 0. \end{cases}$$

Thus, white noise satisfies the conditions of Definition 1.7 and is weakly stationary or stationary. If the white noise variates are also normally distributed or Gaussian, the series is also strictly stationary, as can be seen by evaluating (1.18) using the fact that the noise would also be iid.

Example 1.20 Stationarity of a Moving Average

The three-point moving average process of Example 1.9 is stationary because, from Examples 1.13 and 1.17, the mean and autocovariance functions $\mu_{vt} = 0$, and

$$\gamma_v(h) = \begin{cases} \frac{3}{9}\sigma_w^2 & h = 0, \\ \frac{2}{9}\sigma_w^2 & h = \pm 1, \\ \frac{1}{9}\sigma_w^2 & h = \pm 2, \\ 0 & |h| > 2 \end{cases}$$

are independent of time t, satisfying the conditions of Definition 1.7. Figure 1.12 shows a plot of the autocovariance as a function of lag h with $\sigma_w^2 = 1$. Interestingly, the autocovariance function is symmetric about lag zero and decays as a function of lag.

The autocovariance function of a stationary process has several useful properties (also, see Problem 1.25). First, the value at $h = 0$, namely

$$\gamma(0) = E[(x_t - \mu)^2] \tag{1.24}$$

is the variance of the time series; note that the Cauchy–Schwarz inequality implies

$$|\gamma(h)| \leq \gamma(0).$$

A final useful property, noted in the previous example, is that the autocovariance function of a stationary series is symmetric around the origin; that is,

$$\gamma(h) = \gamma(-h) \tag{1.25}$$

for all h. This property follows because shifting the series by h means that

$$\begin{aligned} \gamma(h) &= \gamma(t + h - t) \\ &= E[(x_{t+h} - \mu)(x_t - \mu)] \\ &= E[(x_t - \mu)(x_{t+h} - \mu)] \\ &= \gamma(t - (t + h)) \\ &= \gamma(-h), \end{aligned}$$

which shows how to use the notation as well as proving the result.

When several series are available, a notion of stationarity still applies with additional conditions.

Definition 1.10 *Two time series, say,* x_t *and* y_t*, are said to be* **jointly stationary** *if they are each stationary, and the cross-covariance function*

$$\gamma_{xy}(h) = \text{cov}(x_{t+h}, y_t) = E[(x_{t+h} - \mu_x)(y_t - \mu_y)] \tag{1.26}$$

is a function only of lag h.

Definition 1.11 *The* **cross-correlation function (CCF)** *of jointly stationary time series* x_t *and* y_t *is defined as*

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_y(0)}}. \tag{1.27}$$

Again, we have the result $-1 \leq \rho_{xy}(h) \leq 1$ which enables comparison with the extreme values -1 and 1 when looking at the relation between x_{t+h} and y_t. The cross-correlation function is not generally symmetric about zero [i.e., typically $\rho_{xy}(h) \neq \rho_{xy}(-h)$]; however, it is the case that

$$\rho_{xy}(h) = \rho_{yx}(-h), \tag{1.28}$$

which can be shown by manipulations similar to those used to show (1.25).

Example 1.21 Joint Stationarity

Consider the two series, x_t and y_t, formed from the sum and difference of two successive values of a white noise process, say,

$$x_t = w_t + w_{t-1}$$

and

$$y_t = w_t - w_{t-1},$$

where w_t are independent random variables with zero means and variance σ_w^2. It is easy to show that $\gamma_x(0) = \gamma_y(0) = 2\sigma_w^2$ and $\gamma_x(1) = \gamma_x(-1) = \sigma_w^2, \gamma_y(1) = \gamma_y(-1) = -\sigma_w^2$. Also,

$$\gamma_{xy}(1) = \text{cov}(x_{t+1}, y_t) = \text{cov}(w_{t+1} + w_t, w_t - w_{t-1}) = \sigma_w^2$$

because only one term is nonzero (recall footnote 3 on page 20). Similarly, $\gamma_{xy}(0) = 0, \gamma_{xy}(-1) = -\sigma_w^2$. We obtain, using (1.27),

$$\rho_{xy}(h) = \begin{cases} 0 & h = 0, \\ 1/2 & h = 1, \\ -1/2 & h = -1, \\ 0 & |h| \geq 2. \end{cases}$$

Clearly, the autocovariance and cross-covariance functions depend only on the lag separation, h, so the series are jointly stationary.

Example 1.22 Prediction Using Cross-Correlation

As a simple example of cross-correlation, consider the problem of determining possible leading or lagging relations between two series x_t and y_t. If the model

$$y_t = Ax_{t-\ell} + w_t$$

holds, the series x_t is said to lead y_t for $\ell > 0$ and is said to lag y_t for $\ell < 0$. Hence, the analysis of leading and lagging relations might be important in predicting the value of y_t from x_t. Assuming, for convenience, that x_t and y_t have zero means, and the noise w_t is uncorrelated with the x_t series, the cross-covariance function can be computed as

$$\gamma_{yx}(h) = \text{cov}(y_{t+h}, x_t) = \text{cov}(Ax_{t+h-\ell} + w_{t+h}, x_t)$$
$$= \text{cov}(Ax_{t+h-\ell}, x_t) = A\gamma_x(h-\ell).$$

The cross-covariance function will look like the autocovariance of the input series x_t, with a peak on the positive side if x_t leads y_t and a peak on the negative side if x_t lags y_t.

The concept of weak stationarity forms the basis for much of the analysis performed with time series. The fundamental properties of the mean and autocovariance functions (1.21) and (1.22) are satisfied by many theoretical models that appear to generate plausible sample realizations. In Examples 1.9 and 1.10, two series were generated that produced stationary looking realizations, and in Example 1.20, we showed that the series in Example 1.9 was, in fact, weakly stationary. Both examples are special cases of the so-called linear process.

Definition 1.12 *A* **linear process**, *x_t, is defined to be a linear combination of white noise variates w_t, and is given by*

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty. \tag{1.29}$$

For the linear process (see Problem 1.11), we may show that the autocovariance function is given by

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j \tag{1.30}$$

for $h \geq 0$; recall that $\gamma(-h) = \gamma(h)$. This method exhibits the autocovariance function of the process in terms of the lagged products of the coefficients. Note that, for Example 1.9, we have $\psi_0 = \psi_{-1} = \psi_1 = 1/3$ and the result in Example 1.20 comes out immediately. The autoregressive series in Example 1.10 can also be put in this form, as can the general autoregressive moving average processes considered in Chapter 3.

Finally, as previously mentioned, an important case in which a weakly stationary series is also strictly stationary is the normal or Gaussian series.

Definition 1.13 *A process, $\{x_t\}$, is said to be a* **Gaussian process** *if the n-dimensional vectors $\boldsymbol{x} = (x_{t_1}, x_{t_2}, \dots, x_{t_n})'$, for every collection of time points $t_1, t_2, \dots, t_n$, and every positive integer n, have a multivariate normal distribution.*

Defining the $n \times 1$ mean vector $E(\boldsymbol{x}) \equiv \boldsymbol{\mu} = (\mu_{t_1}, \mu_{t_2}, \dots, \mu_{t_n})'$ and the $n \times n$ covariance matrix as $\text{var}(\boldsymbol{x}) \equiv \Gamma = \{\gamma(t_i, t_j);\ i, j = 1, \dots, n\}$, which is

LINEAR PROCESSES are the most general form of stationary processes we need in this course. They are formed as linear combinations of a **white noise** $\{w_t\} \sim \mathrm{WN}(0, \sigma_w^2)$.

Moving Average of order q or MA(q) Models:

$$x_t = w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q},$$

where $\theta = (\theta_1, \ldots, \theta_q)$ is the vector of parameters.

What happens when $q = \infty$?
MA(∞) Models or Processes.