STAT 626: Outline of Lecture 15

Multiplicative Seasonal ARIMA (SARIMA) Models (§3.9)

1. **Plot the Data**

2. **Induce Stationarity by Seasonal Differencing or Other Means**

3. **Model Formulation: Use the ACF and PACF to Select** p, q, P, Q

4. **Model Estimation: Find the MLE of the Parameters**

5. **Model Diagnostic: Check the Residuals for Independence**

$$H_0 : \textbf{The model residuals are uncorrelated (WN)}$$

vs.

$$H_a : \textbf{Residuals are correlated.}$$

6. **If Not Happy, Go to Step 2 and Repeat the PROCESS**

Example 3.46: The Federal Reserve Board Production Index

Example 3.43 A Seasonal ARMA Series

$$(1 - \Phi B^{12})x_t = (1 + \Theta B^{12})w_t.$$

What are the connections with ARMA(1,1) models?

Is it causal? Invertible?

Its MA(∞) representation?

Its autocovariance function?

Its ACF?

Its predictors? Prediction error variance?

Example 3.44: A Mixed Seasonal Model

$$x_t = \Phi x_{t-12} + w_t + \theta w_{t-1}.$$

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3.9 Multiplicative Seasonal ARIMA Models

In this section, we introduce several modifications made to the ARIMA model to account for seasonal and nonstationary behavior. Often, the dependence on the past tends to occur most strongly at multiples of some underlying seasonal lag s. For example, with monthly economic data, there is a strong yearly component occurring at lags that are multiples of $s = 12$, because of the strong connections of all activity to the calendar year. Data taken quarterly will exhibit the yearly repetitive period at $s = 4$ quarters. Natural phenomena such as temperature also have strong components corresponding to seasons. Hence, the natural variability of many physical, biological, and economic processes tends to match with seasonal fluctuations. Because of this, it is appropriate to introduce autoregressive and moving average polynomials that identify with the seasonal lags. The resulting pure seasonal autoregressive moving average model, say, ARMA$(P, Q)_s$, then takes the form

$$\Phi_P(B^s)x_t = \Theta_Q(B^s)w_t, \tag{3.155}$$

with the following definition.

Definition 3.12 *The operators*

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_P B^{Ps} \tag{3.156}$$

and

$$\Theta_Q(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \cdots + \Theta_Q B^{Qs} \tag{3.157}$$

are the **seasonal autoregressive operator** *and the* **seasonal moving average operator** *of orders P and Q, respectively, with seasonal period s.*

Analogous to the properties of nonseasonal ARMA models, the pure seasonal ARMA$(P, Q)_s$ is causal only when the roots of $\Phi_P(z^s)$ lie outside the unit circle, and it is invertible only when the roots of $\Theta_Q(z^s)$ lie outside the unit circle.

Example 3.43 A Seasonal ARMA Series

A first-order seasonal autoregressive moving average series that might run over months could be written as

$$(1 - \Phi B^{12})x_t = (1 + \Theta B^{12})w_t$$

or

$$x_t = \Phi x_{t-12} + w_t + \Theta w_{t-12}.$$

This model exhibits the series x_t in terms of past lags at the multiple of the yearly seasonal period $s = 12$ months. It is clear from the above form that estimation and forecasting for such a process involves only straightforward modifications of the unit lag case already treated. In particular, the causal condition requires $|\Phi| < 1$, and the invertible condition requires $|\Theta| < 1$.

Table 3.3. Behavior of the ACF and PACF for Pure SARMA Models

	$\text{AR}(P)_s$	$\text{MA}(Q)_s$	$\text{ARMA}(P, Q)_s$
ACF*	Tails off at lags ks, $k = 1, 2, \ldots,$	Cuts off after lag Qs	Tails off at lags ks
PACF*	Cuts off after lag Ps	Tails off at lags ks $k = 1, 2, \ldots,$	Tails off at lags ks

*The values at nonseasonal lags $h \neq ks$, for $k = 1, 2, \ldots$, are zero.

For the first-order seasonal ($s = 12$) MA model, $x_t = w_t + \Theta w_{t-12}$, it is easy to verify that

$$\begin{aligned} \gamma(0) &= (1 + \Theta^2)\sigma^2 \\ \gamma(\pm 12) &= \Theta\sigma^2 \\ \gamma(h) &= 0, \quad \text{otherwise.} \end{aligned}$$

Thus, the only nonzero correlation, aside from lag zero, is

$$\rho(\pm 12) = \Theta/(1 + \Theta^2).$$

For the first-order seasonal ($s = 12$) AR model, using the techniques of the nonseasonal AR(1), we have

$$\begin{aligned} \gamma(0) &= \sigma^2/(1 - \Phi^2) \\ \gamma(\pm 12k) &= \sigma^2 \Phi^k/(1 - \Phi^2) \quad k = 1, 2, \ldots \\ \gamma(h) &= 0, \quad \text{otherwise.} \end{aligned}$$

In this case, the only non-zero correlations are

$$\rho(\pm 12k) = \Phi^k, \quad k = 0, 1, 2, \ldots \ .$$

These results can be verified using the general result that $\gamma(h) = \Phi\gamma(h - 12)$, for $h \geq 1$. For example, when $h = 1$, $\gamma(1) = \Phi\gamma(11)$, but when $h = 11$, we have $\gamma(11) = \Phi\gamma(1)$, which implies that $\gamma(1) = \gamma(11) = 0$. In addition to these results, the PACF have the analogous extensions from nonseasonal to seasonal models.

As an initial diagnostic criterion, we can use the properties for the pure seasonal autoregressive and moving average series listed in Table 3.3. These properties may be considered as generalizations of the properties for nonseasonal models that were presented in Table 3.1.

In general, we can combine the seasonal and nonseasonal operators into a multiplicative seasonal autoregressive moving average model, denoted by $\text{ARMA}(p, q) \times (P, Q)_s$, and write

$$\Phi_P(B^s)\phi(B)x_t = \Theta_Q(B^s)\theta(B)w_t \tag{3.158}$$

as the overall model. Although the diagnostic properties in Table 3.3 are not strictly true for the overall mixed model, the behavior of the ACF and PACF tends to show rough patterns of the indicated form. In fact, for mixed models, we tend to see a mixture of the facts listed in Tables 3.1 and 3.3. In fitting such models, focusing on the seasonal autoregressive and moving average components first generally leads to more satisfactory results.

Example 3.44 A Mixed Seasonal Model

Consider an ARMA$(0,1) \times (1,0)_{12}$ model

$$x_t = \Phi x_{t-12} + w_t + \theta w_{t-1},$$

where $|\Phi| < 1$ and $|\theta| < 1$. Then, because x_{t-12}, w_t, and w_{t-1} are uncorrelated, and x_t is stationary, $\gamma(0) = \Phi^2\gamma(0) + \sigma_w^2 + \theta^2\sigma_w^2$, or

$$\gamma(0) = \frac{1+\theta^2}{1-\Phi^2}\,\sigma_w^2.$$

In addition, multiplying the model by x_{t-h}, $h > 0$, and taking expectations, we have $\gamma(1) = \Phi\gamma(11) + \theta\sigma_w^2$, and $\gamma(h) = \Phi\gamma(h-12)$, for $h \geq 2$. Thus, the ACF for this model is

$$\begin{aligned}
\rho(12h) &= \Phi^h \quad h = 1, 2, \ldots \\
\rho(12h-1) = \rho(12h+1) &= \frac{\theta}{1+\theta^2}\Phi^h \quad h = 0, 1, 2, \ldots, \\
\rho(h) &= 0, \quad \text{otherwise.}
\end{aligned}$$

The ACF and PACF for this model, with $\Phi = .8$ and $\theta = -.5$, are shown in Figure 3.20. These type of correlation relationships, although idealized here, are typically seen with seasonal data.

To reproduce Figure 3.20 in R, use the following commands:

```
phi = c(rep(0,11),.8)
ACF = ARMAacf(ar=phi, ma=-.5, 50)[-1]      # [-1] removes 0 lag
PACF = ARMAacf(ar=phi, ma=-.5, 50, pacf=TRUE)
par(mfrow=c(1,2))
plot(ACF,  type="h", xlab="lag", ylim=c(-.4,.8));  abline(h=0)
plot(PACF, type="h", xlab="lag", ylim=c(-.4,.8));  abline(h=0)
```

Seasonal nonstationarity can occur, for example, when the process is nearly periodic in the season. For example, with average monthly temperatures over the years, each January would be approximately the same, each February would be approximately the same, and so on. In this case, we might think of average monthly temperature x_t as being modeled as

$$x_t = S_t + w_t,$$

where S_t is a seasonal component that varies slowly from one year to the next, according to a random walk,

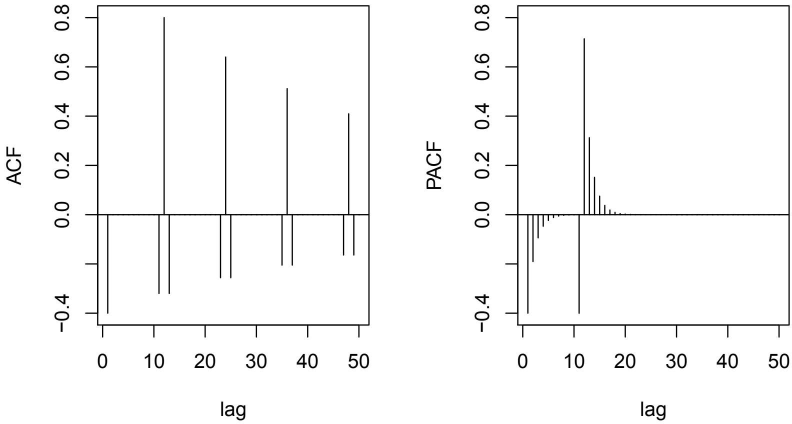


Fig. 3.20. ACF and PACF of the mixed seasonal ARMA model $x_t = .8x_{t-12} + w_t - .5w_{t-1}$.

$$S_t = S_{t-12} + v_t.$$

In this model, w_t and v_t are uncorrelated white noise processes. The tendency of data to follow this type of model will be exhibited in a sample ACF that is large and decays very slowly at lags $h = 12k$, for $k = 1, 2, \ldots$. If we subtract the effect of successive years from each other, we find that

$$(1 - B^{12})x_t = x_t - x_{t-12} = v_t + w_t - w_{t-12}.$$

This model is a stationary $\text{MA}(1)_{12}$, and its ACF will have a peak only at lag 12. In general, seasonal differencing can be indicated when the ACF decays slowly at multiples of some season s, but is negligible between the periods. Then, a seasonal difference of order D is defined as

$$\nabla_s^D x_t = (1 - B^s)^D x_t, \tag{3.159}$$

where $D = 1, 2, \ldots$, takes positive integer values. Typically, $D = 1$ is sufficient to obtain seasonal stationarity. Incorporating these ideas into a general model leads to the following definition.

Definition 3.13 *The multiplicative* **seasonal autoregressive integrated moving average** *model, or* **SARIMA** *model is given by*

$$\Phi_P(B^s)\phi(B)\nabla_s^D\nabla^d x_t = \delta + \Theta_Q(B^s)\theta(B)w_t, \tag{3.160}$$

where w_t is the usual Gaussian white noise process. The general model is denoted as **ARIMA**$(p, d, q) \times (P, D, Q)_s$. *The ordinary autoregressive and moving average components are represented by polynomials $\phi(B)$ and $\theta(B)$ of orders p and q, respectively [see (3.5) and (3.18)], and the seasonal autoregressive and moving average components by $\Phi_P(B^s)$ and $\Theta_Q(B^s)$ [see (3.156) and (3.157)] of orders P and Q and ordinary and seasonal difference components by $\nabla^d = (1 - B)^d$ and $\nabla_s^D = (1 - B^s)^D$.*

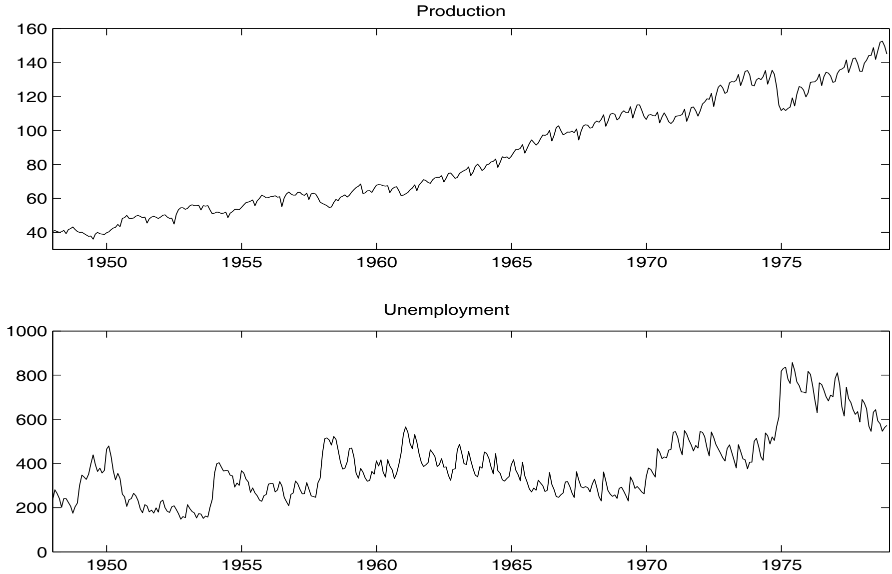


Fig. 3.21. Values of the Monthly Federal Reserve Board Production Index and Unemployment (1948-1978, $n = 372$ months).

Example 3.45 An SARIMA Model

Consider the following model, which often provides a reasonable representation for seasonal, nonstationary, economic time series. We exhibit the equations for the model, denoted by ARIMA$(0, 1, 1) \times (0, 1, 1)_{12}$ in the notation given above, where the seasonal fluctuations occur every 12 months. Then, the model (3.160) becomes

$$(1 - B^{12})(1 - B)x_t = (1 + \Theta B^{12})(1 + \theta B)w_t. \tag{3.161}$$

Expanding both sides of (3.161) leads to the representation

$$(1 - B - B^{12} + B^{13})x_t = (1 + \theta B + \Theta B^{12} + \Theta\theta B^{13})w_t,$$

or in difference equation form

$$x_t = x_{t-1} + x_{t-12} - x_{t-13} + w_t + \theta w_{t-1} + \Theta w_{t-12} + \Theta\theta w_{t-13}.$$

Note that the multiplicative nature of the model implies that the coefficient of w_{t-13} is the product of the coefficients of w_{t-1} and w_{t-12} rather than a free parameter. The multiplicative model assumption seems to work well with many seasonal time series data sets while reducing the number of parameters that must be estimated.

Selecting the appropriate model for a given set of data from all of those represented by the general form (3.160) is a daunting task, and we usually

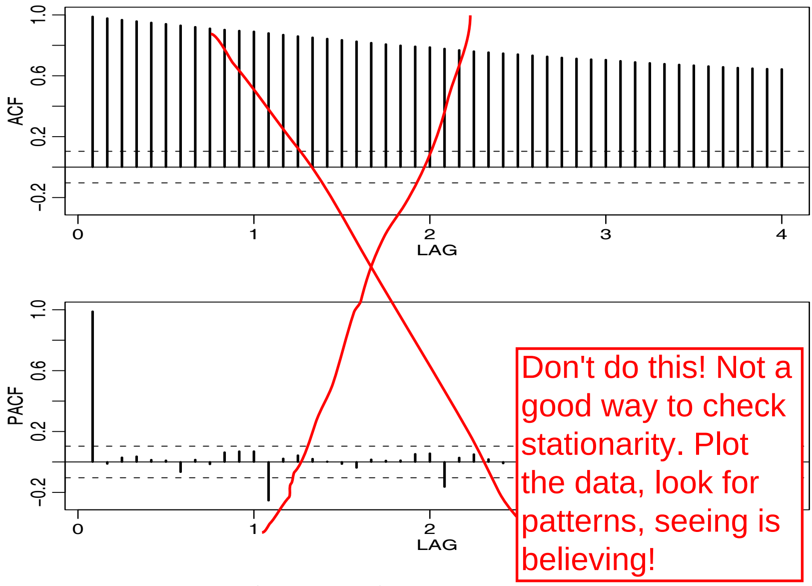


Fig. 3.22. ACF and PACF of the production series.

think first in terms of finding difference operators that produce a roughly stationary series and then in terms of finding a set of simple autoregressive moving average or multiplicative seasonal ARMA to fit the resulting residual series. Differencing operations are applied first, and then the residuals are constructed from a series of reduced length. Next, the ACF and the PACF of these residuals are evaluated. Peaks that appear in these functions can often be eliminated by fitting an autoregressive or moving average component in accordance with the general properties of Tables 3.1 and 3.2. In considering whether the model is satisfactory, the diagnostic techniques discussed in §3.8 still apply.

Example 3.46 The Federal Reserve Board Production Index

A problem of great interest in economics involves first identifying a model within the Box–Jenkins class for a given time series and then producing forecasts based on the model. For example, we might consider applying this methodology to the Federal Reserve Board Production Index shown in Figure 3.21. For demonstration purposes only, the ACF and PACF for this series are shown in Figure 3.22. We note that the trend in the data, the slow decay in the ACF, and the fact that the PACF at the first lag is nearly 1, all indicate nonstationary behavior.

Following the recommended procedure, a first difference was taken, and the ACF and PACF of the first difference

$$\nabla x_t = x_t - x_{t-1}$$

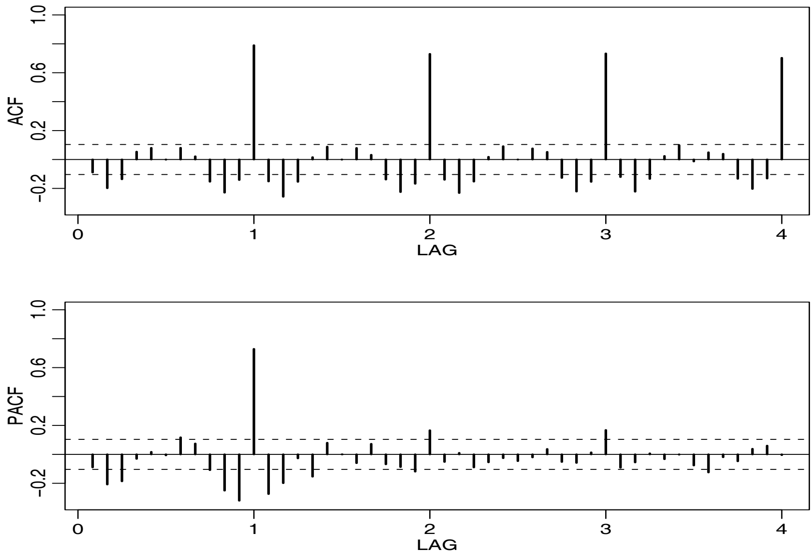


Fig. 3.23. ACF and PACF of differenced production, $(1 - B)x_t$.

are shown in Figure 3.23. Noting the peaks at seasonal lags, $h = 1s, 2s, 3s, 4s$ where $s = 12$ (i.e., $h = 12, 24, 36, 48$) with relatively slow decay suggests a seasonal difference. Figure 3.24 shows the ACF and PACF of the seasonal difference of the differenced production, say,

$$\nabla_{12}\nabla x_t = (1 - B^{12})(1 - B)x_t.$$

1

First, concentrating on the seasonal ($s = 12$) lags, the characteristics of the ACF and PACF of this series tend to show a strong peak at $h = 1s$ in the autocorrelation function, with smaller peaks appearing at $h = 2s, 3s$, combined with peaks at $h = 1s, 2s, 3s, 4s$ in the partial autocorrelation function. It appears that either

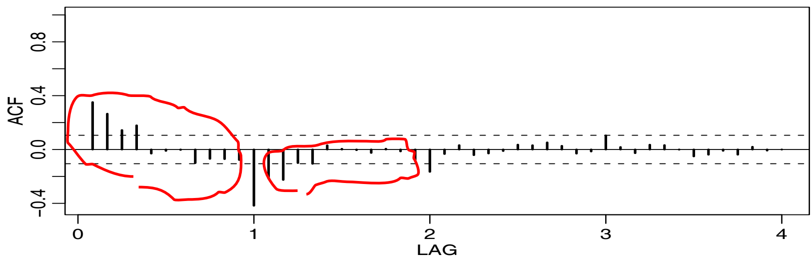
(i) the ACF is cutting off after lag $1s$ and the PACF is tailing off in the seasonal lags,
(ii) the ACF is cutting off after lag $3s$ and the PACF is tailing off in the seasonal lags, or
(iii) the ACF and PACF are both tailing off in the seasonal lags.

Using Table 3.3, this suggests either (i) an SMA of order $Q = 1$, (ii) an SMA of order $Q = 3$, or (iii) ~~an SARMA of orders $P = 2$ (because of the two spikes in the PACF) and $Q = 1$.~~

2

Next, inspecting the ACF and the PACF at the within season lags, $h = 1, \ldots, 11$, it appears that either (a) both the ACF and PACF are tailing off, or (b) that the PACF cuts off at lag 2. Based on Table 3.1, this result indicates that we should either consider fitting a model (a) with both $p > 0$ and $q > 0$ for the nonseasonal components, say $p = 1, q = 1$, or (b) $p =$

Bad reasoning



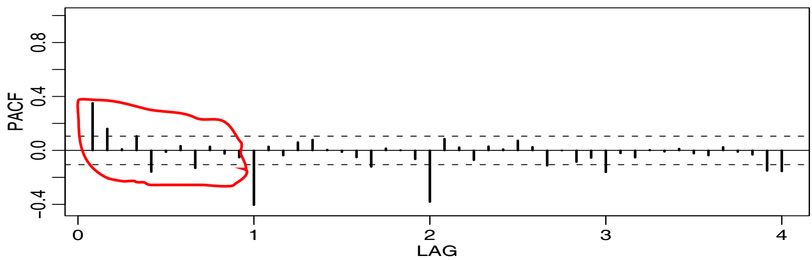


Fig. 3.24. ACF and PACF of first differenced and then seasonally differenced production, $(1-B)(1-B^{12})x_t$.

$2, q = 0$. It turns out that there is little difference in the results for case (a) and (b), but that (b) is slightly better, so we will concentrate on case (b).

Fitting the three models suggested by these observations we obtain:

(i) ARIMA$(2,1,0) \times (0,1,1)_{12}$:

AIC= 1.372, AICc= 1.378, BIC= .404

(ii) ARIMA$(2,1,0) \times (0,1,3)_{12}$:

AIC= 1.299, AICc= 1.305, BIC= .351

(iii) ARIMA$(2,1,0) \times (2,1,1)_{12}$:

AIC= 1.326, AICc= 1.332, BIC= .379

The ARIMA$(2,1,0)\times(0,1,3)_{12}$ is the preferred model, and the fitted model in this case is

$$(1 - .30_{(.05)}B - .11_{(.05)}B^2)\nabla_{12}\nabla\widehat{x}_t$$
$$= (1 - .74_{(.05)}B^{12} - .14_{(.06)}B^{24} + .28_{(.05)}B^{36})\widehat{w}_t$$

with $\widehat{\sigma}^2_w = 1.312$.

The diagnostics for the fit are displayed in Figure 3.25. We note the few outliers in the series as exhibited in the plot of the standardized residuals and their normal Q-Q plot, and a small amount of autocorrelation that still remains (although not at the seasonal lags) but otherwise, the model fits well. Finally, forecasts based on the fitted model for the next 12 months are shown in Figure 3.26.

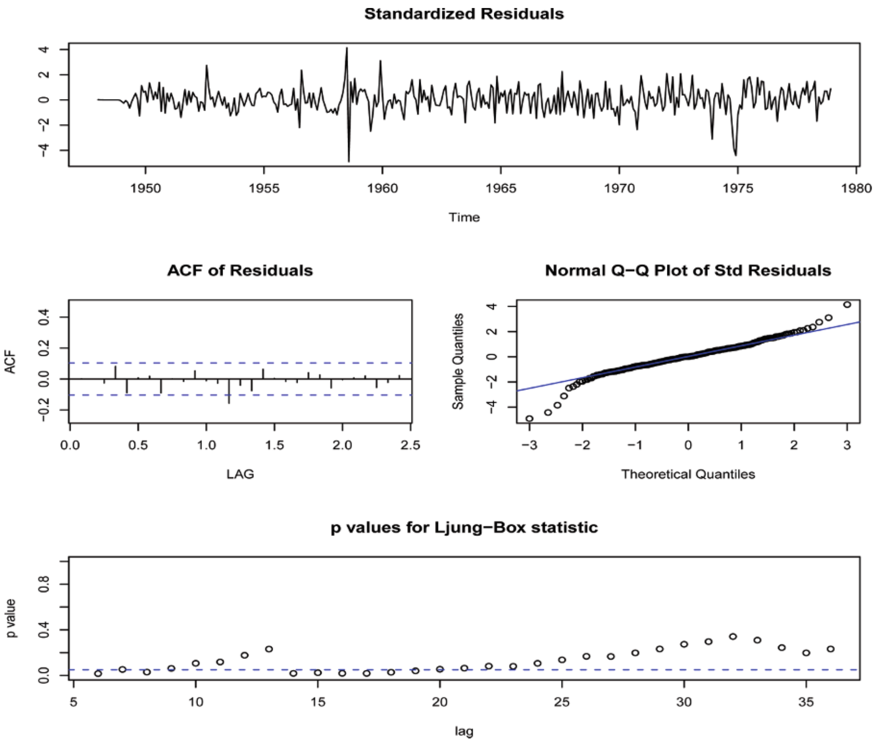


Fig. 3.25. Diagnostics for the ARIMA$(2,1,0) \times (0,1,3)_{12}$ fit on the Production Index.

The following R code can be used to perform the analysis.

```
acf2(prodn, 48)
acf2(diff(prodn), 48)
acf2(diff(diff(prodn), 12), 48)
sarima(prodn, 2, 1, 1, 0, 1, 3, 12)  # fit model (ii)
sarima.for(prodn, 12, 2, 1, 1, 0, 1, 3, 12) # forecast
```

Problems

Section 3.2

3.1 For an MA(1), $x_t = w_t + \theta w_{t-1}$, show that $|\rho_x(1)| \leq 1/2$ for any number θ. For which values of θ does $\rho_x(1)$ attain its maximum and minimum?

3.2 Let w_t be white noise with variance σ_w^2 and let $|\phi| < 1$ be a constant. Consider the process $x_1 = w_1$, and

$$x_t = \phi x_{t-1} + w_t, \quad t = 2, 3, \ldots .$$