Homework 04
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5.1)

```
school1 = scan(file = "../Data/school1.dat")
school2 = scan(file = "../Data/school2.dat")
school3 = scan(file = "../Data/school3.dat")
school = data.frame()

for(i in 1:3) {
  s = paste("school", i, sep = "")

  ## Prior
  mu_0 = 5; s2_0 = 4; k_0 = 1; v_0 = 2

  ## Data
  n = length(eval(parse(text=s))); ybar = mean(eval(parse(text=s)));
  s2 = var(eval(parse(text=s)))

  ## Posterior Inference
  k_n = k_0 + n; v_n = v_0 + n

  ## Posterior mean of theta
  mu_n = (k_0 * mu_0 + n * ybar) / k_n

  ## Posterior mean of sigma
  s2_n = (v_0*s2_0 + (n - 1)*s2 + k_0*n*(ybar-mu_0)^2 / (k_n)) / v_n

  school =
    rbind(school, data.frame(
      School = s,
      mu_n = mu_n,
      s_n = sqrt(s2_n),
      k_n = k_n,
      v_n = v_n,
      mu_95_l = qnorm(p = .025, mean = mu_n, sd = s2_n / k_n),
      mu_95_u = qnorm(p = .975, mean = mu_n, sd = s2_n / k_n),
      s_95_l = sqrt(1 / qgamma(p = .975, shape = v_n / 2, rate = v_n * s2_n / 2)),
      s_95_u = sqrt(1 / qgamma(p = .025, shape = v_n / 2, rate = v_n * s2_n / 2))))
}
## A) Posterior Parameters with 95% Confidence Intervals of Mu and Sigma
school
```

```
   School     mu_n      s_n k_n v_n  mu_95_l   mu_95_u   s_95_l   s_95_u
1 school1 9.292308 3.798019  26  27 8.204908 10.379707 3.002789 5.169623
2 school2 6.948750 4.263589  24  25 5.464225  8.433275 3.343751 5.885496
3 school3 7.812381 3.618490  21  22 6.590346  9.034416 2.798522 5.121435
```

```r
## B)

theta = data.frame(
  theta.1 = rnorm(n = 1E6, mean = 9.292, sd = sqrt(3.798^2 / 26)),
  theta.2 = rnorm(n = 1E6, mean = 6.948, sd = sqrt(4.263^2 / 24)),
  theta.3 = rnorm(n = 1E6, mean = 7.812, sd = sqrt(3.618^2 / 21))
)

theta = cbind(theta, data.frame(p1 = F, p2 = F, p3 = F, p4 = F, p5 = F, p6 = F))
theta$p1[theta$theta.1 < theta$theta.2 & theta$theta.1 < theta$theta.3 &
           theta$theta.2 < theta$theta.3] = T
theta$p2[theta$theta.1 < theta$theta.3 & theta$theta.1 < theta$theta.2 &
           theta$theta.3 < theta$theta.2] = T
theta$p3[theta$theta.2 < theta$theta.1 & theta$theta.2 < theta$theta.3 &
           theta$theta.1 < theta$theta.3] = T
theta$p4[theta$theta.2 < theta$theta.3 & theta$theta.2 < theta$theta.1 &
           theta$theta.3 < theta$theta.1] = T
theta$p5[theta$theta.3 < theta$theta.1 & theta$theta.3 < theta$theta.2 &
           theta$theta.1 < theta$theta.2] = T
theta$p6[theta$theta.3 < theta$theta.2 & theta$theta.3 < theta$theta.1 &
           theta$theta.2 < theta$theta.1] = T

## P1: theta.1 < theta.2 < theta.3
length(which(theta$p1 == TRUE)) / 1E6
```

```
[1] 0.00465
```

```r
## P2: theta.1 < theta.3 < theta.2
length(which(theta$p2 == TRUE)) / 1E6
```

```
[1] 0.002927
```

```r
## P3: theta.2 < theta.1 < theta.3
length(which(theta$p3 == TRUE)) / 1E6
```

```
[1] 0.078753
```

```r
## P4: theta.2 < theta.3 < theta.1
length(which(theta$p4 == TRUE)) / 1E6
```

```
[1] 0.686057
```

```r
## P5: theta.3 < theta.1 < theta.2
length(which(theta$p5 == TRUE)) / 1E6
```

```
[1] 0.012604
```

```r
## P6: theta.3 < theta.2 < theta.1
length(which(theta$p6 == TRUE)) / 1E6
```

```
[1] 0.215009
```

```r
## C)
## Simulate SD from the posterior distribution of sigma
Y1.sd = sqrt(1/rgamma(n = 1E6, shape = 27 / 2, rate = 27 * 3.798^2 / 2) / 26)
Y2.sd = sqrt(1/rgamma(n = 1E6, shape = 25 / 2, rate = 25 * 4.263^2 / 2) / 24)
Y3.sd = sqrt(1/rgamma(n = 1E6, shape = 22 / 2, rate = 22 * 3.618^2 / 2) / 21)

## Simulate predicted Y from the posterior predictive distribution of theta
Y = data.frame(
  y.1 = rnorm(n = 1E6, mean = 9.292, sd = Y1.sd),
  y.2 = rnorm(n = 1E6, mean = 6.487, sd = Y2.sd),
  y.3 = rnorm(n = 1E6, mean = 7.812, sd = Y3.sd)
)

Y = cbind(Y, data.frame(p1 = F, p2 = F, p3 = F, p4 = F, p5 = F, p6 = F))
Y$p1[Y$y.1 < Y$y.2 & Y$y.1 < Y$y.3 & Y$y.2 < Y$y.3] = T
Y$p2[Y$y.1 < Y$y.3 & Y$y.1 < Y$y.2 & Y$y.3 < Y$y.2] = T
Y$p3[Y$y.2 < Y$y.1 & Y$y.2 < Y$y.3 & Y$y.1 < Y$y.3] = T
Y$p4[Y$y.2 < Y$y.3 & Y$y.2 < Y$y.1 & Y$y.3 < Y$y.1] = T
Y$p5[Y$y.3 < Y$y.1 & Y$y.3 < Y$y.2 & Y$y.1 < Y$y.2] = T
Y$p6[Y$y.3 < Y$y.2 & Y$y.3 < Y$y.1 & Y$y.2 < Y$y.1] = T

## P1: y.1 < y.2 < y.3
length(which(Y$p1 == TRUE)) / 1E6
```

```
[1] 0.003003
```

```r
## P2: y.1 < y.3 < y.2
length(which(Y$p2 == TRUE)) / 1E6
```

```
[1] 0.001484
```

```r
## P3: y.2 < y.1 < y.3
length(which(Y$p3 == TRUE)) / 1E6
```

```
[1] 0.089654
```

```r
## P4: y.2 < y.3 < y.1
length(which(Y$p4 == TRUE)) / 1E6
```

```
[1] 0.76976
```

```r
## P5: y.3 < y.1 < y.2
length(which(Y$p5 == TRUE)) / 1E6
```

```
[1] 0.005683
```

```r
## P6: y.3 < y.2 < y.1
length(which(Y$p6 == TRUE)) / 1E6
```

```
[1] 0.130416
```

```
## D)

## Probability that Theta.1 is greather than Theta.2 and Theta.3
theta$p7 = F; theta$p7[theta$theta.1 > theta$theta.2 & theta$theta.1 > theta$theta.3] = T
length(which(theta$p7 == TRUE)) / 1E6
```

```
[1] 0.901066
```

```
## Probability that Theta.1 is greather than Theta.2 and Theta.3
Y$p7 = F; Y$p7[Y$y.1 > Y$y.2 & Y$y.1 > Y$y.3] = T
length(which(Y$p7 == TRUE)) / 1E6
```

```
[1] 0.900176
```

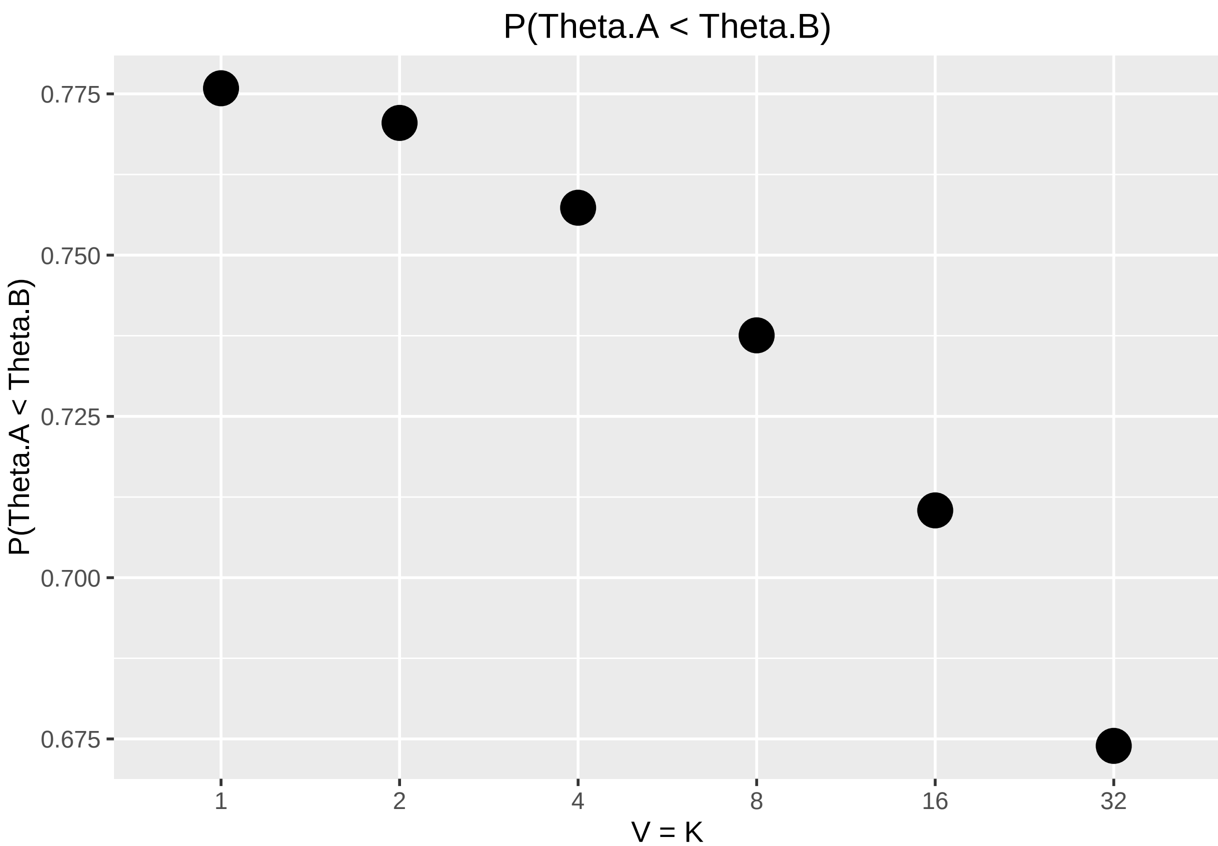
5.2

```
## Prior probabilities
mu_0 = 75; s_0 = 10; vk_0 = c(1, 2, 4, 8, 16, 32)

## Sample data
n = 16; ybar_a = 75.2; s_a = 7.3; ybar_b = 77.5; s_b = 8.1

## Parameters
param = data.frame(mu_0, s_0, vk_0)
param$vk_n = with(param, vk_0 + n)
param$s_a_n = with(param, 1/vk_n * ((vk_0 * s_0^2) + ((n - 1) * s_a^2) +
                                      (vk_0 * n / vk_n) * (ybar_a - mu_0)^2))
param$s_b_n = with(param, 1/vk_n * ((vk_0 * s_0^2) + ((n - 1) * s_b^2) +
                                      (vk_0 * n / vk_n) * (ybar_b - mu_0)^2))
param$mu_a_n = with(param, ((vk_0 * mu_0) + (n*ybar_a)) / vk_n)
param$mu_b_n = with(param, ((vk_0 * mu_0) + (n*ybar_b)) / vk_n)

## Simulate draws from the posterior distributions
results = c()

for (i in 1:6) {
  theta_1 = rnorm(n = 1E6, mean = param$mu_a_n[i], sd = sqrt(param$s_a_n / param$vk_n[i]))
  theta_2 = rnorm(n = 1E6, mean = param$mu_b_n[i], sd = sqrt(param$s_b_n / param$vk_n[i]))
  results = c(results, length(which(theta_1 < theta_2)) / 1E6)
}

(param = cbind(param, thetaA_LT_thetaB = results))
```

```
  mu_0 s_0 vk_0 vk_n    s_a_n    s_b_n   mu_a_n   mu_b_n thetaA_LT_thetaB
1   75  10    1   17 52.90516 64.11955 75.18824 77.35294         0.775869
2   75  10    2   18 55.52340 66.40340 75.17778 77.22222         0.770498
3   75  10    4   20 59.97390 70.20750 75.16000 77.00000         0.757346
4   75  10    8   24 66.64847 75.72847 75.13333 76.66667         0.737565
5   75  10   16   32 74.98969 82.31719 75.10000 76.25000         0.710429
6   75  10   32   48 83.32868 88.55868 75.06667 75.83333         0.673924
```

The increase in V represents having more prior information. Since the prior mu is 75, as V and K increase, the posterior probabilities of Theta A and Theta B move closer to 75. This shows how when you have more data which forms your prior opinion, the prior has a much larger influence on the posterior probabilities.

```
library(ggplot2)
ggplot(aes(x = factor(vk_0), y = thetaA_LT_thetaB), data = param) +
  geom_point(pch = 16, size = 6) +
  scale_x_discrete("V = K") +
  scale_y_continuous("P(Theta.A < Theta.B)") +
  ggtitle("P(Theta.A < Theta.B)")
```



5.5

a)

$$
\begin{aligned}
f(y|\mu,\sigma^2) =&(2\pi\sigma^2)^{-.5}exp[-.5(y-\mu)^2/\sigma] \\
f(y|\mu,\psi=\frac{1}{\sigma^2}) =&\frac{(2\pi)^{-.5}}{\psi}exp[-.5\psi(y-\mu)^2] \\
Lp(y|\theta,\psi) =&\frac{(2\pi)^{-n/2}}{\psi}exp[\frac{-\psi}{2}\sum(y_i-\theta)^2] \\
lp(y|\theta,\psi) =&\frac{-n}{2}log(2\pi)-\frac{n}{2}log(\psi)-\frac{1}{2\psi}\sum(y_i-\theta)^2
\end{aligned}
$$

b)

$$
\begin{aligned}
log[Pu(\theta,\psi)] =&\frac{lP(\theta,\psi|y)}{n}+c \\
log[\psi^{.5}]-\frac{\gamma\psi}{2}(\theta-\mu)^2+(a-1)log[\psi]-b\psi =&\frac{\frac{-n}{2}log(2\pi)-\frac{n}{2}log(1/\psi)-\frac{\psi}{2}\sum(y_i-\theta)^2}{n}+c \\
-\frac{\gamma\psi}{2}(\theta-\mu)^2+(a-1)log[\psi]-b\psi =&\frac{-log[2\pi]}{2}-\frac{\sum(y_i-\bar{y})^2+n(\theta-\bar{y})^2}{2\psi n}+c
\end{aligned}
$$

Im lost at this point

c) The joint density can be considered a posterior density because it integrates to 1 and depends on data only through the sufficient statistic.