STATISTICS 608 Linear Models -EXAM I
April 1, 2014

Student's Name: _____

Student's Email Address: _____

INSTRUCTIONS FOR STUDENTS:

1. There are **14** pages including this cover page.
2. You have exactly 75 minutes to complete the exam.
3. There may be more than one correct answer; choose the best answer.
4. You will not be penalized for submitting too much detail in your answers, but you may be penalized for not providing enough detail.
5. Do not discuss or provide any information to any one concerning any of the questions on this exam or your solutions until I post the solutions next week.
6. You may use one 8.5" X 11" sheet of notes and a calculator.
7. At the end of the exam, leave your sheet of notes with your proctor along with the exam.

I attest that I spent no more than 75 minutes to complete the exam. I used only the materials described above. I did not receive assistance from anyone during the taking of this exam.

Student's Signature: _____

INSTRUCTIONS FOR PROCTOR:

Immediately after the student completes the exam scan it to a pdf file and have student upload to Webassign.

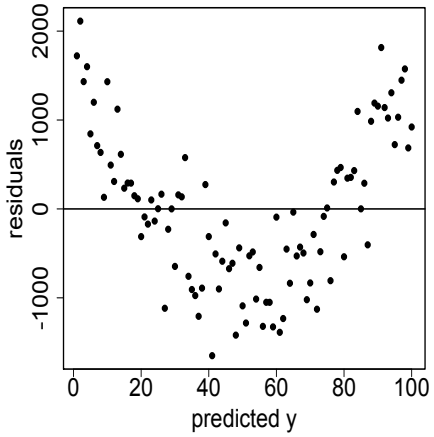
1. I certify that the time at which the student started the exam was _____ and the time at which the student completed the exam was _____.
2. I certify that the student has followed all the **INSTRUCTIONS FOR STUDENTS** listed above.
3. I certify that the exam was scanned in to a pdf and uploaded to Webassign in my presence.
4. I certify that the student has left the exam and sheet of notes with me, to be returned to the student no less than one week after the exam or shredded.

Proctor's Signature: _____

Part I: Multiple choice

1. An added variable plot is most useful for which of the following in a linear regression model?
   (a) **Visually assessing the effect of predictors after adjusting for the effects of the other predictors.
   (b) Visually assessing whether the mean function is modeled appropriately.
   (c) Determining whether the predictors are correlated with each other.
   (d) Determining whether a polynomial term not currently included should be added to the model.
   (e) Estimating an appropriate transformation of the predictors and/or response variable.
2. In a model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + e$, what does it mean for the predictor variable x_1 to have a variance inflation factor of 3?
   (a) **The variance of $\hat{\beta}_1$ is three times higher than it would have been if the predictors hadn't been linearly related.
   (b) The largest standardized residual is 3, indicating we should investigate outliers in this model.
   (c) The variance of the residuals is three times higher for some values of x_1 than for others.
   (d) There is a "bad" outlier, a point with both a large standardized residual and high leverage, in this model that should be investigated.
3. A model $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$ was fit to a data set, where x_1 was a continuous predictor and x_2 was an indicator or dummy variable. If all parameters are non-zero, what is the geometric interpretation of the model?
   (a) A single regression line
   (b) **Parallel lines Notice when the indicator variable equals zero, the intercept equals β_0. When the indicator variable equals 1, the intercept is $\beta_0 + \beta_2$.
   (c) Two lines with the same y-intercept but separate slopes
   (d) Two lines with separate intercepts and separate slopes
   (e) Two parabolas with separate intercepts
4. In which of the following scenarios would we be most likely to attempt fitting a weighted least squares regression model with weights $w_i = n_i$?
   (a) In a sample based on Y_i the average of n_i observations.
   (b) In a model for which the residuals were not normally distributed.
   (c) In a model for which a percent increase interpretation of slope was desired.
   (d) In a data set for which variance inflation factors were high.

5. Suppose a multiple linear regression model has been fit, and the following plot of the residuals against the fitted values is created. Which of the following adjustments is suggested by the plot?



   (a) Transform some of the variables

   (b) Include an interaction term

   (c) Include polynomial terms for one or more predictors

   (d) Remove outliers

   (e) In general, it cannot be determined in what way the model is invalid based on residual plots.

6. A researcher has two models that are being compared:

$$\text{Model 1: } y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e, \text{ with } R^2 = 0.8$$
$$\text{Model 2: } y = \alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 + e, \text{ with } R^2 = 0.7$$

   The response variable is the same in both models, though the predictors used are different. The researcher states that because R^2 is larger in the first model, it is more valid than the second. Do you agree? Why or why not?

   (a) Yes, because the proportion of variability in y explained is greater for Model 1.

   (b) Yes, because Model 1 is more linear than Model 2.

   (c) Yes, because the higher value of R^2 indicates Model 1 has fewer outliers than Model 2.

   (d) No, because R^2 is not invariant under transformation.

   (e) No, because we need to check other plots like residual plots to find out whether model assumptions are met.

Part II: Long Answer

7. A study investigating distractions during lectures randomly assigned 100 students to each of the following groups (for a total of 300 students):

   - Group 1 was permitted to use cell phones during lecture but not eat.
   - Group 2 was permitted to eat but not use cell phones.
   - Group 3 was not permitted to eat or use cell phones.

   The response variable was the grade on a quiz following the lectures. The resulting design matrix used was:

$$\mathbf{X} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

   where each entry was a 100×1 vector. The estimates of the three parameters, average grades on the quiz for each of the three groups, was $\hat{\boldsymbol{\beta}}' = \begin{bmatrix} 42.81, 59.84, 58.67 \end{bmatrix}$.

   (a) Suppose researchers are interested in testing whether the average quiz grade for the first and third groups were equal. Write down the null and alternative hypotheses using vector notation, i.e. define $\mathbf{a}$ for $\mathbf{a}'\boldsymbol{\beta}$.

   In scalar notation, the hypotheses are $H_0 : \beta_1 = \beta_3$ vs. $H_a : \beta_1 \neq \beta_3$, or to re-arrange algebraically, the null hypothesis is $H_0 : \beta_1 - \beta_3 = 0$. Then if our vector $\boldsymbol{\beta}' = [\beta_1, \beta_2, \beta_3]$, $\mathbf{a}' = [1, 0, -1]$ so that $H_0 : \mathbf{a}'\boldsymbol{\beta} = 0$ and $H_a : \mathbf{a}'\boldsymbol{\beta} \neq 0$.

   (b) We saw in the homework that $\text{Var}(\mathbf{a}'\hat{\boldsymbol{\beta}}|\mathbf{X}) = \sigma^2\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{a}$. In our case, $\hat{\sigma}^2 = 105.06$ and $(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix}$.

   Use this to calculate $\text{Var}(\mathbf{a}'\hat{\boldsymbol{\beta}}|\mathbf{X})$ for our problem.

$$\begin{aligned} \text{Var}(\mathbf{a}'\hat{\boldsymbol{\beta}}|\mathbf{X}) &= \sigma^2\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{a} \\ &= 105.06[1,\ 0,\ -1] \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= 105.06(1/1000)(2) \\ &= 0.21012 \end{aligned}$$

Now, the whole point of this is to be able to finish off the hypothesis test: $t = \frac{a'\hat{\beta}}{\sqrt{0.21012}} = \frac{42.81-58.67}{\sqrt{0.21012}} = -34.6$. So we see since our t-value is so far negative, and our sample size is large enough to make that t-value close enough to a z-value, we have strong evidence (p-value very tiny) that Groups 1 and 3 score differently on quizzes, on average.

8. Suppose that $\text{Var}(e_i|x_i) = \sqrt{n_i}\sigma^2$ for the simple linear regression model $y_i = \beta_0 + \beta_1 x_i + e_i$ $(i = 1, 2, \ldots, n)$. Clearly define and write down a formula for a weighted estimate for the parameters β_0 and β_1. There is no need to finish any algebra that you may set up; simply write down equations and define terms not already defined.

   Where: $w_i = 1/\sqrt{n_i}$:

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \mathbf{W} = \begin{bmatrix} 1/\sqrt{n_1} & 0 & \ldots & 0 \\ 0 & 1\sqrt{n_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1/\sqrt{n_n} \end{bmatrix}$$

   then we have $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}\mathbf{Y}$.

   We use this weight because $\text{Var}(\sqrt{w_i}e_i|\mathbf{X}) = w_i\sqrt{n_i}\sigma^2 = \sigma^2$, which is constant across different values of x_i.

9. A clinical trial for hormone therapy was concerned in part with predicting the log of HDL cholesterol level using predictors age and body mass index (BMI). Researchers considered two models. For the first model, they noted BMI was right-skewed, and transformed it to normality using a log transformation. For the second, they cosidered adding a polynomial term in BMI. The two models are shown below; their marginal model plots are shown in the appendix. (Ignore the "c" in the "BMIc" on those plots for now.)

$$\text{Model 1: } \log(HDL) = \beta_0 + \beta_1\, age + \beta_2\, \log(BMI) + e$$
$$\text{Model 2: } \log(HDL) = \beta_0 + \beta_1\, age + \beta_2\, BMI + \beta_3\, BMI^2 + e$$

(a) If you had to choose one of the models based only on their marginal model plots, which would you prefer? Explain.

The nonparametric model (designed to fit the data more perfectly) and the second parametric model agree more than the nonparametric and first parametric models, giving us reason to believe our mean function may be modeled more correctly with the second model.

(b) Researchers decided to use Model 2. Calculate a 95% confidence interval for β_1 from Model 2 and interpret your interval in context.

$0.0034526 \pm 1.96(0.0007219) : (0.00204, 0.00487)$ I am 95% confident that when age increases by 1 year, HDL level increases by between 0.2% and 0.5%, assuming that BMI is held constant.

10. In a study modeling the progression of diabetes (prog), predictors age, BMI, sex, blood pressure (bp), and six serum measurements (s1-s6) were under consideration. The first model considered included all the predictors (output for Model 1 is included in the appendix):

$$\text{Model 1: } prog = \beta_0 + \beta_1\, age + \beta_2\, BMI + \beta_3\, sex + \beta_4\, bp + \beta_5\, s_1 + \beta_6\, s_2 + \beta_7\, s_3 + \beta_8\, s_4 + \beta_9\, s_5 + \beta_{10}\, s_6 + e$$

(a) Is there any problem with multicollinearity for this model? Explain.

There is definitely a problem: our VIF's for this model include values of 39 and 59, far exceeding our cutoff of 5. This means the variances of our parameter estimates are much larget than they ought to be, hinting at a very unstable model. A slight change of the value of a single observation could possibly drastically change our parameter estimates.

We can also note that there is some obvious correlation shown in the scatterplot matrix among the predictors, especially s_1 and s_2, also indicating multicollinearity. We cannot rely only on scatterplots to find problems with multicollinearity because they do not show relationships between all possible linear combinations among the predictors.

(b) A researcher is interested in simply removing all variables in the model that have large p-values at one time. Is this method appropriate? Explain.

No. Multicollinearity indicates that there are some linear combinations of some of the predictors that explain almost the same variability in y that others do, because of high correlation between the predictors' linear combinations. The p-values associated with the parameters in the model measure how much variability in y is explained by that predictor, after the other variables have already been added to the model. That means if a variable's p-value is large, it might not be true that it has no predictive power; instead it might simply be correlated with other predictors that also predict the response variable well. Removing variables one at a time or conducting an F-test of model reduction are improvements on this method.

(c) A second model under consideration is below. Conduct an F-test of model reduction to determine whether the model can be reduced, assuming appropriate model assumptions are met. (The p-value for the test is 0.1508.) Be sure to state your hypotheses, give a test statistic, and write a conclusion in the context of the problem.

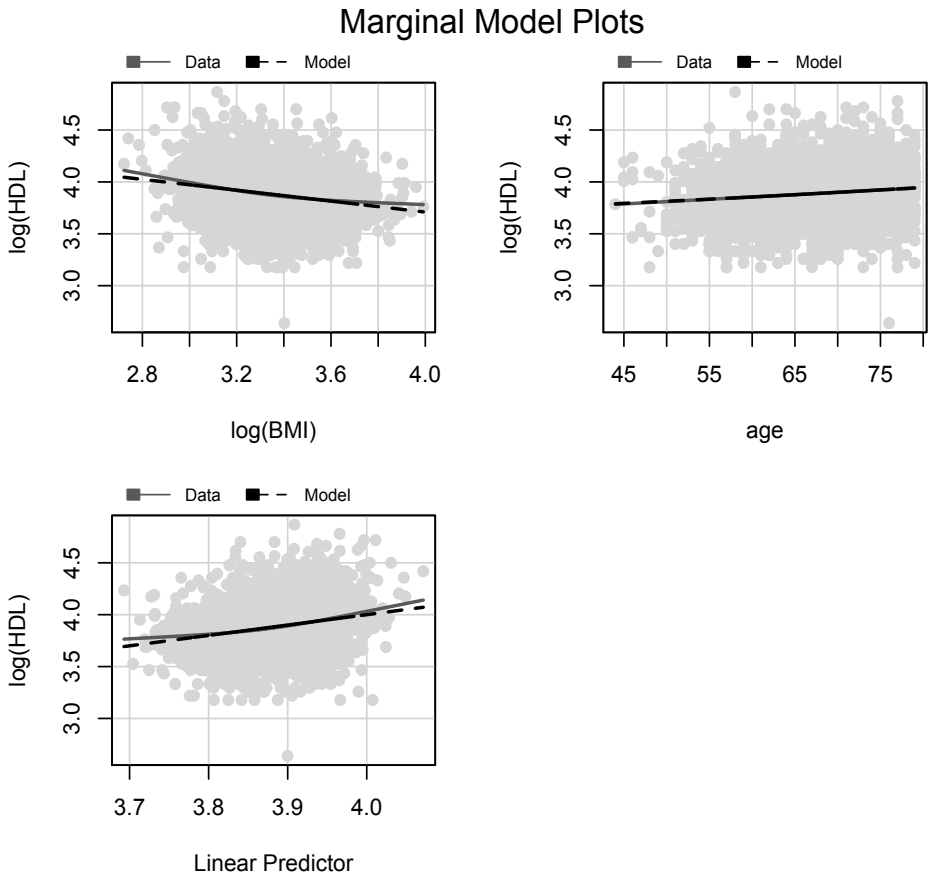
$$\text{Model 2: } prog = \beta_0 + \beta_1\, BMI + \beta_2\, sex + \beta_3\, bp + \beta_4\, s_3 + \beta_5\, s_5 + e$$

$H_0 : \beta_1 = \beta_5 = \beta_6 = \beta_8 = \beta_{10} = 0$ vs. H_a : at least one of the listed β's not zero.
or, H_0 : Model 1 and Model 2 are equivalent vs. H_a : Model 2 is preferable.

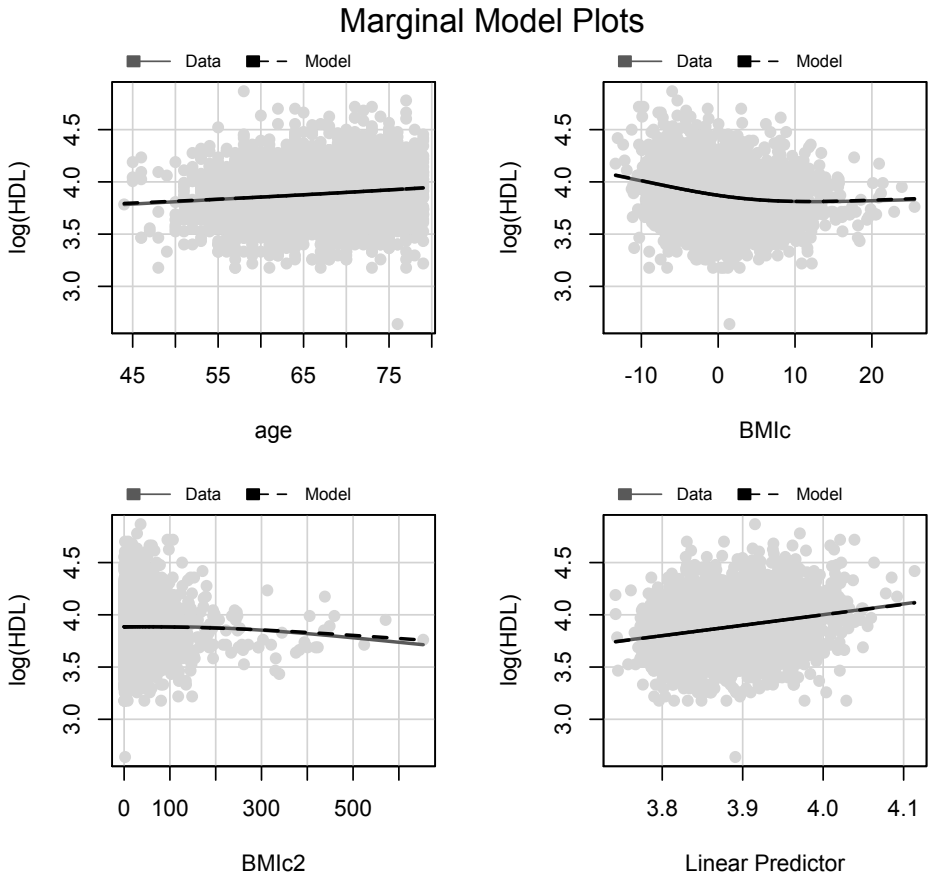
$$F = \frac{(1287881 - 1263986)/(436 - 431)}{1263986/431} = 1.63$$

Because our p-value (0.1508) is so large, we fail to reject the null hypothesis. We don't have evidence that Model 1 is better than Model 2 at explaining the variability in diabetes progression, so we use the simpler, easier model from here.

HDL prediction: Model 1



HDL prediction: Model 2

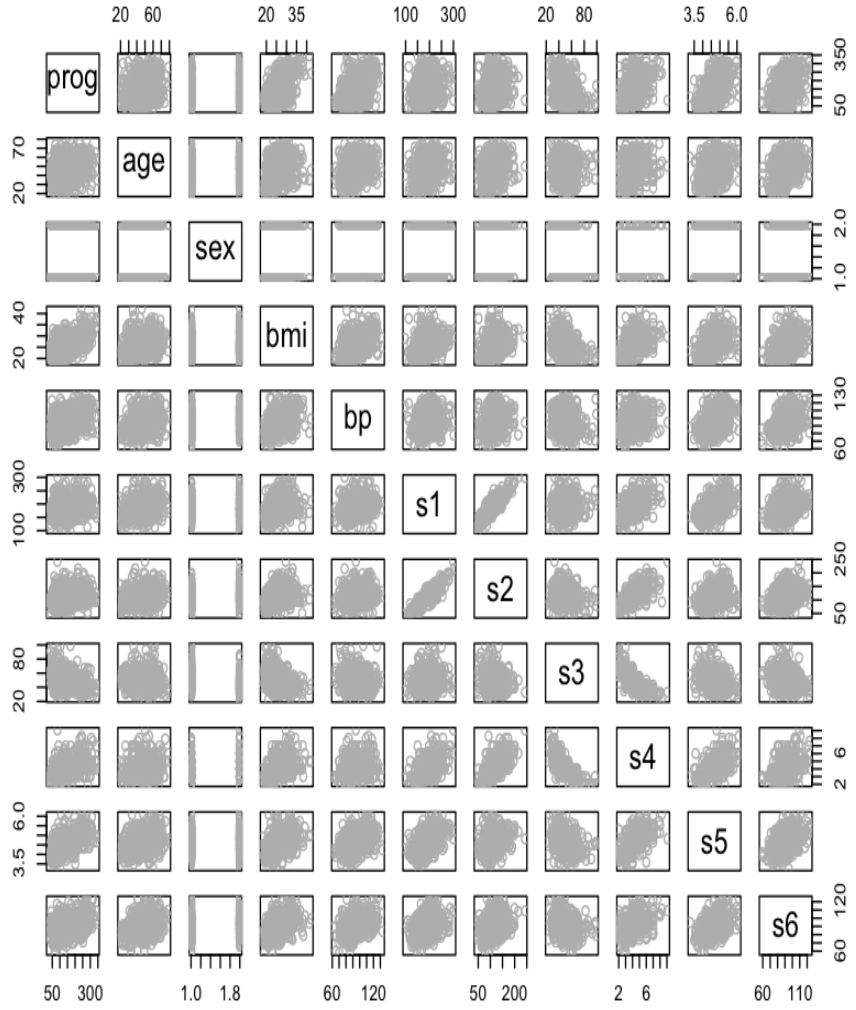


HDL prediction: Model 2

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  3.6417749  0.0484669  75.139  < 2e-16 ***
age          0.0034526  0.0007219   4.783 1.82e-06 ***
BMIc        -0.0096966  0.0009772  -9.923  < 2e-16 ***
BMIc2        0.0004219  0.0001041   4.053 5.21e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.2482 on 2743 degrees of freedom
Multiple R-squared:  0.048,Adjusted R-squared:  0.04696
F-statistic:  46.1 on 3 and 2743 DF,  p-value: < 2.2e-16
```

Diabetes Progression: Scatterplot Matrix



Diabetes Progression: Model 1

```
Coefficients:
             Estimate  Std. Error  t value  Pr(>|t|)     VIF
(Intercept) -334.56709    67.45462   -4.960  1.02e-06
age           -0.03636     0.21704   -0.168  0.867030   1.217
sex          -22.85965     5.83582   -3.917  0.000104   1.278
bmi            5.60296     0.71711    7.813  4.30e-14   1.509
bp             1.11681     0.22524    4.958  1.02e-06   1.459
s1            -1.09000     0.57333   -1.901  0.057948  59.203
s2             0.74645     0.53083    1.406  0.160390  39.193
s3             0.37200     0.78246    0.475  0.634724  15.402
s4             6.53383     5.95864    1.097  0.273459   8.891
s5            68.48312    15.66972    4.370  1.56e-05  10.076
s6             0.28012     0.27331    1.025  0.305989   1.485
---

Residual standard error: 54.15 on 431 degrees of freedom
Multiple R-squared:  0.5177,Adjusted R-squared:  0.5066
F-statistic: 46.27 on 10 and 431 DF,  p-value: < 2.2e-16


Analysis of Variance Table

Response: prog
           Df  Sum Sq Mean Sq  F value    Pr(>F)
age         1   92527   92527  31.5504 3.490e-08 ***
sex         1     293     293   0.1000    0.7519
bmi         1  826955  826955 281.9792 < 2.2e-16 ***
bp          1  129312  129312  44.0934 9.448e-11 ***
s1          1    1791    1791   0.6108    0.4349
s2          1    5058    5058   1.7246    0.1898
s3          1  237329  237329  80.9257 < 2.2e-16 ***
s4          1    1821    1821   0.6210    0.4311
s5          1   58856   58856  20.0690 9.582e-06 ***
s6          1    3080    3080   1.0504    0.3060
Residuals 431 1263986    2933
```

Diabetes Progression: Model 2

```
Coefficients:
             Estimate  Std. Error  t value  Pr(>|t|)    VIF
(Intercept) -217.6849     35.7638   -6.087  2.53e-09
sex          -22.4742      5.7640   -3.899  0.000112  1.238
bmi            5.6431      0.7037    8.019  9.93e-15  1.443
bp             1.1232      0.2172    5.171  3.55e-07  1.347
s3            -1.0644      0.2417   -4.404  1.34e-05  1.459
s5            43.2344      5.9874    7.221  2.32e-12  1.461
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 54.35 on 436 degrees of freedom
Multiple R-squared:  0.5086,Adjusted R-squared:  0.503
F-statistic: 90.26 on 5 and 436 DF,  p-value: < 2.2e-16
```

```
Analysis of Variance Table

Response: prog
           Df  Sum Sq Mean Sq  F value    Pr(>F)
sex         1    4860    4860   1.6454    0.2003
bmi         1  896764  896764 303.5911 < 2.2e-16 ***
bp          1  146250  146250  49.5115 7.690e-12 ***
s3          1  131238  131238  44.4293 7.989e-11 ***
s5          1  154016  154016  52.1406 2.317e-12 ***
Residuals 436 1287881    2954
```