Homework 7 (Written Section)

1. (From Weisberg, 2005.) We are interested in the linear model $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$.

   (a) Suppose we fit the model above to data for which $x_1 = 2.2x_2$ with no error. For example, X_1 could be a weight in pounds, and X_2 the weight of the same object in kg. Describe the appearance of the added-variable plot for x_2 after x_1 had been added to the model. Explain. Assume that Y has a correlation with the predictors that is neither 0 nor 1. Hint: think about what goes on the x-axis and the y-axis of the added variable plot. You should notice something interesting about one of those residual vectors.

   (b) Again referring to the model above, this time suppose that Y and X_1 are perfectly correlated, so $Y = 3X_1$, without any error. Describe the appearance of the added-variable plot for x_2 after x_1 had been added to the model. Explain. Assume this time that the correlation between the predictors is between 0 and 1.

   (c) Refer to the same model, and this time suppose that X_1 and X_2 are independent (orthogonal). In this case, the added variable plot will look exactly like another plot we might have created when checking whether our model is valid. Which one does it look like? Explain why.

2. Suppose we are interested in the linear model $y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + e_i$. Also suppose the columns $\mathbf{x}_1$ and $\mathbf{x}_2$ of the design matrix for this model have mean 0 and *length* 1. (That is, $\mathbf{x}_1'\mathbf{x}_1 = 1$, and the same is true of $\mathbf{x}_2$.) Then if ρ is the correlation between $\mathbf{x}_1$ and $\mathbf{x}_2$, we have the following:

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} n & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix} \text{ and } (\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1/(1-\rho^2) & -\rho/(1-\rho^2) \\ 0 & -\rho/(1-\rho^2) & 1/(1-\rho^2) \end{bmatrix}$$

   (a) Determine what values of ρ will make the variance of $\hat{\beta}_1$ and $\hat{\beta}_2$ large. Explain why, using what you know about the variance of the vector $\hat{\boldsymbol{\beta}}$. (No fair looking at the VIF formulas.)

   (b) In our setup where the predictors have mean 0 and length 1, show that SXX = 1. Use that to show that the VIF formula on page 203 matches $\sigma^2(\mathbf{X}'\mathbf{X})^{-1}$ (above).

3. In a study on weight gain in rabbits, researchers randomly assigned 6 rabbits to 1, 2, or 3 mg of one of dietary supplement A or B (one rabbit to each level of each supplement). Consider the linear model $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$, where x_1 is the dosage level of the supplement, and x_2 is a dummy variable indicating the type of supplement used.

   (a) Compute the variance inflation factor for variable x_1. You should be able to do this completely without the use of statistical software. Explain, using the word "orthogonal," why the variance inflation factor is the value computed.

(b) Now suppose the researcher used levels 1, 2, and 3 for supplement A, and levels 2, 3, and 4 for supplement B. Use software if desired. What is the variance inflation factor for variable x_1 in this case? Is it larger or smaller than in part (a) above? Why?

(c) Now consider a linear model (not for our rabbits) $Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_p x_p + e$. Under what circumstances would the variance inflation factors for all p variables be equal to 1?

(Hint: To get started in part (a), it is perfectly reasonable to use R and the vif() function. You'll have to invent a response vector y; try:

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y <- c(1, 2, 3, 4, 5, 6)
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to get you started. Notice that the VIF is the same no matter what values you use for y. Why? Then you might look at the formulas for VIF, and realize that correlation is part of that formula. Calculate correlations between vectors to see what happens. Then you'll see what is orthogonal to what.)