STAT 659 Spring 2016
Homework 1 Solution

1.2

(c) ordinal
(e) nominal
(f) ordinal

1.3

(a) The distribution of the students' number of correct answers on the exam is a binomial distribution with $n = 100$ and $p = \frac{1}{4}$, where p is the probability of selecting the correct answer for one question.

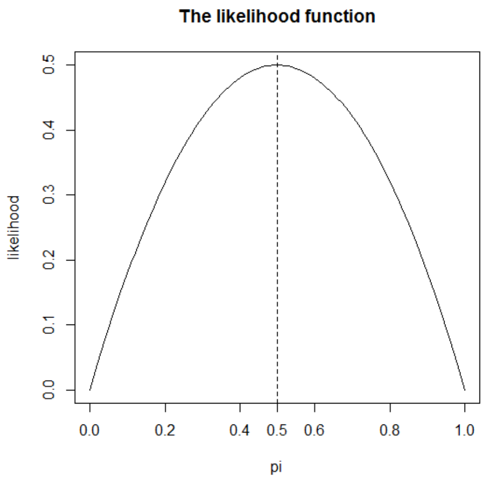
(b) The mean of the number correct responses is $np = 25$ and the standard deviation is $\sqrt{np(1-p)} = 4.33$. You can see the 50 is approximately 6 standard deviation above the mean, so the result is surprising.

1.4

(a) The distribution of possible values of Y obeys the binomial distribution with $n = 2$ and $\pi = 0.5$. Thus $P(Y = 0) = (1 - 0.5)^2 = 0.25$, $P(Y = 1) = 2 * 0.5 * (1 - 0.5) = 0.5$ and $P(Y = 2) = 0.5^2 = 0.25$. The mean of Y is $n\pi = 1$ and the standard deviation of it is $\sqrt{n\pi(1-\pi)} = 0.707$.

(b) If $\pi = 0.6$, then $P(Y = 0) = (1 - 0.6)^2 = 0.16$, $P(Y = 1) = 2 * 0.6 * (1 - 0.6) = 0.48$, $P(Y = 2) = 0.6^2 = 0.36$; if $\pi = 0.4$, then $P(Y = 0) = (1 - 0.4)^2 = 0.36$, $P(Y = 1) = 2 * 0.4 * (1 - 0.4) = 0.48$, $P(Y = 2) = 0.4^2 = 0.16$.

(c) The likelihood is $l(\pi|y = 1) = 2\pi(1-\pi)$, where $0 \leq \pi \leq 1$. When you plot the likelihood, you can find the mode is $\pi = 0.5$.



(d) Since the maximum likelihood estimator maximize the likelihood function, then from part (c) we know $\hat{\pi}_{ML} = 0.5$.

1.5

If $y = 0$, then the likelihood function will be $l(\pi|y = 0) = (1 - \pi)^2$. So it is easy to see $\hat{\pi} = 0$ maximizes it. That means it is impossible to get a head when flipping a coin. Obviously The maximum likelihood estimator for this case is not reasonable.

1.8

(a) The survey result can be used to estimate the population proportion of "yes", which is $344/1170 = 0.294$.

(b) The null hypothesis is $H_0 : \pi_0 = 0.5$ and the alternative is $H_a : \pi_0 \neq 0.5$. The score test statistics is $\frac{\hat{\pi}-\pi_0}{\sqrt{\pi_0(1-\pi_0)/n}} = -14.093$. The p-value is $2 * P(Z < -14.093) \approx 0$, which means we reject null hypothesis under $\alpha = 0.01$.

(c) The 99 percent confidence interval for the population proportion is $0.294 \pm Z_{0.995} * \sqrt{0.294(1 - .294)/1170}$, that is $(0.2597, 0.3283)$. We have 99 percent confidence that the true population proportion who would say "yes" falls into this interval.

1.9

(a) Denote the proportion of the women who will report greater relief from the new analgesic by π_0. The null hypothesis is $\pi_0 = 0.5$ while the alternative is $\pi_0 \neq 0.5$. The score test statistic is $\frac{\hat{\pi}-\pi_0}{\sqrt{\pi_0(1-\pi_0)/100}} = \frac{0.6-0.5}{\sqrt{0.5^2/100}} = 2$ and the p-value is $2P(Z > 2) = 0.0455$ where Z is the standard normal variable.

(b) The 95% Wald interval is $0.6 \pm 1.96\sqrt{(0.6 * 0.4/100)}$ which is $(0.50398, 0.69602)$; The 95% score interval and Clopper-Pearson interval are $(0.502, 0.691)$ and $(0.497, 0.697)$, respectively. The $\hat{p}$ for Agresti-Coull confidence interval is $62/104 = 0.596$ and the corresponding CI is $0.596 \pm 1.96\sqrt{0.596(1-0.596)/104} = (0.50185, 0.6904)$. We can see there is little difference between the above intervals for the comparably large sample size.

1.10

The marginal error for the proportion is $1.96 * \sqrt{0.75(1-0.75)/n}$, so let it be equal to 0.08 we can solve for sample size n. The result is 112.5, so we need the sample size to be 113 to achieve this accuracy.

1.12

(a) Since for $\hat{\pi} = 0$ the standard deviation of the Wald test is zero, thus the test statistic is undefined.

(b) The Wald interval is only one point 0 since its standard deviation is zero, so it is not reliable.

(c) For the score test, the test statistic is $(0-0.5)/\sqrt{0.5 * (1-0.5)/25} = -5$ and the p-value is $2P(Z < -5) = 5.73 * 10^{-7}$.

(d) When testing $H_0 : \pi_0 = 0.133$ against $H_a : \pi_0 \neq 0.133$, the test statistic is $\frac{0-0.133}{\sqrt{0.133(1-0.133)/25}} = -1.958 \approx -1.96$ and the two sided p-value is 0.05. So the score interval has been verified.

1.14

(a) (i)The p-value for $H_a : \pi > 0.5$ equals $P(8) + P(9) + P(10) = 0.055$ according to the table 1.2 (ii) the p-value for $H_a : \pi < 0.5 = \sum_{i=0}^{8} P(i) = 1 - P(9) - P(10) = 0.989$.

(b) (i)The mid p-value for $H_a : \pi > 0.5$ equals $P(8)/2 + P(9) + P(10) = 0.033$ (ii) the mid p-value for $H_a : \pi < 0.5 = \sum_{i=0}^{7} P(i) + P(8)/2 = 1 - P(9) - P(10) - P(8)/2 = 0.967$.

(c) We can see for ordinary p-value, the sum of the one sided p-values equals to $\sum_{i=1}^{10} P(i) + P(8) = 1 + P(8) > 1$, but for mid p-value, the sum of two one sided p-values equals to $\sum_{i=1}^{10} P(i) = 1$.

1.19

$\hat{\pi} = 8/13 = 0.615$, so the Wald interval is $8/13 \pm 1.96\sqrt{\hat{\pi}(1-\hat{\pi})/13}$ which is $(0.351, 0.880)$; For Agresti-Coull interval, $\hat{\pi} = (8+2)/(13+4) = 10/17 = 0.588$, so the interval is $10/17 \pm 1.96\sqrt{0.588 * (1-0.588)/17}$ which is $(0.354, 0.822)$; for the score interval, we apply the result in question 1.18 to obtain it, which is $(0.355, 0.823)$. The Clopper-Pearson interval is $(0.316, 0.861)$. The The width of the Clopper-Pearson interval is 0.545, which is largest among three intervals, this may be natural since the actual coverage probability of the Clopper-Pearson interval is usually grater than $1-\alpha$ due to the discreteness of the binomial distribution; The Agresti-Coull interval and score interval have similar upper and lower bounds and both of them are not symmetric for the estimate 8/13 while the Wald interval is symmetric for the estimate 8/13.

The remaining problems are only for students who have taken STAT 414, 610 or STAT 630.

1.15

(a) For binomial variable Y, its variance is $n\pi(1-\pi)$. So $Var(p) = Var(Y/n) = Var(Y)/n^2 = \pi(1-\pi)/n$, then the standard deviation of p is $\sigma(p) = \sqrt{\pi(1-\pi)/n}$.

(b) Since we use $p = Y/n$ to estimate π, from (a) we can see the standard deviation of p tends to be zero when π is near 0 or 1, thus it is easier to estimate π in this case; But when π is near 0.5, the standard deviation of p is close to its maximum value $\sqrt{1/4n}$ which makes estimating π harder(The estimate has largest standard deviation in this case).

1.18

Denote $z_0 = \frac{p-\pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$. Then $z_0^2 = \frac{(p-\pi_0)^2}{\pi_0(1-\pi_0)/n}$. Hence, $p^2 - 2p\pi_0 + \pi_0^2 = z_0^2\pi_0/n - z_0^2\pi_0^2/n$.
Therefore, $(1+z_0^2/n)\pi_0^2 + (-2p - z_0^2/n)\pi_0 + p^2 = 0$.
Using the formula, we get

$$\begin{aligned}\pi_0 &= \frac{2p + z_0^2/n \pm \sqrt{(2p+z_0^2/n)^2 - 4p^2(1+z_0^2/n)}}{2(1+z_0^2/n)^2} \\ &= \frac{p + z_0^2/2n \pm z_0\sqrt{p(1-p)/n + z_0^2/4n^2}}{1+z_0^2/n}\end{aligned}$$

The 95% confidence interval in Section 1.3.4 is $(0.5959, 0.9821)$.