Homework 1

I. Matrix Algebra Review.

Define matrices A, B, and C as follows:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 2 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 3 & 0 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

1. Calculate A', the transpose of matrix A.

$$A' = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 0 \\ 3 & -2 \end{bmatrix}$$

2. Calculate $A' + B$.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 2 & 0 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 3 & 0 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 2 \\ 4 & 1 \\ 3 & -4 \end{bmatrix}$$

3. Calculate AB, where AB is the matrix product, or matrix multiplication.

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 3 & 0 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ 6 & 5 \end{bmatrix}$$

4. Calculate BA. (Is $AB = BA$?)

$$\begin{bmatrix} 0 & -1 \\ 3 & 0 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 3 & 0 & 6 & 9 \\ 1 & 2 & 4 & 4 \\ 2 & -4 & 0 & 4 \end{bmatrix}$$

No, AB is not equal to BA. Not only are the elements not equal; the matrices aren't even the same size. Matrix multiplication is not commutative.

5. Is the matrix AB singular? Why or why not?

   No; the easiest way to show this is probably $det(AB) = 20 - (-30) = 50 \neq 0$. We could also show that the columns of AB are linearly independent: The only set of a_1, a_2 such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{0}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the column vectors of AB are $a_1 = a_2 = 0$.

6. Calculate $(AB)^{-1}$.

$$AB^{-1} = \frac{1}{20-(-30)} \begin{bmatrix} 5 & 5 \\ -6 & 4 \end{bmatrix} = \begin{bmatrix} -1/10 & 1/10 \\ -3/25 & 2/25 \end{bmatrix}$$

7. Calculate the trace of AB.
   $tr(AB) = 4 + 5 = 9$

8. Write $(BA)'$ in another form (algebraically).
   $(BA)' = A'B'$

9. Write I_2, the 2×2 identity matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

10. What is I_2A? Why?
    $I_2A = A$. I is the identity matrix; identity means when we multiply *any* other matrix M by the identity, it results in our original matrix M.

11. Describe geometrically the space spanned by C. That is, the space spanned by the two vertical vectors in the matrix C. Assume we're working in three dimensional space defined by axes xyz.
    The space spanned by C is the (x, y) plane, the plane where $z = 0$.

12. Calculate the projection matrix for C.
    Since C is already an orthonormal basis, we can simply use $P = CC'$, where P is the projection matrix:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

13. Project the vector $d = [2\,2\,2]'$ onto the space spanned by C.

$$Pd = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

14. Describe geometrically what you did in the previous step.
By projecting d onto the (x, y) plane, we found the vector in the space spanned by C that was closest to d without leaving the sub-space. We might also say that the projection is a "shadow" of the vector d.

15. Are the vectors d defined above and $f = [1\,0\,0]'$ orthogonal? Why or why not? (Talk about a dot product in your answer.)
The vectors $d = [2\,2\,2]'$ and $f = [1\,0\,0]'$ are not orthogonal, because the dot product

$$\begin{bmatrix} 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2$$

is not 0.

16. Calculate the dot product $1 \cdot 1$, where the vector $1 = [1\,1 \ldots 1]'$ is of length n.

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = n$$

17. Calculate the dot product $1 \cdot x$, where 1 is defined as above and the vector $x = [x_1\, x_2 \ldots x_n]'$.

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i$$

18. Calculate the dot product $x \cdot x$, where x is defined as above. Memorize these last three answers (15, 16, and 17) - it's good for you!

$$\begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i^2$$

II. Calculus Review.

Define

$$f(x, y) = 3x^2 + 2xy^2 - y.$$

19. Calculate $\frac{\partial}{\partial x} f(x, y)$.

$$\frac{\partial}{\partial x} f(x, y) = 6x + 2y^2$$

20. Calculate $\frac{\partial}{\partial y} f(x, y)$.

$$\frac{\partial}{\partial y} f(x, y) = 4xy - 1$$

III. Log Review.

21. Calculate $log(e)$. (*Remember statisticians tend to write log instead of ln when they mean log base e!*)
$log(e) = 1$. Also, $e^{log(1)} = 1$. Note that log is also how the statistical program R denotes log base e.

22. Rewrite $log\left(\frac{x}{y}\right)$ in another way.
$log(x) - log(y)$

23. Rewrite $log\left(x^n\right)$ in another way.
$n\,log(x)$

24. Solve $log(x) = y$ for x.
$x = e^y$

IV. Statistics: Linear Regression Review.

After regressing eight patients' weights (in kg) on their height (in cm), a doctor found the following output.

Coefficient	Estimate	Std. Error	t-value	$Pr(>\lvert t\rvert)$
(Intercept)	-129.1667	24.3610	-5.302	0.001826
Height	1.1667	0.1521	?????	0.000257

25. Write down the least squares regression line using $\hat{y}$ = predicted weight and x = height.
$\hat{y} = -129.1667 + 1.1667x$

26. What weight does the model predict for someone who is 160 cm tall?
$-129.1667 + 1.1667(160) = 57.5053kg$

27. Interpret the slope of the line in the context of this model.
If a person's height increases by 1 cm, their weight is expected to increase by 1.1667, according to our model. (Slope is rise over run, so if we make the run = 1 - if the x-variable increases by 1 unit - rise = slope - the y-variable increases by slope units. Notice also that we need a "wiggle word" somewhere in our explanation

of slope. We don't know for sure that if x increases by 1, y increases by slope in the *population*; remember our slope is calculated based on only a sample of data. Appropriate wiggle words and phrases include "expected," "according to our model," "predicted," and "on average.")

28. Interpret the standard error of the slope in the context of this model.
The standard error of the slope, 0.1521, tells us the estimated standard deviation of the sample slope from one sample to another. In repeated sampling, our calculated slope based on a random sample will vary from one sample to another. A typical difference between the calculated sample slope and the population slope is estimated to be 0.1521.

29. Calculate the t-statistic for testing whether there exists a linear relationship between height and weight.
Where β_1 is the hypothesized value of the slope, we have:
$$t_{n-2} = \frac{\hat{\beta}_1 - \beta_1}{se_{\hat{\beta}_1}} = \frac{1.1667 - 0}{0.1521} = 7.671$$
(In our case, we are only interested in whether height and weight are linearly associated. If they aren't, the population slope is 0.)

30. Are height and weight linearly associated? Explain.
Yes, we have statistically significant evidence that height and weight are linearly associated in the population. While the non-zero slope of 1.1667 in our sample tells us that height and weight are associated for the patients whose heights and weights we measured, the p-value of 0.000257 tells us that we can be quite sure the slope is also non-zero among patients in the population as well (assuming appropriate random sampling, among other things). (We'll actually need lots of plots to check assumptions and actually be able to make this conclusion; we'll talk about them in class.)

31. A journal article might report that height is a *significant* predictor of weight. Explain what this means in context, as if to someone with no statistical background.
The word "significant" in statistics actually does not mean important or large. Rather, it means "unlikely to have occurred by chance." In the context of our problem, it is unlikely (more specifically, the probability is less than 1 / 1000) that we would collect a random sample of patients that resulted in a sample slope of 1.1667 or even further from 0, just by chance, if in the population there were no linear association between height and weight.

32. Calculate a 95% confidence interval for the slope.
$\hat{\beta}_1 \pm t^*_{n-2}\, se_{\hat{\beta}_1}$
$1.1667 \pm 2.447\,(0.1521)$
(0.7945, 1.5389)

33. Interpret your interval above in context. I am 95% confident that as height increases by 1 centimeter in the general population of patients, weight increases by between 0.807 and 1.53 kg, on average.