HANDOUT # 9

RANDOM EFFECTS MODELS AND NESTED MODELS

I. Mixed Effects Models: Random and Fixed Factor Levels

a. Experiments with Random Treatment Levels

b. Experiments with Fixed and Random Treatment Levels

c. Derivation of Expected Mean Squares from Random and Mixed Effects Models

d. Estimation of Variance Components

II. Nested Treatment Structure

a. Experiments with Nested Treatment Structure

b. Experiments with Nested and Crossed Treatment Structure

I. Experiments with Random Treatment Levels

Example of Experiment with Random Treatment Levels

A beverage company has encountered problems with the machinery that fills the containers with products. The amount of beverage placed in the containers varies too much around the specified amount listed on the container. A quality control engineer designs the following study to investigate the amount of variability in amount of beverage placed in the container due to the two major suspected sources: the filling machine and the operator of the machine.

Research Question: The company wanted to identify what proportion of the variability in the amount of beverage placed in the container was due to machinery differences and what proportion was due to operator variation.

Treatment Design: The quality control engineer randomly selected 3 filling machines from the 130 machines used by the company and randomly selected 4 workers from its work force to operate the filling machines. The 12 treatments consist of two factors: Machine M_1, M_2, M_3 with 3 random levels and Operator O_1, O_2, O_3, O_4 with 4 random levels.

Experimental Design: Every worker filled 50 containers using each of the four machines in a random order. That is, the treatments consist of the 12 random combinations: (M_i, O_j). A random sample of 10 containers were then randomly assigned to each of these 12 random combinations (M_i, O_j) (treatments). The 10 containers were then filled with the beverage. The response variable Y_{ijk} is the deviation from the nominal amount on container (ijk). The complete experiment was then repeated 5 times with a new randomization during each of the 5 replications. There are a total of 600 responses.

It is important to note that in experiments involving treatments having random levels that there are two randomizations involved in the experiment.

1. First the levels of the treatment are randomly selected from a population of treatments.
2. Second, the EU's are randomly assigned to the randomly selected treatments.
3. In some instances, the order in which the responses from the EU's are obtained is also randomized.

Example of Experiment with Fixed and Random Treatment Levels: Mixed Factor Levels

Design and Analysis of Experiments by Gary W. Dehlert

Dental Fillings Experiment

Dental fillings made from gold can vary in hardness depending on how the metal is treated prior to its placement in the tooth. Two factors thought to influence the hardness: the gold alloy and the condensation method. In addition, some dentists performing the dental work are better at some types of filling than others. Five dentists were randomly selected and agreed to participate in the experiment. Each dentist prepares 24 fillings (in random order), one for each of the combinations of condensation method (three levels) and alloy (eight levels). The fillings were then measured for hardness using the Diamond Pyramid Hardness Number (big scores are better). The data are contained in the following table:

		Alloy							
Dentist	Method	1	2	3	4	5	6	7	8
1	1	792	824	813	792	792	907	792	835
1	2	772	772	782	698	665	1115	835	870
1	3	782	803	752	620	835	847	560	585
2	1	803	803	715	803	813	858	907	882
2	2	752	772	772	782	743	933	792	824
2	3	715	707	835	715	673	698	734	681
3	1	715	724	743	627	752	858	762	724
3	2	792	715	813	743	613	824	847	782
3	3	762	606	743	681	743	715	824	681
4	1	673	946	792	743	762	894	792	649
4	2	657	743	690	882	772	813	870	858
4	3	690	245	493	707	289	715	813	312
5	1	634	715	707	698	715	772	1048	870
5	2	649	724	803	665	752	824	933	835
5	3	724	627	421	483	405	536	405	312

The treatments consist of combining a level of the Factor having random levels with the levels of the Factors having fixed levels.

1. First the levels of the factor having random levels are randomly selected from a population of levels, the Dentist.

2. Second the treatments are constructed by combining the levels of the fixed level factors, Method and Gold Alloy, with the randomly selected levels of the Dentist factor.

3. Third, the EU's are randomly assigned to the treatments. In the above example, the 24 fillings are randomly assigned to the Dentist.

4. In some experiments, the order in which the responses from the EU's are obtained is also randomized (the preparation of 24 fillings by each Dentist was done in a random order).

Statistical Model for CRD With Treatments Constructed From Two Factors

We will consider three cases, both factors having fixed levels, both factors having random levels, and one with random levels and one with fixed levels.

<u>Case 1</u> **Both Factors Having Fixed Levels:** Suppose we have two factors F_1 at a fixed levels and F_2 at b fixed levels. There are r EU's randomly assigned to each of the $t = ab$ treatments.

MODEL: $y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + e_{ijk}$ with

i. e_{ijk}'s iid $N(0, \sigma_e^2)$

ii. $\tau_a = 0, \quad \gamma_b = 0 \quad (\tau\gamma)_{aj} = (\tau\gamma)_{ib} = 0 \quad$ for all $\quad (i,j)$

The above conditions yield the following results: $y_{ijk} \overset{\mathcal{D}}{\sim} N(\mu_{ij},\ \sigma_e^2)$

a. $\mu_{ij} = E[y_{ijk}] = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij}$ many of the terms, $\tau_i, \gamma_j, (\tau\gamma)_{ij}$, are 0

b. $Var[y_{ijk}] = \sigma_e^2$

c. y_{ijk}'s are independent normally distributed random variables with the same variance but are not identically distributed due to different means.

d. Goal: Test hypotheses about the interaction and main effects of the two factors using contrasts in the μ_{ij}

e. Goal: Use multiple comparisons, contrasts, other procedures to investigate differences in the μ_{ij}'s

<u>Case 2</u> **Both Factors Having Random Levels :** Suppose we have two factors F_1 at a randomly selected levels and F_2 at b randomly selected levels. There are r EU's randomly assigned to each of the $t = ab$ treatments.

MODEL: $y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_{ijk}$ with

i. e_{ijk}'s iid $N(0, \sigma_e^2)$

ii. a_i's iid $N(0, \sigma_{F_1}^2)$, b_i's iid $N(0, \sigma_{F_2}^2)$, and $(ab)_{ij}$'s iid $N(0, \sigma_{F_1*F_2}^2)$

iii. The random variables a_i, b_j, $(ab)_{ij}$, and e_{ijk} are all independent

The above conditions yield the following results: $y_{ijk} \overset{\mathcal{D}}{\sim} N(\mu,\ \sigma_y^2)$

a. $\mu_{ij} = E[y_{ijk}] = \mu$

b. $\sigma_y^2 = Var[y_{ijk}] = \sigma_{F_1}^2 + \sigma_{F_2}^2 + \sigma_{F_1*F_2}^2 + \sigma_e^2$

c. y_{ijk}'s are identically distributed but not independent:

$$Cov[y_{ijk}, y_{i'j'k'}] = \begin{cases} \sigma_{F_1}^2 + \sigma_{F_2}^2 + \sigma_{F_1*F_2}^2 + \sigma_e^2 & if \quad i = i', j = j', k = k' \\ \sigma_{F_1}^2 + \sigma_{F_2}^2 + \sigma_{F_1*F_2}^2 & if \quad i = i', j = j', k \neq k' \\ \sigma_{F_1}^2 & if \quad i = i', j \neq j' \\ \sigma_{F_2}^2 & if \quad i \neq i', j = j' \\ 0 & if \quad i \neq i', j \neq j' \end{cases}$$

d. Goal: Test hypotheses about the interaction and main effects of the two factors by testing the hypotheses:

Test of Interaction: $H_o : \sigma_{F_1*F_2} = 0$ versus $H_1 : \sigma_{F_1*F_2} \neq 0$

Test of Main effect of F_1: $H_o : \sigma_{F_1} = 0$ versus $H_1 : \sigma_{F_1} \neq 0$

Test of Main effect of F_2: $H_o : \sigma_{F_2} = 0$ versus $H_1 : \sigma_{F_2} \neq 0$

e. Proportionally allocate the variability in the measurements, σ_y^2 into the four sources of variation based on the relative sizes of $\sigma_{F_1}^2, \sigma_{F_2}^2, \sigma_{F_1*F_2}^2$, and σ_e^2

f. Multiple comparisons and contrasts are not appropriate for factors with random factor levels.

<u>Case 3</u> **Mixed Factors: One Factor has Fixed Levels and one Factor Has Random Levels :** Suppose we have one factor F_1 at a fixed levels and F_2 at b randomly selected levels. There are r EU's randomly assigned to each of the $t = ab$ treatments.

MODEL: $y_{ijk} = \mu + \tau_i + b_j + (\tau b)_{ij} + e_{ijk}$ with

i. e_{ijk}'s iid $N(0, \sigma_e^2)$

ii. τ_i's are population parameters with $\tau_a = 0$, b_i's iid $N(0, \sigma_{F_2}^2)$, and $(\tau b)_{ij}$'s iid $N(0, \sigma_{F_1*F_2}^2)$

iii. The random variables b_j, $(\tau b)_{ij}$, and e_{ijk} are all independent

The above conditions yield the following results: $y_{ijk} \overset{\mathcal{D}}{\sim} N(\mu_{ij},\ \sigma_y^2)$

a. $\mu_{ij} = E[y_{ijk}] = E[\mu + \tau_i + b_j + (\tau b)_{ij} + e_{ijk}] = \mu + \tau_i$

b. $\sigma_y^2 = Var[y_{ijk}] = Var[\mu + \tau_i + b_j + (\tau b)_{ij} + e_{ijk}] = \sigma_{F_2}^2 + \sigma_{F_1*F_2}^2 + \sigma_e^2$

c. y_{ijk}'s are neither identically distributed (if $\tau_i \neq 0$) nor independent :

$$Cov[y_{ijk}, y_{i'j'k'}] = \begin{cases} \sigma_{F_2}^2 + \sigma_{F_1*F_2}^2 + \sigma_e^2 & if \quad i = i', j = j', k = k' \\ \sigma_{F_2}^2 + \sigma_{F_1*F_2}^2 & if \quad i = i', j = j', k \neq k' \\ \sigma_{F_2}^2 & if \quad i \neq i', j = j' \\ 0 & if \quad j \neq j' \end{cases}$$

d. Goal: Test hypotheses about the interaction and main effects of the two factors by testing the hypotheses:

Test of Interaction: $H_o : \sigma_{F_1*F_2} = 0$ versus $H_1 : \sigma_{F_1*F_2} \neq 0$

Test of Main effect of F_1: $H_o : \tau_i = 0$ for all i versus $H_1 : \tau_i \neq 0$ for at least one i

Test of Main effect of F_2: $H_o : \sigma_{F_2} = 0$ versus $H_1 : \sigma_{F_2} \neq 0$

e. Proportionally allocate the variability in the measurements, σ_y^2 into the three sources of variation based on the relative sizes of : $\sigma_{F_2}^2, \sigma_{F_1*F_2}^2$, and σ_e^2

f. Use multiple comparisons and contrasts to investigate differences in the levels of the factor having fixed factor levels, F_1. Because F_2 has random levels, we would evaluate the levels of F_1 averaged over the levels of F_2 even when there is significant evidence of an interaction between factors F_1 and F_2. For example, F_1 is type of Machine and F_2 is Operators of Machine, we would still want to compare the average responses of the different types of machines even if there is a Machine by Operator interaction.

For each of the Three Situations: Fixed Effects, Random Effects, and Mixed Effects, we need to determine the appropriate hypotheses and test statistics. In each of the above three types of experiments, the computational form of the sums of squares for the AOV table are identical when there are an equal number of replications for each of the $t = ab$ treatments. However, the expected Mean Squares are different depending on the randomness of the terms in the corresponding model. In order to identify the proper test statistics, it is necessary to determine the expected values of the Mean Squares.

Model I: Both F_1 and F_2 Fixed

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + brQ_{F_1}$	$H_o : Q_{F_1} = 0$	$\frac{MS_{F_1}}{MSE}$
F_2	MS_{F_2}	$\sigma_e^2 + arQ_{F_2}$	$H_o : Q_{F_2} = 0$	$\frac{MS_{F_2}}{MSE}$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + rQ_{F_1*F_2}$	$H_o : Q_{F_1*F_2} = 0$	$\frac{MS_{F_1*F_2}}{MSE}$
Error	MSE	σ_e^2		

where we define the following parameters

$Q_{F_1} = \frac{1}{a-1} \sum_{i=1}^{a} (\bar{\mu}_{i.} - \bar{\mu}_{..})^2$

$Q_{F_2} = \frac{1}{b-1} \sum_{j=1}^{b} (\bar{\mu}_{.j} - \bar{\mu}_{..})^2$

$Q_{F_1*F_2} = \frac{1}{(a-1)(b-1)} \sum_{i=1}^{a} \sum_{j=1}^{b} (\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..})^2$

Model II: Both F_1 and F_2 Random

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + r\sigma_{F_1*F_2}^2 + br\sigma_{F_1}^2$	$H_o : \sigma_{F_1}^2 = 0$	$\frac{MS_{F_1}}{MS_{F_1*F_2}}$
F_2	MS_{F_2}	$\sigma_e^2 + r\sigma_{F_1*F_2}^2 + ar\sigma_{F_2}^2$	$H_o : \sigma_{F_2}^2 = 0$	$\frac{MS_{F_2}}{MS_{F_1*F_2}}$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + r\sigma_{F_1*F_2}^2$	$H_o : \sigma_{F_1*F_2}^2 = 0$	$\frac{MS_{F_1*F_2}}{MSE}$
Error	MSE	σ_e^2		

Model III: F_1-Fixed and F_2-Random

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + r\sigma_{F_1*F_2}^2 + brQ_{F_1}$	$H_o : Q_{F_1} = 0$	$\frac{MS_{F_1}}{MS_{F_1*F_2}}$
F_2	MS_{F_2}	$\sigma_e^2 + r\sigma_{F_1*F_2}^2 + ar\sigma_{F_2}^2$	$H_o : \sigma_{F_2}^2 = 0$	$\frac{MS_{F_2}}{MS_{F_1*F_2}}$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + r\sigma_{F_1*F_2}^2$	$H_o : \sigma_{F_1*F_2}^2 = 0$	$\frac{MS_{F_1*F_2}}{MSE}$
Error	MSE	σ_e^2		

WARNING: If the conditions on the model parameters in the mixed model are changed from $\tau_a = 0$ to $\sum_{i=1}^a \tau_i = 0$ and $\sum_{i=1}^a (\tau b)_{ij} = 0$ for $j = 1, \ldots, b$, then the above results are changed to the following table:

Model III**: F_1-Fixed and F_2-Random

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + r\sigma_{F_1*F_2}^2 + brQ_{F_1}$	$H_o : Q_{F_1} = 0$	$\frac{MS_{F_1}}{MS_{F_1*F_2}}$
F_2	MS_{F_2}	$\sigma_e^2 + ar\sigma_{F_2}^2$	$H_o : \sigma_{F_2}^2 = 0$	$\frac{MS_{F_2}}{MSE}$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + r\sigma_{F_1*F_2}^2$	$H_o : \sigma_{F_1*F_2}^2 = 0$	$\frac{MS_{F_1*F_2}}{MSE}$
Error	MSE	σ_e^2		

The derivation of the Expected Mean Squares will be illustrated as follows.

Derivation of Expected Mean Squares for a CRD with an $a \times b$ Treatment Structure and r Reps per Treatment (Equally replicated)

Suppose we have two factors F_1 with a levels and F_2 with b levels. There are r observations per treatment. The methodology for computing the Expected Mean Squares will be illustrated by computing $E[MS_{F_1}]$ under four possible Models:

$$MS_{F_1} = \frac{rb}{a-1}\sum_{i=1}^{a}(\bar{y}_{i..} - \bar{y}_{...})^2 \quad \Rightarrow EMS_{F_1} = E[MS_{F_1}] = \frac{rb}{a-1}\sum_{i=1}^{a}E[(\bar{y}_{i..} - \bar{y}_{...})^2]$$

MODEL 1: F_1 - Fixed Factor Levels and F_2 - Fixed Factor Levels

CLAIM: $EMS_{F_1} = brQ_{F_1} + \sigma_e^2$

Proof of CLAIM:

$y_{ijk} = \mu_{ij} + e_{ijk}$ with e_{ijk}'s iid $N(0, \sigma_e^2)$ random variables

$\bar{y}_{i..} = \bar{\mu}_{i.} + \bar{e}_{i..} \qquad \bar{y}_{...} = \bar{\mu}_{..} + \bar{e}_{...} \quad \Rightarrow \quad \bar{y}_{i..} - \bar{y}_{...} = (\bar{\mu}_{i.} - \bar{\mu}_{..}) + (\bar{e}_{i..} - \bar{e}_{...})$

$$\begin{aligned}(a.)\ \ E[(\bar{y}_{i..} - \bar{y}_{...})^2] &= (\bar{\mu}_{i.} - \bar{\mu}_{..})^2 + 2(\bar{\mu}_{i.} - \bar{\mu}_{..})E[(\bar{e}_{i..} - \bar{e}_{...})] + E[(\bar{e}_{i..} - \bar{e}_{...})^2] \\ &= (\bar{\mu}_{i.} - \bar{\mu}_{..})^2 + 0 + Var[\bar{e}_{i..} - \bar{e}_{...}]\end{aligned}$$

$\bar{e}_{i..} - \bar{e}_{...} = \bar{e}_{i..} - \frac{1}{a}\sum_{m=1}^{a}\bar{e}_{m..} = (1 - \frac{1}{a})\bar{e}_{i..} - \frac{1}{a}\sum_{m\neq i}^{a}\bar{e}_{m..} \quad \Rightarrow$

$$\begin{aligned}(b.)\ \ Var[\bar{e}_{i..} - \bar{e}_{...}] &= \left(1 - \frac{1}{a}\right)^2 Var[\bar{e}_{i..}] + \frac{1}{a^2}\sum_{m\neq i}Var[\bar{e}_{m..}] \\ &= \left(1 - \frac{1}{a}\right)^2 \frac{\sigma_e^2}{br} + \frac{1}{a^2}\sum_{m\neq i}\frac{\sigma_e^2}{br} \\ &= \left(\frac{a-1}{a}\right)^2 \frac{\sigma_e^2}{br} + \left(\frac{a-1}{a^2}\right)\frac{\sigma_e^2}{br} = \left(\frac{a-1}{abr}\right)\sigma_e^2\end{aligned}$$

From (a.) and (b.), we have

$$E[(\bar{y}_{i..} - \bar{y}_{...})^2] = (\bar{\mu}_{i.} - \bar{\mu}_{..})^2 + \left(\frac{a-1}{abr}\right)\sigma_e^2 \quad \Rightarrow$$

$$EMS_{F_1} = \frac{rb}{a-1}\sum_{i=1}^{a}E[(\bar{y}_{i..} - \bar{y}_{...})^2] = \frac{br}{a-1}\sum_{i=1}^{a}\left[(\bar{\mu}_{i.} - \bar{\mu}_{..})^2 + \left(\frac{a-1}{abr}\right)\sigma_e^2\right] = brQ_{F_1} + \sigma_e^2$$

MODEL 2: F_1 **- Random Factor Levels and** F_2 **- Random Factor Levels**

CLAIM: $EMS_{F_1} = br\sigma^2_{F_1} + r\sigma^2_{F_1*F_2} + \sigma^2_e$

Proof of CLAIM:

$$y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + e_{ijk}$$

with $a_i, b_j, (ab)_{ij}, e_{ijk}$'s independent normally distributed random variables with expectation equal to 0 and variances $\sigma^2_{F_1}, \sigma^2_{F_2}, \sigma^2_{F_1*F_2}$, and σ^2_e, respectively.

$$\bar{y}_{i..} = \mu + a_i + \bar{b}_. + \bar{(ab)}_{i.} + \bar{e}_{i..} \quad \text{and} \quad \bar{y}_{...} = \mu + \bar{a}_. + \bar{b}_. + \bar{(ab)}_{..} + \bar{e}_{...} \quad \Rightarrow$$

$$\bar{y}_{i..} - \bar{y}_{...} = (a_i - \bar{a}_.) + (\bar{(ab)}_{i.} - \bar{(ab)}_{..}) + (\bar{e}_{i..} - \bar{e}_{...})$$

$$\begin{aligned}
E[(\bar{y}_{i..} - \bar{y}_{...})^2] &= E[(a_i - \bar{a}_.)^2] + E[(\bar{(ab)}_{i.} - \bar{(ab)}_{..})^2] + E[(\bar{e}_{i..} - \bar{e}_{...})^2] \\
&+ 2E[(a_i - \bar{a}_.)]E[(\bar{e}_{i..} - \bar{e}_{...})] + 2E[(\bar{(ab)}_{i.} - \bar{(ab)}_{..})]E[(\bar{e}_{i..} - \bar{e}_{...})] \\
&+ 2E[(a_i - \bar{a}_.)]E[(\bar{(ab)}_{i.} - \bar{(ab)}_{..})] \\
&= Var\left[\left(1 - \frac{1}{a}\right) a_i - \frac{1}{a}\sum_{m \neq i} a_m\right] \\
&+ Var\left[\left(1 - \frac{1}{a}\right) \bar{(ab)}_{i.} - \frac{1}{a}\sum_{m \neq i} \bar{(ab)}_{m.}\right] \\
&+ Var\left[(1 - \frac{1}{a})\bar{e}_{i..} + \frac{1}{a}\sum_{m \neq i} \bar{e}_{m..}\right] + 0 \\
&= \left[\left(\frac{a-1}{a}\right)^2 \sigma^2_{F_1} + \left(\frac{a-1}{a^2}\right)\sigma^2_{F_1}\right] \\
&+ \left[\left(\frac{a-1}{a}\right)^2 \frac{\sigma^2_{F_1*F_2}}{b} + \left(\frac{a-1}{a^2}\right)\frac{\sigma^2_{F_1*F_2}}{b}\right] \\
&+ \left[\left(\frac{a-1}{a}\right)^2 \frac{\sigma^2_e}{br} + \left(\frac{a-1}{a^2}\right)\frac{\sigma^2_e}{br}\right] \\
&= \left(\frac{a-1}{a}\right)\sigma^2_{F_1} + \left(\frac{a-1}{ab}\right)\sigma^2_{F_1*F_2} + \left(\frac{a-1}{abr}\right)\sigma^2_e \quad \Rightarrow
\end{aligned}$$

$$\begin{aligned}
EMS_{F_1} &= \tfrac{rb}{a-1}\textstyle\sum_{i=1}^a E[(\bar{y}_{i..} - \bar{y}_{...})^2] \\
&= \tfrac{br}{a-1}\textstyle\sum_{i=1}^a \left[\left(\frac{a-1}{a}\right)\sigma^2_{F_1} + \left(\frac{a-1}{ab}\right)\sigma^2_{F_1*F_2} + \left(\frac{a-1}{abr}\right)\sigma^2_e\right] \\
&= rb\sigma^2_{F_1} + r\sigma^2_{F_1*F_2} + \sigma^2_e
\end{aligned}$$

MODEL 3: F_1 **- Fixed Factor Levels and** F_2 **- Random Factor Levels**

CLAIM: $EMS_{F_1} = brQ_{F_1} + r\sigma^2_{F_1*F_2} + \sigma^2_e$

Proof of CLAIM:

$$y_{ijk} = \mu + \tau_i + b_j + (\tau b)_{ij} + e_{ijk}$$

with $\tau_a = 0$ and $b_j, (\tau b)_{ij}, e_{ijk}$'s independent normally distributed random variables with expectation equal to 0 and variances $\sigma^2_{F_2}, \sigma^2_{F_1*F_2},$ and σ^2_e, respectively.

$$\bar{y}_{i..} = \mu + \tau_i + \bar{b}_. + (\bar{\tau b})_{i.} + \bar{e}_{i..} \qquad \bar{y}_{...} = \mu + \bar{\tau}_. + \bar{b}_. + (\bar{\tau b})_{..} + \bar{e}_{...} \quad \Rightarrow$$

$$\bar{y}_{i..} - \bar{y}_{...} = (\tau_i - \bar{\tau}_.) + ((\bar{\tau b})_{i.} - (\bar{\tau b})_{..}) + (\bar{e}_{i..} - \bar{e}_{...}) \quad \Rightarrow$$

$$\begin{aligned}
E[(\bar{y}_{i..} - \bar{y}_{...})^2] \quad &= \quad (\tau_i - \bar{\tau}_.)^2 + E[(\bar{\tau b})_{i.} - (\bar{\tau b})_{..})^2] + E[(\bar{e}_{i..} - \bar{e}_{...})^2] \\
&+ \quad 2(\tau_i - \bar{\tau}_.)E[(\bar{e}_{i..} - \bar{e}_{...})] + 2E[((\bar{\tau b})_{i.} - (\bar{\tau b})_{..})]E[(\bar{e}_{i..} - \bar{e}_{...})] \\
&+ \quad 2(\tau_i - \bar{\tau}_.)E[((\bar{\tau b})_{i.} - (\bar{\tau b})_{..})] \\
&= \quad (\tau_i - \bar{\tau}_.)^2 + Var\left[\left(1 - \frac{1}{a}\right)(\bar{\tau b})_{i.} - \frac{1}{a}\sum_{m \neq i}(\bar{\tau b})_{m.}\right] \\
&+ \quad Var\left[(1 - \frac{1}{a})\bar{e}_{i..} + \frac{1}{a}\sum_{m \neq i}\bar{e}_{m..}\right] + 0 \\
&= \quad (\tau_i - \bar{\tau}_.)^2 + \left[\left(\frac{a-1}{a}\right)^2 \frac{\sigma^2_{F_1*F_2}}{b} + \left(\frac{a-1}{a^2}\right)\frac{\sigma^2_{F_1*F_2}}{b}\right] \\
&+ \quad \left[\left(\frac{a-1}{a}\right)^2 \frac{\sigma^2_e}{br} + \left(\frac{a-1}{a^2}\right)\frac{\sigma^2_e}{br}\right] \\
&= \quad (\tau_i - \bar{\tau}_.)^2 + \left(\frac{a-1}{ab}\right)\sigma^2_{F_1*F_2} + \left(\frac{a-1}{abr}\right)\sigma^2_e \quad \Rightarrow
\end{aligned}$$

$$\begin{aligned}
EMS_{F_1} &= \tfrac{rb}{a-1}\textstyle\sum_{i=1}^a E[(\bar{y}_{i..} - \bar{y}_{...})^2] \\
&= \tfrac{br}{a-1}\textstyle\sum_{i=1}^a \left[(\tau_i - \bar{\tau}_.)^2 + \left(\frac{a-1}{ab}\right)\sigma^2_{F_1*F_2} + \left(\frac{a-1}{abr}\right)\sigma^2_e\right] \\
&= rbQ_{F_1} + r\sigma^2_{F_1*F_2} + \sigma^2_e
\end{aligned}$$

MODEL 4: F_1 **- Random Factor Levels and** F_2 **- Fixed Factor Levels**

CLAIM: $EMS_{F_1} = br\sigma^2_{F_1} + r\sigma^2_{F_1*F_2} + \sigma^2_e$

Proof of CLAIM:

$$y_{ijk} = \mu + a_i + \gamma_j + (a\gamma)_{ij} + e_{ijk}$$

with $\gamma_b = 0$ and $a_i, (a\gamma)_{ij}, e_{ijk}$'s independent normally distributed random variables with expectation equal to 0 and variances $\sigma^2_{F_1}, \sigma^2_{F_1*F_2},$ and σ^2_e, respectively.

$$\bar{y}_{i..} = \mu + s_i + \bar{\gamma}_. + (\bar{a\gamma})_{i.} + \bar{e}_{i..} \qquad \bar{y}_{...} = \mu + \bar{a}_. + \bar{\gamma}_. + (\bar{a\gamma})_{..} + \bar{e}_{...} \quad \Rightarrow$$

$$\begin{aligned}
E[(\bar{y}_{i..} - \bar{y}_{...})^2] &= E[(a_i - \bar{a}_.)^2] + E[((\bar{a\gamma})_{i.} - (\bar{a\gamma})_{..})^2] + E[(\bar{e}_{i..} - \bar{e}_{...})^2] \\
&+ 2E[(a_i - \bar{a}_.)]E[(\bar{e}_{i..} - \bar{e}_{...})] + 2E[((\bar{a\gamma})_{i.} - (\bar{a\gamma})_{..})]E[(\bar{e}_{i..} - \bar{e}_{...})] \\
&+ 2E[(a_i - \bar{a}_.)]E[((\bar{a\gamma})_{i.} - (\bar{a\gamma})_{..})] \\
&= Var\left[\left(1 - \frac{1}{a}\right) a_i - \frac{1}{a}\sum_{m \neq i} \bar{a}_{m.}\right] \\
&+ Var\left[\left(1 - \frac{1}{a}\right) (\bar{a\gamma})_{i.} - \frac{1}{a}\sum_{m \neq i} (\bar{a\gamma})_{m.}\right] \\
&+ Var\left[(1 - \frac{1}{a})\bar{e}_{i..} + \frac{1}{a}\sum_{m \neq i} \bar{e}_{m..}\right] + 0 \\
&= \left[\left(\frac{a-1}{a}\right)^2 \sigma^2_{F_1} + \left(\frac{a-1}{a^2}\right)\frac{\sigma^2_{F_1*F_2}}{b}\right] \\
&+ \left[\left(\frac{a-1}{a}\right)^2 \frac{\sigma^2_{F_1*F_2}}{b} + \left(\frac{a-1}{a^2}\right)\sigma^2_{F_1*F_2}\right] \\
&+ \left[\left(\frac{a-1}{a}\right)^2 \frac{\sigma^2_e}{br} + \left(\frac{a-1}{a^2}\right)\frac{\sigma^2_e}{br}\right] \\
&= \left(\frac{a-1}{a}\right)\sigma^2_{F_1} + \left(\frac{a-1}{ab}\right)\sigma^2_{F_1*F_2} + \left(\frac{a-1}{abr}\right)\sigma^2_e \quad \Rightarrow
\end{aligned}$$

$EMS_{F_1} = \frac{rb}{a-1}\sum_{i=1}^a E[(\bar{y}_{i..} - \bar{y}_{...})^2]$

$= \frac{br}{a-1}\sum_{i=1}^a \left[\left(\frac{a-1}{a}\right)\sigma^2_{F_1} + \left(\frac{a-1}{ab}\right)\sigma^2_{F_1*F_2} + \left(\frac{a-1}{abr}\right)\sigma^2_e\right]$

$= rb\sigma^2_{F_1} + r\sigma^2_{F_1*F_2} + \sigma^2_e$

Statistical Model for CRD With Three Factors

We will consider four cases,

1. all factors having fixed levels
2. all factors having random levels
3. one factor with random levels and two with fixed levels
4. two factors with random levels and one with fixed levels

<u>Case 1</u> **All Factors Having Fixed Levels:** Suppose we have three factors F_1 with a fixed levels, F_2 with b fixed levels, and F_3 with c fixed levels. There are r EU's randomly assigned to each of the $t = abc$ treatments.

MODEL: $y_{ijkl} = \mu + \tau_i + \gamma_j + \delta_k + (\tau\gamma)_{ij} + (\tau\delta)_{ik} + (\gamma\delta)_{jk} + (\tau\gamma\delta)_{ijk} + e_{ijkl}$ with

i. e_{ijkl}'s iid $N(0, \sigma_e^2)$

ii. $\tau_a = 0, \quad \gamma_b = 0 \quad \delta_c = 0$

iii. $(\tau\gamma)_{aj} = (\tau\gamma)_{ib} = (\tau\delta)_{ak} = (\tau\delta)_{ic} = (\gamma\delta)_{jc} = (\gamma\delta)_{bk} = 0 \quad \text{for all} \quad (i, j, k)$

iv. $(\tau\gamma\delta)_{ajk} = (\tau\gamma\delta)_{ibk} = (\tau\gamma\delta)_{ijc} = 0 \quad \text{for all} \quad (i, j, k)$

The above conditions yield the following results

a. $\mu_{ijk} = E[y_{ijkl}] = \mu + \tau_i + \gamma_j + \delta_k + (\tau\gamma)_{ij} + (\tau\delta)_{ik} + (\gamma\delta)_{jk} + (\tau\gamma\delta)_{ijk}$ with a large number of these terms equal to 0

b. $Var[y_{ijkl}] = \sigma_e^2$

c. y_{ijkl}'s are independently distributed as $N(\mu_{ijk}, \sigma_e^2)$ r.v.'s, that is, independent normally distributed random variables with the same variance but not identically distributed due to potentially different means, μ_{ijk}.

   That is, $y_{ijkl} \overset{\mathcal{D}}{\sim} N(\mu_{ijk},\ \sigma_e^2)$

<u>Case 2</u> All Factors Having Random Levels : Suppose we have three factors F_1 with a randomly selected levels, F_2 with b randomly selected levels, and F_3 with c randomly selected levels. There are r EU's randomly assigned to each of the $t = abc$ treatments.

MODEL: $y_{ijkl} = \mu + a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} + e_{ijkl}$ with

i. e_{ijkl}'s iid $N(0, \sigma_e^2)$

ii. a_i's iid $N(0, \sigma_{F_1}^2)$, b_j's iid $N(0, \sigma_{F_2}^2)$, and c_k's iid $N(0, \sigma_{F_3}^2)$

iii. $(ab)_{ij}$'s iid $N(0, \sigma_{F_1*F_2}^2)$, $(ac)_{ik}$'s iid $N(0, \sigma_{F_1*F_3}^2)$, $(bc)_{jk}$'s iid $N(0, \sigma_{F_2*F_3}^2)$, $(abc)_{ijk}$'s iid $N(0, \sigma_{F_1*F_2*F_3}^2)$,

iv. The random variables $a_i, b_j, c_k, (ab)_{ij}, (ac)_{ik}, (bc)_{jk}, (abc)_{ijk}$, and e_{ijkl} are all independent

The above conditions yield the following results

a. $\mu_{ijk} = E[y_{ijkl}] = \mu$

b. $\sigma_y^2 = Var[y_{ijkl}] = \sigma_{F_1}^2 + \sigma_{F_2}^2 + \sigma_{F_3}^2 + \sigma_{F_1*F_2}^2 + \sigma_{F_1*F_3}^2 + \sigma_{F_2*F_3}^2 + \sigma_{F_1*F_2*F_3}^2 + \sigma_e^2$

c. y_{ijkl}'s are identically distributed as $N(\mu, \sigma_y^2)$ r.v.'s but they are not independent because of the following correlation:

$$
\begin{aligned}
\text{Cov}(y_{ijk\ell}, y_{i'j'k'\ell'}) &= E[(y_{ijk\ell} - \mu)(y_{i'j'k'ell'} - \mu)] \\
&= E[(a_i + b_j + c_k + (ab)_{ij} + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} + e_{ijkl}) * \\
&\quad (a_{i'} + b_{j'} + c_{k'} + (ab)_{i'j'} + (ac)_{i'k'} + (bc)_{j'k'} + (abc)_{i'j'k'} + e_{i'j'k'l'})] \\
&= E[(a_i)(a_{i'})] + E[(b_j)(b_{j'})] + E[(c_k)(c_{k'})] + E[(ab)_{ij}(ab)_{i'j'}] + E[(ac)_{ik}(ac)_{i'k'}] + \\
&\quad E[(bc)_{jk}(bc)_{j'k'}] + E[(abc)_{ijk}(abc)_{i'j'k'}] + E[(e_{ijkl})(e_{i'j'k'l'})] + 0
\end{aligned}
$$

We thus have the following expression for the covariance for the various combinations of (i, i', j, j', k, k', l, l')

$$\text{Cov}(y_{ijk\ell}, y_{i'j'k'\ell'}) = \begin{cases} \sigma^2_{F_1} + \sigma^2_{F_2} + \sigma^2_{F_3} + \sigma^2_{F_1*F_2} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} + \sigma^2_e & \text{if } i = i', j = j', k = k', l = l' \\ = Var(y_{ijk\ell}) & \\ \sigma^2_{F_1} + \sigma^2_{F_2} + \sigma^2_{F_3} + \sigma^2_{F_1*F_2} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} & \text{if } i = i', j = j', k = k', l \neq l' \\ \sigma^2_{F_2} + \sigma^2_{F_3} + \sigma^2_{F_2*F_3} & \text{if } i \neq i', j = j', k = k' \\ \sigma^2_{F_1} + \sigma^2_{F_3} + \sigma^2_{F_1*F_3} & \text{if } i = i', j \neq j', k = k' \\ \sigma^2_{F_1} + \sigma^2_{F_2} + \sigma^2_{F_1*F_2} & \text{if } i = i', j = j', k \neq k' \\ \sigma^2_{F_3} & \text{if } i \neq i', j \neq j', k = k' \\ \sigma^2_{F_2} & \text{if } i \neq i', j = j', k \neq k' \\ \sigma^2_{F_1} & \text{if } i = i', j \neq j', k \neq k' \\ 0 & \text{if } i \neq i', j \neq j', k \neq k' \end{cases}$$

<u>Case 3</u> **Mixed Factors: One Factor has Random Levels and two Factors Have Fixed Levels :** Suppose we have three factors F_1 with a fixed levels, F_2 with b fixed levels, and F_3 with c randomly selected levels. There are r EU's randomly assigned to each of the $t = abc$ treatments.

MODEL: $y_{ijkl} = \mu + \tau_i + \gamma_j + c_k + (\tau\gamma)_{ij} + (\tau c)_{ik} + (\gamma c)_{jk} + (\tau\gamma c)_{ijk} + e_{ijkl}$ with

i. e_{ijkl}'s iid $N(0, \sigma_e^2)$

ii. τ_i's, γ_j's and $(\tau\gamma)_{ij}$'s are population parameters with

$\tau_a = 0, \quad \gamma_b = 0 \quad (\tau\gamma)_{aj} = (\tau\gamma)_{ib} = 0 \quad \text{for all} \quad (i, j)$

iii. c_k's iid $N(0, \sigma^2_{F_3})$, $(\tau c)_{ij}$'s iid $N(0, \sigma^2_{F_1*F_3})$ $(\gamma c)_{ij}$'s iid $N(0, \sigma^2_{F_2*F_3})$ $(\tau\gamma c)_{ij}$'s iid $N(0, \sigma^2_{F_1*F_2*F_3})$

iii. The random variables $c_k, (\tau c)_{ik}, (\gamma c)_{jk}, (\tau\gamma c)_{ijk},$ and e_{ijkl} are all independent

The above conditions yield the following results

a. $\mu_{ijk} = E[y_{ijk}] = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij}$

b. $\sigma_y^2 = Var[y_{ijk}] = \sigma^2_{F_3} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} + \sigma_e^2$

c. y_{ijkl}'s are distributed $N(\mu_{ijk}, \sigma_y^2)$ and are thus not identically distributed nor independent because of the following correlation:

$$\begin{aligned}
\text{Cov}(y_{ijkl}, y_{i'j'k'l'}) &= E[(y_{ijkl} - \mu - \tau_i - \gamma_j - (\tau\gamma)_{ij})(y_{i'j'k'l'} - \mu - \tau_{i'} - \gamma j' - (\tau\gamma)_{i'j'})] \\
&= E[(c_k + (ac)_{ik} + (bc)_{jk} + (abc)_{ijk} + e_{ijkl}) * \\
&\quad (c_{k'} + (ac)_{i'k'} + (bc)_{j'k'} + (abc)_{i'j'k'} + e_{i'j'k'l'})] \\
&= E[(c_k)(c_{k'})] + E[(ac)_{ik}(ac)_{i'k'}] + E[(bc)_{jk}(bc)_{j'k'}] + \\
&\quad E[(abc)_{ijk}(abc)_{i'j'k'}] + E[(e_{ijkl})(e_{i'j'k'l'})] + 0
\end{aligned}$$

We thus have the following expression for the covariance for the various combinations of (i, i', j, j', k, k', l, l')

$$
\mathrm{Cov}(y_{ijkl}, y_{i'j'k'l'}) = \begin{cases}
\sigma^2_{F_3} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} + \sigma^2_e & \text{if } i = i', j = j', k = k', l = l' \\
\sigma^2_{F_3} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} & \text{if } i = i', j = j', k = k', l \neq l' \\
\sigma^2_{F_3} + \sigma^2_{F_2*F_3} & \text{if } i \neq i', j = j', k = k' \\
\sigma^2_{F_3} + \sigma^2_{F_1*F_3} & \text{if } i = i', j \neq j', k = k' \\
\sigma^2_{F_3} & \text{if } i \neq i', j \neq j', k = k' \\
0 & \text{if } k \neq k'
\end{cases}
$$

Case 4 **Mixed Factors: One Factor has Fixed Levels and two Factors Have Random Levels :** Suppose we have one factor F_1 with a fixed levels, F_2 with b randomly selected levels, and F_3 with c randomly selected levels. There are r EU's randomly assigned to each of the $t = abc$ treatments.

MODEL: $y_{ijkl} = \mu + \tau_i + b_j + c_k + (\tau b)_{ij} + (\tau c)_{ik} + (bc)_{jk} + (\tau bc)_{ijk} + e_{ijkl}$ with

i. e_{ijkl}'s iid $N(0, \sigma_e^2)$

ii. τ_i's are population parameters with $\tau_a = 0$ b_j's iid $N(0, \sigma^2_{F_2})$ c_k's iid $N(0, \sigma^2_{F_3})$

$(\tau b)_{ij}$'s iid $N(0, \sigma^2_{F_1*F_2})$ $(\tau c)_{ij}$'s iid $N(0, \sigma^2_{F_1*F_3})$ $(bc)_{ij}$'s iid $N(0, \sigma^2_{F_2*F_3})$ $(\tau bc)_{ij}$'s iid $N(0, \sigma^2_{F_1*F_2*F_3})$

iii. The random variables $b_j, c_k, (\tau b)_{ij}, (\tau c)_{ik}, (bc)_{jk}, (\tau bc)_{ijk},$ and e_{ijkl} are all independent

The above conditions yield the following results

a. $\mu_{ijk} = E[y_{ijk}] = \mu + \tau_i$

b. $\sigma_y^2 = Var[y_{ijk}] = \sigma^2_{F_2} + \sigma^2_{F_3} + \sigma^2_{F_1*F_2} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} + \sigma_e^2$

c. y_{ijkl}'s are distributed $N(\mu + \tau_i, \sigma_y^2)$ and thus, are not identically distributed nor independent because of the following correlation:

$$
\mathrm{Cov}(y_{ijkl}, y_{i'j'k'l'}) = \begin{cases}
\sigma^2_{F_2} + \sigma^2_{F_3} + \sigma^2_{F_1*F_2} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} + \sigma_e^2 & \text{if } i = i', j = j', k = k', l = l' \\
\sigma^2_{F_2} + \sigma^2_{F_3} + \sigma^2_{F_1*F_2} + \sigma^2_{F_1*F_3} + \sigma^2_{F_2*F_3} + \sigma^2_{F_1*F_2*F_3} & \text{if } i = i', j = j', k = k', l \neq l' \\
\sigma^2_{F_2} + \sigma^2_{F_3} + \sigma^2_{F_2*F_3} & \text{if } i \neq i', j = j', k = k' \\
\sigma^2_{F_3} + \sigma^2_{F_1*F_3} & \text{if } i = i', j \neq j', k = k' \\
\sigma^2_{F_2} + \sigma^2_{F_1*F_2} & \text{if } i = i', j = j', k \neq k' \\
\sigma^2_{F_3} & \text{if } i \neq i', j \neq j', k = k' \\
\sigma^2_{F_2} & \text{if } i \neq i', j = j', k \neq k' \\
0 & \text{if } j \neq j', k \neq k'
\end{cases}
$$

For each of the Three Situations: Fixed Effects, Random Effects, and Mixed Effects, we need to determine the appropriate hypotheses and test statistics. In each of the above four types of experiments, the computational form of the sums of squares for the AOV table are identical when there are an equal number of replications for each of the $t = ab$ treatments. However, the expected Mean Squares are different depending on the randomness of the terms in the corresponding model. In order to identify the proper test statistics, it is necessary to determine the expected values of the Mean Squares.

Model I: The Fixed Levels For F_1, F_2 and F_3

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + bcrQ_{F_1}$	$H_o : Q_{F_1} = 0$	$\frac{MS_{F_1}}{MSE}$
F_2	MS_{F_2}	$\sigma_e^2 + acrQ_{F_2}$	$H_o : Q_{F_2} = 0$	$\frac{MS_{F_2}}{MSE}$
F_3	MS_{F_3}	$\sigma_e^2 + abrQ_{F_3}$	$H_o : Q_{F_3} = 0$	$\frac{MS_{F_3}}{MSE}$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + crQ_{F_1*F_2}$	$H_o : Q_{F_1*F_2} = 0$	$\frac{MS_{F_1*F_2}}{MSE}$
$F_1 * F_3$	$MS_{F_1*F_3}$	$\sigma_e^2 + brQ_{F_1*F_3}$	$H_o : Q_{F_1*F_3} = 0$	$\frac{MS_{F_1*F_3}}{MSE}$
$F_2 * F_3$	$MS_{F_2*F_3}$	$\sigma_e^2 + arQ_{F_2*F_3}$	$H_o : Q_{F_2*F_3} = 0$	$\frac{MS_{F_2*F_3}}{MSE}$
$F_1 * F_2 * F_3$	$MS_{F_1*F_1*F_3}$	$\sigma_e^2 + rQ_{F_1*F_2*F_3}$	$H_o : Q_{F_1*F_2*F_3} = 0$	$\frac{MS_{F_1*F_2*F_3}}{MSE}$
Error	MSE	σ_e^2		

where we define the following parameters

$Q_{F_1} = \frac{1}{a-1}\sum_{i=1}^{a}(\bar{\mu}_{i..} - \bar{\mu}_{...})^2, \qquad Q_{F_2} = \frac{1}{b-1}\sum_{j=1}^{b}(\bar{\mu}_{.j.} - \bar{\mu}_{...})^2, \qquad Q_{F_3} = \frac{1}{b-1}\sum_{j=1}^{b}(\bar{\mu}_{..k} - \bar{\mu}_{...})^2$

$Q_{F_1*F_2} = \frac{1}{(a-1)(b-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{\mu}_{ij.} - \bar{\mu}_{i..} - \bar{\mu}_{.j.} + \bar{\mu}_{...})^2$

$Q_{F_1*F_3} = \frac{1}{(a-1)(c-1)}\sum_{i=1}^{a}\sum_{k=1}^{c}(\bar{\mu}_{ik.} - \bar{\mu}_{i..} - \bar{\mu}_{..k} + \bar{\mu}_{...})^2$

$Q_{F_2*F_3} = \frac{1}{(b-1)(c-1)}\sum_{j=1}^{b}\sum_{k=1}^{c}(\bar{\mu}_{.jk} - \bar{\mu}_{.j.} - \bar{\mu}_{..k} + \bar{\mu}_{...})^2$

$Q_{F_1*F_2*F_3} = \frac{1}{(a-1)(b-1)(c-1)}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{i=1}^{c}(\mu_{ijk} - \bar{\mu}_{ij.} - \bar{\mu}_{i.k} - \bar{\mu}_{.jk} + \bar{\mu}_{i..} + \bar{\mu}_{.j.} + \bar{\mu}_{..k} - \bar{\mu}_{...})^2$

Model II: All Factors, F_1, F_2 and F_3, are Random

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2} + rb\sigma^2_{F_1*F_3} + rbc\sigma^2_{F_1}$	$H_o : \sigma^2_{F_1} = 0$	$*$
F_2	MS_{F_2}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2} + ra\sigma^2_{F_2*F_3} + rac\sigma^2_{F_2}$	$H_o : \sigma^2_{F_2} = 0$	$**$
F_3	MS_{F_3}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3} + ra\sigma^2_{F_2*F_3} + rab\sigma^2_{F_3}$	$H_o : \sigma^2_{F_3} = 0$	$***$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2}$	$H_o : \sigma^2_{F_1*F_2} = 0$	$\frac{MS_{F_1*F_2}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_3$	$MS_{F_1*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3}$	$H_o : \sigma^2_{F_1*F_3} = 0$	$\frac{MS_{F_1*F_3}}{MS_{F_1*F_2*F_3}}$
$F_2 * F_3$	$MS_{F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + ra\sigma^2_{F_2*F_3}$	$H_o : \sigma^2_{F_2*F_3} = 0$	$\frac{MS_{F_2*F_3}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_2 * F_3$	$MS_{F_1*F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3}$	$H_o : \sigma^2_{F_1*F_2*F_3} = 0$	$\frac{MS_{F_1*F_2*F_3}}{MSE}$
Error	$MS_{F_1*F_2*F_3}$	σ_e^2		

Approximate F-tests are obtained by obtaining a linear combination of Mean Squares, M, such that the expected value of M equals the expected value of the MS for corresponding source of variation, e.g., $E[M] = E_{H:_o}[MS_{F_1}]$.

1. For F_1, $* = \frac{MS_{F_1}}{M}$ where $M = MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3}$

2. For F_2, $** = \frac{MS_{F_2}}{M}$ where $M = MS_{F_1*F_2} + MS_{F_2*F_3} - MS_{F_1*F_2*F_3}$

3. For F_3, $*** = \frac{MS_{F_3}}{M}$ where $M = MS_{F_1*F_3} + MS_{F_2*F_3} - MS_{F_1*F_2*F_3}$

Model III: F_1-Fixed, F_2-Random, and F_3-Random

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2} + rb\sigma^2_{F_1*F_3} + rbcQ_{F_1}$	$H_o : Q_{F_1} = 0$	$*$
F_2	MS_{F_2}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2} + ra\sigma^2_{F_2*F_3} + rac\sigma^2_{F_2}$	$H_o : \sigma^2_{F_2} = 0$	$**$
F_3	MS_{F_3}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3} + ra\sigma^2_{F_2*F_3} + rab\sigma^2_{F_3}$	$H_o : \sigma^2_{F_3} = 0$	$***$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2}$	$H_o : \sigma^2_{F_1*F_2} = 0$	$\frac{MS_{F_1*F_2}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_3$	$MS_{F_1*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3}$	$H_o : \sigma^2_{F_1*F_3} = 0$	$\frac{MS_{F_1*F_3}}{MS_{F_1*F_2*F_3}}$
$F_2 * F_3$	$MS_{F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + ra\sigma^2_{F_2*F_3}$	$H_o : \sigma^2_{F_2*F_3} = 0$	$\frac{MS_{F_2*F_3}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_2 * F_3$	$MS_{F_1*F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3}$	$H_o : \sigma^2_{F_1*F_2*F_3} = 0$	$\frac{MS_{F_1*F_2*F_3}}{MSE}$
Error	$MS_{F_1*F_2*F_3}$	σ_e^2		

Approximate F-tests are obtained by obtaining a linear combination of Mean Squares, M, such that the expected value of M equals the expected value of the MS for corresponding source of variation, e.g., $E[M] = E_{H:_o}[MS_{F_1}]$.

1. For F_1, $* = \frac{MS_{F_1}}{M}$ where $M = MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3}$

2. For F_2, $** = \frac{MS_{F_2}}{M}$ where $M = MS_{F_1*F_2} + MS_{F_2*F_3} - MS_{F_1*F_2*F_3}$

3. For F_3, $*** = \frac{MS_{F_3}}{M}$ where $M = MS_{F_1*F_3} + MS_{F_2*F_3} - MS_{F_1*F_2*F_3}$

Model IV: F_1-Fixed, F_2-Fixed, and F_3-Random

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3} + rbcQ_{F_1}$	$H_o : Q_{F_1} = 0$	$\frac{MS_{F_1}}{MS_{F_1*F_3}}$
F_2	MS_{F_2}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + ra\sigma^2_{F_2*F_3} + racQ_{F_2}$	$H_o : Q_{F_2} = 0$	$\frac{MS_{F_2}}{MS_{F_2*F_3}}$
F_3	MS_{F_3}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3} + ra\sigma^2_{F_2*F_3} + rab\sigma^2_{F_3}$	$H_o : \sigma^2_{F_3} = 0$	$*$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rcQ_{F_1*F_2}$	$H_o : Q_{F_1*F_2} = 0$	$\frac{MS_{F_1*F_2}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_3$	$MS_{F_1*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3}$	$H_o : \sigma^2_{F_1*F_3} = 0$	$\frac{MS_{F_1*F_3}}{MS_{F_1*F_2*F_3}}$
$F_2 * F_3$	$MS_{F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + ra\sigma^2_{F_2*F_3}$	$H_o : \sigma^2_{F_2*F_3} = 0$	$\frac{MS_{F_2*F_3}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_2 * F_3$	$MS_{F_1*F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3}$	$H_o : \sigma^2_{F_1*F_2*F_3} = 0$	$\frac{MS_{F_1*F_2*F_3}}{MSE}$
Error	$MS_{F_1*F_2*F_3}$	σ_e^2		

Approximate F-tests are obtained by obtaining a linear combination of Mean Squares, M, such that the expected value of M equals the expected value of the MS for corresponding source of variation, e.g., $E[M] = E_{H:_o}[MS_{F_1}]$.

For F_3, $* = \frac{MS_{F_3}}{M}$ where $M = MS_{F_1*F_3} + MS_{F_2*F_3} - MS_{F_1*F_2*F_3}$

Expected Mean Square Rules when $n_{ijk} \equiv r$

The following rules for determining the expected mean squares apply to equally replicated designs under the set the last term equal to 0 restrictions for all fixed effects terms in the model. The rules will be illustrated for an experiment with

factors F_1 and F_2 having a and b fixed levels, respectively, and factor F_3 having c random levels.

1. Write out the linear model for the experiment:

   MODEL: $y_{ijkl} = \mu + \tau_i + \gamma_j + c_k + (\tau\gamma)_{ij} + (\tau c)_{ik} + (\gamma c)_{jk} + (\tau\gamma c)_{ijk} + e_{l(ijk)}$

   Note that the replication source of variation is nested within the ijk treatment combination.

2. Construct a two-way table with

   a. The first column containing an entry for each source of variation in the model, excluding μ, include the number of levels of each factor, and whether the factor is Fixed "F" or Random "R"

   b. A column for each random variance component, σ or fixed variance component Q

   c. Above each column write "R" if the component is a variance or "F" if the component is a fixed levels treatment difference

Factor F_1-Fixed, Factor F_2-Fixed, Factor F_3-Random

		F	F	R	F	R	R	R	R
SV	Levels	Q_{F_1}	Q_{F_2}	$\sigma^2_{F_3}$	$Q_{F_1*F_2}$	$\sigma^2_{F_1*F_3}$	$\sigma^2_{F_2*F_3}$	$\sigma^2_{F_1*F_2*F_3}$	$\sigma^2_{e(F_1,F_2,F_3)}$
F_1-F	a								
F_2-F	b								
F_3-R	c								
$F_1 * F_2$-F	ab								
$F_1 * F_3$-R	ac								
$F_2 * F_3$-R	bc								
$F_1 * F_2 * F_3$-R	abc								
Error(F_1, F_2, F_3)-R	$abcr$								

3. For each row, place the following values under each variance component:

   a. In the column for a **fixed** variance component place a 0 in all rows **except** for the row where the source of variation exactly matches the subscripts of the **fixed** variance component. For this row divide the number consisting of r times the number of levels of all factors, $rabc$ in this example, by the number of levels of the factors in the subscript of the variance component, then place the resulting number in the row of the SV that exactly matched the subscripts of the **fixed** variance component.

      E.g, For $F_1 * F_2$ where both F_1 and F_2 have fixed levels, divide $abcr$ by ab yielding the value cr. Then place cr in row for $F_1 * F_2$ under $Q_{F_1 * F_2}$

      E.g., For SV F_1, a 0 would be placed in the row for F_1 under $Q_{F_1 * F_2}$ but bcr would be placed in the row for F_1 under Q_{F_1}

   b. If the source of variation (SV) is not a part of the subscript of a variance component, then place a 0 in the row for the SV and column of that variance component.

      E.g., For SV F_1, a 0 would be placed under $\sigma_{F_2 * F_3}$ in the row for F_1

   c. If the source of variation is a part of or the complete subscript of a **random** variance component then divide the number consisting of r times the number of levels of all factors, $rabc$ in this example, by the number of levels of the factors in the subscript of the variance component, then place the resulting number in the row for that SV under the random variance component.

      E.g., For SV F_1 and random variance component $\sigma^2_{F_1 * F_3}$, divide $abcr$ by ac yielding the value br. Then place place br under $\sigma^2_{F_1 * F_3}$ in the row for F_1

   d. Place a 1 in each row of the column under $\sigma^2_e = \sigma^2_{e(F_1, F_2, F_3)}$ because all sources of variation are contained in the subscript and $abcr$ divided by $abcr$ is 1.

4. After completing all entries in the table, the expected mean square for each source of variation is obtained by multiplying the entries in the row corresponding to each source of variation by the corresponding variance component.

Factor F_1-Fixed, Factor F_2-Fixed, Factor F_3-Random

SV	Levels	F Q_{F_1}	F Q_{F_2}	R $\sigma^2_{F_3}$	F $Q_{F_1*F_2}$	R $\sigma^2_{F_1*F_3}$	R $\sigma^2_{F_2*F_3}$	R $\sigma^2_{F_1*F_2*F_3}$	R σ^2_e
F_1	a	bcr	0	0	0	br	0	r	1
F_2	b	0	acr	0	0	0	ar	r	1
F_3	c	0	0	abr	0	br	ar	r	1
$F_1 * F_2$	ab	0	0	0	cr	0	0	r	1
$F_1 * F_3$	ac	0	0	0	0	br	0	r	1
$F_2 * F_3$	bc	0	0	0	0	0	ar	r	1
$F_1 * F_2 * F_3$	abc	0	0	0	0	0	0	r	1
Error	$abcr$	0	0	0	0	0	0	0	1

SV	Expected Mean Square
F_1	$bcrQ_{F_1} + br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_2	$acrQ_{F_2} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_3	$abr\sigma^2_{F_3} + br\sigma^2_{F_1*F_3} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2$	$crQ_{F_1*F_2} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_3$	$br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_2 * F_3$	$ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2 * F_3$	$r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
Error	σ^2_e

Factor F_1-Fixed, Factor F_2-Random, Factor F_3-Random

SV	Levels	.	.	.	.	.	.	.	.
F_1									
F_2									
F_3									
$F_1 * F_2$									
$F_1 * F_3$									
$F_2 * F_3$									
$F_1 * F_2 * F_3$									
Error									

SV	Expected Mean Square
F_1	
F_2	
F_3	
$F_1 * F_2$	
$F_1 * F_3$	
$F_2 * F_3$	
$F_1 * F_2 * F_3$	
Error	

Factor F_1-Fixed, Factor F_2-Random, Factor F_3-Random

SV	Levels	F Q_{F_1}	R $\sigma^2_{F_2}$	R $\sigma^2_{F_3}$	R $\sigma^2_{F_1*F_2}$	R $\sigma^2_{F_1*F_3}$	R $\sigma^2_{F_2*F_3}$	R $\sigma^2_{F_1*F_2*F_3}$	R σ^2_e
F_1	a	bcr	0	0	cr	br	0	r	1
F_2	b	0	acr	0	cr	0	ar	r	1
F_3	c	0	0	abr	0	br	ar	r	1
$F_1 * F_2$	ab	0	0	0	cr	0	0	r	1
$F_1 * F_3$	ac	0	0	0	0	br	0	r	1
$F_2 * F_3$	bc	0	0	0	0	0	ar	r	1
$F_1 * F_2 * F_3$	abc	0	0	0	0	0	0	r	1
Error	$abcr$	0	0	0	0	0	0	0	1

SV	Expected Mean Square
F_1	$bcrQ_{F_1} + cr\sigma^2_{F_1*F_2} + br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_2	$acr\sigma^2_{F_2} + cr\sigma^2_{F_1*F_2} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
F_3	$abr\sigma^2_{F_3} + br\sigma^2_{F_1*F_3} + ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2$	$cr\sigma^2_{F_1*F_2} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_3$	$br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_2 * F_3$	$ar\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
$F_1 * F_2 * F_3$	$r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e$
Error	σ^2_e

Multiple Comparison and Contrasts in the Mixed Model

I. Two Factors F_1 and F_2

Case 1: Both Factors have Fixed Levels

Model: $y_{ijk} = \mu + \tau_i + \gamma_j + (\tau\gamma)_{ij} + e_{ijk}$ for $i = 1, \ldots, a \quad j = 1, \ldots, b \quad k = 1, \ldots, r$

Case 1a: $F_1 * F_2$ not significant

Suppose there is not significant evidence of an interaction between F_1 and F_2, then comparisons in the marginal means would be of interest: $\bar{\mu}_{i.}, \quad i = 1, \ldots, a$ and $\bar{\mu}_{.j}, \quad j = 1, \ldots, b$. This will require modifications to our formulas for testing contrasts and multiple comparisons. Suppose we want to compare means across the levels of factor $F_1 : \bar{\mu}_{i.}$'s with $\hat{\bar{\mu}}_{i.} = \bar{y}_{i..}$

$$Var(\bar{y}_{i..}) = Var(\bar{e}_{i..}) = \frac{\sigma_e^2}{br} \quad \Rightarrow \widehat{SE}(\bar{y}_{i..}) = \sqrt{\frac{MSE}{rb}}$$

$$Var(\bar{y}_{i..} - \bar{y}_{h..}) = Var(\bar{e}_{i..} - \bar{e}_{h..}) = 2Var(\bar{e}_{i..}) = \frac{2\sigma_e^2}{br} \quad \Rightarrow \widehat{SE}(\bar{y}_{i..} - \bar{y}_{h..}) = \sqrt{\frac{2MSE}{rb}}$$

Use the above in the formulas for Multiple Comparisons, Dunnett's Procedure, Hsu's Procedure, and testing contrasts where we are evaluating differences in levels of Factor F_1 averaged over the levels of factor F_2: $\bar{\mu}_{i.}, \quad i = 1, \ldots, a$

1. The F Test: State there is significant evidence that the contrast

   $C = \sum_{i=1}^{a} c_i \bar{\mu}_{i.}$ is not 0 if $\widehat{C} = \sum_{i=1}^{a} c_i \bar{y}_{i..}$ satisfies:

   $$|\widehat{C}| \geq \sqrt{MSE} \quad \sqrt{\sum_{i=1}^{a} \frac{c_i^2}{rb}} \; \sqrt{F_{\alpha,1,df_{MSE}}}$$

2. The Tukey-Kramer's HSD: State there is significant evidence that the marginal means $\bar{\mu}_{i.}$ and $\bar{\mu}_{h.}$ are different if

   $$|\bar{y}_{i..} - \bar{y}_{h..}| \geq q_{\alpha,a,df_{MSE}} \quad \sqrt{\frac{1}{2}\left(\widehat{SE}(\bar{y}_{i..} - \bar{y}_{h..})\right)^2} = q_{\alpha,a,df_{MSE}} \; \sqrt{\frac{MSE}{rb}}$$

3. The Dunnett's HSD: State there is significant evidence that the control mean $\bar{\mu}_{c.}$ is less than $\bar{\mu}_{i.}$ if

   $$(\bar{y}_{i..} - \bar{y}_{c..}) \geq d_{\alpha,a-1,df_{MSE}} \quad \sqrt{\frac{2MSE}{rb}}$$

Similar modifications must be made for the other multiple comparison procedures on the marginal means.

Case 1b: $F_1 * F_2$ significant

If there is a significant interaction, then the comparisons are across the treatment means μ_{ij} globally or across the levels of $\mu_{ij}, \quad i = 1, \dots, a$ for fixed levels of $j = 1, \dots, b$. In this case, the formulas will be modified in a similar manner

$$Var(\bar{y}_{ij.}) = Var(\bar{e}_{ij.}) = \frac{\sigma_e^2}{r} \quad \Rightarrow \widehat{SE}(\bar{y}_{ij.}) = \sqrt{\frac{MSE}{r}}$$

$$Var(\bar{y}_{ij.} - \bar{y}_{hk.}) = Var(\bar{e}_{ij.} - \bar{e}_{hk.}) = 2Var(\bar{e}_{ij.}) = \frac{2\sigma_e^2}{r} \quad \Rightarrow \widehat{SE}(\bar{y}_{ij.} - \bar{y}_{hk.}) = \sqrt{\frac{2MSE}{r}}$$

with the number of treatments being $t = ab$.

For example, suppose we want to compare the levels of factor F_1 separately at each level of factor F_2: For each value of $j = 1, \dots, b$, compare the a levels of Factor F_1:

1. The Bonferroni - F Test: For each value of $j = 1, 2, \dots, b$; State there is significant evidence that the contrast

   $C = \sum_{i=1}^{a} c_i \mu_{ij}$ is not 0 if $\widehat{C} = \sum_{i=1}^{a} c_i \bar{y}_{ij.}$ satisfies :

   $$|\widehat{C}| \geq \sqrt{MSE} \quad \sqrt{\sum_{i=1}^{a} \frac{c_i^2}{r}} \quad \sqrt{F_{\frac{\alpha}{b},1,df_{MSE}}}$$

2. The Bonferroni - Tukey's HSD: For each value of $j = 1, 2, \dots, b$; State there is significant evidence that the marginal means μ_{ij} and μ_{hj} are different if

   $$|\bar{y}_{ij.} - \bar{y}_{hj.}| \geq q_{\frac{\alpha}{b},a,df_{MSE}} \quad \sqrt{\frac{MSE}{r}}$$

   Alternatively, compare the adjusted Tukey p-values from the LSMEANS statement in the SAS output to $\alpha = .05$

   or Compare the unadjusted p-values from the LSMEANS statement in the SAS output to $\alpha_{PC} = .05/M$ where $M = ba(a-1)/2$

3. The Bonferroni - Dunnett's HSD: For each value of $j = 1, 2, \dots, b$; State there is significant evidence that the control mean $\mu_{c.}$ is less than μ_{ij} if

   $$(\bar{y}_{ij.} - \bar{y}_{c..}) \geq d_{\frac{\alpha}{b},a-1,df_{MSE}} \quad \sqrt{\frac{2MSE}{r}}$$

Similar modifications must be made for the other multiple comparison procedures on the marginal means.

Case 2: F_1 - Fixed Levels and F_2 - Random Levels

Case 2a: $F_1 * F_2$ not significant

If there is not significant evidence of an interaction between F_1 and F_2, then comparisons of differences in the marginal treatment means $\bar{\mu}_{i.}$ across the levels of F_1 are of interest:

$y_{ijk} = \mu + \tau_i + b_j + (\tau b)_{ij} + e_{ijk}$; for $i = 1, \dots, a \quad j = 1, \dots, b \quad k = 1, \dots, r$

$$\begin{aligned} Var(\bar{y}_{i..} - \bar{y}_{h..}) &= Var((\bar{b}_{.} + (\bar{\tau b})_{i.} + \bar{e}_{i..}) - (\bar{b}_{.} + (\bar{\tau b})_{h.} + \bar{e}_{h..})) \\ &= 2Var((\bar{\tau b})_{i.}) + 2Var(\bar{e}_{i..}) \\ &= \frac{2\sigma^2_{F_1*F_2}}{b} + \frac{2\sigma^2_e}{rb} = \frac{2(r\sigma^2_{F_1*F_2} + \sigma^2_e)}{rb} = \frac{2(EMS_{F_1*F_2})}{rb} \quad \Rightarrow \end{aligned}$$

Using the Method of Moments AOV estimators,
$\widehat{SE}(\bar{y}_{i..} - \bar{y}_{h..}) = \sqrt{\frac{2(MS_{F_1*F_2})}{rb}}$, with df=(a-1)(b-1)

Use the above in the formulas for Multiple Comparisons, Dunnett's Procedure, Hsu's Procedure, and testing contrasts.

1. The F Test: State there is significant evidence that the contrast

   $C = \sum_{i=1}^a c_i \bar{\mu}_{i.}$ is not 0 if $\widehat{C} = \sum_{i=1}^a c_i \bar{y}_{i..}$ satisfies :

   $|\widehat{C}| \geq \sqrt{MS_{F_1*F_2}} \ \sqrt{\sum_{i=1}^a \frac{c_i^2}{rb}} \ \sqrt{F_{\alpha,1,\nu}}$ where $\nu = df_{MS_{F_1*F_2}} = (a-1)(b-1)$
2. The Tukey-Kramer's HSD: State there is significant evidence that the marginal means $\bar{\mu}_{i.}$ and $\bar{\mu}_{h.}$ are different if

   $|\hat{\bar{\mu}}_{i.} - \hat{\bar{\mu}}_{h.}| = |\bar{y}_{i..} - \bar{y}_{h..}| \geq q_{\alpha,a,\nu} \ \sqrt{MS_{F_1*F_2}} \ \sqrt{\frac{1}{rb}}$ where $\nu = df_{MS_{F_1*F_2}} = (a-1)(b-1)$
3. The Dunnett's HSD: State there is significant evidence that the control mean $\bar{\mu}_{c.}$ is less than $\bar{\mu}_{i.}$ if

   $(\hat{\bar{\mu}}_{i.} - \hat{\bar{\mu}}_{c.}) = (\bar{y}_{i..} - \bar{y}_{c..}) \geq d_{\alpha,a-1,\nu} \ \sqrt{\frac{2MS_{F_1*F_2}}{rb}}$ where $\nu = df_{MS_{F_1*F_2}} = (a-1)(b-1)$

Similar modifications must be made for the other multiple comparison procedures on the marginal means.

Case 2b: $F_1 * F_2$ significant

If there is a significant interaction, then because F_2 has random levels, the same results that were obtained in Case 2a would be appropriate. The levels of F_1 would be compared by averaging over the levels of F_2 using the same procedures as in the case where the $F_1 * F_2$ interaction was not significant.

Case 3: F_1 - Random Levels and F_2 - Random Levels

Multiple comparisons and contrasts in either the treatment means or the marginal means are not of interest due to the random nature of the levels of both factors.

II. Three Factors F_1, F_2 and F_3

Case 1: All Factors have Random Levels

Multiple comparisons and contrasts are not appropriate due to the random nature of the levels of all three factors.

Case 2: All Factors have Fixed Levels

Straight forward modifications to the multiple comparison procedures and tests of contrasts with

A. MSE used as the estimator of σ and $\nu = df_{MSE}$ used in looking up critical values in tables

B. In the formulas, replace t with the appropriate number of means being compared

$t = abc$ for comparison of μ_{ijk}

$t = ab$ for comparisons of $\bar{\mu}_{ij.}$

$t = a$ for comparisons of $\bar{\mu}_{i..}$;

C. In the formulas, replace r or n_i with the appropriate sample size which is determined by the number of terms averaged over to obtain the point estimator. For example,

$n_i = r$ for comparisons of μ_{ijk}

$n_i = rc$ for comparisons of $\mu_{ij.}$

$n_i = rbc$ for comparisons of $\mu_{i..}$

Situation 1: When Three-way Interaction $F_1 * F_2 * F_3$ Is Significant

For each combination $(j,\ k)$ of the levels of $(F_2,\ F_3)$, conduct comparisons across the levels of F_1 using $\hat{\mu}_{ijk}$ in Tukey HSD or using Contrasts in μ_{ijk}.

Situation 2: When Three-way Interaction $F_1 * F_2 * F_3$ Is Not Significant

i. **Both $F_1 * F_2$ and $F_1 * F_3$ are Significant**

a. For each level of j of F_2 conduct comparisons of the levels of F_1 using $\hat{\mu}_{ij.}$ in Tukey HSD or using Contrasts in $\mu_{ij.}$

b. For each level of k of F_3 conduct comparisons of the levels of F_1 using $\hat{\mu}_{i.k}$ in Tukey HSD or using Contrasts in $\mu_{i.k}$

ii. **If $F_1 * F_2$ is Significant but $F_1 * F_3$ is Not Significant**

For each level of j of F_2 conduct comparisons of the levels of F_1 using $\hat{\mu}_{ij.}$ in Tukey HSD or using Contrasts in $\mu_{ij.}$

iii. **If Both $F_1 * F_2$ and $F_1 * F_3$ are Not Significant**

Conduct comparisons of the levels of F_1 using $\hat{\mu}_{i..}$ in Tukey HSD or using Contrasts in $\mu_{i..}$

Case 3: F_1 - Fixed Levels, F_2 - Random Levels and F_3 - Random Levels

Because the levels of F_2 and F_3 are random, only comparisons of differences in the levels of F_1 are of interest, that is, differences in the marginal treatment means $\bar{\mu}_{i..}$ are of interest:

$y_{ijkl} = \mu + \tau_i + b_j + c_k + (\tau b)_{ij} + (\tau c)_{ik} + (bc)_{jk} + (\tau bc)_{ijk} + e_{ijkl} \quad \Rightarrow$

$$
\begin{aligned}
Var(\bar{y}_{i...}) &= Var[(\bar{b}_. + \bar{c}_. + (\bar{\tau b})_{i.} + (\bar{\tau c})_{i.} + (\bar{bc})_{..} + (\bar{\tau bc})_{i..} + \bar{e}_{i...}) \\
&= Var(\bar{b}_.) + Var(\bar{c}_.) + Var((\bar{\tau b})_{i.}) + Var((\bar{\tau c})_{i.}) + Var((\bar{\tau bc})_{i..}) + Var(\bar{e}_{i...}) \\
&= \\
&= \frac{\sigma^2_{F_2}}{b} + \frac{\sigma^2_{F_3}}{c} + \frac{\sigma^2_{F_1*F_2}}{b} + \frac{\sigma^2_{F_1*F_3}}{c} + \frac{\sigma^2_{F_1*F_2*F_3}}{bc} + \frac{\sigma^2_e}{rbc} \\
&= \\
&= \frac{cr\sigma^2_{F_2} + br\sigma^2_{F_3} + cr\sigma^2_{F_1*F_2} + br\sigma^2_{F_1*F_3} + r\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e}{bcr} \\
&= \left(\frac{1}{abcr}\right)(EMS_{F_2} + EMS_{F_3} + (a-1)EMS_{F_1*F_2} + (a-1)EMS_{F_1*F_3} - EMS_{F_2*F_3} \\
&\quad +(1-a)EMS_{F_1*F_2*F_3}) \quad \Rightarrow
\end{aligned}
$$

Using the Method of Moments AOV estimators,

$$\widehat{SE}(\bar{y}_{i...}) = \sqrt{\left(\frac{1}{abcr}\right)(MS_{F_2} + MS_{F_3} + (a-1)MS_{F_1*F_2} + (a-1)MS_{F_1*F_3} - MS_{F_2*F_3} + (1-a)EMS_{F_1*F_2*F_3})}$$

Similarly, we compute:

$$\begin{aligned}
Var(\bar{y}_{i...} - \bar{y}_{h...}) &= Var[(\bar{b}_{.} + \bar{c}_{.} + (\bar{\tau b})_{i.} + (\bar{\tau c})_{i.} + (\bar{bc})_{..} + (\bar{\tau bc})_{i..} + \bar{e}_{i...}) \\
&\quad -(\bar{b}_{.} + \bar{c}_{.} + (\bar{\tau b})_{h.} + (\bar{\tau c})_{h.} + (\bar{bc})_{..} + (\bar{\tau bc})_{h..} + \bar{e}_{h...})] \\
&= 2Var((\bar{\tau b})_{i.}) + 2Var((\bar{\tau c})_{i.}) + 2Var((\bar{\tau bc})_{i..}) + 2Var(\bar{e}_{i...}) \\
&= \frac{2(cr\sigma^2_{F_1*F_2} + br\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e)}{bcr} \\
&= \left(\frac{2}{bcr}\right)(EMS_{F_1*F_2} + EMS_{F_1*F_3} - EMS_{F_1*F_2*F_3}) \quad \Rightarrow
\end{aligned}$$

Using the Method of Moments AOV estimators,

$$\widehat{SE}(\bar{y}_{i...} - \bar{y}_{h...}) = \sqrt{\left(\frac{2}{bcr}\right)(MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3})}$$

The *df* associated with the estimator of the difference in marginal means is given by the Satterthwaite approximation:

$$df = \frac{(MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3})^2}{\frac{(MS_{F_1*F_2})^2}{df_{MS_{F_1*F_2}}} + \frac{(MS_{F_1*F_3})^2}{df_{MS_{F_1*F_3}}} + \frac{(MS_{F_1*F_2*F_3})^2}{df_{MS_{F_1*F_2*F_3}}}}$$

Use the above in the formulas for Multiple Comparisons, Dunnett's Procedure, Hsu's Procedure, and testing contrasts.

1. The F Test: State there is significant evidence that the contrast

   $C = \sum_{i=1}^{a} c_i\bar{\mu}_{i..}$ is not 0 if if $\widehat{C} = \sum_{i=1}^{a} c_i\bar{y}_{i...}$ satisfies :

   $|\widehat{C}| \geq \sqrt{M} \quad \sqrt{\sum_{i=1}^{a} \frac{c_i^2}{rabc}} \quad \sqrt{F_{\alpha,1,\nu}}$

   $M = MS_{F_2} + MS_{F_3} + (a-1)MS_{F_1*F_2} + (a-1)MS_{F_1*F_3} - MS_{F_2*F_3} + (1-a)EMS_{F_1*F_2*F_3}$

   ν is computed from the Satterthwaite Approximation.

2. The Tukey-Kramer's HSD: State there is significant evidence that the marginal means $\bar{\mu}_{i.}$ and $\bar{\mu}_{h.}$ are different if

   $|\bar{y}_{i..} - \bar{y}_{h..}| \geq q_{\alpha,a,\nu} \quad \sqrt{MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3}} \quad \sqrt{\frac{1}{rbc}}$

   ν is given above from the Satterthwaite Approximation.

3. The Dunnett's HSD: State there is significant evidence that the control mean $\bar{\mu}_{c.}$ is less than $\bar{\mu}_{i.}$ if

$$(\bar{y}_{i..} - \bar{y}_{c..}) \geq d_{\alpha,a-1,\nu} \ \sqrt{\left(\frac{2}{rbc}\right)\left(MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3}\right)}$$

ν is given above from the Satterthwaite Approximation.

Similar modifications must be made for the other multiple comparison procedures on the marginal means.

Case 4: F_1 - Fixed Levels, F_2 - Fixed Levels and F_3 - Random Levels

Case 4a : $F_1 * F_2$ Not Significant

If there is not significant evidence of a two-way interaction $F_1 * F_2$, then comparisons of differences in the marginal treatment means $\bar{\mu}_{i..}$ across the levels of F_1 or the marginal treatment means $\bar{\mu}_{.j.}$ across the levels of F_2 are of interest:

$$y_{ijkl} = \mu + \tau_i + \gamma_j + c_k + (\tau\gamma)_{ij} + (\tau c)_{ik} + (\gamma c)_{jk} + (\tau\gamma c)_{ijk} + e_{ijkl}$$

1. Comparing the F_1 marginal treatment means $\bar{\mu}_{1..}, \ldots, \bar{\mu}_{a..}$

$$\begin{aligned}
Var(\bar{y}_{i...} - \bar{y}_{h...}) &= Var[(\bar{c}_{.} + \overline{(\tau c)}_{i.} + \overline{(\gamma c)}_{..} + \overline{(\tau\gamma c)}_{i..} + \bar{e}_{i...}) \\
&\qquad -(\bar{c}_{.} + \overline{(\tau c)}_{h.} + \overline{(\gamma c)}_{..} + \overline{(\tau\gamma c)}_{h..} + \bar{e}_{h...})] \\
&= 2Var(\overline{(\tau c)}_{i.}) + 2Var(\overline{(\tau\gamma c)}_{i..}) + 2Var(\bar{e}_{i...}) \\
&= \frac{2\sigma^2_{F_1*F_3}}{c} + \frac{2\sigma^2_{F_1*F_2*F_3}}{bc} + \frac{2\sigma^2_e}{bcr} \\
&= \frac{2(rb\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e)}{bcr} \\
&= \left(\frac{2}{bcr}\right)(EMS_{F_1*F_3}) \quad \Rightarrow
\end{aligned}$$

Using the Method of Moments AOV estimators,

$$\widehat{SE}(\bar{y}_{i...} - \bar{y}_{h...}) = \sqrt{\left(\frac{2}{bcr}\right)(MS_{F_1*F_3})}$$

with $df = df_{MS_{F_1*F_3}} = (a-1)(c-1)$

Use the above in the formulas for Multiple Comparisons, Dunnett's Procedure, Hsu's Procedure, and testing contrasts.

2. A similar argument yields:

Comparing the F_2 marginal treatment means $\bar{\mu}_{.1.}, \ldots, \bar{\mu}_{.b.}$

$$Var(\bar{y}_{.j..} - \bar{y}_{.k..}) = \left(\frac{2}{acr}\right)(EMS_{F_2*F_3}) \quad \Rightarrow$$

Using the Method of Moments AOV estimators,

$$\widehat{SE}(\bar{y}_{.j..} - \bar{y}_{.k..}) = \sqrt{\left(\frac{2}{acr}\right)(MS_{F_2*F_3})}$$

with $df = df_{MS_{F_2*F_3}} = (b-1)(c-1)$

Case 4b : $F_1 * F_2$ is Significant

If there is significant evidence of a two-way interaction $F_1 * F_2$, then comparisons of differences in the interaction means $\bar{\mu}_{ij.}$ may be of interest.

1. Comparing the treatment means $\bar{\mu}_{ij.}$ across the levels of F_1 separately at each level of F_2:

$$\bar{\mu}_{1j.}, \ldots, \bar{\mu}_{aj.} \text{ for } j = 1, \ldots, b$$

$$y_{ijkl} = \mu + \tau_i + \gamma_j + c_k + (\tau\gamma)_{ij} + (\tau c)_{ik} + (\gamma c)_{ij} + (\tau\gamma c)_{ijk} + e_{ijkl}$$

$$\begin{aligned}
Var(\bar{y}_{ij..} - \bar{y}_{hj..}) &= Var[(\bar{c}_{.} + (\bar{\tau c})_{i.} + (\bar{\gamma c})_{j.} + (\bar{\tau\gamma c})_{ij.} + \bar{e}_{ij..}) \\
&\qquad -(\bar{c}_{.} + (\bar{\tau c})_{h.} + (\bar{\gamma c})_{j.} + (\bar{\tau\gamma c})_{hj.} + \bar{e}_{hj..})] \\
&= 2Var((\bar{\tau c})_{i.}) + 2Var((\bar{\tau\gamma c})_{ij.}) + 2Var(\bar{e}_{ij..}) \\
&= \frac{2\sigma^2_{F_1*F_3}}{c} + \frac{2\sigma^2_{F_1*F_2*F_3}}{c} + \frac{2\sigma^2_e}{rc} \\
&= \frac{2(r\sigma^2_{F_1*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e)}{rc} \\
&= \left(\frac{2}{brc}\right)(EMS_{F_1*F_3} + (b-1)EMS_{F_1*F_2*F_3}) \quad \Rightarrow
\end{aligned}$$

Using the Method of Moments AOV estimators,

$$\widehat{SE}(\bar{y}_{ij..} - \bar{y}_{hj..}) = \sqrt{\left(\frac{2}{brc}\right)(MS_{F_1*F_3} + (b-1)MS_{F_1*F_2*F_3})}$$

The *df* associated with the estimator of the difference in marginal means is given by the Satterthwaite approximation:

$$df = \frac{\left(MS_{F_1*F_3} + (b-1)MS_{F_1*F_2*F_3}\right)^2}{\frac{(MS_{F_1*F_3})^2}{df_{MS_{F_1*F_3}}} + \frac{((b-1)MS_{F_1*F_2*F_3})^2}{df_{MS_{F_1*F_2*F_3}}}}$$

2. Comparing the treatment means $\bar{\mu}_{ij.}$ across the levels of F_2 separately at each level of F_1:

$$\bar{\mu}_{i1.}, \ldots, \bar{\mu}_{ib.} \text{ for } i = 1, \ldots, a$$

$$Var(\bar{y}_{ij..} - \bar{y}_{ik..}) = \left(\frac{2}{acr}\right)(EMS_{F_2*F_3} + (a-1)EMS_{F_1*F_2*F_3}) \quad \Rightarrow$$

Using the Method of Moments AOV estimators,

$$\widehat{SE}(\bar{y}_{ij..} - \bar{y}_{ik..}) = \sqrt{\left(\frac{2}{acr}\right)(MS_{F_2*F_3} + (a-1)MS_{F_1*F_2*F_3})}$$

The *df* associated with the estimator of the difference in marginal means is given by the Satterthwaite approximation:

$$df = \frac{\left(MS_{F_2*F_3} + (a-1)MS_{F_1*F_2*F_3}\right)^2}{\frac{(MS_{F_2*F_3})^2}{df_{MS_{F_2*F_3}}} + \frac{((a-1)MS_{F_1*F_2*F_3})^2}{df_{MS_{F_1*F_2*F_3}}}}$$

3. Comparing all $ab(ab-1)/2$ pairs of treatment means $\bar{\mu}_{ij.}$ versus $\bar{\mu}_{i'j'.}$:

$$\begin{aligned}
Var(\bar{y}_{ij..} - \bar{y}_{hk..}) &= Var[(\bar{c}_{.} + (\bar{\tau c})_{i.} + (\bar{\gamma c})_{j.} + (\bar{\tau\gamma c})_{ij.} + \bar{e}_{ij..}) \\
&\qquad -(\bar{c}_{.} + (\bar{\tau c})_{h.} + (\bar{\gamma c})_{k.} + (\bar{\tau\gamma c})_{hk.} + \bar{e}_{hk..})] \\
&= 2Var((\bar{\tau c})_{i.}) + 2Var((\bar{\gamma c})_{j.}) + 2Var((\bar{\tau\gamma c})_{ij.}) + 2Var(\bar{e}_{ij..}) \\
&= \frac{2\sigma^2_{F_1*F_3}}{c} + \frac{2\sigma^2_{F_2*F_3}}{c} + \frac{2\sigma^2_{F_1*F_2*F_3}}{c} + \frac{2\sigma^2_e}{rc} \\
&= \frac{2(r\sigma^2_{F_1*F_3} + r\sigma^2_{F_2*F_3} + r\sigma^2_{F_1*F_2*F_3} + \sigma^2_e)}{rc} \\
&= \left(\frac{2}{abrc}\right)(a\ EMS_{F_1*F_3} + b\ EMS_{F_2*F_3} + (ab-b-a)\ EMS_{F_1*F_2*F_3}) \quad \Rightarrow
\end{aligned}$$

Using the Method of Moments AOV estimators,

$$\widehat{SE}(\bar{y}_{ij..} - \bar{y}_{hk..}) = \sqrt{\left(\frac{2}{abcr}\right)(a\ MS_{F_1*F_3} + b\ MS_{F_2*F_3} + (ab-b-a)\ MS_{F_1*F_2*F_3})}$$

The *df* associated with the estimator of the difference in marginal means is given by the Satterthwaite approximation:

$$df = \frac{(a\ MS_{F_1*F_3} + b\ MS_{F_2*F_3} + (ab-b-a)\ MS_{F_1*F_2*F_3})^2}{\frac{(a\ MS_{F_1*F_3})^2}{df_{MS_{F_1*F_3}}} + \frac{(b\ MS_{F_2*F_3})^2}{df_{MS_{F_2*F_3}}} + \frac{((ab-b-a)\ MS_{F_1*F_2*F_3})^2}{df_{MS_{F_1*F_2*F_3}}}}$$

Estimation of Variance Components

I. Using AOV-Method of Moments

The AOV-MOM estimation procedure simply equates observed Mean Squares with Expected Values of the Mean Squares and solves for all relevant variance components. These estimators have the limitation that was discussed previously in Handout # 6.

For example, suppose we have a mixed model with factors F_1 and F_2 random. We then have three variance components to estimate: $\sigma^2_{F_2}$, $\sigma^2_{F_1*F_2}$, and σ^2_e. From the AOV table we have the expected Mean Squares as follows:

F_1-Fixed, F_2-Random

Source of Variation	MS	Expected MS
F_1	MS_{F_1}	$\sigma^2_e + r\sigma^2_{F_1*F_2} + rbQ_{F_1}$
F_2	MS_{F_2}	$\sigma^2_e + r\sigma^2_{F_1*F_2} + ra\sigma^2_{F_2}$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma^2_e + r\sigma^2_{F_1*F_2}$
Error	MSE	σ^2_e

Equating Expected MS to MS from the above table, the estimators are obtained as follows:

$$MSE = \hat{\sigma}^2_e \quad \Rightarrow \quad \hat{\sigma}^2_e = MSE$$

$$MS_{F_1*F_2} = \hat{\sigma}^2_e + r\hat{\sigma}^2_{F_1*F_2} \quad \Rightarrow \quad \hat{\sigma}^2_{F_1*F_2} = \frac{MS_{F_1*F_2} - \hat{\sigma}^2_e}{r} = \frac{MS_{F_1*F_2} - MSE}{r}$$

$$MS_{F_2} = \hat{\sigma}^2_e + r\hat{\sigma}^2_{F_1*F_2} + ra\hat{\sigma}^2_{F_2} \quad \Rightarrow \quad \hat{\sigma}^2_{F_2} = \frac{MS_{F_2} - (\hat{\sigma}^2_e + r\hat{\sigma}^2_{F_1*F_2})}{ra} = \frac{MS_{F_2} - MS_{F_1*F_2}}{ra}$$

To proportionally allocate the variability in the responses σ^2_y, first compute $\sigma^2_y = Var(y_{ijk}) = \sigma^2_{F_1*F_2} + \sigma^2_{F_2} + \sigma^2_e$, then estimate the proportional allocation as follows:

Variance Component	Proportion of Total
F_2	$\frac{\hat{\sigma}^2_{F_2}}{\hat{\sigma}^2_y}$
$F_1 * F_2$	$\frac{\hat{\sigma}^2_{F_1*F_2}}{\hat{\sigma}^2_y}$
$Error$	$\frac{\hat{\sigma}^2_e}{\hat{\sigma}^2_y}$

II. Likelihood Based Estimators

The mixed model is written as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$$

where $\mathbf{y}$ denotes the vector of observed responses, $\mathbf{X}$ is the known Design matrix related to the fixed factor effects, $\boldsymbol{\beta}$ is vector of fixed-effects parameters, $\mathbf{Z}$ is the known Design matrix related to the random effects and $\mathbf{u}$ is a vector of random-effects. $\mathbf{X}$ contains indicator variables constructed from the fixed effects and $\mathbf{Z}$ contains indicator variables constructed from random effects. Finally, $\mathbf{e}$ is the unobserved vector of independent and identically distributed Gaussian random errors.

Assume that $\mathbf{u}$ and $\mathbf{e}$ are independent Gaussian random variables that are uncorrelated and have expectations 0 and variance-covariance matrices $\mathbf{G}$ and $\mathbf{R}$, respectively:

$$E\begin{bmatrix}\mathbf{u}\\ \mathbf{e}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} \qquad\qquad Var\begin{bmatrix}\mathbf{u}\\ \mathbf{e}\end{bmatrix} = \begin{bmatrix}\mathbf{G} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}$$

The above conditions yield that $\mathbf{y}$ has a Gaussian distribution with

$$E[\mathbf{y}] = \mathbf{X}\boldsymbol{\beta} \qquad\qquad Var[\mathbf{y}] = \mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R}$$

where $\mathbf{G}$ contains the variance components and $\mathbf{R} = \sigma_e^2\mathbf{I}_n$

In the mixed model it is necessary to estimate not only $\boldsymbol{\beta}$ but also the unknown parameters in $\mathbf{G}$ and $\mathbf{R}$. Because the responses y are no longer independent, that is, $\mathbf{V}$ is no longer a diagonal matrix with σ_e^2 along the diagonal, Least Squares estimation is no longer the best method of estimation. For the estimation of $\boldsymbol{\beta}$, Generalized Least Squares (GLS) is the more appropriate method of estimation, that is, find $\hat{\boldsymbol{\beta}}$ which minimizes

$$(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'\mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \quad \Rightarrow \quad \hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}(\mathbf{X}'\mathbf{V}^{-1})\mathbf{y}$$

The use of GLS forces the researcher to supply the values of $\mathbf{V}$, hence the values for $\mathbf{G}$ and $\mathbf{R}$. In most situations, the values for $\mathbf{G}$ and $\mathbf{R}$ are unknown. An alternative is to use estimated GLS, in which efficient estimators of $\mathbf{G}$ and $\mathbf{R}$ are used. This then provides estimates but not very efficient estimates of $\boldsymbol{\beta}$.

In most settings, the better approach is to use **likelihood-based** estimation procedures under the condition that $\mathbf{u}$ and $\mathbf{e}$ are restricted to have Gaussian distributions, a restriction that is not required in LSE and GLE. However, likelihood-based estimators yield more efficient estimators in the cases where $\mathbf{V}$ does not equal $\sigma_e^2\mathbf{I}_n$

Matrix Representation of Mixed Model

Mixed model with factor A having 4 fixed levels, factor B having 3 random levels, and 2 reps :

$$Y_{ijk} = \mu + \tau_i + b_j + (\tau b)_{ij} + e_{ijk}; \quad i = 1,2,3,4; \quad j = 1,2,3; \quad k = 1,2$$

with $\tau_4 = 0$, b_j iid $N(0, \sigma_B^2)$, $(\tau b)_{ij}$ iid $N(0, \sigma_{A*B}^2)$, e_{ijk} iid $N(0, \sigma_e^2)$, all r.v.'s are independent
In matrix form, the above model can be represented as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\epsilon},$$

where $\mathbf{Y}$ - responses, $\mathbf{X}$ - design matrix fixed effects, $\mathbf{Z}$ - design matrix random effects, $\boldsymbol{\beta}$ - fixed effects parameters, $\mathbf{u}$ - random effects, and $\boldsymbol{\epsilon}$ - residuals.

$$\mathbf{Y} = \begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \\ Y_{231} \\ Y_{232} \\ Y_{311} \\ Y_{312} \\ Y_{321} \\ Y_{322} \\ Y_{331} \\ Y_{332} \\ Y_{411} \\ Y_{412} \\ Y_{421} \\ Y_{422} \\ Y_{431} \\ Y_{432} \end{bmatrix}_{24x1} \qquad \mathbf{X} = \begin{bmatrix} 1_6 & 1_6 & 0_6 & 0_6 \\ 1_6 & 0_6 & 1_6 & 0_6 \\ 1_6 & 0_6 & 0_6 & 1_6 \\ 1_6 & 0_6 & 0_6 & 0_6 \end{bmatrix}_{24x4} \qquad \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}_{4x1} \qquad \mathbf{e} = \begin{bmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{131} \\ e_{132} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \\ e_{231} \\ e_{232} \\ e_{311} \\ e_{312} \\ e_{321} \\ e_{322} \\ e_{331} \\ e_{332} \\ e_{411} \\ e_{412} \\ e_{421} \\ e_{422} \\ e_{431} \\ e_{432} \end{bmatrix}_{24x1}$$

$$\mathbf{Z} = \begin{bmatrix} 1_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 1_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 \\ 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 \\ 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 & 0_2 \\ 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 & 0_2 \\ 0_2 & 0_2 & 1_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 0_2 & 1_2 \end{bmatrix}_{24x15} \qquad \mathbf{u} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ (\tau b)_{11} \\ (\tau b)_{12} \\ (\tau b)_{13} \\ (\tau b)_{21} \\ (\tau b)_{22} \\ (\tau b)_{23} \\ (\tau b)_{31} \\ (\tau b)_{32} \\ (\tau b)_{33} \\ (\tau b)_{41} \\ (\tau b)_{42} \\ (\tau b)_{43} \end{bmatrix}_{15x1}$$

Maximum Likelihood Estimation:

Select $\hat{\boldsymbol{\beta}}$, $\hat{\mathbf{G}}$, and $\hat{\mathbf{R}}$ to maximize the likelihood $L(\boldsymbol{\beta}, \mathbf{G}, \mathbf{R})$ or

minimize $l(\boldsymbol{\beta}, \mathbf{G}, \mathbf{R}) = -2log(L(\boldsymbol{\beta}, \mathbf{G}, \mathbf{R}))$:

$$l(\boldsymbol{\beta}, \mathbf{G}, \mathbf{R}) = nlog(2\pi) + log|\mathbf{V}| + (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})' \mathbf{V}^{-1}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})$$

where $\mathbf{V} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R}$.

Typically, obtaining the MLE involves repeated iteration between obtaining an estimate of $\boldsymbol{\beta}$ and then an estimate of $\mathbf{V}$ (for example, Newton-Raphson algorithm). There are serious numerical challenges in obtaining the estimates when $\mathbf{V}$ is of a complex form. The MLE's are biased estimators and the size of the bias can be serious in the case of several random factors. An alternative procedure is the restricted maximum likelihood estimators:

Restricted Maximum Likelihood Estimates (REML):

The -2log-likelihood function is partitioned into two components. The first component does not involve $\boldsymbol{\beta}$:

$$(n-r)log(2\pi) + log(|\mathbf{V}|) + log(|\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}|) + (n-r)log(\mathbf{P}'\mathbf{V}^{-1}\mathbf{P})$$

where r is the rank of

$$\mathbf{P} = \mathbf{y} - \mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\mathbf{y}$$

Maximizing the above yields the REML estimator of the variance components, $\mathbf{V}$, $\hat{\mathbf{V}}$. This involves an iteration procedure as in the MLE.

The value of $\hat{\mathbf{V}}$ is then used to obtain an estimator of the fixed effects parameters, $\boldsymbol{\beta}$:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{X})^{-1}\mathbf{X}'\hat{\mathbf{V}}^{-1}\mathbf{y}$$

and for the random effects $\mathbf{u}$

$$\hat{\mathbf{u}} = \hat{\mathbf{G}}\mathbf{Z}'\hat{\mathbf{V}}^{-1}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})$$

Statistical Properties of REML's

If $\mathbf{G}$ and $\mathbf{R}$ are known then $\hat{\boldsymbol{\beta}}$ is the Best Linear Unbiased Estimator (BLUE) of $\boldsymbol{\beta}$ and $\hat{\mathbf{u}}$ is the Best Linear Unbiased Predictor (BLUP) of $\mathbf{u}$, where best means minimum mean square error. The problem is that $\mathbf{G}$ and $\mathbf{R}$ are generally unknown. In this case, asymptotic properties of the estimators are used.

In most situations, REML are the preferred estimators of variance components. This is the default option in PROC MIXED when using SAS to estimate variance components and test hypotheses in mixed models.

RESIDUALS FROM MIXED MODEL

With the mixed model written as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$$

there are a number of ways of considering the residuals from the mixed model. The **marginal and conditional means** from this model are respectively,

$$E[\mathbf{y}] = \mathbf{X}\boldsymbol{\beta} \qquad \text{and} \qquad E[\mathbf{Y}|\mathbf{u}] = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u}$$

There are two possible sets of residuals depending on whether we use the marginal or conditional residuals.

The **Marginal Residual Vector** is given by

$$\mathbf{r}_m = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}$$

The **Conditional Residual Vector** is given by

$$\mathbf{r}_c = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}} - \mathbf{Z}\hat{\mathbf{u}} = \mathbf{r}_m - \mathbf{Z}\hat{\mathbf{u}}$$

From the above two definitions and considering two different approaches to standardizing the residuals we obtain the following six possible residuals from the fitted mixed model:

Residual Type	Marginal	Conditional
Raw	$r_{mi} = y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}$	$r_{ci} = y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}} - \mathbf{z}_i'\hat{\mathbf{u}}$
Studentized	$rs_{mi} = \frac{r_{mi}}{\widehat{SE}(r_{mi})}$	$rs_{ci} = \frac{r_{ci}}{\widehat{SE}(r_{ci})}$
Pearson	$rp_{mi} = \frac{r_{mi}}{\widehat{SE}(y_i)}$	$rp_{ci} = \frac{r_{ci}}{\widehat{SE}(y_i|\mathbf{u})}$

Which of the residuals is more effective in assessing model conditions is open to debate. Both the Studentized residual adjusts the residual for its estimated variance and hence is a more meaningful measure of model fit than is the Raw residuals. The Pearson residual selects a slightly different value for scaling the residuals than the value selected in the Studentized residuals. The choice may depend on computational ease more than any other reason.

APPROXIMATE F-TESTS

In a number of the experiments involving random factor levels that we have discussed, the form of the F-test for testing the null hypothesis that a variance component is 0 is not evident after examining the Expected Mean Squares. For example, consider a three factor experiment with the levels F_1-Fixed, F_2-Random, and F_3-Random. We obtained the Expected Mean Squares but there was no exact F-test for testing the variance components corresponding to the main effects of the three factors:

Model III: F_1-Fixed, F_2-Random, and F_3-Random

Source of Variation	MS	Expected MS	Null Hypothesis	F-Statistic
F_1	MS_{F_1}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2} + rb\sigma^2_{F_1*F_3} + rbcQ_{F_1}$	$H_o : Q_{F_1} = 0$	$*$
F_2	MS_{F_2}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2} + ra\sigma^2_{F_2*F_3} + rac\sigma^2_{F_2}$	$H_o : \sigma^2_{F_2} = 0$	$**$
F_3	MS_{F_3}	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3} + ra\sigma^2_{F_2*F_3} + rab\sigma^2_{F_3}$	$H_o : \sigma^2_{F_3} = 0$	$***$
$F_1 * F_2$	$MS_{F_1*F_2}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2}$	$H_o : \sigma^2_{F_1*F_2} = 0$	$\frac{MS_{F_1*F_2}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_3$	$MS_{F_1*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3}$	$H_o : \sigma^2_{F_1*F_3} = 0$	$\frac{MS_{F_1*F_3}}{MS_{F_1*F_2*F_3}}$
$F_2 * F_3$	$MS_{F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + ra\sigma^2_{F_2*F_3}$	$H_o : \sigma^2_{F_2*F_3} = 0$	$\frac{MS_{F_2*F_3}}{MS_{F_1*F_2*F_3}}$
$F_1 * F_2 * F_3$	$MS_{F_1*F_2*F_3}$	$\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3}$	$H_o : \sigma^2_{F_1*F_2*F_3} = 0$	$\frac{MS_{F_1*F_2*F_3}}{MSE}$
Error	$MS_{F_1*F_2*F_3}$	σ_e^2		

There are two solutions for obtaining the F-tests:

1. **Solution I: Use the Satterthwaite Approximation:**

   Approximate F-tests are obtained by obtaining a linear combination of Mean Squares, M, such that the expected value of M equals the expected value of the MS, under H_o, for the corresponding source of variation, e.g., $E[M] = E_{H_o}[MS_{F_1}]$.

   1. For F_1, $* = \frac{MS_{F_1}}{M}$ where $M = MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3}$

      $$\begin{aligned} E[M] &= EMS_{F_1*F_2} + EMS_{F_1*F_3} - EMS_{F_1*F_2*F_3} \\ &= (\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2}) + (\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rb\sigma^2_{F_1*F_3}) - (\sigma_e^2 + r\sigma^2_{F_1*F_2*F_3}) \\ &= \sigma_e^2 + r\sigma^2_{F_1*F_2*F_3} + rc\sigma^2_{F_1*F_2} + rb\sigma^2_{F_1*F_3} \\ &= E_{H_o}[MS_{F_1}] \end{aligned}$$

      The degrees of freedom for the F-test are (a-1) and ν, where ν is obtained using the Satterthwaite Aproximation:

      $$df = \frac{(MS_{F_1*F_2} + MS_{F_1*F_3} - MS_{F_1*F_2*F_3})^2}{\frac{(MS_{F_1*F_2})^2}{df_{MS_{F_1*F_2}}} + \frac{(MS_{F_1*F_3})^2}{df_{MS_{F_1*F_3}}} + \frac{(MS_{F_1*F_2*F_3})^2}{df_{MS_{F_1*F_2*F_3}}}}$$

      In a similar fashion, obtain the other two F-tests.

   2. For F_2, $** = \frac{MS_{F_2}}{M}$ where $M = MS_{F_1*F_2} + MS_{F_2*F_3} - MS_{F_1*F_2*F_3}$

   3. For F_3, $*** = \frac{MS_{F_3}}{M}$ where $M = MS_{F_1*F_3} + MS_{F_2*F_3} - MS_{F_1*F_2*F_3}$

2. **Solution II: Pool Mean Squares for Non-significant Interactions:**

In the situation just encountered with F_1 fixed levels and F_2, F_3 having random levels, suppose our test of $H_o : \sigma_{F_1*F_2*F_3} = 0$ is not rejected.

The test of $H_o : \sigma_{F_1*F_2} = 0$ versus $H_1 : \sigma_{F_1*F_2} \neq 0$ raises a question: "Should we consider $\sigma_{F_1*F_2*F_3} = 0$ and hence use the F-test $F = \frac{MS_{F_1*F_2}}{MSE}$ or take into account that a Type II error may have occurred and use the F-test $F = \frac{MS_{F_1*F_2}}{MS_{F_1*F_2*F_3}}$?

Bancroft(1964) *Biometrics*, pp. 427-442, suggests the following solution: Consider the general situation:

we have 3 variance components: $\sigma_1, \sigma_2, \sigma_3$ with associated Mean Squares: MS_1, MS_2, MS_3 having $EMS_1 = \sigma_1^2, EMS_2 = \sigma_1^2 + \sigma_2^2, EMS_3 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$.

We want to test $H_o : \sigma_3 = 0$ versus $H_1 : \sigma_3 \neq 0$. However, based on an earlier test $\sigma_2 = 0$ was not rejected. Thus, should we use $\frac{MS_3}{MS_2}$ or $\frac{MS_3}{MS_{pooled}}$ as the test statistic,

where $MS_{pooled} = \frac{df_1 MS_1 + df_2 MS_2}{df_1 + df_2}$?

(a) Test $H_o : \sigma_2 = 0$ versus $H_1 : \sigma_2 \neq 0$ using $F = \frac{MS_2}{MS_1}$ at level α_1

(b) If $H_o : \sigma_2 = 0$ is rejected then use $F = \frac{MS_3}{MS_2}$ at level α_2 to test $H_o : \sigma_3 = 0$ versus $H_1 : \sigma_3 \neq 0$

(c) If $H_o : \sigma_2 = 0$ is not rejected then use $F = \frac{MS_3}{MS_{pooled}}$ at level α_3 to test $H_o : \sigma_3 = 0$ versus $H_1 : \sigma_3 \neq 0$

Question: What are the appropriate values for α_1, α_2, and α_3, so that the Probability of a Type I error for testing $H_o : \sigma_3 = 0$ versus $H_1 : \sigma_3 \neq 0$ is α?

The exact value of α depends on **8** values:

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad df_{MS_1}, \quad df_{MS_2}, \quad df_{MS_3}, \quad \theta_1 = \frac{\sigma_2^2}{\sigma_1^2}, \quad \theta_2 = \frac{\sigma_2^2}{\sigma_3^2}$$

The problem is that only $df_{MS_1}, df_{MS_2}, df_{MS_3}$ are known values. The values of $\alpha_1, \alpha_2, \alpha_3$ are restricted because we want the value of $P[TypeIError] = \alpha$ for testing $H_o : \sigma_{F_1*F_2} = 0$ versus $H_1 : \sigma_{F_1*F_2} \neq 0$.

The paper by Bancroft suggests the following options:

Option 1: When $\theta_1 = \frac{\sigma_2^2}{\sigma_1^2}$ is small, say, $1 \leq \theta_1 \leq 2$,

then select $\alpha_1 = .50$ if $df_{MS_3} \geq df_{MS_2}$ and $df_{MS_1} \geq 5df_2$

otherwise, select $\alpha_1 = .25$.

In both cases, then, select $\alpha_2 = \alpha_3 = \alpha$.

Option 2: If θ_1 is large or totally unknown, select $\alpha_1 = .01$ and use $\alpha_2 = \alpha_3 = \alpha$.

Option 3: If θ_1 is large or totally unknown, always use $F = \frac{MS_3}{MS_2}$ at level α.

In our example with F_1 having fixed levels, F_2, F_3 having random levels,

1. Test $H_o : \sigma_{F_1*F_2*F_3} = 0$ versus $H_1 : \sigma_{F_1*F_2*F_3} \neq 0$ using $F = \frac{MS_{F_1*F_2*F_3}}{MSE}$ at level α_1
2. If $H_o : \sigma_{F_1*F_2*F_3} = 0$ is rejected then use $F = \frac{MS_{F_1*F_2}}{MS_{F_1*F_2*F_3}}$ at level α_2 to test $H_o : \sigma_{F_1*F_2} = 0$ versus $H_1 : \sigma_{F_1*F_2} \neq 0$
3. If $H_o : \sigma_{F_1*F_2*F_3} = 0$ is not rejected then use $F = \frac{MS_{F_1*F_2}}{MS_{pooled}}$ at level α_3 to test $H_o : \sigma_{F_1*F_2} = 0$ versus $H_1 : \sigma_{F_1*F_2} \neq 0$

Question: What are the appropriate values for α_1, α_2, and α_3, so that the Probability of a Type I error for testing $H_o : \sigma_{F_1*F_2} = 0$ versus $H_1 : \sigma_{F_1*F_2} \neq 0$ is α?

Option 1: When $\theta_1 = \frac{\sigma^2_{F_1*F_2*F_3}}{\sigma_e^2}$ is small, say, $1 \leq \theta_1 \leq 2$,

then select $\alpha_1 = .50$ if $df_{MS_{F_1*F_2}} \geq df_{MS_{F_1*F_2*F_3}}$ and $df_{MSE} \geq 5df_{F_1*F_2*F_3}$

otherwise, select $\alpha_1 = .25$.

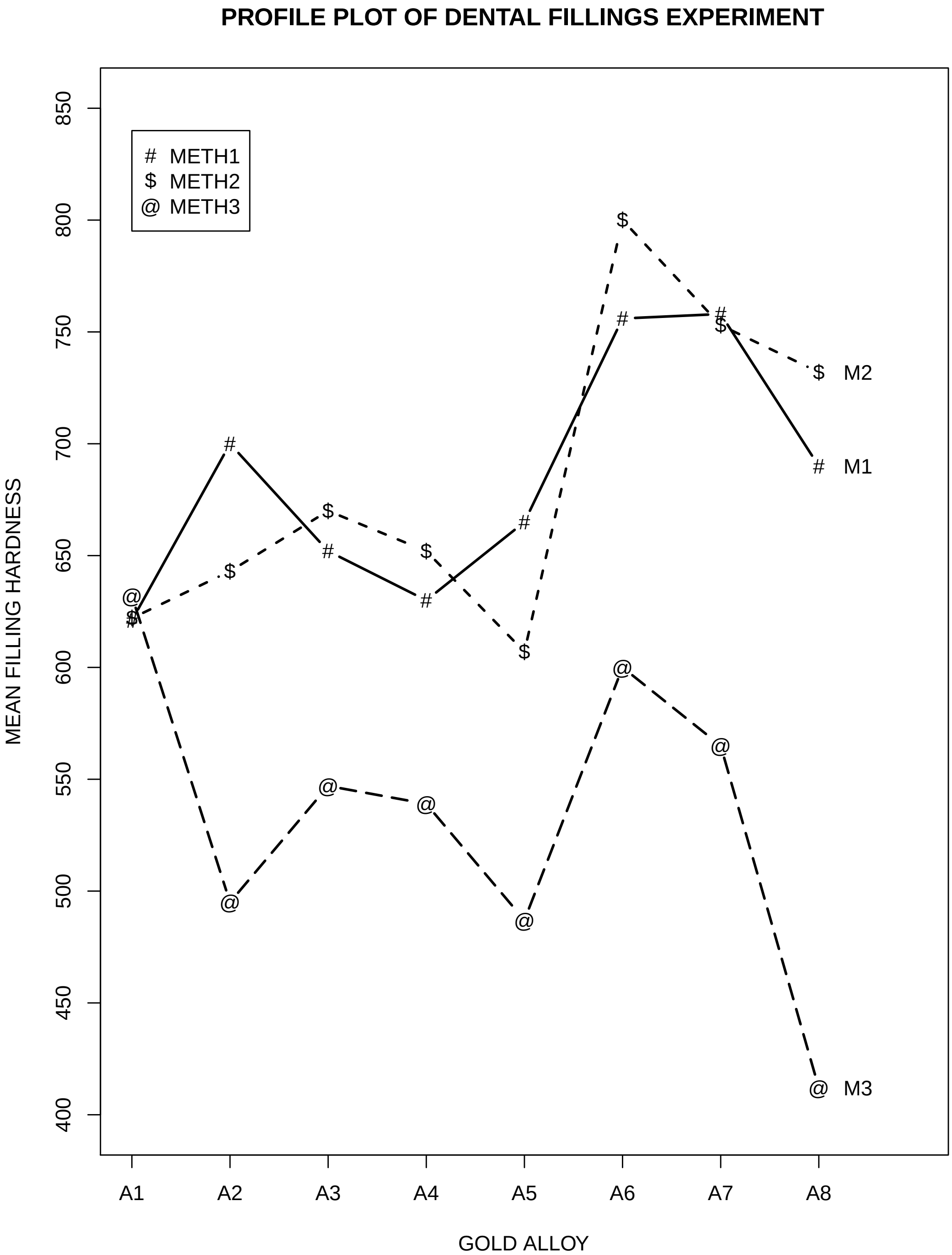
In both cases, then, select $\alpha_2 = \alpha_3 = \alpha$.

Option 2: If θ_1 is large or totally unknown, select $\alpha_1 = .01$ and use $\alpha_2 = \alpha_3 = \alpha$.

Option 3: If θ_1 is large or totally unknown, use $F = \frac{MS_{F_1*F_2}}{F_1*F_2*F_3}$ at level α.

Example of the Analysis of a Mixed Factors Experiment

We will now illustrate the analysis of the Dental Fillings Experiment in which we had three factors - one with random levels and two factors with fixed levels. The following SAS program and SAS output will be used to assess the significance of the three factors: Dentist, Condensation Method, and Gold Alloy on the hardness of the fillings. The following plot illustrates the relationship between type of alloy and condensation method:



```
* random3factors_mixed.sas;
ODS HTML; ODS GRAPHICS ON;
option ls=80 ps=50 nocenter nodate;
TITLE 'AOV - MIXED FACTOR LEVELS';
DATA RAW;
INPUT D $ M $ @@;
DO G = 1 TO 8;
INPUT Y @@; OUTPUT; END;
LABEL G='GOLD ALLOW' D = 'DENTIST'  M = 'CONDENSATION METHOD';
cards;
1           1    792 824 813 792 792 907  792 835
 1           2    772 772 782 698 665 1115 835 870
 1           3    782 803 752 620 835 847 560 585
 2           1    803 803 715 803 813 858 907 882
 2           2    752 772 772 782 743 933 792 824
 2           3    715 707 835 715 673 698 734 681
 3           1    715 724 743 627 752 858 762 724
 3           2    792 715 813 743 613 824 847 782
 3           3    762 606 743 681 743 715 824 681
 4           1    673 946 792 743 762 894 792 649
 4           2    657 743 690 882 772 813 870 858
 4           3    690 245 493 707 289 715 813 312
 5           1    634 715 707 698 715 772 1048 870
 5           2    649 724 803 665 752 824 933 835
 5           3    724 627 421 483 405 536 405 312
RUN;
PROC MIXED METHOD=REML CL;
CLASS D M G;
MODEL Y = M G M*G;
RANDOM D D*M D*G/ CL ALPHA=.05;
LSMEANS M G M*G/CL ADJUST=TUKEY;
RUN;
proc glimmix data=raw;
CLASS D M G;
MODEL Y = M G M*G;
lsmeans M G M*G /  plot = meanplot;
run;
ODS GRAPHICS OFF; ODS HTML CLOSE;
```

SAS OUTPUT:

AOV - MIXED FACTOR LEVELS

The Mixed Procedure

Class	Levels	Values
D	5	1 2 3 4 5
M	3	1 2 3
G	8	1 2 3 4 5 6 7 8

Number of Observations Read	120
Number of Observations Used	120

Covariance Parameter Estimates

Cov Parm	Estimate	Alpha	Lower	Upper
D	894.69	0.05	133.33	4.1733E8
D*M	2973.69	0.05	1082.22	23204
D*G	0	.	.	.
Residual	9132.04	0.05	6895.68	12671

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
M	2	8	9.08	0.0088
G	7	28	3.45	0.0087
M*G	14	56	1.64	0.0964

Least Squares Means

Effect	CONDEN METHOD	GOLD	Est.	St.Err	DF	t	Pr>|t|	95% C. I. Lower	95% C. I. Upper
M	1		786.15	31.654	8	24.84	<.0001	713.16	859.14
M	2		786.95	31.654	8	24.86	<.0001	713.96	859.94
M	3		636.85	31.654	8	20.12	<.0001	563.86	709.84
G		1	727.47	31.4004	28	23.17	<.0001	663.15	791.79
G		2	715.07	31.4004	28	22.77	<.0001	650.75	779.39
G		3	724.93	31.4004	28	23.09	<.0001	660.61	789.25
G		4	709.27	31.4004	28	22.59	<.0001	644.95	773.59
G		5	688.27	31.4004	28	21.92	<.0001	623.95	752.59
G		6	820.6	31.4004	28	26.13	<.0001	756.28	884.92
G		7	794.27	31.4004	28	25.29	<.0001	729.95	858.59
G		8	713.33	31.4004	28	22.72	<.0001	649.01	777.65
M*G	1	1	723.4	50.991	56	14.19	<.0001	621.25	825.55
M*G	1	2	802.4	50.991	56	15.74	<.0001	700.25	904.55
M*G	1	3	754.0	50.991	56	14.79	<.0001	651.85	856.15
M*G	1	4	732.6	50.991	56	14.37	<.0001	630.45	834.75
M*G	1	5	766.8	50.991	56	15.04	<.0001	664.65	868.95
M*G	1	6	857.8	50.991	56	16.82	<.0001	755.65	959.95
M*G	1	7	860.2	50.991	56	16.87	<.0001	758.05	962.35
M*G	1	8	792.0	50.991	56	15.53	<.0001	689.85	894.15
M*G	2	1	724.4	50.991	56	14.21	<.0001	622.25	826.55
M*G	2	2	745.2	50.991	56	14.61	<.0001	643.05	847.35
M*G	2	3	772.0	50.991	56	15.14	<.0001	669.85	874.15
M*G	2	4	754.0	50.991	56	14.79	<.0001	651.85	856.15
M*G	2	5	709.0	50.991	56	13.90	<.0001	606.85	811.15
M*G	2	6	901.8	50.991	56	17.69	<.0001	799.65	1003.95
M*G	2	7	855.4	50.991	56	16.78	<.0001	753.25	957.55
M*G	2	8	833.8	50.991	56	16.35	<.0001	731.65	935.95
M*G	3	1	734.6	50.991	56	14.41	<.0001	632.45	836.75
M*G	3	2	597.6	50.991	56	11.72	<.0001	495.45	699.75
M*G	3	3	648.8	50.991	56	12.72	<.0001	546.65	750.95
M*G	3	4	641.2	50.991	56	12.57	<.0001	539.05	743.35
M*G	3	5	589.0	50.991	56	11.55	<.0001	486.85	691.15
M*G	3	6	702.2	50.991	56	13.77	<.0001	600.05	804.35
M*G	3	7	667.2	50.991	56	13.08	<.0001	565.05	769.35
M*G	3	8	514.2	50.991	56	10.08	<.0001	412.05	616.35

Differences of Least Squares Means

Effect	CONDENSATION METHOD	GOLD ALLOY	CONDENSATION METHOD	GOLD ALLOW	Adjustment	Adj P	Alpha
M	1		2		Tukey-Kramer	0.9998	0.05
M	1		3		Tukey-Kramer	0.0153	0.05
M	2		3		Tukey-Kramer	0.0148	0.05
G		1		2	Tukey-Kramer	1.0000	0.05
G		1		3	Tukey-Kramer	1.0000	0.05
G		1		4	Tukey-Kramer	0.9994	0.05
G		1		5	Tukey-Kramer	0.9460	0.05
G		1		6	Tukey-Kramer	0.1742	0.05
G		1		7	Tukey-Kramer	0.5530	0.05
G		1		8	Tukey-Kramer	0.9999	0.05
G		2		3	Tukey-Kramer	1.0000	0.05
G		2		4	Tukey-Kramer	1.0000	0.05
G		2		5	Tukey-Kramer	0.9935	0.05
G		2		6	Tukey-Kramer	0.0856	0.05
G		2		7	Tukey-Kramer	0.3441	0.05
G		2		8	Tukey-Kramer	1.0000	0.05
G		3		4	Tukey-Kramer	0.9998	0.05
G		3		5	Tukey-Kramer	0.9617	0.05
G		3		6	Tukey-Kramer	0.1517	0.05
G		3		7	Tukey-Kramer	0.5076	0.05
G		3		8	Tukey-Kramer	1.0000	0.05
G		4		5	Tukey-Kramer	0.9986	0.05
G		4		6	Tukey-Kramer	0.0597	0.05
G		4		7	Tukey-Kramer	0.2637	0.05
G		4		8	Tukey-Kramer	1.0000	0.05
G		5		6	Tukey-Kramer	0.0146	0.05
G		5		7	Tukey-Kramer	0.0832	0.05
G		5		8	Tukey-Kramer	0.9957	0.05
G		6		7	Tukey-Kramer	0.9942	0.05
G		6		8	Tukey-Kramer	0.0770	0.05
G		7		8	Tukey-Kramer	0.3186	0.05

Effect	CONDENSATION METHOD	GOLD ALLOY	CONDENSATION METHOD	GOLD ALLOW	Adjustment	Adj P	Alpha
M*G	1	1	1	2	Tukey-Kramer	0.9996	0.05
M*G	1	1	1	3	Tukey-Kramer	1.0000	0.05
M*G	1	1	1	4	Tukey-Kramer	1.0000	0.05
M*G	1	1	1	5	Tukey-Kramer	1.0000	0.05
M*G	1	1	1	6	Tukey-Kramer	0.8414	0.05
M*G	1	1	1	7	Tukey-Kramer	0.8203	0.05
M*G	1	1	1	8	Tukey-Kramer	1.0000	0.05
M*G	1	2	1	3	Tukey-Kramer	1.0000	0.05
M*G	1	2	1	4	Tukey-Kramer	0.9999	0.05
M*G	1	2	1	5	Tukey-Kramer	1.0000	0.05
M*G	1	2	1	6	Tukey-Kramer	1.0000	0.05
M*G	1	2	1	7	Tukey-Kramer	1.0000	0.05
M*G	1	2	1	8	Tukey-Kramer	1.0000	0.05
M*G	1	3	1	4	Tukey-Kramer	1.0000	0.05
M*G	1	3	1	5	Tukey-Kramer	1.0000	0.05
M*G	1	3	1	6	Tukey-Kramer	0.9857	0.05
M*G	1	3	1	7	Tukey-Kramer	0.9816	0.05
M*G	1	3	1	8	Tukey-Kramer	1.0000	0.05
M*G	1	4	1	5	Tukey-Kramer	1.0000	0.05
M*G	1	4	1	6	Tukey-Kramer	0.9086	0.05
M*G	1	4	1	7	Tukey-Kramer	0.8932	0.05
M*G	1	4	1	8	Tukey-Kramer	1.0000	0.05
M*G	1	5	1	6	Tukey-Kramer	0.9972	0.05
M*G	1	5	1	7	Tukey-Kramer	0.9961	0.05
M*G	1	5	1	8	Tukey-Kramer	1.0000	0.05
M*G	1	6	1	7	Tukey-Kramer	1.0000	0.05
M*G	1	6	1	8	Tukey-Kramer	1.0000	0.05
M*G	1	7	1	8	Tukey-Kramer	1.0000	0.05
M*G	2	1	2	2	Tukey-Kramer	1.0000	0.05
M*G	2	1	2	3	Tukey-Kramer	1.0000	0.05
M*G	2	1	2	4	Tukey-Kramer	1.0000	0.05
M*G	2	1	2	5	Tukey-Kramer	1.0000	0.05
M*G	2	1	2	6	Tukey-Kramer	0.3691	0.05
M*G	2	1	2	7	Tukey-Kramer	0.8688	0.05
M*G	2	1	2	8	Tukey-Kramer	0.9746	0.05
M*G	2	2	2	3	Tukey-Kramer	1.0000	0.05
M*G	2	2	2	4	Tukey-Kramer	1.0000	0.05
M*G	2	2	2	5	Tukey-Kramer	1.0000	0.05
M*G	2	2	2	6	Tukey-Kramer	0.6069	0.05
M*G	2	2	2	7	Tukey-Kramer	0.9726	0.05
M*G	2	2	2	8	Tukey-Kramer	0.9981	0.05

Effect	CONDENSATION METHOD	GOLD ALLOY	CONDENSATION METHOD	GOLD ALLOW	Adjustment	Adj P	Alpha
M*G	2	3	2	4	Tukey-Kramer	1.0000	0.05
M*G	2	3	2	5	Tukey-Kramer	1.0000	0.05
M*G	2	3	2	6	Tukey-Kramer	0.8777	0.05
M*G	2	3	2	7	Tukey-Kramer	0.9992	0.05
M*G	2	3	2	8	Tukey-Kramer	1.0000	0.05
M*G	2	4	2	5	Tukey-Kramer	1.0000	0.05
M*G	2	4	2	6	Tukey-Kramer	0.7084	0.05
M*G	2	4	2	7	Tukey-Kramer	0.9891	0.05
M*G	2	4	2	8	Tukey-Kramer	0.9996	0.05
M*G	2	5	2	6	Tukey-Kramer	0.2287	0.05
M*G	2	5	2	7	Tukey-Kramer	0.7238	0.05
M*G	2	5	2	8	Tukey-Kramer	0.9111	0.05
M*G	2	6	2	7	Tukey-Kramer	1.0000	0.05
M*G	2	6	2	8	Tukey-Kramer	1.0000	0.05
M*G	2	7	2	8	Tukey-Kramer	1.0000	0.05
M*G	3	1	3	2	Tukey-Kramer	0.8185	0.05
M*G	3	1	3	3	Tukey-Kramer	0.9988	0.05
M*G	3	1	3	4	Tukey-Kramer	0.9961	0.05
M*G	3	1	3	5	Tukey-Kramer	0.7325	0.05
M*G	3	1	3	6	Tukey-Kramer	1.0000	0.05
M*G	3	1	3	7	Tukey-Kramer	1.0000	0.05
M*G	3	1	3	8	Tukey-Kramer	0.0796	0.05
M*G	3	2	3	3	Tukey-Kramer	1.0000	0.05
M*G	3	2	3	4	Tukey-Kramer	1.0000	0.05
M*G	3	2	3	5	Tukey-Kramer	1.0000	0.05
M*G	3	2	3	6	Tukey-Kramer	0.9844	0.05
M*G	3	2	3	7	Tukey-Kramer	1.0000	0.05
M*G	3	2	3	8	Tukey-Kramer	0.9992	0.05
M*G	3	3	3	4	Tukey-Kramer	1.0000	0.05
M*G	3	3	3	5	Tukey-Kramer	1.0000	0.05
M*G	3	3	3	6	Tukey-Kramer	1.0000	0.05
M*G	3	3	3	7	Tukey-Kramer	1.0000	0.05
M*G	3	3	3	8	Tukey-Kramer	0.8397	0.05
M*G	3	4	3	5	Tukey-Kramer	1.0000	0.05
M*G	3	4	3	6	Tukey-Kramer	1.0000	0.05
M*G	3	4	3	7	Tukey-Kramer	1.0000	0.05
M*G	3	4	3	8	Tukey-Kramer	0.8972	0.05
M*G	3	5	3	6	Tukey-Kramer	0.9639	0.05
M*G	3	5	3	7	Tukey-Kramer	0.9997	0.05
M*G	3	5	3	8	Tukey-Kramer	0.9998	0.05
M*G	3	6	3	7	Tukey-Kramer	1.0000	0.05
M*G	3	6	3	8	Tukey-Kramer	0.2680	0.05
M*G	3	7	3	8	Tukey-Kramer	0.6492	0.05

Nested Treatment Factors

Experiments in which the treatments involve nested factors arise in certain factorial arrangements of the levels of the factors or in experiments in which the randomization of Experimental Units to the treatments are restrictive. In many cases, these restrictions on the randomization are dictated by technical considerations of the implementation of the treatment assignments or methods by which the responses are obtained. For example, in an experiment in which the optimal amount of fertilizer to be applied for several different varieties of wheat is being explored, it may be possible to hand plant the different varieties in small rows in order to achieve a very high degree of precision, whereas the application of the fertilizer must be done using a machinery that is only appropriate for a large field.

Definition: Two factors F_1 with a levels and F_2 with b levels are said to be **crossed** if the physical properties of the b levels of F_2 remain the same for all levels of F_1.

Example: An experiment is designed to study the factors which affect the time to fabricate an automobile part from a specimen of metal. Three factors are identified:

1. F_1- Type of Alloy: A1, A2, A3
2. F_2- Porosity of Alloy: L, M, H
3. F_3- Amount of Lubricant Used in Cutting Machine: L1, L2, L3, L4

A complete $3 \times 3 \times 4$ factorial experiment with all 3 factors crossed would have the levels of F_2 exactly the same for all levels of both factors F_1 and F_3. Also, the levels of F_3 would be the same for all levels of F_1 and F_2. That is, the three Porosity levels would be the same for all three types of Alloys.

Definition: Factor F_2 is said to be **nested within** the levels of factor F_1 if the physical properties of the levels of factor F_2 vary depending on which level of factor F_1 is used.

Example: For many Alloys, the possible porosity levels may vary. For example, suppose that the three possible porosity levels are given as follows:

	Porosity Level		
Alloy Type	Low	Medium	High
A1	.2	.3	.4
A2	.3	.5	.7
A3	.5	.8	.95

Thus, the definition of L, M, H for the factor Porosity varies depending of the level of the factor Alloy. Thus, factor P, porosity is Nested Within the levels of factor A, alloy type. P nested within A is denoted as P(A).

Example: A company is designing a new product. There are 3 producers of raw material. The goal of a study was to evaluate the consistency of the raw material's physical properties across multiply batches of material from each producer and then variation within each batch. The experiment was designed to randomly select 6 batches of material from each of the 3 producers. From each of the 18 batches, 4 samples are taken and the physical properties are determined in the company's lab. The factors are

F_1- Producer: P1, P2, P3

F_2- Batch: B1, B2, B3, B4, B5, B6

F_3- Sample: S1, S2, S3, S4

The levels of Batch only having meaning if we know from which Producer the batches are selected. Thus, the factor Batch is nested within the levels of factor Producer: B(P). Similarly, the levels of Sample only having meaning if we know from which producer and batch the sample is selected. Thus, the factor Sample is nested within both the levels of factor Producer and the factor Batch: S(B,P).

Example: A social scientist is studying the use of drugs by high school students. The factors of interest are Type of High School (private-nonreligious, private-religious,public); Particular Schools within each Type of School; Individual Students from each School with each Type of School.

Example: An animal scientist is studying the level of infestation of ticks on cattle. There are four breeds of cattle; five cows of each Breed; measurements are taken at four locations on each cow: head, neck, flank, and tail.

Construction of Hierarchically Nested Designs

Suppose we have factor F_1 having fixed levels, F_2 nested within the levels of F_1 having random levels, and F_3 nested within the levels of Factor F_2 having random levels.

1. List the factors to be included in the experiment:

   F_1, F_2, F_3

2. Determine the hierarchy of the factors:

   F_2 nested within the levels of F_1

   F_3 nested within the levels of F_2

3. Determine whether factors are fixed or random

   F_1 - Fixed; F_2 - Random F_3 - Random

4. Randomly select, if possible, an equal number of levels for each factor within the levels of the factor for which it is nested

   Randomly select b levels of F_2 within each of the a levels of F_1

   Randomly select c levels of F_3 within each of the b levels of F_2

5. Randomize the run order or the assignment of the factor-level combinations of the EU's

Example The following example from Snedecor and Cochran's book, *Statistical Methods*, describes an experiment in which the variability of calcium concentration in turnip green leaves is investigated. Four turnip green plants were randomly selected. From each plant, three leaves were randomly selected. Each leaf was then processed into two 100-mg samples from which a determination of the calcium content was made. Thus, we have three factors: Plant (P) with four random levels, Leaf (L) with three random levels within each level of P, and Determination (D) with two random levels within each level of L. The data is given below:

		Determination				
Plant	Leaf	1	2	$\bar{y}_{ij.}$	$\bar{y}_{i..}$	$\bar{y}_{...}$
1	1	3.28	3.09	3.185		
	2	3.52	3.48	3.500		
	3	2.88	2.80	2.845	3.175	
2	1	2.46	2.44	2.450		
	2	1.87	1.92	1.895		
	3	2.19	2.19	2.190	2.178	
3	1	2.77	2.66	2.715		
	2	3.74	3.44	3.590		
	3	2.55	2.55	2.550	2.952	
4	1	3.78	3.87	3.825		
	2	4.07	4.12	4.095		
	3	3.31	3.31	3.310	3.743	3.012

The formulas for the sum of squares decomposition in a nested treatment structure with three hierarchically nested factors with levels $F_1 : i = 1, \ldots, a;\ \ F_2(F_1) : j = 1, \ldots, r_i;\ \ F_3(F_1, F_2) : k = 1, \ldots, n_{ij}$ (In many cases, $F_3(F_1, F_2)$ is the error term.) are given here:

1. Total SS: $SS_{TOT} = \sum_{i=1}^{a} \sum_{j=1}^{r_i} \sum_{k=1}^{n_{ij}} (y_{ijk} - \bar{y}_{...})^2$

2. SS due to F_1: $SS_{F_1} = \sum_{i=1}^{a} n_{i.} (\bar{y}_{i..} - \bar{y}_{...})^2$

   with $df = (a - 1)$

3. SS due to F_2 nested within F_1: $SS_{F_2(F_1)} = \sum_{i=1}^{a} \sum_{j=1}^{r_i} n_{ij} (\bar{y}_{ij.} - \bar{y}_{i..})^2$

   with $df = \sum_{i=1}^{a} (r_i - 1)$

4. SS due to F_3 nested within $F_2(F_1)$: $SS_{F_3(F_1,F_2)} = \sum_{i=1}^{a} \sum_{j=1}^{r_i} \sum_{k=1}^{n_{ij}} (y_{ijk} - \bar{y}_{ij.})^2$

   with $df = \sum_{i=1}^{a} \sum_{j=1}^{r_i} (n_{ij} - 1)$

For our example, we have the following calculations: ($a = 4, r_i = r = 3, n_{ij} = n = 2$)

$$\begin{aligned}
SS_{TOT} &= \sum_{i=1}^{a}\sum_{j=1}^{r}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{...})^2 \\
&= \sum_{i=1}^{4}\sum_{j=1}^{3}\sum_{k=1}^{2}(y_{ijk} - 3.012)^2 \\
&= [(3.28 - 3.012)^2 + (3.09 - 3.012)^2 + (3.52 - 3.012)^2 + \cdots + (3.31 - 3.012)^2] \\
&= 10.27035 \quad \text{with} \quad df = 24 - 1 = 23
\end{aligned}$$

$$\begin{aligned}
SS_{F_1} &= \sum_{i=1}^{a} n(\bar{y}_{i..} - \bar{y}_{...})^2 \\
&= (3)(2)[(3.175 - 3.012)^2 + (2.178 - 3.012)^2 + (2.952 - 3.012)^2 + (3.743 - 3.012)^2] \\
&= 7.5603 \quad \text{with} \quad df = 4 - 1 = 3
\end{aligned}$$

$$\begin{aligned}
SS_{F_2(F_1)} &= \sum_{i=1}^{a}\sum_{j=1}^{r} n(\bar{y}_{ij.} - \bar{y}_{i..})^2 \\
&= 2[(3.185 - 3.175)^2 + (3.50 - 3.175)^2 + (2.845 - 3.175)^2 \\
&\quad +(2.45 - 2.178)^2 + (1.895 - 2.178)^2 + (2.19 - 2.178)^2 \\
&\quad +(2.715 - 2.952)^2 + (3.59 - 2.952)^2 + (2.55 - 2.952)^2 \\
&\quad +(3.825 - 3.743)^2 + (4.095 - 3.743)^2 + (3.31 - 3.743)^2] \\
&= 2.6302 \quad \text{with} \quad df = 4(3 - 1) = 8
\end{aligned}$$

$$\begin{aligned}
SS_{F_3(F_1,F_2)} &= \sum_{i=1}^{a}\sum_{j=1}^{r}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij.})^2 \\
&= [(3.28 - 3.185)^2 + (3.09 - 3.185)^2 + (3.52 - 3.50)^2 + (3.48 - 3.50)^2 \\
&\quad +(2.88 - 2.84)^2 + (2.80 - 2.84)^2 + \cdots + (4.07 - 4.095)^2 + (4.12 - 4.095)^2 \\
&\quad +(3.31 - 3.31)^2 + (3.31 - 3.31)^2] \\
&= 0.07985 \quad \text{with} \quad df = (4)(3)(2 - 1) = 12
\end{aligned}$$

Which tests of hypotheses and multiply comparisons to conduct depend on whether the factors are fixed or random.

Case 1: F_1-Fixed Levels, F_2 fixed levels within the levels of F_1,

F_3 random levels within the levels of F_2

Model: $y_{ijk} = \mu + \tau_i + \beta_{j(i)} + c_{k(ij)} \quad i = 1, \ldots, a; \quad j = 1, \ldots, b; \quad k = 1, \ldots, n_{ij}$
with model conditions:

1. $\tau_a = 0; \quad \beta_{r_i(i)} = 0 \quad$ for all $\quad i = 1, \ldots, a$

2. $c_{j(i,j)}$'s are iid $N(0, \sigma^2_{F_3(F_1,F_2)})$

Expected Mean Squares are given in the following table for the balanced case ($r_i = b, n_{ij} = n$):

Source	df	Expected Mean Squares
F_1	$a-1$	$nbQ_{F_1} + \sigma^2_{F_3(F_1,F_2)}$
$F_2(F_1)$	$a(b-1)$	$nQ_{F_2(F_1)} + \sigma^2_{F_3(F_1,F_2)}$
$F_3(F_1,F_2)$	$ab(n-1)$	$\sigma^2_{F_3(F_1,F_2)}$
Total	abn-1	

From the above table, we can determine the appropriate test statistics:

1. To Test no differences across the levels of Factor F_1:

   $H_o : \mu_1 = \mu_2 = \cdots = \mu_a$

   $F = \frac{MS_{F_1}}{MS_{F_3(F_1,F_2)}}$

2. If H_o is rejected, then construct contrasts and multiple comparisons across the levels of F_1

3. To Test no differences across the levels of Factor F_2 nested within the levels of F_1:

   $H_o : \mu_{1(i)} = \mu_{2(i)} = \cdots = \mu_{b(i)}$ for all $i = 1, \ldots, a$

   $F = \frac{MS_{F_2(F_1)}}{MS_{F_3(F_1,F_2)}}$

4. If H_o is rejected, then construct contrasts and multiple comparisons across the levels of F_2 separately for each level of F_1

Case 2: F_1-Fixed Levels, F_2 random levels within the levels of F_1,

F_3 random levels within the levels of F_2

Model: $y_{ijk} = \mu + \tau_i + b_{j(i)} + c_{k(ij)} \quad i = 1, \ldots, a; \quad j = 1, \ldots, b; \quad k = 1, \ldots, n_{ij}$

with model conditions:

1. $\tau_a = 0$

2. $b_{j(i)}$ are iid $N(0, \sigma^2_{F_2(F_1)})$ r.v.'s independent of $c_{j(i,j)}$'s which are iid $N(0, \sigma^2_{F_3(F_1,F_2)})$ r.v.'s

Expected Mean Squares are given in the following table for the balanced case ($r_i = b, n_{ij} = n$):

Source	df	Expected Mean Squares
F_1	$a-1$	$nbQ_{F_1} + n\sigma^2_{F_2(F_1)} + \sigma^2_{F_3(F_1,F_2)}$
$F_2(F_1)$	$a(b-1)$	$n\sigma^2_{F_2(F_1)} + \sigma^2_{F_3(F_1,F_2)}$
$F_3(F_1,F_2)$	$ab(n-1)$	$\sigma^2_{F_3(F_1,F_2)}$
Total	abn-1	

From the above table, we can determine the appropriate test statistics:

1. To Test no differences across the levels of Factor F_1:

   $H_o : \mu_1 = \mu_2 = \cdots = \mu_a$

   $F = \frac{MS_{F_1}}{MS_{F_2(F_1)}}$

2. If H_o is rejected, then construct contrasts and multiple comparisons across the levels of F_1

3. To Test no differences across the levels of Factor F_2 nested within the levels of F_1:

   $H_o : \sigma_{F_2(F_1)} = 0$

   $F = \frac{MS_{F_2(F_1)}}{MS_{F_3(F_1,F_2)}}$

4. If H_o is rejected, no further comparisons are relevant

Case 3: F_1-Random Levels, F_2 random levels within the levels of F_1,
F_3 random levels within the levels of F_2

Model: $y_{ijk} = \mu + a_i + b_{j(i)} + c_{k(ij)} \quad i = 1, \ldots, a; \quad j = 1, \ldots, b; \quad k = 1, \ldots, n_{ij}$
with model conditions:

1. a_i are iid $N(0, \sigma^2_{F_1})$ r.v.'s
2. $b_{j(i)}$ are iid $N(0, \sigma^2_{F_2(F_1)})$ r.v.'s ; $c_{j(i,j)}$'s which are iid $N(0, \sigma^2_{F_3(F_1,F_2)})$ r.v.'s
3. a_i's, $b_{j(i)}$'s; and $c_{j(i,j)}$'s are all independent

Expected Mean Squares are given in the following table for the balanced case ($r_i = b, n_{ij} = n$):

Source	df	Expected Mean Squares
F_1	$a-1$	$nb\sigma^2_{F_1} + n\sigma^2_{F_2(F_1)} + \sigma^2_{F_3(F_1,F_2)}$
$F_2(F_1)$	$a(b-1)$	$n\sigma^2_{F_2(F_1)} + \sigma^2_{F_3(F_1,F_2)}$
$F_3(F_1,F_2)$	$ab(n-1)$	$\sigma^2_{F_3(F_1,F_2)}$
Total	abn-1	

From the above table, we can determine the appropriate test statistics:

1. To Test no differences across the levels of Factor F_1:

   $H_o : \sigma_{F_1} = 0$

   $F = \frac{MS_{F_1}}{MS_{F_2(F_1)}}$

2. If H_o is rejected, no further comparison are relevant

3. To Test no differences across the levels of Factor F_2 nested within the levels of F_1:

   $H_o : \sigma_{F_2(F_1)} = 0$

   $F = \frac{MS_{F_2(F_1)}}{MS_{F_3(F_1,F_2)}}$

4. If H_o is rejected, no further comparisons are relevant

The SAS code needed to analyze our Calcium Content Example is given here: Plant, Leaf(Plant), and Determination are all random.

```
*nested_equalreps.sas;
* This is an example from Snedecor and Cochran of a
Nested Design involving an experiment
to investigate the variability of calcium concentration
in the leaves of turnip greens.
Four plants were randomly selected
and then three leaves were randomly
selected from each plant. Two 100-mg samples
from each leaf. The amount of calcium is determined
by microchemical methods;

ods html;  ods graphics on;

options pagesize=55 linesize=80;
Title 'Nested Design - Equal Sample Sizes';
data raw;
input plant leaf sample X @@;
cards;
1 1 1  3.28  1 1 2  3.09  1 2 1  3.52  1 2 2  3.48
1 3 1  2.88  1 3 2  2.80
2 1 1  2.46  2 1 2  2.44  2 2 1  1.87  2 2 2  1.92
2 3 1  2.19  2 3 2  2.19
3 1 1  2.77  3 1 2  2.66  3 2 1  3.74  3 2 2  3.44
3 3 1  2.55  3 3 2  2.55
4 1 1  3.78  4 1 2  3.87  4 2 1  4.07  4 2 2  4.12
4 3 1  3.31  4 3 2  3.31

;
proc glm;
class plant leaf sample;
model X = plant leaf(plant);
random plant leaf(plant)/test;
run;

*anaylsis using proc mixed;

proc mixed cl alpha=.05 COVTEST;
class plant leaf sample;
model X = /residuals;
random plant leaf(plant);
run;
ods graphics off;  ods html close;
```

SAS OUTPUT

```
                  Nested Design - Equal Sample Sizes
                          The GLM Procedure

                       Class Level Information

                   Class         Levels    Values

                   plant              4    1 2 3 4
                   leaf               3    1 2 3
                   sample             2    1 2

               Number of Observations Read          24
               Number of Observations Used          24

Dependent Variable: X

                                          Sum of
 Source                      DF          Squares     Mean Square    F Value    Pr > F
 Model                       11      10.19054583      0.92641326     139.22    <.0001
 Error                       12       0.07985000      0.00665417
 Corrected Total             23      10.27039583

 Source                      DF      Type III SS     Mean Square    F Value    Pr > F

 plant                        3       7.56034583      2.52011528     378.73    <.0001
 leaf(plant)                  8       2.63020000      0.32877500      49.41    <.0001


   Source                   Type III Expected Mean Square

   plant                    Var(Error) + 2 Var(leaf(plant)) + 6 Var(plant)
   leaf(plant)              Var(Error) + 2 Var(leaf(plant))

          Tests of Hypotheses for Random Model Analysis of Variance

 Source                      DF      Type III SS     Mean Square    F Value    Pr > F
 plant                        3         7.560346        2.520115       7.67    0.0097
 Error                        8         2.630200        0.328775
 Error: MS(leaf(plant))


 Source                      DF      Type III SS     Mean Square    F Value    Pr > F
 leaf(plant)                  8         2.630200        0.328775      49.41    <.0001
 Error: MS(Error)            12         0.079850        0.006654
```

The Mixed Procedure

Covariance Structure	Variance Components
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
plant	4	1 2 3 4
leaf	3	1 2 3
sample	2	1 2

Number of Observations

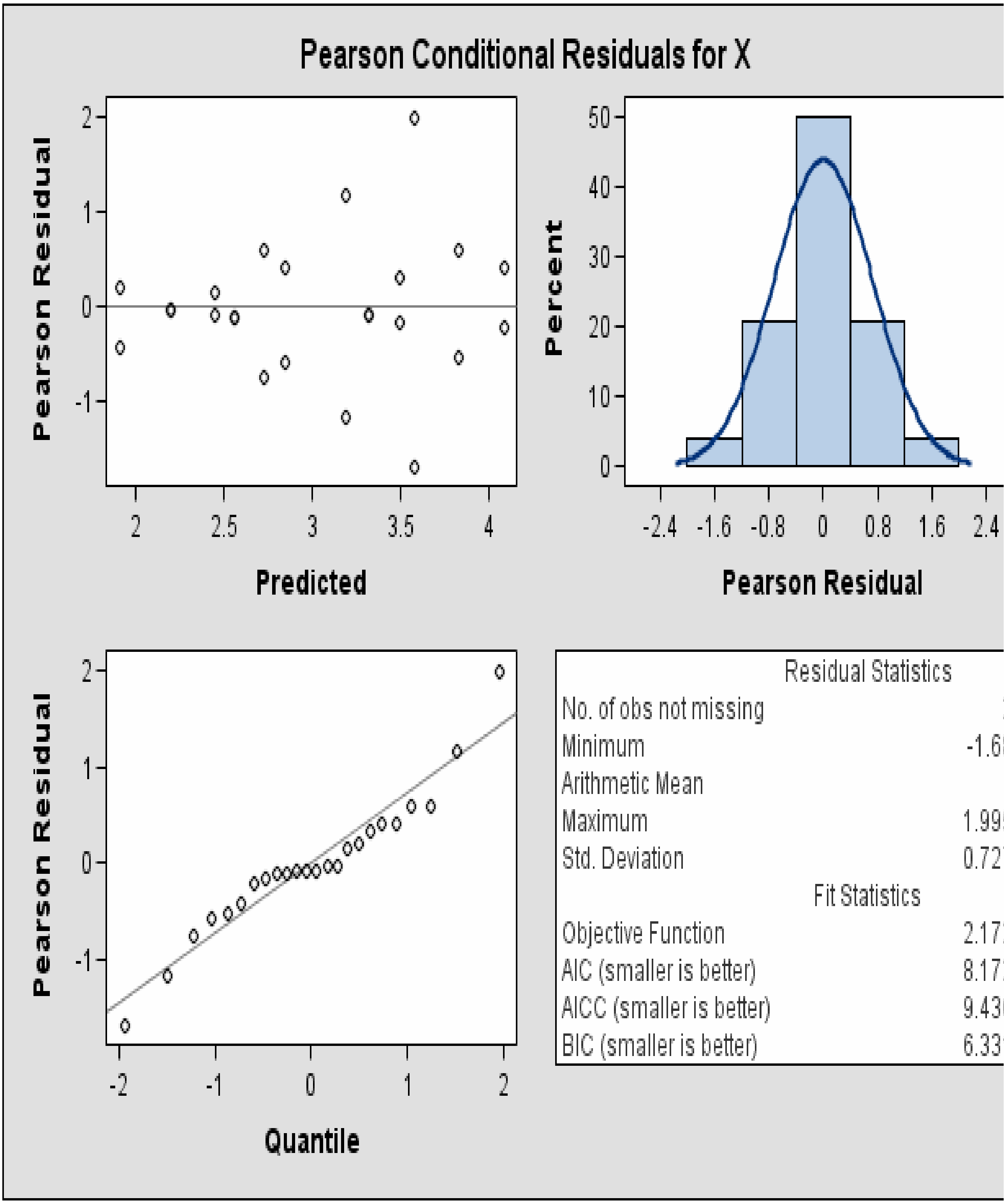
Number of Observations Read	24
Number of Observations Used	24
Number of Observations Not Used	0

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z	Alpha	Lower	Upper
plant	0.3652	0.3440	1.06	0.1442	0.05	0.1042	10.1188
leaf(plant)	0.1611	0.08220	1.96	0.0250	0.05	0.07253	0.6127
Residual	0.006654	0.002717	2.45	0.0072	0.05	0.003422	0.01813

Fit Statistics

-2 Res Log Likelihood	2.2
AIC (smaller is better)	8.2
AICC (smaller is better)	9.4
BIC (smaller is better)	6.3


Pearson Conditional Residuals for X
Pearson Residual
Predicted
Percent
Pearson Residual
Pearson Residual
Quantile
Residual Statistics
No. of obs not missing
Minimum
Arithmetic Mean
Maximum
Std. Deviation
Fit Statistics
Objective Function
AIC (smaller is better)
AICC (smaller is better)
BIC (smaller is better)

Unbalanced Nested Design

Example: A study of the variation of the tensile strength of fiber optic cable is to be designed. There are three major manufacturers (M) of the cable. From each manufacturer, large rolls (R) of wire are randomly selected. From each roll, random samples (S) of 12 inch lengths of wire are taken. The tensile strength of each of the 12 in lengths of wire is then determined in a laboratory.

The factors are M - Fixed levels, R(M) - Random levels, S(M,R) - Random levels.

The data is given here:

	Manufacturer (M)							
	M_1			M_2			M_3	
R(M)	1	2	3	1	2	3	1	2
	110	130	50	130	45	120	100	130
y_{ijk}	90	115	75	45	55	50	200	80
	120	105	85	50	65	150	90	70
			40	40			70	80
							90	150
$\bar{y}_{ij.}$	106.67	116.67	62.5	66.25	55	106.67	110	102
n_{ij}	3	3	4	4	3	3	5	5
r_i		10		10			10	
$\bar{y}_{i..}$		92.0		75.0			106.0	
$\bar{y}_{...}$				91.0				

For our example, we have the following calculations:

$$
\begin{aligned}
SS_{TOT} &= \sum_{i=1}^{a}\sum_{j=1}^{r_i}\sum_{k=1}^{n_{ij}}(y_{ijk}-\bar{y}_{...})^2 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{r_i}\sum_{k=1}^{n_{ij}}(y_{ijk}-91.0)^2 \\
&= [(110-91.0)^2+(90-91.0)^2+(120-91.0)^2+\cdots+(150-91.0)^2] \\
&= 44170.0 \quad \text{with} \quad df = 30-1=29
\end{aligned}
$$

$$
\begin{aligned}
SS_M &= \sum_{i=1}^{a} n_{i.}(\bar{y}_{i..}-\bar{y}_{...})^2 \\
&= (10)[(92.0-91.0)^2]+(10)[(75.0-91.0)^2]+(10)[106.0-91.0)^2] \\
&= 4820.0 \quad \text{with} \quad df = 3-1=2
\end{aligned}
$$

$$
\begin{aligned}
SS_{R(M)} &= \sum_{i=1}^{a}\sum_{j=1}^{r_i} n_{ij}(\bar{y}_{ij.} - \bar{y}_{i..})^2 \\
&= (3)(106.67 - 92.0)^2 + (3)(116.67 - 92.0)^2 + (4)(62.5 - 92.0)^2 \\
&\quad +(4)(66.25 - 75.0)^2 + (3)(55.0 - 75.0)^2 + (3)(106.67 - 75)^2 \\
&\quad +(5)(110 - 106)^2 + (5)(102 - 106)^2 \\
&= 10626.25 \quad \text{with} \quad df = (3-1) + (3-1) + (2-1) = 5
\end{aligned}
$$

$$
\begin{aligned}
SS_{S(M,R)} &= \sum_{i=1}^{a}\sum_{j=1}^{r_i}\sum_{k=1}^{n_{ij}} (y_{ijk} - \bar{y}_{ij.})^2 \\
&= [(110 - 106.67)^2 + (90 - 106.67)^2 + \cdots + (80 - 102)^2 + (150 - 102)^2 \\
&= 28723.75 \quad \text{with} \quad df = 29 - (2+5) = 22
\end{aligned}
$$

The SAS code to analyze the above experiment is given here:

```
* nested_unequalreps.sas;
*This is an example of a nested experiment with
MAN-fixed factor, ROLL and SAMPLE-random factors.
There is a unequal number of levels in the nested
factors. Three different methods of analysis are
considered.;
ods html; ods graphics on;
option ls=80 ps=55 nocenter nodate;
title 'NESTED DESIGN - UNEQUAL SAMPLE SIZES';
data strength;
INPUT MAN $ ROLL SAMPLE Y @@;
LABEL MAN = 'MANUFACTURER' Y= 'STRENGTH';
cards;
M1 1 1 110 M1 1 2  90 M1 1 3 120
M1 2 1 130 M1 2 2 115 M1 2 3 105
M1 3 1  50 M1 3 2  75 M1 3 3  85 M1 3 4 40
M2 1 1 130 M2 1 2  45 M2 1 3  50 M2 1 4 40
M2 2 1  45 M2 2 2  55 M2 2 3  65
M2 3 1 120 M2 3 2  50 M2 3 3 150
M3 1 1 100 M3 1 2 200 M3 1 3  90 M3 1 4 70 M3 1 5 90
M3 2 1 130 M3 2 2  80 M3 2 3  70 M3 2 4 80 M3 2 5 150
RUN;

TITLE ANALYSIS USING PROC GLM;
PROC GLM;
CLASS MAN ROLL SAMPLE;
MODEL  Y = MAN ROLL(MAN)/SSI E1;
RANDOM ROLL(MAN)/TEST;
LSMEANS MAN/STDERR PDIFF ADJUST=TUKEY;
RUN;

TITLE ANALYSIS USING PROC MIXED-REML;
PROC MIXED METHOD=REML CL ALPHA=.05 COVTEST;
CLASS MAN ROLL SAMPLE;
MODEL Y = MAN/RESIDUALS;
RANDOM ROLL(MAN);
LSMEANS MAN/ADJUST=TUKEY;
RUN;

ods graphics off; ods html close;
```

```
NESTED DESIGN - UNEQUAL SAMPLE SIZES
ANALYSIS USING PROC GLM

The GLM Procedure

     Class Level Information

Class         Levels    Values
MAN                3    M1 M2 M3
ROLL               3    1 2 3
SAMPLE             5    1 2 3 4 5

Number of Observations Read          30
Number of Observations Used          30
NESTED DESIGN - UNEQUAL SAMPLE SIZES

Dependent Variable: Y   STRENGTH

                                      Sum of
Source                      DF       Squares    Mean Square   F Value   Pr > F
Model                        7    15446.25000    2206.60714      1.69   0.1634
Error                       22    28723.75000    1305.62500
Corrected Total             29    44170.00000

Source                      DF       Type I SS   Mean Square   F Value   Pr > F
MAN                          2     4820.00000    2410.00000      1.85   0.1815
ROLL(MAN)                    5    10626.25000    2125.25000      1.63   0.1943


Source                 Type I Expected Mean Square

MAN                    Var(Error) + 3.9333 Var(ROLL(MAN)) + Q(MAN)
ROLL(MAN)              Var(Error) + 3.64 Var(ROLL(MAN))

Tests of Hypotheses for Mixed Model Analysis of Variance

Dependent Variable: Y   STRENGTH

Source                      DF    Type  I SS    Mean Square   F Value   Pr > F

MAN                          2    4820.000000    2410.000000      1.05   0.4187
Error                   4.5502    9970.795734    2191.300366
Error: 1.0806*MS(ROLL(MAN)) - 0.0806*MS(Error)




Source                      DF      Type I SS     Mean Square   F Value   Pr > F

ROLL(MAN)                    5          10626    2125.250000      1.63   0.1943
Error: MS(Error)            22          28724    1305.625000
NESTED DESIGN - UNEQUAL SAMPLE SIZES
ANALYSIS USING PROC GLM
```

Least Squares Means
Adjustment for Multiple Comparisons: Tukey-Kramer

MAN	Y LSMEAN	Standard Error	Pr > |t|	LSMEAN Number
M1	95.277778	11.531710	<.0001	1
M2	75.972222	11.531710	<.0001	2
M3	106.000000	11.426395	<.0001	3

Least Squares Means for effect MAN
Pr > |t| for H0: LSMean(i)=LSMean(j)

Dependent Variable: Y

i/j	1	2	3
1		0.4750	0.7885
2	0.4750		0.1772
3	0.7885	0.1772	

NESTED DESIGN - UNEQUAL SAMPLE SIZES
ANALYSIS USING PROC MIXED-REML

The Mixed Procedure

Model Information

Data Set	WORK.STRENGTH
Dependent Variable	Y
Covariance Structure	Variance Components
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class	Levels	Values
MAN	3	M1 M2 M3
ROLL	3	1 2 3
SAMPLE	5	1 2 3 4 5

Number of Observations

Number of Observations Read	30
Number of Observations Used	30
Number of Observations Not Used	0

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z	Alpha	Lower	Upper
ROLL(MAN)	213.12	405.55	0.53	0.2996	0.05	32.4288	54210256
Residual	1318.52	402.00	3.28	0.0005	0.05	784.73	2665.59

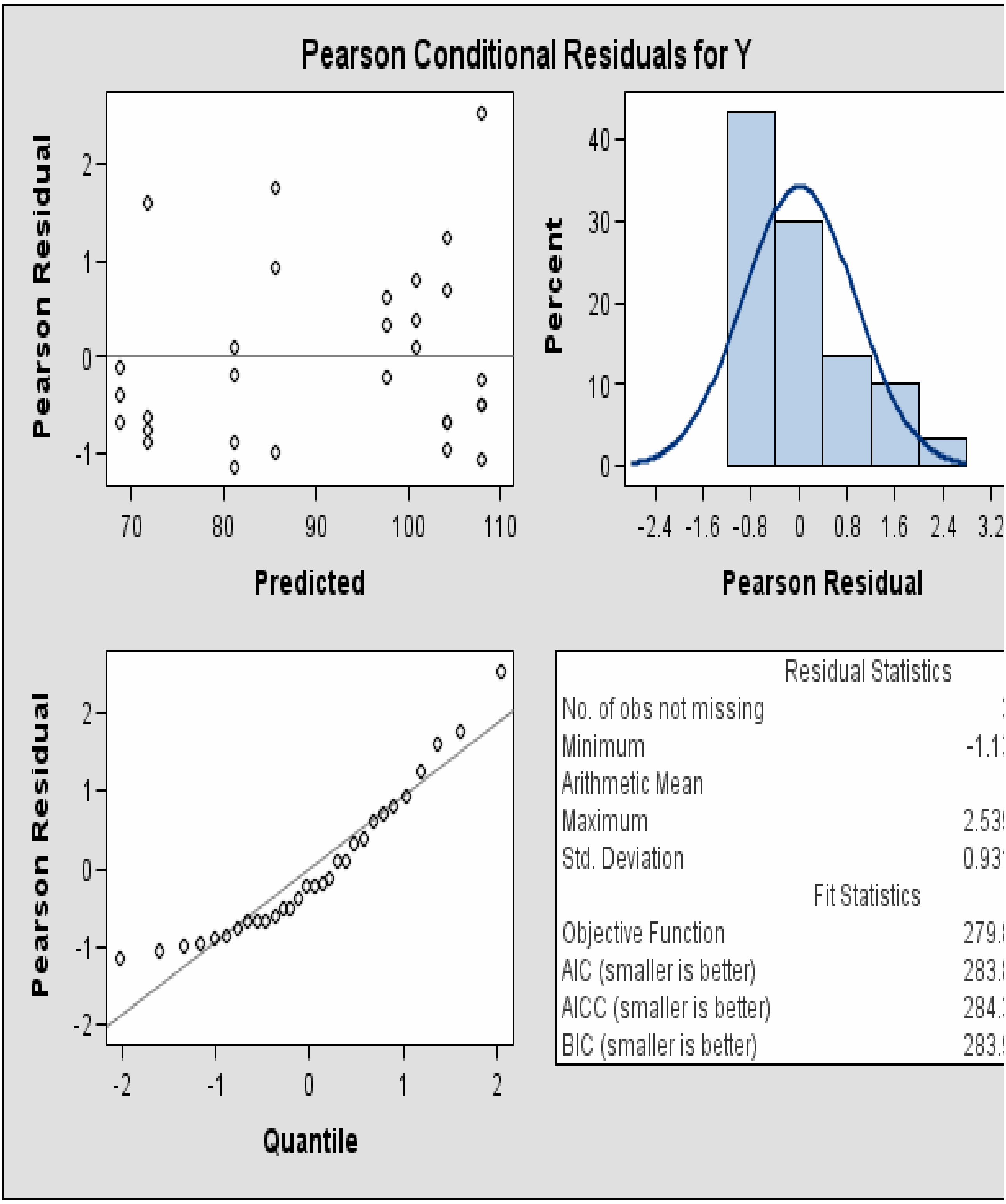
Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
MAN	2	5	1.08	0.4064

Least Squares Means

Effect	MAN	Estimate	Standard Error	DF	t Value	Pr > |t|
MAN	M1	93.2057	14.2755	5	6.53	0.0013
MAN	M2	75.3576	14.2755	5	5.28	0.0032
MAN	M3	106.00	15.4406	5	6.87	0.0010

MAN	MAN	Estimate	Standard Error	DF	t Value	Pr > |t|	Adjustment	Adj P
M1	M2	17.8481	20.1887	5	0.88	0.4171	Tukey-Kramer	0.6724
M1	M3	-12.7943	21.0287	5	-0.61	0.5695	Tukey-Kramer	0.8220
M2	M3	-30.6424	21.0287	5	-1.46	0.2049	Tukey-Kramer	0.3841


Pearson Conditional Residuals for Y
Pearson Residual
Predicted
Percent
Pearson Residual
Pearson Residual
Quantile
Residual Statistics
No. of obs not missing
Minimum
Arithmetic Mean
Maximum
Std. Deviation
Fit Statistics
Objective Function
AIC (smaller is better)
AICC (smaller is better)
BIC (smaller is better)

The Pearson Conditional Residuals indicate that the distribution of the residuals is right skewed with slightly unequal variances. Thus, a log-transformation of the data was done and the transformed data was then analyzed.

The SAS code to analyze a log transformation of the tensile strength from the above experiment is given here:

```
* nested_unequalreps.sas;
*This is an example of a nested experiment with
MAN-fixed factor, ROLL and SAMPLE-random factors.
There is a unequal number of levels in the nested
factors. Three different methods of analysis are
considered.;
ods html; ods graphics on;
option ls=80 ps=55 nocenter nodate;
title 'NESTED DESIGN - UNEQUAL SAMPLE SIZES';
data strength;
INPUT MAN $ ROLL SAMPLE Y @@;
LY = LOG(Y);
LABEL MAN = 'MANUFACTURER' LY= 'LOG-STRENGTH';
cards;
M1 1 1 110 M1 1 2  90 M1 1 3 120
M1 2 1 130 M1 2 2 115 M1 2 3 105
M1 3 1  50 M1 3 2  75 M1 3 3  85 M1 3 4 40
M2 1 1 130 M2 1 2  45 M2 1 3  50 M2 1 4 40
M2 2 1  45 M2 2 2  55 M2 2 3  65
M2 3 1 120 M2 3 2  50 M2 3 3 150
M3 1 1 100 M3 1 2 200 M3 1 3  90 M3 1 4 70 M3 1 5 90
M3 2 1 130 M3 2 2  80 M3 2 3  70 M3 2 4 80 M3 2 5 150
RUN;

TITLE ANALYSIS USING PROC GLM;
PROC GLM;
CLASS MAN ROLL SAMPLE;
MODEL  LY = MAN ROLL(MAN)/SSI E1;
RANDOM ROLL(MAN)/TEST;
LSMEANS MAN/STDERR PDIFF ADJUST=TUKEY;
RUN;

TITLE ANALYSIS USING PROC MIXED-REML;
PROC MIXED METHOD=REML CL ALPHA=.05 COVTEST;
CLASS MAN ROLL SAMPLE;
MODEL LY = MAN/RESIDUALS;
RANDOM ROLL(MAN);
LSMEANS MAN/ADJUST=TUKEY;
RUN;

ods graphics off; ods html close;
```

```
NESTED DESIGN - UNEQUAL SAMPLE SIZES
ANALYSIS USING PROC GLM

The GLM Procedure

     Class Level Information

Class         Levels    Values
MAN                3    M1 M2 M3
ROLL               3    1 2 3
SAMPLE             5    1 2 3 4 5

Number of Observations Read          30
Number of Observations Used          30
NESTED DESIGN - UNEQUAL SAMPLE SIZES

Dependent Variable: LY   LOG-STRENGTH

                                      Sum of
Source                     DF        Squares    Mean Square   F Value    Pr > F
Model                       7     2.39527261     0.34218180      2.40    0.0550
Error                      22     3.13609755     0.14254989
Corrected Total            29     5.53137016

Source                     DF       Type I SS    Mean Square   F Value    Pr > F
MAN                         2     0.85229656     0.42614828      2.99    0.0710
ROLL(MAN)                   5     1.54297604     0.30859521      2.16    0.0953


Source                 Type I Expected Mean Square

MAN                    Var(Error) + 3.9333 Var(ROLL(MAN)) + Q(MAN)
ROLL(MAN)              Var(Error) + 3.64 Var(ROLL(MAN))

Tests of Hypotheses for Mixed Model Analysis of Variance

Dependent Variable: LY   LOG-STRENGTH

Source                     DF    Type  I SS    Mean Square   F Value   Pr > F

MAN                         2     0.852297      0.426148       1.32    0.3506
Error                  4.5502     1.500469      0.321976
Error: 1.0806*MS(ROLL(MAN)) - 0.0806*MS(Error)



Source                     DF       Type I SS     Mean Square   F Value   Pr > F

ROLL(MAN)                   5        1.542976      0.308595       2.16    0.0953
Error: MS(Error)           22        3.136098      0.142550
```

NESTED DESIGN - UNEQUAL SAMPLE SIZES
ANALYSIS USING PROC MIXED-REML

The Mixed Procedure

Model Information

Data Set	WORK.STRENGTH
Dependent Variable	LY
Covariance Structure	Variance Components
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class	Levels	Values
MAN	3	M1 M2 M3
ROLL	3	1 2 3
SAMPLE	5	1 2 3 4 5

Number of Observations

Number of Observations Read	30
Number of Observations Used	30
Number of Observations Not Used	0

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z	Alpha	Lower	Upper
ROLL(MAN)	0.04638	0.05793	0.80	0.2117	0.05	0.01033	11.0582
Residual	0.1434	0.04352	3.30	0.0005	0.05	0.08552	0.2888

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
MAN	2	5	1.29	0.3531

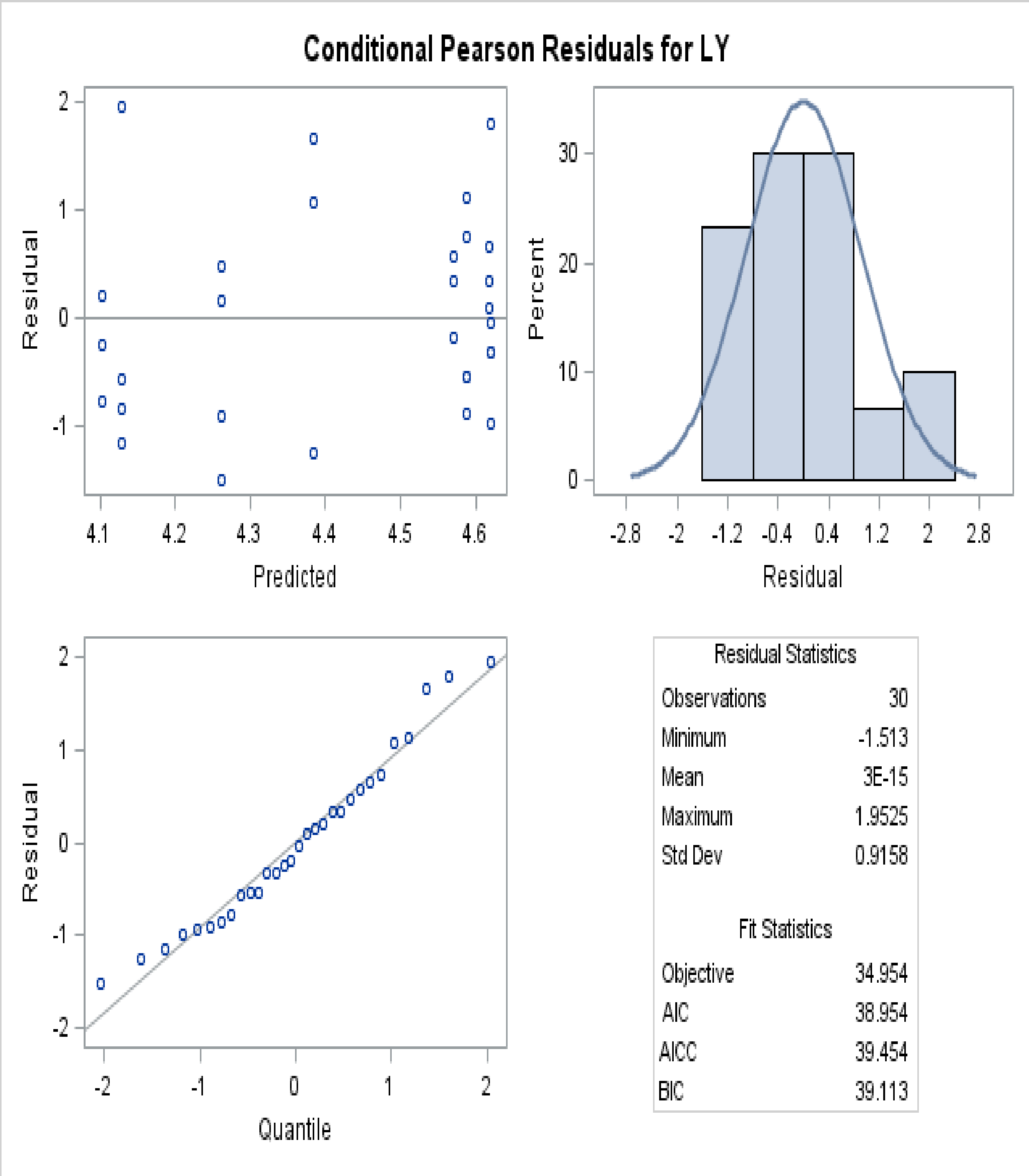
Least Squares Means

Effect	MAN	Estimate	Standard Error	DF	t Value	Pr > |t|
MAN	M1	4.4837	0.1730	5	25.91	<.0001
MAN	M2	4.2051	0.1730	5	24.30	<.0001
MAN	M3	4.6042	0.1937	5	23.77	0.0010

MAN	MAN	Estimate	Standard Error	DF	t Value	Pr > |t|	Adjustment	Adj P
M1	M2	0.2787	0.2447	5	1.14	0.3064	Tukey-Kramer	0.5341
M1	M3	-0.1205	0.2598	5	-0.46	0.6622	Tukey-Kramer	0.8906
M2	M3	-0.3992	0.2598	5	-1.54	0.1850	Tukey-Kramer	0.3519

The Pearson Conditional Residuals on the next page show very little indication of non-normality or heterogeneity in the variances.

In examining the p-values of the various tests, there is very little difference between the results from the original data and the transformed data. This is an indication of the robustness of the F-test to deviations from normally distributed residuals. Heterogeneity in the variances has a much greater impact on the p-values than does deviations from normality.


Conditional Pearson Residuals for LY
Residual
Predicted
Percent
Residual
Residual
Quantile
Residual Statistics
Observations 30
Minimum -1.513
Mean 3E-15
Maximum 1.9525
Std Dev 0.9158
Fit Statistics
Objective 34.954
AIC 38.954
AICC 39.454
BIC 39.113

EXAMPLE OF NESTED AND CROSSED DESIGN

An experiment was designed to investigate the cutoff times of automatic safety switches on lawnmowers produced by three manufacturers, M_1, M_2, and M_3. Three lawnmowers are randomly selected from each of three manufacturers. The lawnmowers are evaluated at two different engine speeds, S_1 and S_2. At both of these speeds, the lawnmowers are evaluated twice. The cutoff times in 10^{-2} seconds are given in the following table. The model for this experiment with y_{ijkm} the cutoff time for the mth run of mower j from manufacturer i conducted at speed k is given by

$$y_{ijkm} = \mu + \tau_i + L_{j(i)} + \beta_k + (\tau\beta)_{ik} + (L * \beta)_{j(i),k} + e_{ijkm}$$

		Speed(S)	
Manufacturer(M)	Lawnmower(L)	L	H
M_1	1	211,230	278,278
	2	184,188	249,272
	3	216,232	275,271
M_2	4	205,217	247,251
	5	169,168	239,252
	6	200,187	261,242
M_3	7	195,207	227,231
	8	199,198	259,282
	9	203,197	251,222

Source	DF	MS	EXPECTED MEAN SQUARE
M	2	MS_M	$\sigma^2_{e(L,M,S)} + 2\sigma^2_{L(M)*S} + 4\sigma^2_{L(M)} + 12Q_M$
L(M)	6	$MS_{L(M)}$	$\sigma^2_{e(L,M,S)} + 2\sigma^2_{L(M)*S} + 4\sigma^2_{L(M)}$
S	1	MS_S	$\sigma^2_{e(L,M,S)} + 2\sigma^2_{L(M)*S} + 18Q_S$
M*S	2	MS_{M*S}	$\sigma^2_{e(L,M,S)} + 2\sigma^2_{L(M)*S} + 6Q_{M*S}$
L(M)*S	6	$MS_{L(M)*S}$	$\sigma^2_{e(L,M,S)} + 2\sigma^2_{L(M)*S}$
Error	18	MSE	$\sigma^2_{e(L,M,S)}$
Total	35		

```
*nested,lawn.sas;
OPTIONS LS=90 PS=55;
DATA LAWN;
INPUT M $ L S $ SUB Y @@;
CARDS;
1 1 L 1 211  1 1 L 2 230  1 1 H 1 278  1 1 H 2 278
1 2 L 1 184  1 2 L 2 188  1 2 H 1 249  1 2 H 2 272
1 3 L 1 216  1 3 L 2 232  1 3 H 1 275  1 3 H 2 271
2 4 L 1 205  2 4 L 2 217  2 4 H 1 247  2 4 H 2 251
2 5 L 1 169  2 5 L 2 168  2 5 H 1 239  2 5 H 2 252
3 6 L 1 200  2 6 L 2 187  2 6 H 1 261  2 6 H 2 242
3 7 L 1 195  3 7 L 2 207  3 7 H 1 227  3 7 H 2 231
3 8 L 1 199  3 8 L 2 198  3 8 H 1 259  3 8 H 2 282
3 9 L 1 203  3 9 L 2 197  3 9 H 1 251  3 9 H 2 222
RUN;


PROC GLM;
CLASS S M L SUB;
MODEL Y = M L(M) S M*S S*L(M)/SS3;
RANDOM L(M) S*L(M)/TEST;
LSMEANS M S M*S/ADJUST=TUKEY;
RUN;


PROC MIXED CL ALPHA=.05 COVTEST;
CLASS S M L SUB;
MODEL Y = M S M*S;
RANDOM L(M) S*L(M);
LSMEANS M S M*S/ADJUST=TUKEY;
RUN;



proc GLIMMIX data=LAWN;
CLASS S M L DEM;
MODEL Y = M S M*S/ddfm=SAT;
RANDOM L(M) S*L(M);
lsmeans M S M*S /  plot = meanplot cl;
RUN;
```

The GLM Procedure

Class Level Information

Class	Levels	Values
S	2	H L
M	3	1 2 3
L	9	1 2 3 4 5 6 7 8 9
SUB	2	1 2

Number of Observations Read 36
Number of Observations Used 36

Dependent Variable: Y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	17	35992.25000	2117.19118	21.14	<.0001
Error	18	1802.50000	100.13889		
Corrected Total	35	37794.75000			

Source	DF	Type III SS	Mean Square	F Value	Pr > F
M	2	2971.50000	1485.75000	14.84	0.0002
S	1	26732.25000	26732.25000	266.95	<.0001
S*M	2	375.16667	187.58333	1.87	0.1824
L(M)	6	3726.00000	621.00000	6.20	0.0011
S*L(M)	6	2187.33333	364.55556	3.64	0.0153

The GLM Procedure

Source	Type III Expected Mean Square
M	Var(Error) + 2 Var(S*L(M)) + 4 Var(L(M)) + Q(M,S*M)
S	Var(Error) + 2 Var(S*L(M)) + Q(S,S*M)
S*M	Var(Error) + 2 Var(S*L(M)) + Q(S*M)
L(M)	Var(Error) + 2 Var(S*L(M)) + 4 Var(L(M))
S*L(M)	Var(Error) + 2 Var(S*L(M))

Least Squares Means

M	Y LSMEAN	Standard Error	Pr > |t|	LSMEAN Number
1	240.333333	2.888755	<.0001	1
2	219.833333	2.888755	<.0001	2
3	222.583333	2.888755	<.0001	3

```
       Least Squares Means for effect M
     Pr > |t| for H0: LSMean(i)=LSMean(j)

            Dependent Variable: Y

i/j             1             2             3
   1                      <.0001        0.0004
   2        <.0001                      0.5094
   3        0.0004        0.5094

                                                      H0:LSMean1=
                                Standard    H0:LSMEAN=0    LSMean2
S         Y LSMEAN                 Error       Pr > |t|       Pr > |t|

H      254.833333        2.358659         <.0001         <.0001
L      200.333333        2.358659         <.0001


ANALYSIS USING PROC MIXED-REML

The Mixed Procedure

Data Set                     WORK.LAWN
Dependent Variable           Y
Covariance Structure         Variance Components
Estimation Method            REML
Residual Variance Method     Profile
Fixed Effects SE Method      Model-Based
Degrees of Freedom Method    Containment


             Class Level Information

Class    Levels    Values
S             2    H L
M             3    1 2 3
L             9    1 2 3 4 5 6 7 8 9
SUB           2    1 2

          Number of Observations

Number of Observations Read              36
Number of Observations Used              36
Number of Observations Not Used           0
```

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z	Alpha	Lower	Upper
L(M)	64.1111	103.94	0.62	0.2687	0.05	11.2676	540614
S*L(M)	132.21	106.55	1.24	0.1073	0.05	42.8709	1742.49
Residual	100.14	33.3796	3.00	0.0013	0.05	57.1743	219.00

Type 3 Tests of Fixed Effects

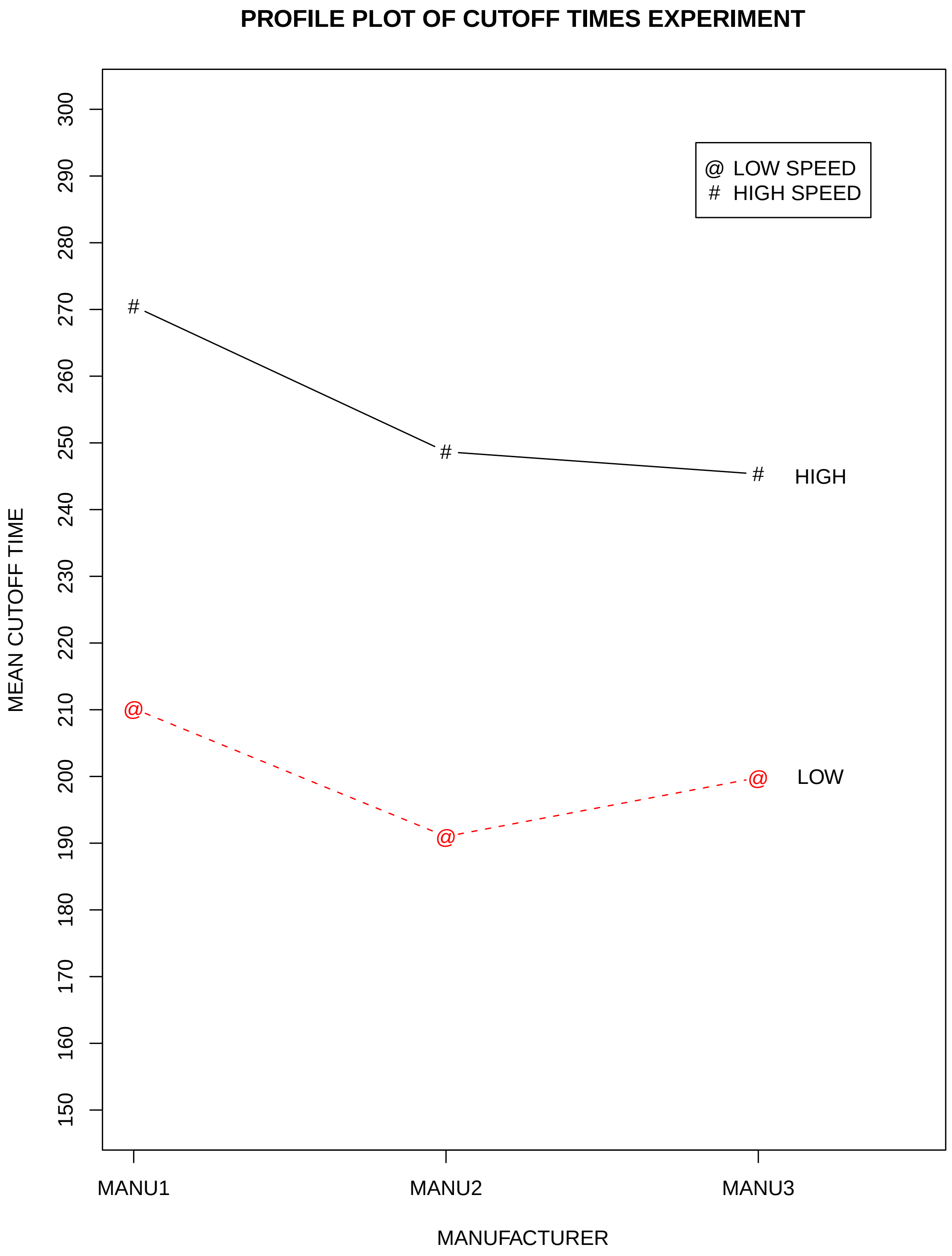
Effect	Num DF	Den DF	F Value	Pr > F
M	2	6	2.39	0.1722
S	1	6	73.33	0.0001
S*M	2	6	0.51	0.6219

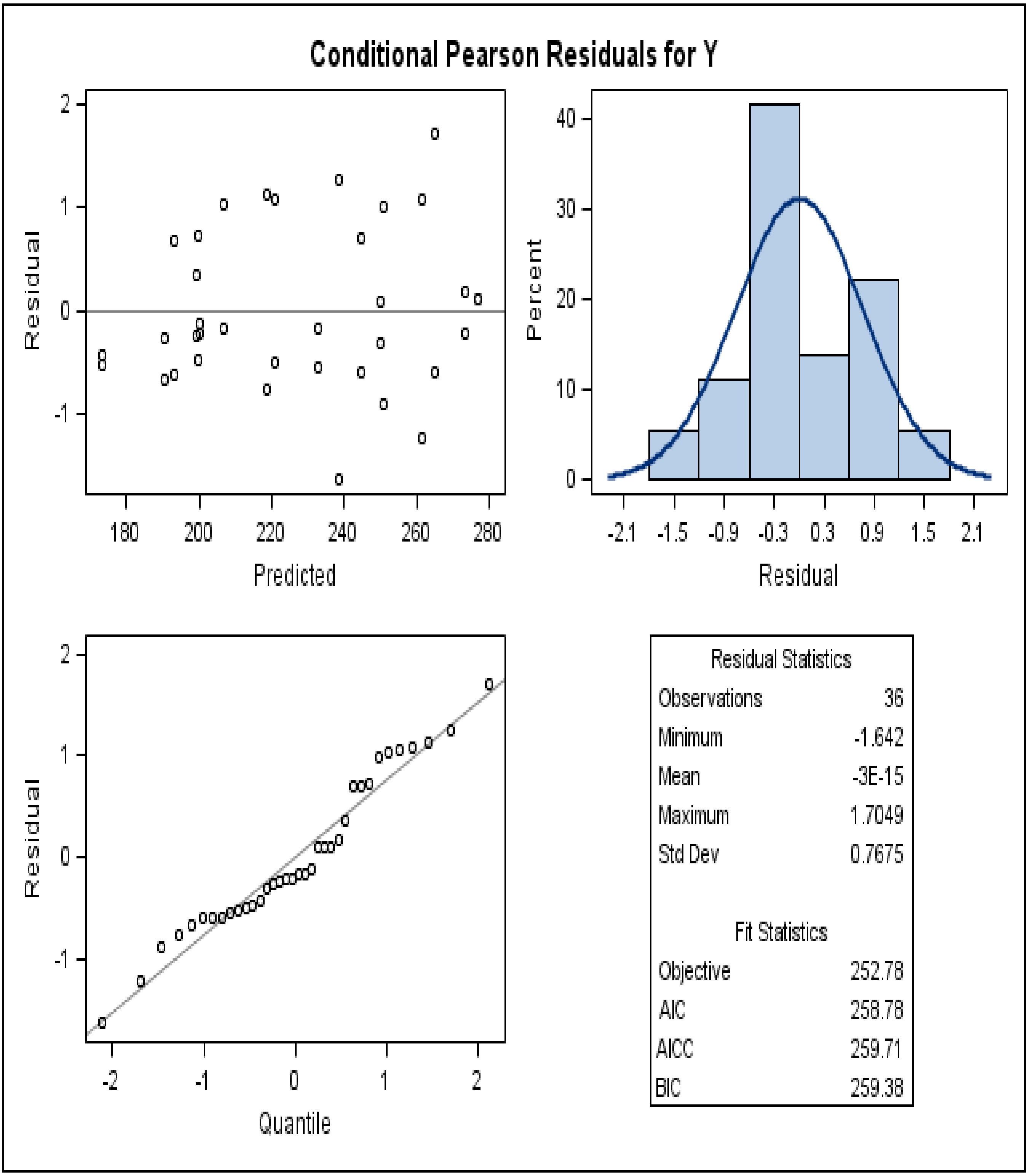
Least Squares Means

Effect	S	M	Estimate	Standard Error	DF	t Value	Pr > |t|
M		1	240.33	7.1937	6	33.41	<.0001
M		2	219.83	7.1937	6	30.56	<.0001
M		3	222.58	7.1937	6	30.94	<.0001
S	H		254.83	5.2323	11.2	48.70	<.0001
S	L		200.33	5.2323	11.2	38.29	<.0001
S*M	H	1	270.50	9.0625	11.2	29.85	<.0001
S*M	H	2	248.67	9.0625	11.2	27.44	<.0001
S*M	H	3	245.33	9.0625	11.2	27.07	<.0001
S*M	L	1	210.17	9.0625	11.2	23.19	<.0001
S*M	L	2	191.00	9.0625	11.2	21.08	<.0001
S*M	L	3	199.83	9.0625	11.2	22.05	<.0001

Differences of Least Squares Means

Effect	S	M	_S	_M	Estimate	Standard Error	DF	t Value	Pr > |t|	Adjustment	Adj P
M		1		2	20.5000	10.1735	6	2.02	0.0905	Tukey	0.1893
M		1		3	17.7500	10.1735	6	1.74	0.1316	Tukey	0.2652
M		2		3	-2.7500	10.1735	6	-0.27	0.7960	Tukey	0.9608
S	H		L		54.5000	6.3644	6	8.56	0.0001	Tukey-Kramer	0.0001
S*M	H	1	H	2	21.8333	12.8164	11.2	1.70	0.1159	Tukey-Kramer	0.5736
S*M	H	1	H	3	25.1667	12.8164	11.2	1.96	0.0748	Tukey-Kramer	0.4502
S*M	H	1	L	1	60.3333	11.0235	6	5.47	0.0016	Tukey-Kramer	0.0116
S*M	H	1	L	2	79.5000	12.8164	11.2	6.20	<.0001	Tukey-Kramer	0.0062
S*M	H	1	L	3	70.6667	12.8164	11.2	5.51	0.0002	Tukey-Kramer	0.0112
S*M	H	2	H	3	3.3333	12.8164	11.2	0.26	0.7995	Tukey-Kramer	0.9997
S*M	H	2	L	1	38.5000	12.8164	11.2	3.00	0.0117	Tukey-Kramer	0.1459
S*M	H	2	L	2	57.6667	11.0235	6	5.23	0.0020	Tukey-Kramer	0.0144
S*M	H	2	L	3	48.8333	12.8164	11.2	3.81	0.0028	Tukey-Kramer	0.0599
S*M	H	3	L	1	35.1667	12.8164	11.2	2.74	0.0188	Tukey-Kramer	0.1953
S*M	H	3	L	2	54.3333	12.8164	11.2	4.24	0.0013	Tukey-Kramer	0.0381
S*M	H	3	L	3	45.5000	11.0235	6	4.13	0.0062	Tukey-Kramer	0.0428
S*M	L	1	L	2	19.1667	12.8164	11.2	1.50	0.1623	Tukey-Kramer	0.6796
S*M	L	1	L	3	10.3333	12.8164	11.2	0.81	0.4368	Tukey-Kramer	0.9560
S*M	L	2	L	3	-8.8333	12.8164	11.2	-0.69	0.5047	Tukey-Kramer	0.9767


PROFILE PLOT OF CUTOFF TIMES EXPERIMENT
@ LOW SPEED
# HIGH SPEED
MEAN CUTOFF TIME
300
290
280
270
260
250
240
230
220
210
200
190
180
170
160
150
HIGH
LOW
MANU1
MANU2
MANU3
MANUFACTURER


Conditional Pearson Residuals for Y
Residual
Predicted
Percent
Residual
Residual
Quantile
Residual Statistics
Observations 36
Minimum -1.642
Mean -3E-15
Maximum 1.7049
Std Dev 0.7675
Fit Statistics
Objective 252.78
AIC 258.78
AICC 259.71
BIC 259.38

Lawnmower Example
Calculations of Estimated Errors of Difference in Marginal and Treatment Means

1. Compute the estimated standard error of the difference in the means for

   Manufacturer 1 and 2, $\widehat{SE}(\hat{\mu}_{1..} - \hat{\mu}_{2..})$:

$$y_{ijkm} = \mu + \tau_i + L_{j(i)} + \beta_k + (\tau\beta)_{ik} + (L*\beta)_{j(i)k} + e_{ijkm} \Rightarrow$$

$$\overline{y}_{1...} = \mu + \tau_1 + \overline{L}_{.(1)} + \overline{\beta}_{.} + \overline{(\tau\beta)}_{1.} + \overline{(L*\beta)}_{.(1).} + \overline{e}_{1...}$$

$$\overline{y}_{2...} = \mu + \tau_2 + \overline{L}_{.(2)} + \overline{\beta}_{.} + \overline{(\tau\beta)}_{2.} + \overline{(L*\beta)}_{.(2).} + \overline{e}_{2...} \Rightarrow$$

$$Var[\overline{y}_{1...} - \overline{y}_{2...}] = Var\left([\overline{L}_{.(1)} - \overline{L}_{.(2)}]\right) + Var\left([\overline{(L*\beta)}_{.(1).} - \overline{(L*\beta)}_{.(2).}]\right) + Var\left([\overline{e}_{1...} - \overline{e}_{2...}]\right)$$

Therefore, we have

$$\begin{aligned} Var[\bar{y}_{1...} - \bar{y}_{2...}] &= \frac{2\sigma^2_{L(M)}}{3} + \frac{2\sigma^2_{L(M)*S}}{(3)(2)} + \frac{2\sigma^2_e}{(3)(2)(2)} \\ &= \frac{2}{12}\left[4\sigma^2_{L(M)} + 2\sigma^2_{L(M)*S} + \sigma^2_e\right] \\ &= \frac{1}{6}\left[EMS_{L(M)}\right] \end{aligned}$$

An estimate of the standard error and the degrees of freedom of the estimate are given as follows.

$$\widehat{SE}(\hat{\mu}_{1..} - \hat{\mu}_{2..}) = \sqrt{\tfrac{1}{6}[MS_{L(M)}]} = \sqrt{\tfrac{621}{6}} = 10.1735.$$

- The degrees of freedom associated with this estimate is $df_{L(M)} = 6$.

- Compute the value of Tukey-Kramer's HSD with $\alpha = .05$ that would be used to determine which pairs of means across the levels of the factor Manufacture are different:

$$HSD = (q_{.05,3,6})\widehat{SE}(\hat{\mu}_{1..} - \mu_{2..})/\sqrt{2} = (4.339)(10.1735)/\sqrt{2} = 31.213$$

2. Compute the estimated standard error of the difference in the means for Speed 1 and 2, $\widehat{SE}(\hat{\mu}_{..1} - \hat{\mu}_{..2})$:

$\bar{y}_{..1.} = \mu + \bar{\tau}_{.} + \bar{L}_{.(.)} + \beta_1 + \overline{(\tau\beta)}_{.1} + \overline{(L*\beta)}_{.(.)1} + \bar{e}_{..1.}$

$\bar{y}_{..2.} = \mu + \bar{\tau}_{.} + \bar{L}_{.(.)} + \beta_2 + \overline{(\tau\beta)}_{.2} + \overline{(L*\beta)}_{.(.)2} + \bar{e}_{..2.} \Rightarrow$

$Var[\bar{y}_{..1.} - \bar{y}_{..2.}] = Var\left([\bar{L}_{.(.)} - \bar{L}_{.(.)}]\right) + Var\left([\overline{(L*\beta)}_{.(.)1} - \overline{(L*\beta)}_{.(.)2}]\right) + Var\left([\bar{e}_{..1.} - \bar{e}_{..2.}]\right)$

Therefore, we have

$$\begin{aligned} Var[\bar{y}_{..1.} - \bar{y}_{..2.}] &= \frac{2\sigma^2_{L(M)*S}}{(3)(3)} + \frac{2\sigma^2_e}{(3)(3)(2)} \\ &= \frac{2}{18}\left[2\sigma^2_{L(M)*S} + \sigma^2_e\right] \\ &= \frac{1}{9}\left[EMS_{L(M)*S}\right] \end{aligned}$$

An estimate of the standard error and the degrees of freedom of the estimate are given as follows.

$\widehat{SE}(\hat{\mu}_{..1} - \hat{\mu}_{..2}) = \sqrt{\frac{1}{9}[MS_{L(M)*S}]} = \sqrt{\frac{364.56}{9}} = 6.3644.$

- The degrees of freedom associated with this estimate is $df_{L(M)*S} = 6$.
- Compute the value of Tukey's HSD with $\alpha = .05$ that would be used to determine which pairs of means across the levels of the factor Speed are different:

  $HSD = (q_{.05,2,6})\widehat{SE}(\hat{\mu}_{..1} - \mu_{..2})/\sqrt{2} = (3.46)(6.3644)/\sqrt{2} = 15.571$

3. Compute the estimated standard error of the difference in the means for (Man 1,Speed 1) and (Man 2, Speed 2) $\widehat{SE}(\hat{\mu}_{1.1} - \hat{\mu}_{2.2})$:

$\bar{y}_{1.1.} = \mu + \tau_1 + \bar{L}_{.(1)} + \beta_1 + (\tau\beta)_{11} + \overline{(L*\beta)}_{.(1)1} + \bar{e}_{1.1.}$

$\bar{y}_{2.2.} = \mu + \tau_2 + \bar{L}_{.(2)} + \beta_2 + (\tau\beta)_{22} + (L*\bar{\beta})_{.(2)2} + \bar{e}_{2.2.} \Rightarrow$

$Var[\bar{y}_{1.1.} - \bar{y}_{2.2.}] = Var\left([\bar{L}_{.(1)} - \bar{L}_{.(2)}]\right) + Var\left([\overline{(L*\beta)}_{1(.)1} - \overline{(L*\beta)}_{2(.)2}]\right) + Var\left([\bar{e}_{1.1.} - \bar{e}_{2.2.}\right)$

Therefore, we have

$$\begin{aligned}
Var[\bar{y}_{1.1.} - \bar{y}_{2.2.}] &= \frac{2\sigma^2_{L(M)}}{3} + \frac{2\sigma^2_{L(M)*S}}{3} + \frac{2\sigma^2_e}{(3)(2)} \\
&= \frac{2}{6}\left[2\sigma^2_{L(M)} + 2\sigma^2_{L(M)*S} + \sigma^2_e\right] \\
&= \frac{1}{3}\left[\frac{1}{2}EMS_{L(M)} + \frac{1}{2}EMS_{L(M)*S}\right] \\
&= \frac{1}{6}\left[EMS_{L(M)} + EMS_{L(M)*S}\right]
\end{aligned}$$

An estimate of the standard error and the degrees of freedom of the estimate are given as follows.

$$\widehat{SE}(\hat{\mu}_{1.1} - \hat{\mu}_{2.2}) = \sqrt{\tfrac{1}{6}[MS_{L(M)} + MS_{L(M)*S}]} = \sqrt{\tfrac{621+364.56}{6}} = 12.8164.$$

- The degrees of freedom associated with this estimate is obtained using the Satterthwaite approximation:

$$df \approx \frac{(MS_{L(M)} + MS_{L(M)*S})^2}{\frac{(MS_{L(M)})^2}{df_{MS_{L(M)}}} + \frac{(MS_{L(M)*S})^2}{df_{MS_{L(M)*S}}}} = \frac{(621 + 364.56)^2}{\frac{(621)^2}{6} + \frac{(364.56)^2}{6}} = 11.24$$

.

- Compute the value of Tukey's HSD with $\alpha = .05$ that would be used to determine which pairs of means across the levels of the treatment combinations of (Manufacture, Speed) are different:

$$HSD = (q_{.05,6,11.24})\widehat{SE}(\hat{\mu}_{1.1} - \mu_{2.2})/\sqrt{2} = (4.804)(12.8164)/\sqrt{2} = 43.536$$