STAT 408/608 Homework 9 Solutions: Written Section

April 22, 2015

1. (a)

$$\begin{aligned}
&\log(\frac{\theta(x)}{1-\theta(x)}) = \beta_0 + \beta_1 x \\
&\Rightarrow \frac{\theta(x)}{1-\theta(x)} = \exp(\beta_0 + \beta_1 x) \\
&\Rightarrow \theta(x) = \exp(\beta_0 + \beta_1 x) - \theta(x)\exp(\beta_0 + \beta_1 x) \\
&\Rightarrow \theta(x) = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x)}
\end{aligned}$$

   (b) From (a),

$$\begin{aligned}
&\theta(x) = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x)} \\
&\Rightarrow \theta(x) = \frac{\exp(\beta_0 + \beta_1 x)}{1 + \exp(\beta_0 + \beta_1 x)} \frac{\exp(-(\beta_0 + \beta_1 x))}{\log\exp(-(\beta_0 + \beta_1 x))} \\
&\Rightarrow \frac{1}{1 + \exp(-(\beta_0 + \beta_1 x))}
\end{aligned}$$

2. When we work with least squares regression we would check the residuals to make sure the assumptions met. In logistic regression we want there to be a linear relationship as well. Therefore, to make certain that the predictors are linearly related to the log odds of the response. We need the transformations for the predictors that will help us to find a fit which has a better linear relationship.

3. (a) $P(X|Y=j) = \pi_j^x(1-\pi_j)^{1-x}$

$$\frac{\theta(x)}{1-\theta(x)} = \frac{P(Y=1|X)}{P(Y=0|X)} = \frac{P(Y=1, X=x)P(X=x)}{P(Y=0, X=x)P(X=x)} = \frac{P(Y=1)P(X=x|Y=1)}{P(Y=0)P(X=x|Y=0)}$$

$$\begin{aligned}
\log(\frac{\theta(x)}{1-\theta(x)}) &= \log(\frac{P(Y=1)}{P(Y=0)}) + \log(\frac{P(X=x|Y=1)}{P(X=x|Y=0)}) \\
&= \log(\frac{P(Y=1)}{P(Y=0)}) + \log(\frac{\pi_1^x(1-\pi_1)^{1-x}}{\pi_0^x(1-\pi_0)^{1-x}}) \\
&= \log(\frac{P(Y=1)}{P(Y=0)}) + x\log(\frac{\pi_1}{\pi_0}) + (1-x)\log(\frac{1-\pi_1}{1-\pi_0}) \\
&= \log(\frac{P(Y=1)}{P(Y=0)}) + \log(\frac{1-\pi_1}{1-\pi_0}) + x(\log(\frac{\pi_1}{\pi_0}) - \log(\frac{1-\pi_1}{1-\pi_0})) \\
&= \log(\frac{P(Y=1)P(X=0|Y=1)}{P(Y=0)P(X=0|Y=0)}) + x\log(\frac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)}) \\
&= a + bx
\end{aligned}$$

(b) The intercept is $\log(\dfrac{P(Y=1)}{P(Y=0)}) + \log(\dfrac{1-\pi_1}{1-\pi_0})$;

the slop is $\log(\dfrac{\pi_1/(1-\pi_1)}{\pi_0/(1-\pi_0)})$