STAT 408/608 Homework 6 Solutions: Written Section

March 18, 2015

1.
$$
\begin{aligned}
Var(\hat{\beta}|X) &= Var((X'X)^{-1}X'Y|X) \\
&= (X'X)^{-1}X'Var(Y|X)X(X'X)^{-1} \\
&= (X'X)^{-1}X'\sigma^2 IX(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}X'X(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}
\end{aligned}
$$

2. (a) Yes. Because number of pens in control group is different from the treatment groups. Therefore, I suggest WLS estimator in this experiment, where number of pens is used as a weight.

   (b) Yes. Because the plot shows straight line will not be a good fit. I suggest a second order polynomial model.

   (c) I will use one second order polynomial model for three group with the same intercept and without interaction. Because the plot does not show much cross relationship between dosage and source, I will ignore the interaction term. Also, when dosage=0, the weight gain for three sources are the same. Therefore, β_0 will be the baseline parameter with dosage 0 for 3 sources.

   (d) Let:
   $iSource1 = 1$ represent being methionine from Source 1 ; 0 otherwise;
   $iSource2 = 1$ represent being methionine from Source 2 ; 0 otherwise;
   $iSource3 = 1$ represent being methionine from Source 3 ; 0 otherwise.
   x represent dosage values, then:
   $x_1 = iSource1 \times x$ represent dosage value from Source 1;
   $x_2 = iSource2 \times x$ represent dosage value from Source 2;
   $x_3 = iSource3 \times x$ represent dosage value from Source 3;
   The suggested model will be,
   $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{33} x_3^2 + e$

3. (a) $H_0 : \frac{\mu_1}{2} + \frac{\mu_2}{2} - \mu_3 = 0$, then $A = \begin{bmatrix} 0.5 & 0.5 & -1 \end{bmatrix}_{r\times(p+1)}$.

Design matrix $X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\beta = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}_{(p+1)\times 1}$, then, $r = 1, p = 2$.

$$\begin{aligned} F &= \frac{(A\hat{\beta} - h)'(A(X'X)^{-1}A')^{-1}(A\hat{\beta} - h)/r}{SSE/(n-p-1)} \\ &= \frac{(A \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \hat{\mu}_3 \end{bmatrix})'(A \begin{bmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} A')^{-1}(A \begin{bmatrix} \hat{\mu}_1 \\ \hat{\mu}_2 \\ \hat{\mu}_3 \end{bmatrix})}{SSE/(12-2-1)} \\ &= \frac{24(0.5\hat{\mu}_1 + 0.5\hat{\mu}_2 - \hat{\mu}_3)^2}{SSE} \end{aligned}$$

where $A = \begin{bmatrix} 0.5 & 0.5 & -1 \end{bmatrix}$.

(b) Given $\hat{\mu}_1 = 5.6, \hat{\mu}_2 = 7.9, \hat{\mu}_3 = 6.1, SSE = 12.8, \alpha = 0.05$,
$F = \dfrac{24(0.5 \times 5.6 + 0.5 \times 7.9 - 6.1)^2}{12.8} = 0.792 < F_{1,9} = 5.12$, which fails to reject H_0.
There is insufficient evidence to suggest the contrast $\dfrac{\mu_1}{2} + \dfrac{\mu_2}{2} - \mu_3$ differ from 0.

4. (a) $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + e$
$beta_0$ is the mean response for group A.
$beta_1$ is the mean difference between group B and A.
$beta_2$ is the mean difference between group C and A.
$beta_3$ is the mean difference between group D and A.

(b) The errors must be iid normal with mean 0 and constant variance.

(c) $\beta_1 = \mu_B - \mu_A$. The confidence interval for β_1:

$$\begin{aligned} \hat{\beta}_1 \pm t^*_{n-p-1} se(\hat{\beta}_1) &= -11.5 \pm t_{196,0.025} \sqrt{19.45^2 \times 0.04} \\ &= -11.5 \pm 1.97 \times 3.89 \\ &= (-19.17, -3.83) \end{aligned}$$

(d) $A\hat{\beta} = \hat{\beta}_0 + \hat{\beta}_1$, where $A = [1\ 1\ 0\ 0]$.
$Var(A\hat{\beta}) = AVar(\hat{\beta})A' = A\sigma^2(X'X)^{-1}A' = 0.02\hat{\sigma}^2$.
The confidence interval for $\beta_0 + \beta_1$:

$$\begin{aligned}
\hat{\beta}_0 + \hat{\beta}_1 \pm t^*_{n-p-1} se(\hat{\beta}_0 + \hat{\beta}_1) &= 37.5 - 11.5 \pm t_{196,0.025}\sqrt{19.45^2 \times 0.02} \\
&= 26 \pm 1.97 \times 2.75 \\
&= (20.58, 31.42)
\end{aligned}$$