STAT 630 Fall 2014
Homework 6 Solution

3.5.4

From the joint distribution of X and Y in question 3.5.3, we can obtain: $p_Y(2) = \frac{2}{11}, p_Y(3) = \frac{3}{11}, p_Y(7) = \frac{5}{11}, p_Y(13) = \frac{1}{11}$ and $p_Y(y) = 0$ otherwise.

(a) $E[X|Y = 2] = -4 * 1/11/(2/11) + 6 * 1/11/(2/11) = -2 + 3 = 1$

(b) $E[X|Y = 3] = -4 * 2/11/(3/11) + 6 * 1/11/(3/11) = -\frac{8}{3} + 2 = -\frac{2}{3}$.

(c) $E[X|Y = 7] = -4 * 4/11/(5/11) + 6 * 1/11/(5/11) = -\frac{16}{5} + \frac{6}{5} = -2$

(d) $E[X|Y = 13] = 6 * 1/11/(1/11) = 6$

(e) $E[X|Y] = 1$ if $Y = 2$, $E[X|Y] = -\frac{2}{3}$ if $Y = 3$, $E[X|Y] = -2$ if $Y = 7$ and $E[X|Y] = 6$ if $Y = 13$.

3.5.11

(a) $f_X(x) = \int_0^1 \frac{6}{19}(x^2 + y^3)dy = \frac{6}{19}(x^2y + y^4/4)|_0^1 = \frac{6}{19}x^2 + \frac{3}{38}$. So

$$\begin{aligned} E[X] &= \int_0^2 x \cdot f_X(x)dx \\ &= \int_0^2 \left(\frac{6}{19}x^3 + \frac{3}{38}x\right) dx \\ &= \frac{3}{38}x^4 + \frac{3}{76}x^2|_0^2 \\ &= \frac{24}{19} + \frac{6}{38} = \frac{27}{19} \end{aligned}$$

(c) $f_Y(y) = \int_0^2 \frac{6}{19}(x^2 + y^3)dx = \frac{6}{19}\left(\frac{x^3}{3} + y^3x\right)|_0^2 = \frac{16}{19} + \frac{12}{19}y^3$. Then $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{3(x^2+y^3)}{8+6y^3}$. Thus $E[X|Y = y] = \int_0^2 \frac{3x(x^2+y^3)}{8+6y^3}dx = \frac{3x^4/4+3x^2y^3/2}{8+6y^3}|_0^2 = \frac{6+3y^3}{4+3y^3}$. So you substitute Y for y, you can obtain $E[X|Y] = \frac{6+3Y^3}{4+3Y^3}$.
(e) $E[E[X|Y = y]] = \int_0^1 \frac{6+3y^3}{4+3y^3} * \frac{4}{19}(3y^3 + 4)dy = \frac{4}{19}(6y + 3y^4/4)|_0^1 = \frac{27}{19} = E[X]$.

3.5.16

Since X given Y=y has a gamma distribution , then $E[X|Y] = \frac{\alpha}{Y}$. Also because $1/Y$ has an exponential distribution with parameter λ, thus $E[X] = E[E[X|Y]] = E[\alpha/Y] = \alpha * E[1/Y] = \frac{\alpha}{\lambda}$.

3.6.3

(a) Since X has a Geometric distribution, thus $E[X] = (1 - 1/2)/(1/2) = 1$. Then $P(X \geq 9) \leq \frac{E[X]}{9} = \frac{1}{9}$.

(b) Similarly $P(X \geq 2) \leq \frac{E[X]}{2} = \frac{1}{2}$.

(c) Since $E[X] = 1$, $P(|X - 1| \geq 1) \leq Var(X)/1^2 = 2$.

(d) We can find the upper bound in (b) is smaller, thus it is more useful.

(e) $P(X \geq 9) = 1 - P(X \leq 8) = 0.001953125$; $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 0.25$; $P(|X - 1| \geq 1) = P(X \geq 2) + P(X = 0) = 0.25 + 0.5 = 0.75$.

3.6.11

(a) $E(W) = \int_0^2 z \cdot z^3/4dz = z^5/20|_0^2 = \frac{8}{5}$.

(b) To obtain the bound by Chebychev's inequality, we need to obtain the variance of W. Since $E(Z^2) = \int_0^2 z^2 \cdot z^3/4dz = z^6/24|_0^2 = \frac{8}{3}$. So $\mathrm{Var}(Z) = E(Z^2) - (E(Z))^2 = \frac{8}{75}$. Then $P(|Z - E(Z)| \geq 1/2) \leq \frac{\mathrm{Var}(W)}{(1/2)^2} = \frac{32}{75}$

(c) The exact probability is $P(|Z - 8/5| \geq 1/2) = P(Z \geq 21/10) + P(Z \leq 11/10)$. Since $P(Z \geq 2) = 0$ and $P(Z \leq 11/10) = \int_0^{11/10} z^3/4dz = 11^4/160000$, thus the exact probability is 0.09150625 which is smaller than the bound in part (b).

Additional Problem A

$$
\begin{aligned}
P(X = x) &= \sum_{n=x}^{\infty} P(X = x|N = n) \cdot P(N = n) \\
&= \sum_{n=x}^{\infty} \binom{n}{x} \theta^x (1-\theta)^{n-x} \frac{e^{-\lambda}\lambda^n}{n!} \\
&= \frac{\theta^x e^{-\lambda}}{x!} \sum_{n=x}^{\infty} \frac{(1-\theta)^{n-x}\lambda^n}{(n-x)!}
\end{aligned}
$$

The above we use the density function of X given N and the density function of N. Next we will try to form the summation of series with limit $e^{\lambda(1-\theta)}$.

$$
\begin{aligned}
P(X=x) &= \frac{(\theta\lambda)^x e^{-\lambda}}{x!}\sum_{n=x}^{\infty}\frac{(\lambda(1-\theta))^{n-x}}{(n-x)!} \\
&= \frac{(\theta\lambda)^x e^{-\lambda}}{x!}\sum_{n'=0}^{\infty}\frac{(\lambda(1-\theta))^{n'}}{n'!} \\
&= \frac{(\theta\lambda)^x e^{-\lambda}}{x!}\cdot e^{\lambda(1-\theta)} \\
&= \frac{(\theta\lambda)^x e^{-\lambda\theta}}{x!}
\end{aligned}
$$

where $n' = n - x$. Thus X has a poisson distribution with parameter $\lambda\theta$.

Additional Problem B

$E[U] = E[E[U|T]] = E[T/2] = \frac{1}{2\lambda}$. $E[U^2] = E[E[U^2|T]] = E[T^2/3] = E[T^2]/3$. Since T has an exponential distribution with parameter λ, then $E[T^2] = \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$. Thus $E[U^2] = \frac{2}{3\lambda^2}$ and $var(U) = \frac{2}{3\lambda^2} - \frac{1}{(2\lambda)^2} = \frac{5}{12\lambda^2}$.

4.1.11

(a) The mean of this sample is 1.551659 and the standard deviation of this sample is 0.5767168. The following is the code:

```
B=1000
xmax=array(0,B)
for(i in 1:B)
 {
   x=rnorm(10)
   xmax[i]=max(x)
 }
 mean(xmax)
 sd(xmax)
```

(b) The result is showed in figure 1. It is right skewed.

```
hist(xmax,freq=FALSE,breaks=20)
```

(c) You just change the sample size to 20 in the above code. I got the sample mean 1.869534 and sample standard deviation 0.541859.

(d) The result is showed in figure 2. It looks less right skewed and more concentrated.

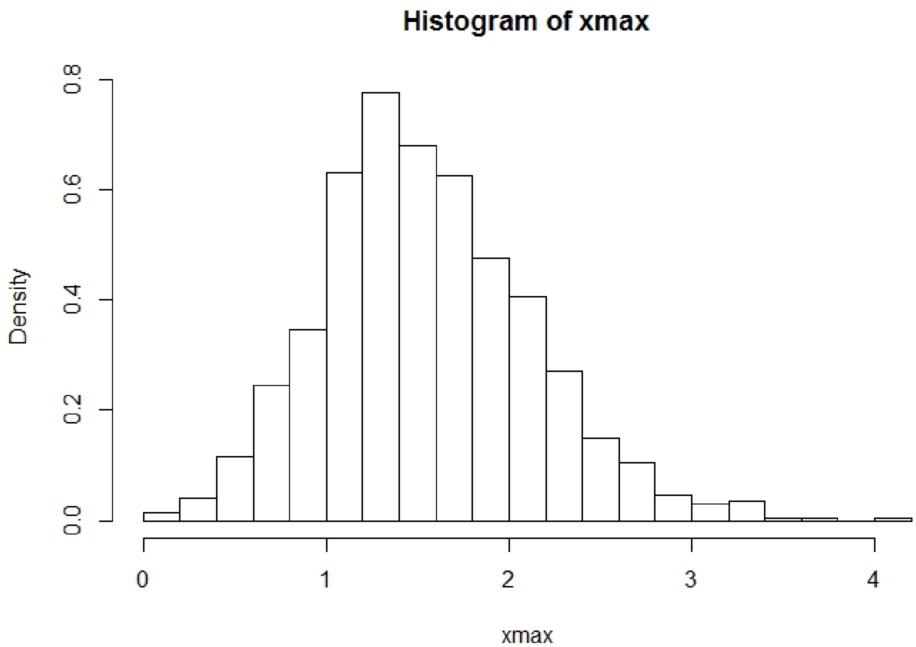

Figure 1: Histogram of maxima with sample size=10.

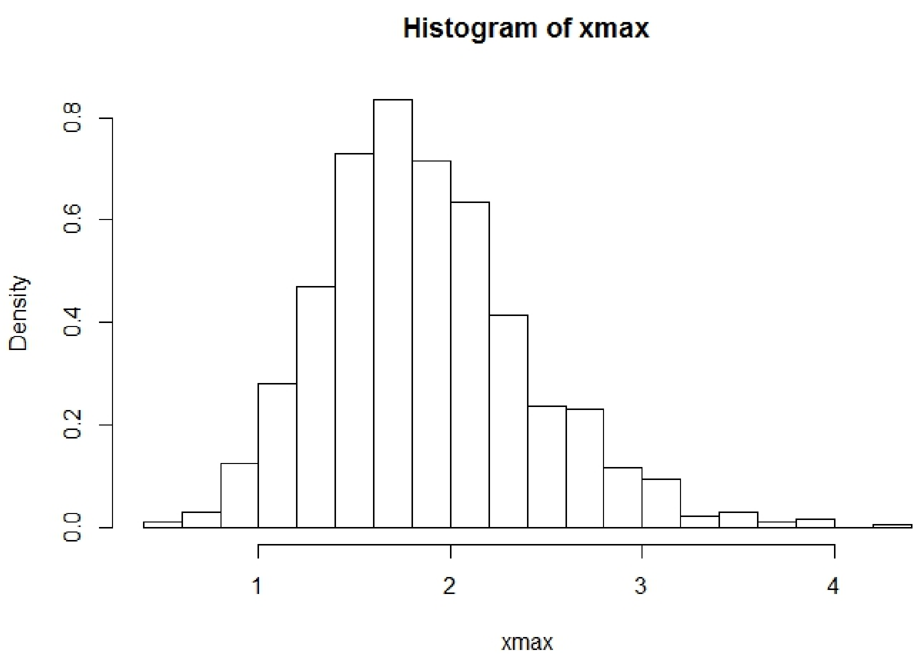

Figure 2: Histogram of maxima with sample size=20.

4.2.2

For any $\epsilon > 0$,

$$P(|X_n| \geq \epsilon) = P(|Y^n| \geq \epsilon) = P(|Y| > \epsilon^{1/n}) = \begin{cases} 1 - \epsilon^{1/n} \to 0, & \text{if} \epsilon < 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence, $X_n \to 0$ in probability.

4.2.10

Let X_i be the squared number showing on the ith rolling. We known $X_1, X_2, \cdots, X_n$ are independent and identically distributed. Therefore due to the weak law of large numbers, $\frac{Z_n}{n} = \frac{\sum\limits_{i=1}^{n} X_i}{n} \to E[X_i]$. Since $E[X_i] = \sum\limits_{i=1}^{6} i^2 \cdot \frac{1}{6} = \frac{91}{6}$, then we know $m = \frac{91}{6}$.

4.2.12

My result is 0.17739 when $n = 20$ and 0.27428 when $n = 50$. So as the sample size n increases, the concentration of the distribution of M_n between this limits also increases. The following is the code:

```
B=10^5
count=0
mn=array(0,B)
for(i in 1:10^5)
{
   x=rexp(50,5)
   mn[i]=mean(x)
   if(mn[i]>=0.19&&mn[i]<=0.21)
      count=count+1

}
p=count/B
```

4.4.4

Let $W \sim U[0,1]$, $0 < W < 1$. Then $P(W_n \leq w) = \int_0^w \frac{1+x/n}{1+1/2n} dx = \frac{w+w^2/2n}{1+1/2n}$ which will converge to w as $n \to \infty$. Also, $P(W \leq w) = w$. Hence, $\lim_{n\to\infty} P(W_n \leq w) = P(W \leq w)$ for all w, so $W_n \to W$ in distribution.

4.4.6

We have $mean(Z_i) = -5, var(Z_i) = \frac{1}{12}(10 - (-20))^2 = 75$. With the central limit theorem, we have $\frac{\sum_{i=1}^{900} Z_i - (-5\times 900)}{\sqrt{900 \times 75}} = \frac{\sum_{i=1}^{900} Z_i + 4500}{30\sqrt{75}} \xrightarrow{\text{D}} N(0,1)$. Hence, $P(\sum_{i=1}^{900} Z_i \geq -4470) =$

$P(\frac{\sum_{i=1}^{900} Z_i+4500}{30\sqrt{75}} \geq \frac{-4470+4500}{30\sqrt{75}} = 1/\sqrt{75}) \approx 1-\Phi(1/\sqrt{75}) = 1-\Phi(0.11547)$. If you use a software, your result should be $P(\sum_{i=1}^{900} Z_i \geq -4470) = 1-\Phi(0.11547) = 0.4540$; if you use the table at the back of the book, $P(\sum_{i=1}^{900} Z_i \geq -4470) = 1-\Phi(0.11547) \approx 1-\Phi(0.12) = 0.4522$.

4.4.12

Since the service time has an exponential distribution with parameter $\lambda = \frac{1}{2}$, we know the mean of service time is $1/\lambda = 2$ and the variance of the service time is $1/\lambda^2 = 4$. Thus due to the central limit theorem, $M_n \sim N(2, \frac{4}{n})$.

(a) When $n = 16$, $P(M_n < 2.5) = P((M_n - \mu)/\sqrt{(\sigma^2/n)} < (2.5-2)/\sqrt{(4/16)}) = P(Z < 1) = 0.8413447$.

(b) When $n = 36$, $P(M_n < 2.5) = P(Z < (2.5-2)/\sqrt{4/36}) = P(Z < \frac{3}{2}) = 0.9331928$.

(c) When $n = 100$, $P(M_n < 2.5) = P(Z < \frac{5}{2}) = 0.9937903$.

The moment generating function for exponential distribution with parameter λ is $\left(1-\frac{t}{\lambda}\right)^{-1}$. Let X_i be the service time for ith custom, then since $X_1, X_2, \cdots, X_n$ are independent and identically distributed, thus the moment generating function for $\sum\limits_{i=1}^{n} X_i$ is $\left(1-\frac{t}{\lambda}\right)^{-n}$ which is the moment generating function for gamma distribution with parameters n and λ. Thus $P(M_n \leq 2.5) = P(\sum\limits_{i=1}^{n} X_i \leq 2.5n) = P(\text{gamma}(n, \frac{1}{2}) \leq 2.5n)$. By using function pgamma in R, we can obtain

(d) $P(M_n \leq 2.5) = 0.8434869$ when $n = 16$.

(e) $P(M_n \leq 2.5) = 0.9257825$ when $n = 36$.

(f) $P(M_n \leq 2.5) = 0.9906209$ when $n = 100$.

The results of true distributions are very close to the results of central limit theorem approximation.

4.4.16

The mean of X_i is -5 and the variance of X_i is 75. Thus by the central limit theorem $P(M_{30} \leq -5) = P(Z \leq (-5+5)/(\sqrt{75/30})) = P(Z \leq 0) = \frac{1}{2}$. The simulation result is 0.50275. We can see the simulation result is very close to the result by central limit theorem approximation. R Code is as follows:

```
B=10^4
prop=array(0,B)
for(i in 1:B){
 x=runif(n=30, min=-20, max=10)
 prop[i]=sum(x <=-5)/30
}
mean(prop)
```

4.5.14

This integral can be seen as the expectation of $cos(x^3)sin(x^4)$ where x has a uniform distribution on $[0, 1]$. The simulation result is 0.1482725. Code is as follows:

```
B=10^5
x=runif(B)
y=cos(x^3)*sin(x^4)
mean(y)
# confidence interval
Lower.limit=mean(y)-3*sd(y)/sqrt(n)
Upper.limit=mean(y)+3*sd(y)/sqrt(n)
```