HANDOUT # 6

Experiments Investigating Variance Components

1. Random Factor Levels

2. Examples

3. Random Effects Model

4. Point Estimators of Variance Components

5. How to Deal with Negative Estimates of Variances

6. C.I. for Variances, Equal Sample Sizes

7. Test Statistics, Equal Sample Sizes

8. Power and Sample Size Determination for Random Effects Models

9. Example: Manufacture of Fish Nets

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Random Effects

In the previous models we examined, the only random term was e_{ij} with the remaining terms fixed, but unknown population parameters:

1. the treatment means, μ_i, in the Cell Means Model: $y_{ij} = \mu_i + e_{ij}$

2. the mean and treatment effects, μ and τ_i, in the Effects Model: $y_{ij} = \mu + \tau_i + e_{ij}$

3. σ_e in both Cell Means and Effects Model: $e_{ij} \sim N(0, \sigma_e^2)$

In this handout we will examine various sources of variation that arise from the manner in which the experiment was conducted. The random terms in the model, that represent the various sources of variation in the experiment, will alter the manner in which we analyze the data and the types of inferences that are made from the data to the population.

EXAMPLE 1: A survey is to be designed to estimate the variability in yield per acre of cotton across farms in Texas. Twenty counties are randomly selected from the 254 counties in Texas. Next, several farms are randomly selected within each county. At each farm, several plots of 1-acre each are randomly selected to determine the yield. The individual measurements of yield on each plot would depend on the variability across counties and the variability across farms within a county. This type of experimental design provides inferences which are considerably different from the inferences that would be made to compare cotton production at a few designated counties within Texas.

EXAMPLE 2: A company produces fiber optic cable for the communication industry. This industry needs cable which has a very consistent tension across long lengths of cable. The company is planning on purchasing new machinery used to bind the cable. There are three major manufacturers of the machines. The fiber optic company wants to determine if there is major differences in the quality of the product from the three manufacturers. A study is designed by randomly selecting 5 binding machines from each each of the manufacturers and then producing 10 cables of length 1000 yards from each of the 15 machines. The quality of the signal transmitted through the cable is then measured. Research Question: Is there a difference in quality due to

1. Manufacturer Differences

2. Variability in Machines From the Same Manufacturer

3. Variability in Cable Produced From the Same Machine

How do the inferences differ in the above study from a study in which a single machine is obtained from each manufacturer?

EXAMPLE 3: There are hundreds of laboratories that are federally qualified to produce assessments of the level of *e. coli* in meat. A consumer research group wants to determine if there is a difference in the accuracy of the determinations across the many laboratories or are the laboratories all essentially the same in the quality of the determinations. A study is designed by

1. Randomly selecting 10 laboratories from the list of all laboratories.

2. Randomly assigning 20 meat samples with a known level of *e. coli* to each of the selected laboratories.

3. Measure the difference between the known value of *e. coli* and the determination made by the laboratory.

How would the inferences differ in this study from the possible inferences running the study at a few preselected laboratories?

Definition A factor has **random effects** if the levels of the factor to be used in the experiment or study are randomly selected from a population of potential levels.

The researcher is interested in all levels of the factor but because of time or economical constraints it is not possible to examine all possible levels. Thus, the responses are observed on a possibly small subset of all possible factor levels (treatments) but inferences are to be about all the levels (treatments).

In Example 1, the researchers are interested in all counties in Texas but will base their inferences on a sample of counties, farms, and plots within these farms.

In Example 2, the company would like to have information about all machines at each of the three Manufacturers but will make inferences based on a limited sample of such machines. If they find large differences in the selected machines, then they may decide to expand the sample to include a larger proportion of the population of machines at each of the three manufactures.

In Example 3, the researchers are interested in the variability across the population of Laboratories. If there is very little difference across the selected Laboratories, then the consumer group may be willing to conclude that government certification has resulted in a homogeneous set of laboratories with very little difference in their measurement error. Whereas, if large differences are found, they may expand their study to determine why the laboratories are producing such a range of differences in the accuracy of their determination of the level of *e. coli* and to determine if these types of differences persist across the population of all such laboratories.

Model for Random Effects:

A random effects experiment (study) has the following elements:

1. Randomly select t treatments (factor levels) from a population of all treatments of interest.

2. Randomly assign n_i experimental units (EU's) to the ith treatment or randomly select r_i EU's from the ith treatment population.

3. Observe or record y_{ij}, the response from the jth EU receiving the ith treatment or the response from the jth EU randomly selected from the ith treatment population.

Random Effects Model: $\quad y_{ij} = \mu + A_i + e_{ij} \quad i = 1, 2, \ldots, t; \quad j = 1, 2, \ldots, n_i,$

where A_i is the the ith random treatment effect with A_i distributed $N(0, \sigma_A^2)$, and μ is the mean response over all treatments in the treatment population. The e_{ij}'s are iid $N(0, \sigma_e^2)$ with A_i's and e_{ij}'s independently distributed. Thus, A_i and e_{ij} are random variables, but μ, σ_A and σ_e are fixed but unknown population parameters which will be estimated using the observed data.
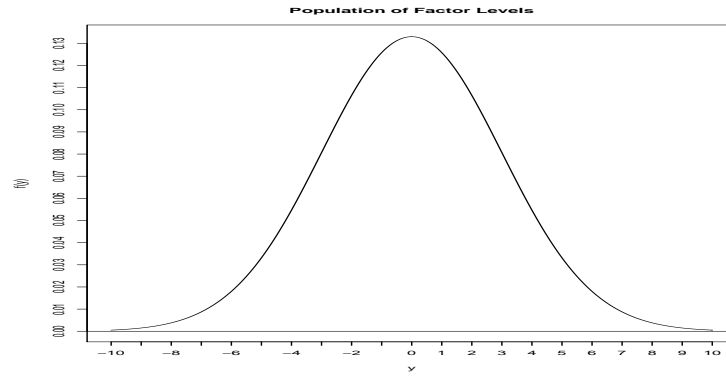
The goal of the experiment is determine if there are treatment differences in the population of treatments, not just in the treatments observed in the study:

$$H_o : \quad \text{No Treatment Differences} \quad \text{versus} \quad H_1 : \quad \text{Treatment Differences Exist}$$

In terms of the population parameters, this is equivalent to testing:

$$H_o : \sigma_A = 0 \quad \text{Versus} \quad H_1 : \sigma_A > 0$$

If $\sigma_A = 0$, then $A_i = 0$ for all treatments and not just the t treatments selected for the given experiment. Also, if $\sigma_A = 0$, then $y_{ij} = \mu + e_{ij}$. Hence, there is a single value in the population of treatment means, μ. Hence, all treatments have the same mean response, μ.

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Note that the mean of the ith treatment is $\mu_i = \mu + A_i$. However, there is very little interest in comparing the treatment means associated with the t treatments selected for the experiment: μ_i versus μ_k, for $i, k = 1, \ldots, t$. The t treatment means are just the means of a small randomly selected group of treatments, whereas the researcher is interested in the difference in the mean responses across all treatments in the population of treatments.

For example, the consumer group is **not** interested in the difference between the accuracy of measuring *e. coli* in just the 10 Laboratories randomly selected for the experiment. They are interested in difference in **all labs** under federal certification.

Next, we will examine how we can interpret the Random Effects Model as a Fixed Effects Model:

The Random Effects Model:

$$y_{ij} = \mu + A_i + e_{ij} \quad i = 1, \ldots, t; \quad j = 1, \ldots, n_i \quad \text{with} \quad A_i \text{ and } e_{ij} \text{ independent}$$

can be interpreted conditionally as a

Cell Means Model:

$$y_{ij} = M_i + e_{ij} \quad i = 1, \ldots, t; \quad j = 1, \ldots, n_i \quad \text{with} \quad M_i \text{ and } e_{ij} \text{ independent}$$

Let $M_1, M_2, \ldots, M_t$ be the population mean responses associated with the t randomly selected treatments.

The M_i are iid $N(\mu, \sigma_\mu^2)$ random variables where μ is the average of all treatments means in the population of treatments and σ_μ is the standard deviation in the population of means.

Thus, conditional on the value of M_i, we have the following

1. $E[y_{ij}|M_i] = E[M_i + e_{ij}|M_i] = E[M_i|M_i] + E[e_{ij}|M_i] = M_i + 0 = M_i$ and

2. $Var(y_{ij}|M_i) = Var(M_i + e_{ij}|M_i) = Var(M_i|M_i) + Var(e_{ij}|M_i) = 0 + Var(e_{ij}) = \sigma_e^2 \quad \Rightarrow$

3. Summarizing, y_{ij}'s **conditional on** M_i are distributed as $N(M_i, \sigma_e^2)$ random variable.

   That is, conditional on the values of the treatment means $M_1, M_2, \ldots, M_t$,

   the y_{ij}'s are independently distributed as $N(M_i, \sigma_e^2)$ r.v.'s.

4. The unconditional distribution of the y_{ij}'s is given by

   - $E[y_{ij}] = E_{M_i}[E[y_{ij}|M_i]] = E_{M_i}[M_i] = \mu$

   - $Var(y_{ij}) = Var_{M_i}(E[y_{ij}|M_i]) + E_{M_i}[Var(y_{ij}|M_i)] = Var_{M_i}(M_i) + E_{M_i}[\sigma_e^2] = \sigma_\mu^2 + \sigma_e^2$

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In regards to the types of inferences that are generally desired, it is more intuitive to use the **Random Effects Model:**

$$y_{ij} = \mu + A_i + e_{ij}$$

1. A_i's are distributed iid $N(0, \sigma_A^2)$ independently of e_{ij}'s which are distributed $N(0, \sigma_e^2)$

2. $E[y_{ij}] = \mu$ but conditional on the selected treatments we have $E[y_{ij}|A_i] = \mu + A_i$

3. $\sigma_y^2 = Var(y_{ij}) = Var(A_i + e_{ij}) = \sigma_A^2 + \sigma_e^2$

4. y_{ij}'s are distributed $N(\mu, \sigma_y^2)$

5. For $j \neq h$,

$$
\begin{aligned}
Cov(y_{ij}, y_{ih}) &= Cov(\mu + A_i + e_{ij}, \mu + A_i + e_{ih}) \\
&= Cov(A_i, A_i) + Cov(A_i, e_{ih}) + Cov(e_{ij}, A_i) + Cov(e_{ij}, e_{ih}) \\
&= Var(A_i) + 0 + 0 + 0 = \sigma_A^2 > 0
\end{aligned}
$$

Thus, the n_i observations from a given treatment are positively correlated.

6. For $j \neq h$ and $i \neq k$,

$$
\begin{aligned}
Cov(y_{ij}, y_{kh}) &= Cov(\mu + A_i + e_{ij}, \mu + A_k + e_{kh}) \\
&= Cov(A_i, A_k) + Cov(A_i, e_{kh}) + Cov(e_{ij}, A_k) + Cov(e_{ij}, e_{kh}) \\
&= 0 + 0 + 0 + 0 = 0
\end{aligned}
$$

Thus, observations on different treatments are independent (uncorrelated jointly normally distributed r.v.'s are independent).

7. The IntraClass Correlation Coefficient for $j \neq h$ is

$$Corr(y_{ij}, y_{ih}) = \frac{Cov(y_{ij}, y_{ih})}{\sqrt{Var(y_{ij})Var(y_{ih})}} = \frac{\sigma_A^2}{\sqrt{\sigma_A^2 + \sigma_e^2}\sqrt{\sigma_A^2 + \sigma_e^2}} = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_e^2} = \frac{\tau}{1+\tau} \text{ where } \tau = \frac{\sigma_A^2}{\sigma_e^2}$$

This is the correlation between two observations receiving the same treatment (or two observations from the same treatment population).

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How do the Random Effects and Fixed Effects Experiments/Models Differ?

1. In Fixed Effects Experiments, there are only t treatments of interest to the researcher.

   In the Random Effect Experiments, there is a population (large number) of treatments, from which the researcher randomly selects t treatments for use in the experiment. The researcher is interested in making inferences concerning the population of treatments, not just the t treatments used in the experiment.

2. In Fixed Effects Model, the Treatment Effects, τ_i are fixed population parameters with $\mu_i = E[y_{ij}] = \mu + \tau_i$. Thus, H_o : No Treatment Difference is equivalent to

   $H_o : \mu_1 = \cdots = \mu_t$ which is equivalent to $H_o : \tau_1 = \cdots = \tau_t = 0$

   In Random Effects Model, the Treatment Effects, A_i are random variables with σ_A reflecting the difference in the treatment means across all treatments in the population of treatments. Thus, H_o : No Treatment Difference is equivalent to $H_o : \sigma_A = 0$

3. In the Fixed Effects Model, the $n = n_1 + \cdots + n_t$ observations y_{ij} are independent random variables.

   In the Random Effects Model, there is independence between observations from different treatments but the n_i observations from the same treatment are correlated, with value ρ_{IC}.

4. To simulate observations from the Fixed Effects Model:

   (a) Specify values for $\mu_1, \ldots, \mu_t$ and σ_e,

   (b) Generate n independent observations from a $N(0,1)$ distribution, and designate them as e_{ij}^* for $i = 1, \ldots, t$ and $h = 1, \ldots, n_i$.

   (c) Finally, let $y_{ij} = \mu_i + \sigma_e e_{ij}^* = \mu_i + e_{ij}$

   (d) Alternatively, generate n independent observations from a $N(0, \sigma_e^2)$ distribution, designate them as e_{ij} for $i = 1, \ldots, t$ and $h = 1, \ldots, n_i$, and let $y_{ij} = \mu_i + e_{ij}$.

   (e) For the Fixed Effects Model, specify values for μ and $\tau_1, \ldots, \tau_t$ and σ_e, generate n independent observations from a $N(0, \sigma_e^2)$ distribution, designate them as e_{ij} for $i = 1, \ldots, t$ and $h = 1, \ldots, n_i$, then let $y_{ij} = \mu + \tau_i + e_{ij}$

5. To simulate observations from the Random Effects Model:

   (a) Specify values for μ, σ_A and σ_e

   (b) Generate t independent observations from a $N(0, \sigma_A^2)$ distribution: $A_1, \ldots, A_t$

   (c) Generate n independent observations from a $N(0, \sigma_e^2)$ distribution: $e_{11}, e_{1,2}, \ldots, e_{tn_t}$,

   where A_i's and e_{ij}'s are independently generated

   (d) Let $y_{ij} = \mu + A_i + e_{ij}$

   Note, A_i is common to $y_{i1}, y_{i2}, \ldots, y_{in_i}$.

   Thus, $y_{i1}, y_{i2}, \ldots, y_{in_i}$ are correlated.

The following two plots display 100 realizations of the pair (e_{11}, e_{12}) distributed $N(0, 1)$ and the corresponding observations (y_{11}, y_{12}) in a CRD having fixed effects with an overall treatment mean of 5 and fixed treatment effect of 5: (observations from same treatment)

$$y_{11} = \mu + \tau_1 + e_{11} = 5 + 5 + e_{11} \qquad \text{and} \qquad y_{12} = \mu + \tau_1 + e_{12} = 5 + 5 + e_{12}$$
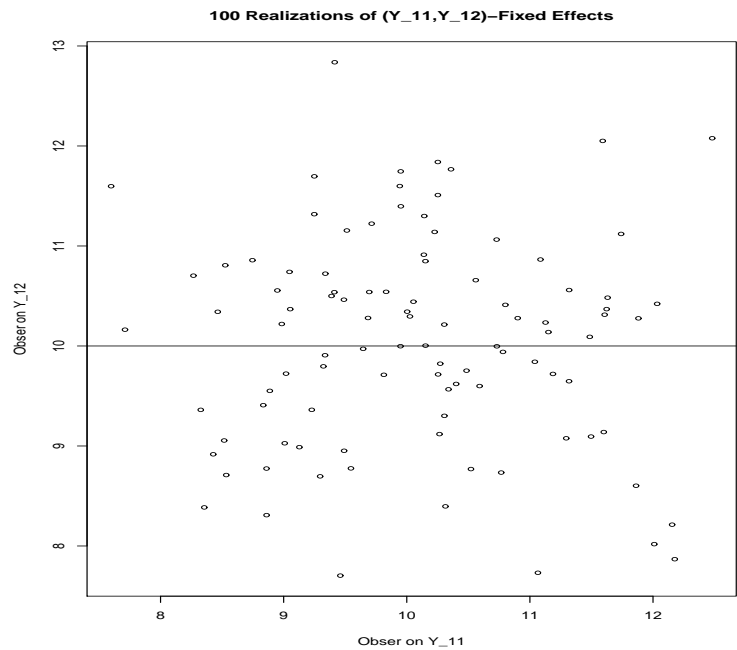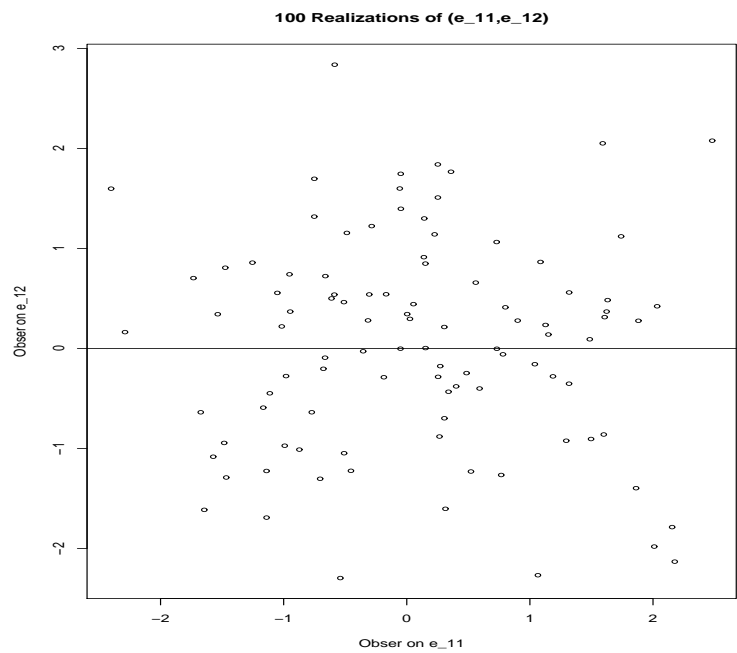
The following two plots display 100 realizations of the pair (e_{11}, e_{12}) distributed $N(0, 1)$ and the corresponding observations (y_{11}, y_{12}) in a CRD having random effects with a treatment mean of 5 and random effect A_1 distributed $N(0, 5^2)$.
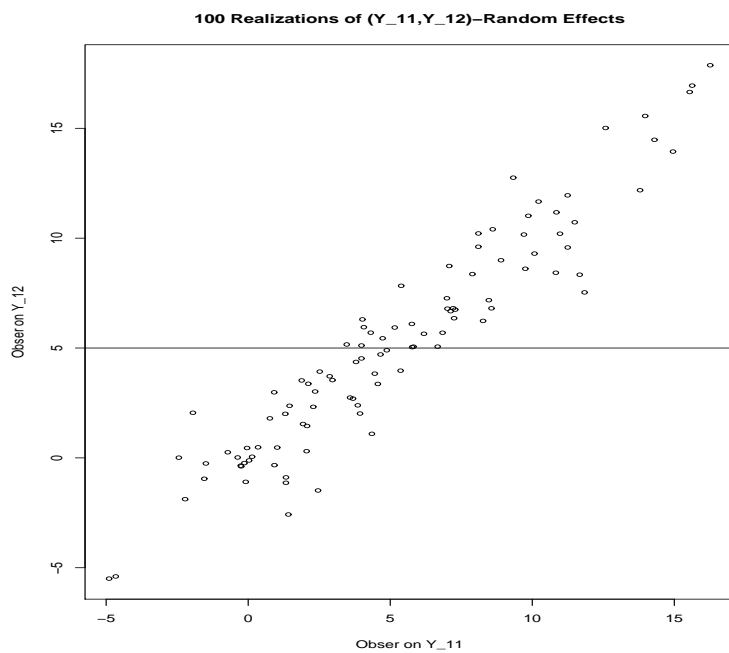
$$y_{11} = \mu + A_1 + e_{11} = 5 + A_1 + e_{11} \qquad \text{and} \qquad y_{12} = \mu + A_1 + e_{12} = 5 + A_1 + e_{12}$$

100 Realizations of (e_11,e_12)

100 Realizations of (Y_11,Y_12)−Random Effects
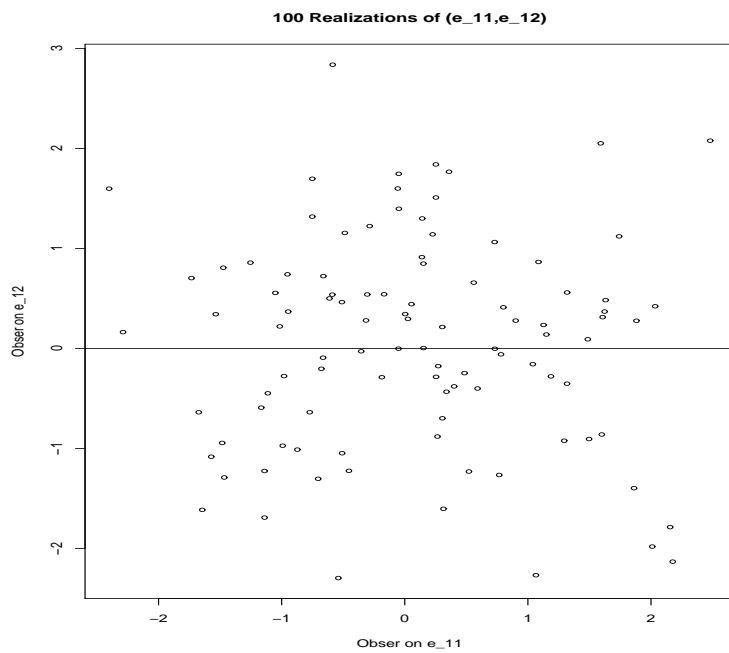
Estimation of Variance Components

In the random effects model: $y_{ij} = \mu + A_i + e_{ij}$ there are two components to the overall variance of the y_{ij}'s

Treatment Variance: $\quad Var(A_i) = \sigma_A^2 \quad$ and Error Variance: $\quad Var(e_{ij} = \sigma_e^2$

Overall Variance: $\quad \sigma_y^2 = Var(y_{ij}) = Var(A_i + e_{ij}) = \sigma_A^2 + \sigma_e^2$

Note: A naive estimator of σ_y^2 would be $\hat{\sigma}_y^2 = \frac{1}{n-1} \sum_{i=1}^{t} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{..})^2$

However, this estimator would not take into account that we have multiple treatments and thus multiple population distributions for the y_{ij}'s.

Point Estimation - Using Method of Moments (MOM)

The partition of SS_{TOT} into two components: SS_{TRT} and SSE will use the same formulas as were used in the Fixed Effects Model:

$$MSE = \frac{1}{n-t} \sum_{i=1}^{t} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_{i.})^2 \quad \text{and} \quad MS_{TRT} = \frac{1}{t-1} \sum_{i=1}^{t} n_i (\bar{y}_{i.} - \bar{y}_{..})^2$$

Using the principles of MOM, the expected values of MS_{TRT} and MSE will be equated to their realizations and the resulting equations will yield the MOM estimates of the variance components: σ_A^2 and σ_e^2.

1. $E[MSE] = \sigma_e^2 \quad \Rightarrow \quad MSE = \hat{\sigma}_e^2$

    Derivation of the above result:

    $y_{ij} = \mu + A_i + e_{ij} \quad \Rightarrow \quad \bar{y}_{i.} = \mu + A_i + \bar{e}_{i.} \quad \Rightarrow$

    $y_{ij} - \bar{y}_{i.} = e_{ij} - \bar{e}_{i.} = (1 - \frac{1}{n_i}) e_{ij} - \frac{1}{n_i} \sum_{k \neq j} e_{ik} \quad \Rightarrow$

$$
\begin{aligned}
E[(y_{ij} - \bar{y}_{i.})^2] = Var(e_{ij} - \bar{e}_{i.}) &= Var\left(\left(1 - \frac{1}{n_i}\right) e_{ij} - \frac{1}{n_i} \sum_{k \neq j}^{n_i} e_{ik} \right) \\
&= \left(1 - \frac{1}{n_i}\right)^2 Var(e_{ij}) + \frac{1}{n_i^2} Var\left(\sum_{k \neq j}^{n_i} e_{ik} \right) \\
&= \left(1 - \frac{1}{n_i}\right)^2 \sigma_e^2 + \frac{1}{n_i^2}(n_i - 1)\sigma_e^2 = \left(1 - \frac{1}{n_i}\right) \sigma_e^2
\end{aligned}
$$

$$E[MSE] \;=\; E\left[\frac{1}{n-t}\sum_{1=i}^{t}\sum_{j=1}^{n_i}(y_{ij}-\bar{y}_{i.})^2\right]$$

$$=\;\frac{1}{n-t}\sum_{1=i}^{t}\sum_{j=1}^{n_i}E[(y_{ij}-\bar{y}_{i.})^2]$$

$$=\;\frac{1}{n-t}\sigma_e^2\sum_{i=1}^{t}\sum_{j=1}^{n_i}(1-\frac{1}{n_i})$$

$$=\;\frac{1}{n-t}\sigma_e^2(n-t)=\sigma_e^2$$

2. $E[MS_{TRT}] = r_o\sigma_A^2 + \sigma_e^2 \;\;\Rightarrow\;\; MS_{TRT} = r_o\hat{\sigma}_A^2 + \hat{\sigma}_e^2 \;\;\Rightarrow$

$\hat{\sigma}_A^2 = \frac{1}{r_o}(MS_{TRT} - MSE)$

where $r_o = \frac{1}{t-1}\left(n - \sum_{i=1}^{t}\frac{n_i^2}{n}\right)$

If $n_1 = n_2 = \cdots = n_t = r$, then $r_o = r$

Derivation of the above result: From the model $y_{ij} = \mu + A_i + e_{ij} \;\;\Rightarrow$

$\bar{y}_{i.} = \mu + A_i + \bar{e}_{i.}$ and $\bar{y}_{..} = \mu + \bar{A}_. + \bar{e}_{..}$, with $\bar{A}_. = \frac{1}{n}\sum_{i=1}^{t}n_iA_i \;\;\Rightarrow$

$\bar{y}_{i.} - \bar{y}_{..} = (A_i - \bar{A}_.) + (\bar{e}_{i.} - \bar{e}_{..})$ and

$(A_i - \bar{A}_.) = \left((1-\frac{n_i}{n})A_i - \frac{1}{n}\sum_{k\neq i}^{t}n_kA_k\right)$

$(\bar{e}_{i.} - \bar{e}_{..}) = \left((\frac{1}{n_i} - \frac{1}{n})\sum_{j=1}^{n_i}e_{ij} - \frac{1}{n}\sum_{k\neq i}^{t}\sum_{j=1}^{n_k}e_{kj}\right) \;\;\Rightarrow$

$$E[(A_i - \bar{A}_.)^2] = Var(A_i - \bar{A}_.) \quad = \quad Var\left(\left(1 - \frac{n_i}{n}\right)A_i - \frac{1}{n}\sum_{k \neq i}^{t} n_k A_k\right)$$

$$= \quad \left(1 - \frac{n_i}{n}\right)^2 Var(A_i) + \frac{1}{n^2}\sum_{k \neq i}^{t} n_k^2 Var(A_k)$$

$$= \quad \left[\left(1 - \frac{n_i}{n}\right)^2 + \frac{1}{n^2}\sum_{k \neq i}^{t} n_k^2\right]\sigma_A^2$$

$$= \quad \left(1 - \frac{2n_i}{n} + \frac{1}{n^2}\sum_{k=1}^{t} n_k^2\right)\sigma_A^2$$

$$E[(\bar{e}_{i.} - \bar{e}_{..})^2] = Var(e_{i.} - \bar{e}_{..}) \quad = \quad Var\left(\left(\frac{1}{n_i} - \frac{1}{n}\right)\sum_{j=1}^{n_i} e_{ij} - \frac{1}{n}\sum_{k \neq i}^{t}\sum_{j=1}^{n_k} e_{kj}\right)$$

$$= \quad \left(\frac{1}{n_i} - \frac{1}{n}\right)^2\sum_{j=1}^{n_i} Var(e_{ij}) + \frac{1}{n^2}\sum_{k \neq i}^{t}\sum_{j=1}^{n_k} Var(e_{kj})$$

$$= \quad \left(\frac{1}{n_i} - \frac{1}{n}\right)^2 n_i \sigma_e^2 + \frac{1}{n^2}\sum_{k \neq i}^{t} n_k \sigma_e^2$$

$$= \quad \left[\left(\frac{1}{n_i} - \frac{1}{n}\right)^2 n_i + \frac{1}{n^2}\sum_{k \neq i}^{t} n_k\right]\sigma_e^2$$

$$= \quad \left(\frac{1}{n_i} - \frac{1}{n}\right)\sigma_e^2$$

$$E[MS_{TRT}] \quad = \quad \frac{1}{t-1}\sum n_i E[(\bar{y}_{i.} - \bar{y}_{..})^2]$$

$$= \quad \frac{1}{t-1}\sum_{i=1}^{t} n_i \left(E[(A_i - \bar{A}_.)^2] + E[(\bar{e}_{i.} - \bar{e}_{..})^2] + 2E[(A_i - \bar{A}_.)(\bar{e}_{i.} - \bar{e}_{..})]\right)$$

$$= \quad \frac{1}{t-1}\sum_{i=1}^{t} n_i \left(\left(1 - \frac{2n_i}{n}\right) + \frac{1}{n^2}\sum_{k=1}^{t} n_k^2\right)\sigma_A^2 + \frac{1}{t-1}\sum_{i=1}^{t} n_i \left(\frac{1}{n_i} - \frac{1}{n}\right)\sigma_e^2 + 0$$

$$= \quad \frac{1}{t-1}\left(n - \frac{1}{n}\sum_{i=1}^{t} n_i^2\right)\sigma_A^2 + \frac{(t-1)\sigma_e^2}{t-1} = r_o \sigma_A^2 + \sigma_e^2$$

One of the problems with the MOM estimator of σ_A^2 is that it could be negative. The following are suggestions from Kuehl, *Design of Experiments* for dealing with this situation:

Courses of Action for Dealing with Negative Variance Estimates

1. Accept the estimate as evidence of a true value of zero and use zero as the estimate, recognizing that the estimator is no longer unbiased.

- Bad Choice. A negative estimate is an indication of problems in the data or the procedure.

2. Retain the negative estimate, recognizing that subsequent calculations using the results may not make much sense

- Ridiculous suggestion: A negative estimate of a parameter which is non-negative.

3. Interpret the negative estimate as an indication of an incorrect model and/or outliers in the data set

4. Utilize a method different from the MOM for estimation of variance components

- Use MLE or REML methods (See Handout # 9)

5. Collect more data and analyze the new data separately or in conjunction with the existing data and hope that the increased information will yield positive estimates.

- Not a good suggestion. What if you continue to obtain negative estimates. When does the process end?????

Confidence Intervals for Variance Components: σ_A^2, and σ_e^2:

In the random effects model: $y_{ij} = \mu + A_i + e_{ij}$ we have

A_i's iid $N(0, \sigma_A^2)$ r.v.s and e_{ij}'s iid $N(0, \sigma_e^2)$ r.v.s with point estimators:

$$\hat{\sigma}_A^2 = \frac{1}{r_o}\left(MS_{TRT} - MSE\right) \quad \text{and} \quad \hat{\sigma}_e^2 = MSE$$

1. $100(1-\alpha)\%$ Confidence Interval for σ_e^2:

$$\left[\frac{SSE}{\chi_{\alpha/2,n-t}^2}, \frac{SSE}{\chi_{1-\alpha/2,n-t}^2}\right]$$

where $\chi_{\alpha/2,n-t}^2$ and $\chi_{1-\alpha/2,n-t}^2$ are the upper and lower $\alpha/2$ percentiles, respectively, from the Chi-square distribution with $df = df_E = n - t$

This follows from the Pivot: $\frac{1}{\sigma_e^2}SSE$

which has a Chi-square distribution with $df = df_E = n - t$

Derivation:

We have shown that $\frac{1}{\sigma_e^2}SSE$ has a Chi-square distribution with $df = df_E = n-t$. Therefore,

$$1 - \alpha = P\left[\chi_{1-\alpha/2}^2 \leq \frac{1}{\sigma_e^2}SSE \leq \chi_{\alpha/2,n-t}^2\right] \quad \Rightarrow$$

$$1 - \alpha = P\left[\frac{1}{\chi_{1-\alpha/2}^2} \geq \frac{1}{SSE}\sigma_e^2 \geq \frac{1}{\chi_{\alpha/2,n-t}^2}\right] \quad \Rightarrow$$

$$1 - \alpha = P\left[\frac{SSE}{\chi_{1-\alpha/2}^2} \geq \sigma_e^2 \geq \frac{SSE}{\chi_{\alpha/2,n-t}^2}\right] \quad \Rightarrow$$

2. Conservative $100(1 - \alpha)\%$ Confidence Interval for σ_A^2:

$$\left[\frac{\frac{1}{r}SS_{TRT}\left(1 - \frac{F_U}{F_o}\right)}{C_U}, \quad \frac{\frac{1}{r}SS_{TRT}\left(1 - \frac{F_L}{F_o}\right)}{C_L} \right],$$

- Interval requires equal sample sizes, $n_1 = \cdots = n_t = r$,

- Interval is conservative (true coverage probability at least $100(1 - \alpha)\%$)

- $F_U = F_{\alpha/4, t-1, n-t}$, and $F_L = F_{1-\alpha/4, t-1, n-t} = \frac{1}{F_{\alpha/4, n-t, t-1}}$ are

- upper and lower $\alpha/4$ percentiles, respectively, from the F-distribution with $df = t-1, n-t$,

- $F_o = \frac{MS_{TRT}}{MSE}$ from the AOV,

- $C_U = \chi^2_{\alpha/4, t-1}$, and $C_L = \chi^2_{1-\alpha/4, t-1}$ are the upper and lower $\alpha/4$ percentiles, respectively,

  from the Chi-square distribution with $df = df_{TRT} = t - 1$

  Derviation:

  (a) $\frac{1}{\sigma_e^2 + r\sigma_A^2}SS_{TRT}$ has a Chi-square distribution with $df = t - 1$

  $$1 - \frac{\alpha}{2} = P\left[C_L \leq \frac{SS_{TRT}}{\sigma_e^2 + r\sigma_A^2} \leq C_U\right] \quad \Rightarrow$$

  $$1 - \frac{\alpha}{2} = P\left[\frac{1}{C_L} \geq \frac{\sigma_e^2 + r\sigma_A^2}{SS_{TRT}} \geq \frac{1}{C_U}\right] \quad \Rightarrow$$

  $$1 - \frac{\alpha}{2} = P\left[\frac{SS_{TRT}}{C_L} \geq \sigma_e^2 + r\sigma_A^2 \geq \frac{SS_{TRT}}{C_U}\right] \quad \Rightarrow$$

  $$\left[\frac{SS_{TRT}}{C_U}, \frac{SS_{TRT}}{C_L}\right] \text{ is a } 100(1 - \alpha/2)\% \text{ confidence interval for } \sigma_e^2 + r\sigma_A^2$$

  (b)

  $$F_o\frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} = \frac{MS_{TRT}}{MSE}\frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} = \frac{\left(\frac{SS_{TRT}}{\sigma_e^2 + r\sigma_A^2}\right)/(t - 1)}{\left(\frac{SSE}{\sigma_e^2}\right)/(n - t)}$$

  has an F-distribution with $df = t - 1, n - t$

  $$P\left[F_L \leq F_o\frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} \leq F_U\right] = 1 - (\alpha/4 + \alpha/4) \quad \Rightarrow$$

  $$1 - \frac{\alpha}{2} = P\left[\frac{1}{F_L} \geq \frac{1}{F_o}\frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2} \geq \frac{1}{F_U}\right] \quad \Rightarrow$$

  $$1 - \frac{\alpha}{2} = P\left[\frac{F_o}{F_L} \geq \frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2} \geq \frac{F_o}{F_U}\right] \quad \Rightarrow$$

  $$\left[\frac{F_o}{F_U}, \frac{F_o}{F_L}\right] \text{ is a } 100(1 - \alpha/2) \text{ confidence interval for } \frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2}$$

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(c) Define the event $A = \left\{ \frac{SS_{TRT}}{C_U} \leq \sigma_e^2 + r\sigma_A^2 \leq \frac{SS_{TRT}}{C_L} \right\}$, then

$$P[A] = 1 - \alpha/2$$

(d) Define the event $B = \left\{ \frac{F_o}{F_U} \leq \frac{\sigma_e^2 + r\sigma_A^2}{\sigma_e^2} \leq \frac{F_o}{F_L} \right\}$, then

$$P[B] = 1 - \alpha/2$$

$$B = \left\{ \frac{F_U}{F_o} \geq \frac{\sigma_e^2}{\sigma_e^2 + r\sigma_A^2} \geq \frac{F_L}{F_o} \right\}$$

$$B = \left\{ 1 - \frac{F_U}{F_o} \leq \frac{r\sigma_A^2}{\sigma_e^2 + r\sigma_A^2} \leq 1 - \frac{F_L}{F_o} \right\}$$

$$B = \left\{ \frac{1}{r}(\sigma_e^2 + r\sigma_A^2)\left(1 - \frac{F_U}{F_o}\right) \leq \sigma_A^2 \leq \frac{1}{r}(\sigma_e^2 + r\sigma_A^2)\left(1 - \frac{F_L}{F_o}\right) \right\}$$

(e) $A \cap B \subseteq \left\{ \frac{1}{r}\frac{SS_{TRT}}{C_U}\left(1 - \frac{F_U}{F_o}\right) \leq \sigma_A^2 \leq \frac{1}{r}\frac{SS_{TRT}}{C_L}\left(1 - \frac{F_L}{F_o}\right) \right\} = C.I.$

(f) $P[C.I.] \geq P[A \cap B] = 1 - P[A^c \cup B^c] \geq 1 - (P[A^c] + P[B^c]) = 1 - \left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) = 1 - \alpha$

3. In *PROC MIXED*, SAS provides an approximate (large sample) C.I. when the n_i's are unequal.

Test of $H_o : \sigma_A = 0$ versus $H_1 : \sigma_A > 0$

Using our results from Quadratic Forms and the F-Distribution, we have the following:

If W_1 and W_2 are independent random variables such that $W_1/E[W_1]$ and $W_2/E[W_2]$ have chi-square distributions with $df = \nu_1$ and ν_2, respectively, then

$$F = \frac{\left(\frac{W_1}{E[W_1]}\right)/\nu_1}{\left(\frac{W_2}{E[W_2]}\right)/\nu_2}$$

has an F-distribution with $df = \nu_1, \nu_2$.

Using our results on Quadratic Forms, if $n_1 = n_2 = \cdots = n_t = r$, then

1. $SS_{TRT}/(\sigma_e^2 + r\sigma_A^2)$ has a chi-square distribution with $df = t - 1$

2. SSE/σ_e^2 has a chi-square distribution with $df = n - t$

3. SS_{TRT} and SSE are independent

4. Therefore,

$$F = \frac{\frac{SS_{TRT}}{(t-1)(\sigma_e^2+r\sigma_A^2)}}{\frac{SSE}{(n-t)\sigma_e^2}} = \frac{MS_{TRT}}{MSE}\frac{\sigma_e^2}{\sigma_e^2+r\sigma_A^2} = \frac{MS_{TRT}}{MSE}\frac{1}{1+r\tau} \quad \text{with} \quad \tau = \frac{\sigma_A^2}{\sigma_e^2}$$

   has an F-distribution with $df = t - 1, n - t$.

5. Under $H_o : \sigma_A^2 = 0 \Rightarrow \tau = 0 \Rightarrow \frac{1}{1+r\tau} = 1$ which implies that

   $\frac{MS_{TRT}}{MSE}$ has a central F-distribution with $df = t - 1, n - t$.

6. Under $H_1 : \sigma_A^2 > 0$, $\frac{1}{1+r\tau} \neq 1$ which implies that

   $\frac{MS_{TRT}}{MSE}\frac{1}{1+r\tau}$ has a central F-distribution with $df = t - 1, n - t$ and hence we need to know the value of τ in order to compute power.

7. Decision Rule: Reject H_o at level α if $F_o = \frac{MS_{TRT}}{MSE} \geq F_{\alpha, t-1, n-t}$

8. $p - value = P[F \geq F_o] = 1 - G(F_o),$

   where $G(\cdot)$ is the cdf of a central F-distribution with $df = t - 1, n - 1$ and F_o is the computed value of $\frac{MS_{TRT}}{MSE}$.

   We are able to compute the p-value because this computation is done under the condition that H_o is true, that is, $\tau = 0$.

9. With $\tau = \sigma_A^2/\sigma_e^2$, the test of $H_o : \sigma_A = 0$ versus $H_1 : \sigma_A > 0$ has power function given by

$$
\begin{aligned}
\text{Power at specified } \tau = \gamma(\tau) &= P[\text{Reject } H_o \text{ for specified } \tau] \\
&= P\left[\frac{MS_{TRT}}{MSE} \geq F_{\alpha,t-1,n-t}\right] \\
&= P\left[\frac{MS_{TRT}}{MSE}\frac{1}{1+r\tau} \geq \frac{1}{1+r\tau}F_{\alpha,t-1,n-t}\right] \\
&= 1 - G\left(\frac{1}{1+r\tau}F_{\alpha,t-1,n-t}\right) \\
&= 1 - pf\left(\frac{1}{1+r\tau}F_{\alpha,t-1,n-t}\right) \quad \text{using R} \\
&= 1 - PROBF\left(\frac{1}{1+r\tau}F_{\alpha,t-1,n-t}\right) \quad \text{using SAS}
\end{aligned}
$$

where $G(\cdot)$ is the cdf of a **central** F-distribution with $df = t-1, n-t$

10. For specified values of α and τ, the central F-distribution can be used to calculate power and sample sizes.

EXAMPLE: Suppose $t = 5$, $r = 4$, $\alpha = .05$, and from previous studies $\sigma_e \approx 2.18$

Determine the power of an $\alpha = .05$ F-test of $H_o : \sigma_A = 0$ versus $H_1 : \sigma_A > 0$

to determine that $\sigma_A \geq 1.5$:

Need to calculate $\tau = \frac{\sigma_A^2}{\sigma_e^2}$.

From the above information, we have

$\tau = \frac{\sigma_A^2}{\sigma_e^2} = \frac{(1.5)^2}{(2.18)^2} = 0.4734$ and $F_{.05,5-1,20-5} = F_{.05,4,15} = 3.056$

$$
\begin{aligned}
\gamma(\sigma_A = 1.5) = \gamma(\tau = .4734) &= 1 - G_{t-1,n-t}\left(\frac{1}{1+(4)(.4734)}3.056\right) \\
&= 1 - G_{4,15}(1.056) = 1 - .588 = .412
\end{aligned}
$$

$G_{4,15}(1.056) = pf(1.056, 4, 15)$ using R or
$G_{4,15}(1.056) = PROBF(1.056, 4, 15)$ using SAS

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The graph given on the next page is Table X from the textbook.

We can use this graph to find the power:

1. Compute $\nu_1 = t - 1 = 5 - 1 = 4$ and $\nu_2 = n - t = (5)(4) - 5 = 15$

2. Compute $\lambda = \sqrt{1 + \frac{r\sigma_A^2}{\sigma_e^2}} = \sqrt{1 + r\tau} = \sqrt{1 + (4)(.4734)} = 1.70$

3. Use the set of graphs with $\nu_1 = 4$, locate graph for $\alpha = .05$ and $\nu_2 = 15$

4. Intersect a vertical line at $\lambda = 1.70$ on the $\alpha = .05$ horizontal scale with

   the $\nu_2 = 15$ line and read the power=0.412 on the vertical scale.

Table X *F* test power curves for random effects model analysis of variance

Power $(1 - \beta)$ vs λ (for $\alpha = 0.01$) $\rightarrow$ and λ (for $\alpha = 0.05$), with $\nu_1 = 4$.
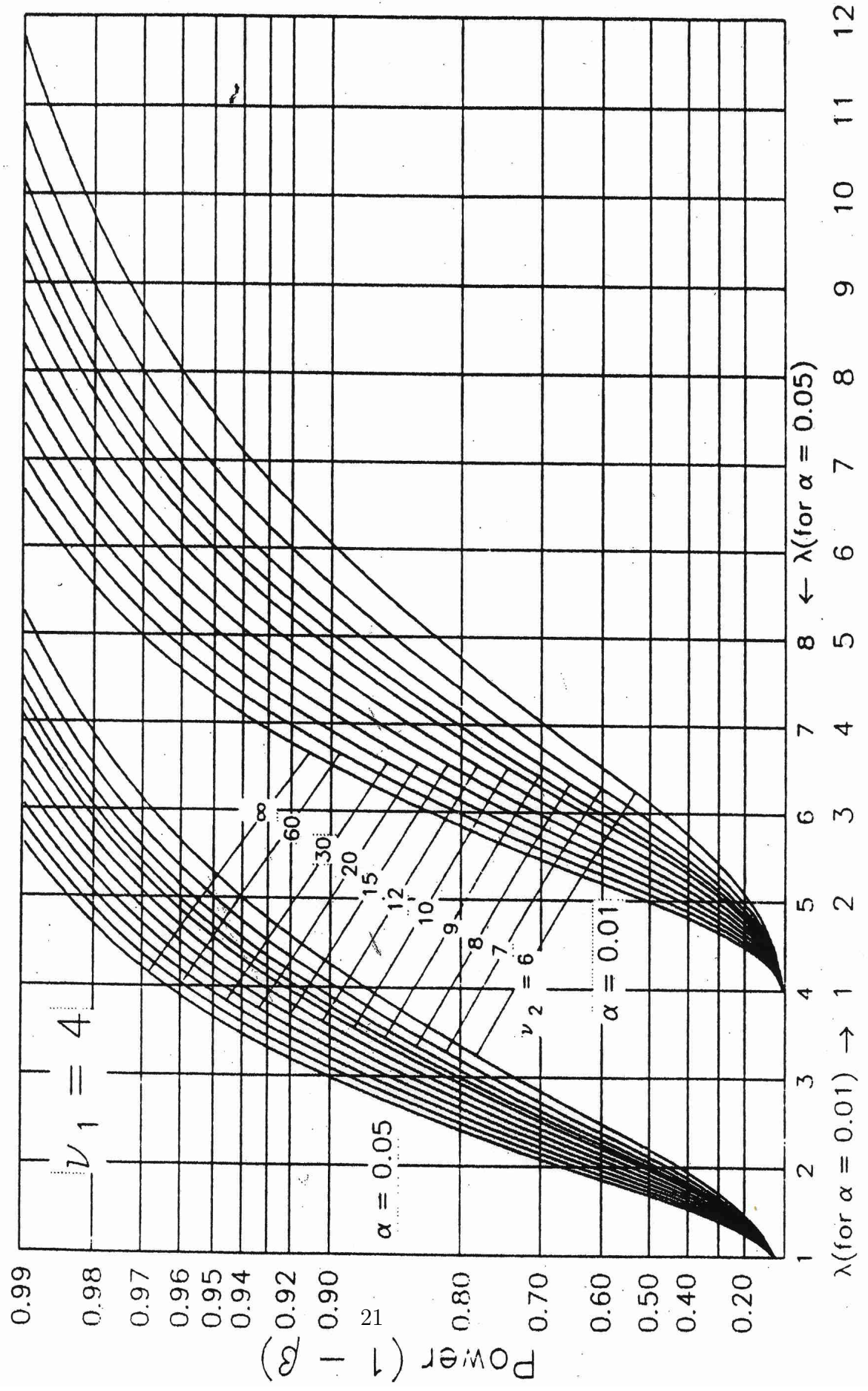
Determine the Minimum Number of Reps, r, to Achieve a Specified Power for Fixed Number of Treatments, t

For a fixed number of treatments t, determine the minimum r such that $\gamma(\tau) \geq \gamma_o$ whenever $\tau \geq \tau_o$.

That is, determine the minimum number of replications such that a level α test of $H_o : \sigma_A = 0$ vs $H_1 : \sigma_A > 0$ will have power of at least γ_o to detect that $\sigma_A \geq \sigma_e \sqrt{\tau_o}$.

That is, determine the minimum number of replications such that a level α test will have probability of Type II error of at most $\beta_o = 1 - \gamma_o$ whenever $\sigma_A \geq \sigma_e \sqrt{\tau_o}$.

1. Specify a value of t

2. Specify a value of α

3. Specify a value for γ_o

4. Specify a value for τ_o or

5. Specify a value for σ_A and an estimate of σ_e

Using an iterative procedure, determine the minimum value of r such that $\gamma(\tau) \geq \gamma_o$ whenever $\tau \geq \tau_o$.

The SAS or R program that was used in the fixed effects case can be easily modified for the random effects.

Just use the central F-cdf in place of the noncentral F-cdf.

How to select the value of τ_o

In a Quality Control Study, the process engineer is interested in finding Factors A such that σ_A is large relative to σ_y.

That is, σ_A is the variability that is Detectable (Special Cause Variability) or variability due to an *Identifiable* problem in the process.

The other component in $\sigma_y^2 = \sigma_A^2 + \sigma_e^2$ is σ_e, which is referred to as *Common Cause* variability or the variability that is just an inherent component in the design of the process.

Now if $\frac{\sigma_A}{\sigma_y} > C$, then the engineer wants to have a strong chance of detecting that Factor A is an important component in σ_y.

Thus, we have

$$\frac{\sigma_A}{\sigma_y} > C \quad \Rightarrow \quad \sigma_A^2 > C^2 \sigma_y^2 = C^2(\sigma_A^2 + \sigma_e^2) \quad \Rightarrow$$

$$\sigma_A^2(1 - C^2) > C^2 \sigma_e^2 \quad \Rightarrow$$

$$\tau = \frac{\sigma_A^2}{\sigma_e^2} > \frac{C^2}{1 - C^2} = \tau_o.$$

Thus, we can take $\tau_o = \frac{C^2}{1-C^2}$

SAS Code for Sample Size Calculations

```
options ps=55 ls=70 nocenter nodate;
SAS Program to compute sample size for experiments with random effects with
Fixed number of treatments,t, we want to determine number of reps, r for specified power;
*repsize,randomeffects,fixedt.SAS;
data;
input  r @@;
sigma_e=2.7566;
sigma_a=2.5;
t=5;
u1=t-1; u2=t*(r-1);
gamma=.85;
alpha=.05;
tau=(sigma_a/sigma_e)**2;
lambda=sqrt(1+(r)*(tau));
Fcr=finv(1-alpha,u1,u2);
C=(1/lambda**2)*Fcr;
power=1-probf(C,u1,u2);
cards;
5 6 7 8 9
run;
proc print; var r u1 u2 C Fcr lambda power;
run;
```

EXAMPLE: Suppose $t = 5$, $\alpha = .05$, $\gamma_o = .85$, and $\tau_o = .8225$

That is, for a Random Effects Experiment with $t = 5$ treatments, determine the minimum number of replications per treatment such that an $\alpha = .05$ test with have at least an 85% chance to reject H_o and declare $\sigma_A > 0$ whenever $\sigma_A^2 > .8225\sigma_e^2$.

Using Table X, the following specifications are needed:

1. $t = 5$

2. $\alpha = .05$ and $\gamma_o = .85$

3. $\nu_1 = t - 1 = 5 - 1 = 4$ and $\nu_2 = n - t = rt - t = t(r-1) = 5(r-1)$

4. $\lambda_o^2 = 1 + r\tau_o = 1 + .8225r$

5. Let $G_{t-1,r(t-1)}(\cdot)$ be the cdf of a central F-distribution with df $= t-1, r(t-1)$

6. $\gamma(\tau_o) = 1 - G_{t-1,t(r-1)}\left(\frac{1}{1+r\tau_o}F_{\alpha,t-1,t(r-1)}\right) = 1 - G_{t-1,r(t-1)}\left(\frac{1}{\lambda_o^2}F_{\alpha,t-1,t(r-1)}\right)$

7. Iteratively, determine the smallest value of r such that $\gamma(.8225) \geq .85$

From Table X or SAS Output, we obtain the following values:

r	$\nu_1 = 4$	$\nu_2 = 5(r-1)$	$F_{.05,4,5(r-1)}$	$\lambda = \sqrt{1 + .8225r}$	$\gamma(.8225)$
5	4	20	2.8661	2.26	$1 - G_{4,20}(.5606) = .6939$
6	4	25	2.7587	2.44	$1 - G_{4,25}(.4648) = .7609$
7	4	30	2.6896	2.60	$1 - G_{4,30}(.3980) = .8085$
8	4	35	2.6415	2.75	$1 - G_{4,35}(.3485) = .8433$
9	4	40	2.6060	2.90	$1 - G_{4,40}(.3101) = .8695$

Thus, the minimum value of r is 9. Thus, an $\alpha = .05$ test of the difference in $t = 5$ randomly selected treatments would require 9 reps per treatment to achieve a power of at least 0.85 whenever $\sigma_A^2 > .8225\sigma_e^2$.

Determine the Minimum Number of Treatments, t, to Achieve Specified Power for Fixed Number of Reps, r

For a fixed number of reps r, determine the minimum number of treatments t to randomly select such that $\gamma(\tau) \geq \gamma_o$ whenever $\tau \geq \tau_o$.

1. Specify a value of r

2. Specify a value of α

3. Specify a value for γ_o

4. Specify a value for τ_o or

5. Specify a value for σ_A and an estimate of σ_e

Using an iterative procedure, determine the minimum value of t such that $\gamma(\tau) \geq \gamma_o$ whenever $\tau \geq \tau_o$.

SAS Code for Sample Size Calculations

```
SAS Program) to compute sample size for experiments with random effects
*Fix number of reps, r, then determine number of treatments, t for Specified Power;
*trtsize,randomeffects,fixedr.SAS;
data;
input  t @@;
r=5;
u1=t-1;
u2=t*(r-1);
sigma_e=2.7566;
sigma_a=2.5;
gamma=.85;
alpha=.05;
tau=(sigma_a/sigma_e)**2;
lambda=sqrt(1+(r)*(tau));
Fcr=finv(1-alpha,u1,u2);
C=(1/lambda**2)*Fcr;
power=1-probf(C,u1,u2);
cards;
3 4 5 6 7 8 9
run;
proc print; var t u1 u2 Fcr lambda power;
run;
```

EXAMPLE: Suppose $r = 5$, $\alpha = .05$, $\gamma_o = .85$, and $\tau_o = .8225$

That is, for a Random Effects Experiment with $r = 5$ replications, determine the minimum number of treatments such that an $\alpha = .05$ test with have at least an 85% chance to reject H_o and declare $\sigma_A > 0$ whenever $\sigma_A^2 > .8225\sigma_e^2$.

Using Table X, the following specifications are needed:

1. $r = 5$

2. $\alpha = .05$ and $\gamma_o = .85$

3. $\nu_1 = t - 1$ and $\nu_2 = n - t = rt - t = t(r - 1) = t(5 - 1) = 4t$

4. $\lambda^2 = 1 + r\tau = 1 + (.8225)(5) = 5.1125 \quad \Rightarrow \lambda = 2.26$

5. $\gamma(\tau_o) = 1 - G_{t-1,r(t-1)}\left(\frac{1}{1+r\tau_o}F_{\alpha,t-1,t(r-1)}\right) = 1 - G_{t-1,r(t-1)}\left(\frac{1}{\lambda^2}F_{\alpha,t-1,t(r-1)}\right)$

6. Iteratively, determine the smallest value of t such that $\gamma(.8225) \geq .85$

From Table X or SAS Output, we obtain the following values:

t	$\nu_1 = t - 1$	$\nu_2 = 4t$	$F_{.05,t-1,4t)}$	$\lambda = 2.26$	$\gamma(.8225)$
3	2	12	3.8853	2.26	$1 - G_{2,12}(.7560) = .4889$
4	3	16	3.3289	2.26	$1 - G_{3,16}(.6335) = .6041$
5	4	20	2.8661	2.26	$1 - G_{4,20}(.5606) = .6939$
6	5	24	2.6208	2.26	$1 - G_{5,24}(.5126) = .7640$
7	6	28	2.4453	2.26	$1 - G_{6,28}(.4783) = .8187$
8	7	32	2.3127	2.26	$1 - G_{7,32}(.4524) = .8611$
9	8	36	2.2085	2.26	$1 - G_{8,36}(.4320) = .8939$

Thus, the minimum value of t is 8. Thus, an $\alpha = .05$ test would require randomly selecting $t = 8$ treatments along with $r = 5$ reps per treatment to achieve a power of at least 0.85 whenever $\sigma_A^2 > .8225\sigma_e^2$.

EXAMPLE: Fishing Nets

A manufacturer of commercial fishing nets produces the net material on a large number of machines. The company would like the machines to be as homogeneous as possible in order to produce netting that has a very uniform strength. The company is concerned that there is considerable amount of material that must be discarded, reworked, or downgraded to a lower quality product. This results in loss of revenue to the company. The process engineer, after examining the maintenance records of the machines, suspects that there may be a larger variation in material strength between machines relative to the usual variation in material produced from the same machine. The two sources of variation are measured by σ_M^2, the detectable variation between machines, referred to as the "Special Cause" variability, the variability due to Machine Differences. The second source of variability is measured by σ_e^2, the background noise or variability in the process, referred to as the "Common Cause" variability. The engineer decides to run a small pilot study by randomly selecting four machines at random and then selecting five samples of material from each of the selected machines for strength determinations. The 20 samples of material were then analyzed in random order. The results are given here.

Machine	Strength Measurements: y_{ij}					Mean:$\bar{y}_{i.}$
	1	2	3	4	5	
1	128	127	129	126	128	127.6
2	121	120	123	122	125	122.2
3	126	125	127	125	124	125.4
4	125	126	129	128	127	127.0

The above data was analyzed using the following SAS code (fishnet-random.sas):

```
* fishnet-random.sas;
ods html; ods graphics on;
option ls=80 ps=55 nocenter nodate;
title 'One-way ANOVA with Random Factor';

data fishnet;
array Y Y1-Y5;
input MACH $ Y1-Y5; do over Y; S=Y; output; end;
      drop Y1-Y5;
      label MACH = 'MACHINE' S = 'STRENGTH OF MATERIAL';
cards;
M1 128 127 129 126 128
M2 121 120 123 122 125
M3 126 125 127 125 124
M4 125 126 129 128 127
RUN;


title 'ANALYSIS USING PROC GLM';

PROC GLM;
CLASS MACH;
MODEL  S = MACH /E3 SS3 ;
RANDOM MACH / TEST;
RUN;


title 'ANALYSIS USING PROC MIXED-REML';

PROC MIXED METHOD=REML CL;
CLASS MACH;
MODEL  S = /S COVB ;
RANDOM MACH/CL ALPHA=.05;
RUN;
ods graphics off; ods html close;
```

ANALYSIS USING PROC GLM

        Class Level Information

Class   Levels   Values
MACH   4   M1 M2 M3 M4

Number of Observations Read   20
Number of Observations Used   20

Dependent Variable: S   STRENGTH OF MATERIAL

                              Sum of
Source   DF   Squares   Mean Square   F Value   Pr > F
Model   3   87.7500000   29.2500000   13.30   0.0001
Error   16   35.2000000   2.2000000
Corrected Total   19 122.9500000

Source   Type III Expected Mean Square
MACH   Var(Error) + 5 Var(MACH)

ANALYSIS USING PROC MIXED-REML

                    Class Level Information

Class   Levels   Values
MACH   4   M1 M2 M3 M4

                  Number of Observations

Number of Observations Read   20
Number of Observations Used   20
Number of Observations Not Used   0

              Covariance Parameter Estimates

Cov Parm   Estimate   Alpha   Lower   Upper

MACH   5.4100   0.05   1.6283   107.22
Residual   2.2000   0.05   1.2203   5.0958

                Solution for Fixed Effects

                        Standard
Effect   Estimate   Error   DF   t Value   Pr > |t|

Intercept   125.55   1.2093   3   103.82   <.0001

From the above, there is significant $(p-value < .0001)$ evidence that $\sigma_M > 0$. Furthermore, $\hat{\sigma}_M$ is considerably larger than $\hat{\sigma}_e$. An estimate of the variation in the strength measurements can be computed as

$$\hat{\sigma}_y^2 = \hat{\sigma}_M^2 + \hat{\sigma}_e^2 = 5.41 + 2.2 = 7.61.$$

This implies that the proportion of the total variation in the strength measurements attributed to the differences in the machines and the proportion attributed to the within machine variation are

Proportion Due to Machine Differences: $\frac{\hat{\sigma}_M^2}{\hat{\sigma}_y^2} = \frac{5.41}{7.61} = 71.1\%$

Proportion Due to Within Machine Differences: $\frac{\hat{\sigma}_e^2}{\hat{\sigma}_y^2} = \frac{2.2}{7.61} = 28.9\%$

Thus, the major source of variation is due to differences in the machines. This may be caused by faulty setup of the machine, ineffective supervision of the operators of the machine, infrequent or faulty maintenance, or poorly trained operators. Based on this pilot study the process engineer would like to design a more extensive study of the hundreds of machines used to produce the fish net material. A sample size calculation is needed to determine how many machines to sample and how many replications per machine. The engineer has decided on the number of replications needed per machine but wants assistance in determining how many machines to select.

C.I.'s and Tests of Hypotheses when n_i's are unequal

When the sample sizes are unequal, the F-statistic $\frac{MS_{TRT}}{MSE}$ does not have a multiple times a F-distribution, as in the equal sample size case. The random variables, MS_{TRT} and MSE are still independent and $\frac{1}{\sigma_e^2}MSE$ still has a chi-square distribution, but MS_{TRT} is no longer a constant times a chi-square distributed random variable. However, it is still possible to use the statistic $F = \frac{MS_{TRT}}{MSE}$ to test of $H_o : \sigma_A = 0$ versus $H_1 : \sigma_A > 0$ because under H_o Wald(1940) has shown that $\frac{MS_{TRT}}{MSE}$ has a Central F-Distribution with $df = t - 1, n - t$. Wald defines the following statistic:

$$w_i = \frac{\sigma_e^2}{Var(\bar{y}_{i.})} = \frac{\sigma_e^2}{\sigma_A^2 + \frac{\sigma_e^2}{n_i}} = \frac{n_i}{1 + n_i\tau} = \frac{1}{\frac{1}{n_i} + \tau}$$

$$h(\tau) = \sum_{i=1}^{t} w_i \left(\bar{y}_{i.} - \frac{\sum_{i=1}^{t} w_i \bar{y}_{i.}}{\sum_{i=1}^{t} w_i} \right)^2$$

Wald then shows that

$$F^* = \frac{h(\tau)/(t-1)}{MSE}$$

has an F-distribution with $df = t - 1, n - t$.

However, under $H_o : \sigma_A = 0 \Rightarrow \tau = \sigma_A^2/\sigma_e^2 = 0$,

$$\tau = 0 \quad \Rightarrow \quad w_i = \frac{n_i}{1 + n_i\tau} = n_i \quad \Rightarrow \quad h(\tau) = SS_{TRT} \quad \Rightarrow F^* = \frac{MS_{TRT}}{MSE}$$

Thus, under H_o, $F^* = F$. Thus, we can use the following decision rule:

Reject $H_o : \sigma_A = 0$ if $F_o \geq F_{\alpha, t-1, n-t}$ or

Reject $H_o : \sigma_A = 0$ if $p - value = 1 - G(F_o) \leq \alpha$ where $G(\cdot)$ is the cdf of a central F-distribution with $df = t - 1, n - t$ and F_o is the observed value of $\frac{MS_{TRT}}{MSE}$

However, the power function of the test is intractable, because the distribution of $\frac{MS_{TRT}}{MSE}$ under $H_1 : \sigma_A > 0$ is unknown. Simulations could be used to obtain approximations to the power function for given specifications of the sample sizes and parameters.

Confidence Intervals for σ_e^2 and σ_A^2:

1. $100(1-\alpha)\%$ Confidence interval for σ_e^2:

$$\left[\frac{SSE}{\chi^2_{\alpha/2,n-t}}, \frac{SSE}{\chi^2_{1-\alpha/2,n-t}}\right]$$

The above interval is an exact $100(1-\alpha)\%$ confidence interval even for unequal sample sizes, that is, n_is not all the same value.

2. $100(1-\alpha)\%$ Approximate Confidence interval for σ_A^2:

$$\left[\frac{kL(MS^*_{TRT})}{f_1(1+kL)}, \frac{kU(MS^*_{TRT})}{f_3(1+kU)}\right]$$

$$f_1 = \chi^2_{\alpha/2,t-1}, \quad f_2 = F_{\alpha/2,t-1,n-t}, \quad f_3 = \chi^2_{1-\alpha/2,t-1}, \quad f_4 = F_{1-\alpha/2,t-1,n-t}$$

$$m = min(n_1,\ldots,n_t), \quad M = max(n_1,\ldots,n_t), \quad k = \left(\sum_{i=1}^{t}\frac{1}{n_i}\right)^{-1}$$

$$MS^*_{TRT} = \frac{1}{t-1}\sum_{i=1}^{t}\left(\bar{y}_{i.} - \frac{1}{t}\sum_{i=1}^{t}\bar{y}_{i.}\right)^2, \quad L = \frac{MS^*_{TRT}}{f_2 MSE} - \frac{1}{m}, \quad U = \frac{MS^*_{TRT}}{f_4 MSE} - \frac{1}{M}$$

For details, see Burdick & Eckman(1986), *Journal of Statistical Computation and Simulation*, Vol. 25, pp. 259-272.

Subsampling in a Fixed Effects Experiment

In some experiments the researcher wants to obtain additional information concerning the variability of the response within the individual EU's. The EU may be subdivided and/or random samples of information are obtained on the each EU.

EXAMPLE: Turf Grass Investigation

A national golf association is interested in determining if the four leading root growth stimulators generate different mean root growth. A test area on a golf course is selected and twenty-four homogeneous plots of grass of equal size were identified. The four stimulators were then applied to six plots of grass each. After two months of treatment with the stimulators, three cores of a uniform size were randomly selected from each of the 24 plots. The cores were then analyzed in a random order for root weight in grams. The cores were selected to determine the amount of variability of the root growth over the plots. A large number of cores were misplaced and the experiment became unbalanced in that there are not six plots per stimulator and there are not three cores per plot. The reason that some of the cores are missing was not related to the treatments.

| Stimulator | Plots | | | | | | Mean $\bar{y}_{i..}$ |
	1	2	3	4	5	6	
S1	3.3,3.4,3.5	3.1,3.5,3.0	3.2,3.1,3.4	3.3,2.9,3.0	3.3,3.3,3.1	.	3.2267
(means)	(3.4)	(3.2)	(3.233)	(3.067)	(3.233)	.	
S2	3.8,3.7,4.0	3.5,3.8,3.9	3.6,3.4,3.8	3.4,3.7	3.6,3.7,3.6	3.5,3.9	3.6779
(means)	(3.833)	(3.733)	(3.6)	(3.55)	(3.633)	(3.7)	
S3	3.8,3.9,4.0	3.6,3.7,3.8	3.3,3.4	3.6,3.7	3.5,3.9	3.4,3.7	3.6516
(means)	(3.9)	(3.7)	(3.35)	(3.65)	(3.7)	(3.55)	
S4	4.3,4.3,4.4	4.1,3.9,3.8	4.2,4.1,3.9	3.7,3.9,4.0	.	.	4.05
(means)	(4.333)	(3.933)	(4.067)	(3.867)	.	.	

There are four treatments, the root growth stimulators, a fixed number of levels.

The EU's are the 21 (24-3) plots of land, a random effect.

The 57 (72-15) cores are subsamples, a random factor.

If the four root growth stimulators were randomly selected from a long list of root growth stimulators, then this would be a random effect also.

Thus, the sources of variation in the root weights are the four Stimulators (fixed effect), the Plots nested within Stimulators (random effect), and the Cores nested within Plots within Stimulators (random effect).

Models for Subsampling Experiments

Case 1: Fixed Treatment Effect:

Suppose that there are t treatments which are the only treatments of interest to the researcher, there are n_i E.U.'s randomly assigned to treatment i, there are m_{ij} subsamples on the jth E.U. receiving treatment i, and y_{ijk} is the response on the kth subsample from the jth E.U. receiving treatment i.

$$y_{ijk} = \mu + \tau_i + e_{ij} + d_{ijk}, \quad \text{for} \quad i = 1, \ldots, t; \quad j = 1, \ldots, n_i \quad k = 1, \ldots, m_{ij}$$

1. μ is the overall treatment mean;

2. τ_i is the effect of ith treatment;

3. e_{ij} is random effect of (i, j) E.U.;

4. d_{ijk} is random effect of (i, j, k) subsample;

5. Set $\tau_t = 0 \quad \Rightarrow \quad \mu = \mu_t; \quad \tau_i = \mu_i - \mu_t; \quad i = 1, \ldots, t - 1$,

6. e_{ij}'s are iid $N(0, \sigma_e^2)$

7. d_{ijk} are iid $N(0, \sigma_d^2)$

8. e_{ij}'s and d_{ijk}'s are independent

9. $\mu_i = E[y_{ijk}] = \mu + \tau_i$

10. $\sigma_y^2 = Var[y_{ijk}] = \sigma_e^2 + \sigma_d^2$

11. The multiple responses on the same EU are correlated:

$$\begin{aligned} Cov(y_{ijk}, y_{i'j'k'}) &= E[(y_{ijk} - \mu - \tau_i)(y_{i'j'k'} - \mu - \tau_i)] \\ &= E[e_{ij}e_{i'j'}] + E[d_{ijk}d_{i'j'k'}] + E[e_{ij}d_{i'j'k'}] + E[e_{i'j'}d_{ijk}] \end{aligned}$$

Thus, we have $Cov(y_{ijk}, y_{i'j'k'}) = E[e_{ij}e_{i'j'}] + E[d_{ijk}d_{i'j'k'}] + 0 \Rightarrow$

$$Cov(y_{ijk}, y_{i'j'k'}) = \begin{cases} 0 & if \quad i \neq i' \text{ or } j \neq j' \quad \text{(different EU's)} \\ \\ \sigma_e^2 & if \quad i = i', j = j', k \neq k' \quad \text{(same EU's, different Obs)} \end{cases}$$

Goal: Test for differences in the treatments

Cell Means Model: Test $H_o : \mu_1 = \mu_2 = \cdots = \mu_t$ versus H_1 : not all μ_i are equal

Effects Model: Test $H_o : \tau_1 = \tau_2 = \cdots = \tau_t = 0$ versus H_1 : at least one $\tau_i \neq 0$

Both Models: Test $H_o : \theta_\tau = 0$ versus $H_1 : \theta_\tau \neq 0$, where

$\theta_\tau = \frac{1}{t-1} \sum_{i=1}^{t} m_{i.}(\mu_i - \mu)^2$ for Cell Means Model

$\theta_\tau = \frac{1}{t-1} \sum_{i=1}^{t} m_{i.}(\tau_i)^2$ for Effects Model

where $m_{i.} = \sum_{j=1}^{n_i} m_{ij}$, the number of subsamples for the ith treatment.

Case 2: Random Treatment Effects:

Treatments are randomly selected from a population of treatments.

Model: $\quad y_{ijk} = \mu + A_i + e_{ij} + d_{ijk}, \quad$ for $\quad i = 1, \ldots, t; \quad j = 1, \ldots, n_i \quad k = 1, \ldots, m_{ij}$

1. μ is the overall treatment mean;

2. A_i is the effect of ith treatment;

3. A_i's are distributed iid $N(0, \sigma_A^2)$;

4. e_{ij} is random effect of (i, j) E.U.;

5. e_{ij}'s are distributed iid $N(0, \sigma_e^2)$,

6. d_{ijk} is random effect of (i, j, k) subsample;

7. d_{ijk} are distributed iid $N(0, \sigma_d^2)$

8. A_i's, e_{ij}'s and d_{ijk}'s are mutually independent

9. $\mu_i = E[y_{ijk}] = \mu$

10. $\sigma_y^2 = Var[y_{ijk}] = Var[\mu + A_i + e_{ij} + d_{ijk}] = \sigma_A^2 + \sigma_e^2 + \sigma_d^2$

11. $Cov(y_{ijk}, y_{i'j'k'}) = E[(y_{ijk} - \mu)(y_{i'j'k'} - \mu)] = E[A_i A_{i'}] + E[e_{ij} e_{i'j'}] + E[d_{ijk} d_{i'j'k'}] + 0 \Rightarrow$

$$Cov(y_{ijk}, y_{i'j'k'}) = \begin{cases} 0 & if \quad i \neq i' \quad \text{(different TRT's)} \\[2mm] \sigma_A^2 & if \quad i = i', j \neq j' \quad \text{(same TRT's, different EU's)} \\[2mm] \sigma_A^2 + \sigma_e^2 & if \quad i = i', j = j', k \neq k' \quad \text{(same TRT's, same EU's, diff. Obs)} \end{cases}$$

Goal: Obtain estimates of the variance components and evaluate the relative sizes of

$$\sigma_A^2, \qquad \sigma_e^2, \qquad \sigma_d^2$$

Sum of Squares for Both Fixed and Random Effects Models

Define the following Sum of Squares with the following notation: t = number of treatments,

n_i = number of EU's assigned to Trt i, $n. = \sum_{i=1}^{t} n_i$ = total number of EU's

m_{ij} = number of subsamples taken on the jth EU assigned to Trt i,

$m_{i.} = \sum_{j=1}^{n_i} m_{ij}$ = number of data values obtained from Trt i

1. Corrected Sum of Squares Total: SS_{TOT}

- Pooled Variation of y_{ijk}'s ignoring treatments, reps, and subsamples

$$SS_{TOT} = \sum_{i=1}^{t} \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} (y_{ijk} - \bar{y}_{...})^2$$

with $df = \sum_{ij} m_{ij} - 1 = n - 1$

2. Sum of Squares Subsample Within Rep: Sub(Rep,Trt)

- Pooled Variation Within Subsamples for Same EU -

   Variation Within EU or Due to Measurement Error

$$SS_{S(R,T)} = \sum_{i=1}^{t} \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} (y_{ijk} - \bar{y}_{ij.})^2$$

with $df = \sum_{i=1}^{t} \sum_{j=1}^{n_i} (m_{ij} - 1) = n - n.$

3. Sum of Squares Rep within Treatment: R(T)

- Pooled Variation Between Replications Within Same Treatment

$$SS_{R(T)} = \sum_{i=1}^{t} \sum_{j=1}^{n_i} m_{ij} (\bar{y}_{ij.} - \bar{y}_{i..})^2$$

with $df = \sum_{i=1}^{t} (n_i - 1) = n. - t$ = Number of EU's - Number of Treatments

4. Sum of Squares Treatment: TRT

- Variation Between the t Treatments

$$SS_{TRT} = \sum_{i=1}^{t} m_{i.} (\bar{y}_{i..} - \bar{y}_{...})^2$$

with $df = t - 1$

To determine the proper test statistic, it is necessary to derive the Expected Mean Squares:

Expected Mean Squares:

1. $E[MS_{TRT}] = \sigma_d^2 + C_1\sigma_e^2 + C_2\theta_\tau$ (Fixed Treatment Effects)

   $E[MS_{TRT}] = \sigma_d^2 + C_1\sigma_e^2 + C_2\sigma_A^2$ (Random Treatment Effects)

2. $E[MS_{R(T)}] = \sigma_d^2 + C_3\sigma_e^2$ (Random and Fixed Treatment Effects)

3. $E[MS_{S(R,T)}] = \sigma_d^2$ (Random and Fixed Treatment Effects)

The constants C_i are defined as:

$$C_1 = \frac{1}{t-1}\left(A - \frac{B}{n}\right), \qquad C_2 = \frac{1}{t-1}\left(n - \frac{D}{n}\right), \qquad C_3 = \frac{1}{n_. - t}\left(n - A\right)$$

where $A = \sum_{i=1}^{t}\sum_{j=1}^{n_i}\frac{m_{ij}^2}{m_{i.}}, \qquad B = \sum_{i=1}^{t}\sum_{j=1}^{n_i}m_{ij}^2, \qquad D = \sum_{i=1}^{t}m_{i.}^2$

Under $H_o : \theta_\tau = 0$,

$E[MS_{TRT}] = \sigma_d^2 + C_1\sigma_e^2$

$E[MS_{R(T)}] = \sigma_d^2 + C_3\sigma_e^2$

$E[MS_{S(R,T)}] = \sigma_d^2$

Case 1: Equal Number of Subsamples

When $m_{ij} = m$ for all (i, j), equal number of subsamples for all reps,

$A = \sum_{i=1}^{t} \sum_{i=1}^{n_i} \frac{m^2}{mn_i} = tm$

$B = \sum_{i=1}^{t} \sum_{j=1}^{n_i} m^2 = \sum_{i=1}^{t} n_i m^2 = mn$ because $n = \sum_{i=1}^{t} \sum_{j=1}^{n_i} m = m \sum_{i=1}^{t} n_i$

$D = \sum_{i=1}^{t} m_{i.}^2 = \sum_{i=1}^{t} (mn_i)^2 = m^2 \sum_{i=1}^{t} n_i^2$

$C_1 = \frac{1}{t-1} \left(A - \frac{B}{n} \right) = \frac{1}{t-1} \left(tm - \frac{mn}{n} \right) = m$

$C_2 = \frac{1}{t-1} \left(n - \frac{D}{n} \right) = \frac{1}{t-1} \left(trm - \frac{t(rm)^2}{n} \right) = rm$ (If also $n_i = r$)

$C_3 = \frac{1}{n_{.}-t} \left(n - A \right) = \frac{1}{n_{.}-t} \left(mn_{.} - tm \right) = m$

and hence $C_1 = C_3$ which implies that under H_o, $E[MS_{TRT}] = E[MS_{R(T)}]$.

Thus, $F = \frac{MS_{TRT}}{MS_{R(T)}}$ is a test statistic for testing

either $H_o : \theta_\tau = 0$ vs $H_1 : \theta_\tau > 0$ or testing $H_o : \sigma_A = 0$ vs $H_1 : \sigma_A > 0$

In either case, $F = \frac{MS_{TRT}}{MS_{R(T)}}$ has an F-distribution with $df = t - 1, n_{.} - t$ under H_o.

Thus, reject $H_o : \theta_\tau = 0$ if $F_o \geq F_{\alpha, t-1, n_{.}-t}$

and reject $H_o : \sigma_A = 0$ if $F_o \geq F_{\alpha, t-1, n_{.}-t}$

Also, we can compute $p - value = 1 - G(F_o)$, where $G(\cdot)$ is the cdf of an F-distribution with $df = t - 1, n_{.} - t$

and F_o is the computed value of $\frac{MS_{TRT}}{MS_{R(T)}}$

Thus, reject H_o is p-value $\leq \alpha$.

Under $H_1 : \theta_\tau > 0$, $F = \frac{MS_{TRT}}{MS_{R(T)}}$ has a non-central F-distribution with df = t-1, t(n.-1) and non-centrality parameter

$$\Delta = \frac{mr \sum_{i=1}^{t} (\mu_i - \bar{\mu})^2}{\sigma_d^2 + r\sigma_e^2} = \frac{mr \sum_{i=1}^{t} (\mu_i - \bar{\mu})^2}{\sigma_{S(R,T)}^2 + r\sigma_{R(T)}^2}$$

This result will be used to determine the power function and in computing sample size when we have subsampling. The power function is given by

$$\gamma(\Delta) = 1 - G(F_{\alpha, t-1, t(n.-1)})$$

where G is the cdf of a non-central F-distribution with df = t-1, t(n.-1) and non-centrality parameter

$\Delta = \frac{mr \sum_{i=1}^{t} (\mu_i - \bar{\mu})^2}{\sigma_d^2 + r\sigma_e^2}$, when we restrict $m_{ij} = m$, $n_i = r \Rightarrow n = trm$

Case 2: Unequal Number of Subsamples

When n_{ij}'s are not all equal, then $C_1 \neq C_3$ and $\frac{MS_{TRT}}{MS_{R(T)}}$ does not have an F-distribution.

An approximate test statistic can be constructed by considering a linear combination of $MS_{R(T)}$ and $MS_{S(R)}$:

$M = a_1 MS_{R(T)} + a_2 MS_{S(R)}$ such that under H_o we have $E[M] = E[MS_{TRT}]$. That is, select a_1 and a_2 such that

$a_1 E[MS_{R(T)}] + a_2 E[MS_{S(R)}] = E[MS_{TRT}] \implies$

$a_1 (\sigma_d^2 + C_3 \sigma_e^2) + a_2 \sigma_d^2 = \sigma_d^2 + C_1 \sigma_e^2 \implies$

$a_1 + a_2 = 1$ and $a_1 C_3 = C_1 \implies a_1 = \frac{C_1}{C_3} \quad a_2 = 1 - \frac{C_1}{C_3}$

Thus, a potential test statistic is $F = \frac{MS_{TRT}}{M}$.

We need to determine the distribution of F under H_o:

Satterthwaite Approximation:

Let $M_1, \ldots, M_k$ be independent Mean Squares from the AOV having $df = \nu_1, \ldots, \nu_k$.

Let $M = \sum_{i=1}^{k} a_i M_i$ and $\nu = \frac{M^2}{\sum_{i=1}^{k} \frac{(a_i M_i)^2}{\nu_i}}$

Then $\frac{\nu M}{E[M]}$ has an approximate chi-square distribution with $df = \nu$

Applying the Satterthwaite Approximation to $F = \frac{MS_{TRT}}{M}$ yields the result that

F has under H_o an approximate F-distribution with $df = t - 1, \nu$, where

$$\nu = \frac{M^2}{\frac{\left(\frac{C_1}{C_3}\right)^2 (MS_{R(T)})^2}{(n_.-t)} + \frac{\left(1 - \frac{C_1}{C_3}\right)^2 (MS_{S(R)})^2}{(n - n_.)}}$$

This result follows from

1. $\frac{SS_{TRT}}{E[MS_{TRT}]}$ has a chi-square distribution with $df = t - 1$

2. $\frac{\nu M}{E[M]}$ has approximately a chi-square distribution with $df = \nu$

3. MS_{TRT} is independent of $MS_{R(T)}$ and $MS_{S(R)}$ and hence of M

4. Under H_o, $E[M] = E[MS_{TRT}]$

5. Under H_o, $F = \frac{MS_{TRT}}{M} = \frac{SS_{TRT}/(t-1)E[MS_{TRT}]}{\nu M/\nu E[M]}$

   has approximately an F-distribution with $df = t - 1, \nu$

Thus, reject $H_o : \theta_\tau = 0$ or reject $H_o : \sigma_A = 0$ if $F \geq F_{\alpha, t-1, \nu}$ or

with $p - value \approx 1 - G(F)$, where $G(\cdot)$ is the cdf of an F-distribution with $df = t - 1, \nu$

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Estimation of μ_i in Fixed Effects Model

When the treatments have fixed levels, then estimates of μ_i's are given by

The LSE of μ_i is $\hat{\mu}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \bar{y}_{ij.}$

Note that when we have unequal numbers of subsamples, i.e., m_{ij}'s are not equal,

$\hat{\mu}_i$ is not equal to $\bar{y}_{i..} = \frac{1}{m_{i.}} \sum_{j=1}^{n_i} \sum_{k=1}^{m_{ij}} y_{ijk}$

This results from the fact that the subsamples are not valid replications of the treatment means. In fact, the AOV F-test for testing differences in the treatment means is equivalent to running an AOV on the means over the subsamples: $\bar{y}_{ij.}$

To obtain a C.I. for μ_i we need to compute the variance of $\hat{\mu}_i$

$$
\begin{aligned}
Var(\hat{\mu}_i) &= Var\left(\frac{1}{n_i} \sum_{j=1}^{n_i} \bar{y}_{ij.} \right) \\
&= Var\left(\frac{1}{n_i} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} (\mu + \tau_i + e_{ij} + d_{ijk}) \right) \\
&= Var\left(\bar{e}_{i.} + \frac{1}{n_i} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} d_{ijk} \right) \\
&= Var(\bar{e}_{i.}) + Var\left(\frac{1}{n_i} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} \sum_{k=1}^{m_{ij}} d_{ijk} \right) \\
&= \frac{\sigma_e^2}{n_i} + \frac{1}{n_i^2} \sum_{j=1}^{n_i} \frac{1}{m_{ij}^2} \sum_{k=1}^{m_{ij}} \sigma_d^2 = \frac{\sigma_e^2}{n_i} + \frac{\sigma_d^2}{n_i^2} \sum_{j=1}^{n_i} \frac{1}{m_{ij}}
\end{aligned}
$$

When we have equal number of subsamples and equal number of reps, that is, $m_{ij} = m$ and $n_i = r$, we have

$Var(\hat{\mu}_i) = \frac{\sigma_e^2}{n_i} + \frac{\sigma_d^2}{n_i^2} \sum_{j=1}^{n_i} \frac{1}{m_{ij}} = \frac{\sigma_e^2}{r} + \frac{\sigma_d^2}{r^2} \sum_{j=1}^{r} \frac{1}{m} = \frac{\sigma_d^2 + m\sigma_e^2}{rm} = \frac{EMS_{R(T)}}{rm}$

Therefore, in the case when $m_{ij} = m$ and $n_i = r$, our estimator of the standard error of $\hat{\mu}_i$ would be

$\widehat{SE}(\hat{\mu}_i) = \sqrt{\frac{MS_{R(T)}}{rm}}$

When we have an unequal number of subsamples and/or an unequal number of reps, that is, $m_{ij} \neq m$ and/or $n_i \neq r$, then to obtain an estimate of $Var(\hat{\mu}_i)$, it is necessary to first obtain estimates of σ_e^2 and σ_d^2:

Using an AOV-MOM procedure, we have

$$E[MS_{S(R)}] = \sigma_d^2 \quad \Rightarrow \quad \hat{\sigma}_d^2 = MS_{S(R)}$$

$$E[MS_{R(T)}] = \sigma_d^2 + C_3\sigma_e^2 \quad \Rightarrow \quad \hat{\sigma}_e^2 = \frac{MS_{R(T)} - MS_{S(R)}}{C_3}$$

An approximate $100(1-\alpha)\%$ confidence interval for μ_i is given by

$$\hat{\mu}_i \pm t_{\alpha/2,\nu}\widehat{SE}(\hat{\mu}_i) \quad \text{where}$$

$\widehat{SE}(\hat{\mu}_i) = \sqrt{\frac{\hat{\sigma}_e^2}{n_i} + \frac{\hat{\sigma}_d^2}{n_i^2}\sum_{j=1}^{n_i}\frac{1}{m_{ij}}}$ and
ν is obtained from the Satterthwaite Approximation.:

$$\nu = \frac{\left(MS_{R(T)} - MS_{S(R)}\right)^2}{\frac{(MS_{R(T)})^2}{n_. - t} + \frac{(MS_{S(R)})^2}{n - n_.}}$$

Multiple comparisons of the t treatment means can be obtained using an approximate Tukey procedure after noting that $Var\left(\bar{y}_{k.} - \bar{y}_{h.}\right) = \sigma_e^2\left(\frac{1}{n_k} + \frac{1}{n_h}\right) + \sigma_d^2\left(\frac{1}{n_k^2}\sum_{j=1}^{n_k}\frac{1}{m_{kj}} + \frac{1}{n_h^2}\sum_{j=1}^{n_h}\frac{1}{m_{hj}}\right)$

Approximate Tukey-Kramer HSD Procedure: Use Proc Mixed with the command: LSMEANS TRT/ADJUST=TUKEY

1. State there is significant evidence at level α_o that the treatment means μ_k and μ_h are different if

$$|\hat{\mu}_k - \hat{\mu}_h| \geq HSD(t,\nu)$$

2. Simultaneous $100(1-\alpha_o)\%$ confidence intervals for the absolute difference in all pairs of treatment means $\mu_k - \mu_h$ are given by

$$|\bar{y}_{k.} - \bar{y}_{h.}| \pm HSD(t,\nu) \text{ where } HSD(t,\nu) \approx q(\alpha_o,t,\nu)\sqrt{\frac{1}{2}[\widehat{SE}(\bar{y}_{k..} - \bar{y}_{h..})]^2}$$

$$
\begin{aligned}
HSD(t,\nu) &= q(\alpha_o,t,\nu)\sqrt{\frac{1}{2}\left(\frac{\hat{\sigma}_e^2}{n_k} + \frac{\hat{\sigma}_d^2}{n_k^2}\sum_{j=1}^{n_k}\frac{1}{m_{kj}} + \frac{\hat{\sigma}_e^2}{n_h} + \frac{\hat{\sigma}_d^2}{n_h^2}\sum_{j=1}^{n_h}\frac{1}{m_{hj}}\right)} \\
&= q(\alpha_o,t,\nu)\sqrt{m\hat{\sigma}_e^2 + \hat{\sigma}_d^2}\sqrt{\frac{1}{2}\left(\frac{1}{mn_k} + \frac{1}{mn_h}\right)} \quad \text{when} \quad m_{kj} = m_{hj} = m \\
&= q(\alpha_o,t,\nu)\sqrt{m\hat{\sigma}_e^2 + \hat{\sigma}_d^2}\sqrt{\frac{1}{mn}} \quad \text{when} \quad m_{kj} = m_{hj} = m, \quad n_k = n_h = n
\end{aligned}
$$

In general, ν is the df from the Satterthwaite Approximation and $\nu = n_. - t$ when $m_{ij} = m$
$q(\alpha_o,t,\nu)$ is given in Table VII in Kuehl or using R: $q(\alpha_o,t,\nu) = qtukey(1-\alpha_o,t,\nu)$

These results will be illustrated through our turfgrass example on page 34. The following SAS code will be applied to this example.

```
* turfgrass_sub.sas;
option ls=72 ps=55 nocenter nodate;
title 'One-way ANOVA with Subsampling Design';


DATA TURF; ARRAY X X1-X3;
INPUT STIM $ PLOT X1-X3 @@;
DO OVER X;Y=X;OUTPUT; END;DROP X1-X3;
LABEL STIM ='STIMULATOR' Y= 'ROOT WEIGHT';
cards;
S1 1 3.3  3.4  3.5 S1 2 3.1  3.5  3.0
S1 3 3.2  3.1  3.4 S1 4 3.3  2.9 3.0
S1 5 3.3  3.3  3.1 S1 6  .    .    .
S2 1 3.8  3.7  4.0 S2 2 3.5  3.8  3.9
S2 3 3.6  3.4  3.8 S2 4 3.4  3.7  .
S2 5 3.6  3.7  3.6 S2 6 3.5  3.9  .
S3 1 3.8  3.9  4.0 S3 2 3.6  3.7  3.8
S3 3 3.3 3.4  .   S3 4 3.6  .    3.7
S3 5 3.5  .    3.9 S3 6 3.4  .    3.7
S4 1 4.3  4.3  4.4 S4 2 4.1  3.9  3.8
S4 3 4.2  4.1  3.9 S4 4 3.7  3.9  4.0
S4 5  .    .    . S4 6  .    .    .
RUN;
TITLE 'ANALYSIS USING PROC GLM';
PROC GLM;
CLASS STIM  PLOT;
MODEL Y = STIM PLOT(STIM)/SS1 e1;
RANDOM PLOT(STIM)/TEST;
MEANS STIM PLOT(STIM);
LSMEANS STIM/STDERR PDIFF ADJUST=TUKEY;
RUN;




TITLE 'ANALYSIS USING PROC MIXED-REML';
PROC  MIXED METHOD=REML CL;
CLASS STIM  PLOT;
MODEL Y =STIM;
RANDOM PLOT(STIM)/CL ALPHA=.05;
LSMEANS STIM/CL ADJUST=TUKEY;
RUN;
```

ANALYSIS USING PROC GLM

        Class Level Information

Class   Levels   Values

STIM   4   S1 S2 S3 S4
PLOT   6   1 2 3 4 5 6
Number of Observations Read   72
Number of Observations Used   57
ANALYSIS USING PROC GLM

Dependent Variable: Y   ROOT WEIGHT

                            Sum of
Source   DF   Squares   Mean Square   F Value   Pr > F
Model   20   5.70315789   0.28515789   10.40   <.0001
Error   36   0.98666667   0.02740741
Corrected Total   56   6.68982456

Source   DF   Type I SS   Mean Square   F Value   Pr > F
STIM   3   4.61397337   1.53799112   56.12   <.0001
PLOT(STIM)   17   1.08918452   0.06406968   2.34   0.0159

Source   Type I Expected Mean Square
STIM   Var(Error) + 2.7964 Var(PLOT(STIM)) + Q(STIM)
PLOT(STIM)   Var(Error) + 2.6954 Var(PLOT(STIM))

Tests of Hypotheses for Mixed Model Analysis of Variance

Dependent Variable: Y   ROOT WEIGHT

Source	DF	Type III SS	Mean Square	F Value	Pr > F
STIM	3	4.613973	1.529866	23.50	<.0001
Error	16.477	1.078306	0.065443		

Error: 1.0375*MS(PLOT(STIM)) - 0.0375*MS(Error)

Source	DF	Type III SS	Mean Square	F Value	Pr > F
PLOT(STIM)	17	1.089185	0.064070	2.34	0.0159
Error: MS(Error)	36	0.986667	0.027407		

Least Squares Means
Adjustment for Multiple Comparisons: Tukey-Kramer

STIM	Y LSMEAN	Standard Error	Pr > |t|	LSMEAN Number
S1	3.22666667	0.04274530	<.0001	1
S2	3.67500000	0.04214743	<.0001	2
S3	3.64166667	0.04505750	<.0001	3
S4	4.05000000	0.04779070	<.0001	4

Least Squares Means for effect STIM
Pr > |t| for H0: LSMean(i)=LSMean(j)

Dependent Variable: Y

i/j	1	2	3	4
1		<.0001	<.0001	<.0001
2	<.0001		0.9486	<.0001
3	<.0001	0.9486		<.0001
4	<.0001	<.0001	<.0001	

Statement from SAS Documentation:

PROC GLM uses only the information pertaining to expected mean squares
when you specify the TEST option in the RANDOM statement and,
even then, only in the extra F tests produced by the RANDOM statement.
Other features in the GLM procedure –
including the results of the LSMEANS and ESTIMATE statements –
assume that all effects are fixed, so that all tests and
Pestimability checks for these statements are based on a fixed effects model,
even when you use a RANDOM statement.
Therefore, you should use the MIXED procedure to compute tests involving these features
that take the random effects into account;
see the section "PROC GLM versus PROC MIXED for Random Effects Analysis" and Chapter 46,
"The MIXED Procedure," for more information.

PROC GLM versus PROC MIXED for Random-Effects Analysis
Other SAS procedures that can be used to analyze models with random effects include the
MIXED and VARCOMP procedures. Note that, for these procedures, the random-effects
specification is an integral part of the model, affecting how both random and fixed
effects are fit; for PROC GLM, the random effects are treated in a post hoc fashion
after the complete fixed-effect model is fit. This distinction affects other features
in the GLM procedure, such as the results of the LSMEANS and ESTIMATE statements.
These features assume that all effects are fixed, so that all tests and
estimability checks for these statements are based on a fixed-effects model,
even when you use a RANDOM statement. Standard errors for estimates and
LS-means based on the fixed-effects model might be significantly smaller
than those based on a true random-effects model; in fact, some functions that are
estimable under a true random-effects model might not even be estimable under the
fixed-effects model. Therefore, you should use the MIXED procedure to compute tests
involving these features that take the random effects into account;

Class Level Information

Class	Levels	Values
STIM	4	S1 S2 S3 S4
PLOT	6	1 2 3 4 5 6

Number of Observations

Number of Observations Read	72
Number of Observations Used	57
Number of Observations Not Used	15

Covariance Parameter Estimates

Cov Parm	Estimate	Alpha	Lower	Upper
PLOT(STIM)	0.01362	0.05	0.005376	0.07839
Residual	0.02648	0.05	0.018710	0.04036

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
STIM	3	17	22.50	<.0001

```
                      Least Squares Means

                                Standard
Effect    STIMULATOR    Estimate    Error    DF    Lower     Upper

STIM      S1            3.2267      0.06743  17    3.1380    3.3153
STIM      S2            3.6779      0.06334  17    3.5892    3.7608
STIM      S3            3.6516      0.06529  17    3.5654    3.7489
STIM      S4            4.0500      0.07539  17    3.9509    4.1491
```

```
Effect    STIMULATOR    Estimate    Error    DF    Lower     Upper
```

```
                  Differences of Least Squares Means

                                      Standard
Effect  STIMULATOR  STIMULATOR  Estimate    Error   DF  t Value  Pr > |t|
STIM    S1          S2          -0.4513   0.09251   17   -7.67    <.0001
STIM    S1          S3          -0.4249   0.09386   17   -7.12    <.0003
STIM    S1          S4          -0.8233   0.10110   17  -13.06    <.0001
STIM    S2          S3          0.02637   0.09097   17    0.30    0.7754
STIM    S2          S4          -0.3721   0.09846   17   -6.03    0.0015
STIM    S3          S4          -0.3984   0.09973   17   -6.14    0.0009


Effect  STIMULATOR  STIMULATOR  Adjustment     Adj P   Alpha    Lower    Upper
STIM    S1          S2          Tukey-Kramer  <.0007    0.05  -0.6465  -0.2561
STIM    S1          S3          Tukey-Kramer  <.0015    0.05  -0.6229  -0.2269
STIM    S1          S4          Tukey-Kramer  <.0001    0.05  -1.0367  -0.6099
STIM    S2          S3          Tukey-Kramer  0.9912    0.05  -0.1656   0.2183
STIM    S2          S4          Tukey-Kramer  0.0074    0.05  -0.5798  -0.1643
STIM    S3          S4          Tukey-Kramer  0.0047    0.05  -0.6088  -0.1880

                                      Adj       Adj
Effect  STIMULATOR  STIMULATOR      Lower     Upper
STIM    S1          S2            -0.7142   -0.1883
STIM    S1          S3            -0.6917   -0.1581
STIM    S1          S4            -1.1108   -0.5358
STIM    S2          S3            -0.2322    0.2849
STIM    S2          S4            -0.6519   -0.09217
STIM    S3          S4            -0.6819   -0.1149
```

From the above we can conclude that the only pair of Stimulators which are not significantly different are S2 and S3. Thus we have the following groupings:

Simulator	S1	S3	S2	S4
Groupings	a	b	b	c