STAT 636, Fall 2015 - Assignment 2
Due Monday, September 21, 3:00pm Central
Online Students: Submit your assignment through WebAssign.
On-Campus Students: Email your assignment to the TA.

1. Consider the matrix
$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$
Without using a computer:
   (a) Find the eigenvalues and normalized eigenvectors of $\mathbf{A}$.
   (b) Write the spectral decomposition of $\mathbf{A}$.
   (c) Verify that the determinant of $\mathbf{A}$ equals the product of its eigenvalues.
   (d) The trace of a square matrix equals the sum of its diagonal elements. Verify that the trace of $\mathbf{A}$ equals the sum of its eigenvalues.
   (e) Is $\mathbf{A}$ orthogonal? Why or why not?
   (f) Is $\mathbf{A}$ positive definite? Why or why not?
   (g) Find $\mathbf{A}^{-1}$ and determine its eigenvalues and normalized eigenvectors.

2. Consider the matrices
$$\mathbf{A} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4.000 & 4.001 \\ 4.001 & 4.002001 \end{bmatrix}$$
These matrices are identical except for a small difference in the (2, 2) position. Also, the columns of $\mathbf{A}$ and $\mathbf{B}$ are nearly linearly dependent. Show that $\mathbf{A}^{-1} \approx (-3)\mathbf{B}^{-1}$. So, small changes - perhaps due to rounding - can result in substantially different inverses.

3. Derive expressions for the means and variances of the following linear combinations in terms of the means and covariances of the random variables X_1, X_2, and X_3.
   (a) $X_1 - 2X_2$.
   (b) $X_1 + 2X_2 - X_3$.
   (c) $3X_1 - 4X_2$ if X_1 and X_2 are independent (so, $\sigma_{12} = 0$).

4. Let $\boldsymbol{\mu}' = [1, 1]$, and consider the following covariance matrices
$$\mathbf{\Sigma}_1 = \begin{bmatrix} 1.00 & 0.80 \\ 0.80 & 1.00 \end{bmatrix} \quad \mathbf{\Sigma}_2 = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \quad \mathbf{\Sigma}_3 = \begin{bmatrix} 1.00 & -0.80 \\ -0.80 & 1.00 \end{bmatrix}$$
$$\mathbf{\Sigma}_4 = \begin{bmatrix} 1.00 & 0.40 \\ 0.40 & 0.25 \end{bmatrix} \quad \mathbf{\Sigma}_5 = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 0.25 \end{bmatrix} \quad \mathbf{\Sigma}_6 = \begin{bmatrix} 1.00 & -0.40 \\ -0.40 & 0.25 \end{bmatrix}$$
$$\mathbf{\Sigma}_7 = \begin{bmatrix} 0.25 & 0.40 \\ 0.40 & 1.00 \end{bmatrix} \quad \mathbf{\Sigma}_8 = \begin{bmatrix} 0.25 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \quad \mathbf{\Sigma}_9 = \begin{bmatrix} 0.25 & -0.40 \\ -0.40 & 1.00 \end{bmatrix}$$
For each covariance matrix:

(a) Draw the ellipse consisting of all points $\mathbf{x}' = [x_1, x_2]$ for which

$$(\mathbf{x} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\mathbf{x} - \boldsymbol{\mu}) \leq \chi_2^2(0.05)$$

where $\chi_2^2(0.05)$ is the 95th percentile of the chi square distribution with $p = 2$ degrees of freedom. You can draw it by hand if you want, as long as you label the axis tick marks carefully. Alternatively, you can use the `draw.ellipse` function from the `plotrix` package.

(b) Simulate 5000 realizations from the corresponding bivariate normal distribution using `rmvnorm` function from the `mvtnorm` package and compute the proportion that are inside the ellipse you just drew.

For an arbitrary multivariate normal random vector $\mathbf{X} = [X_1, X_2, \ldots, X_p]$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, what would you guess $P\left((\mathbf{X} - \boldsymbol{\mu})' \, \boldsymbol{\Sigma}^{-1} \, (\mathbf{X} - \boldsymbol{\mu}) \leq \chi_p^2(\alpha)\right)$ equals?

5. Consider the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}' = [4, 3, 2, 1]$ and covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

and consider the linear combinations $\mathbf{A}\mathbf{X}^{(1)}$ and $\mathbf{B}\mathbf{X}^{(2)}$. Find the following:

(a) $E\left(\mathbf{X}^{(1)}\right)$.

(b) $E\left(\mathbf{B}\mathbf{X}^{(2)}\right)$.

(c) $\text{Cov}\left(\mathbf{A}\mathbf{X}^{(1)}\right)$.

(d) $\text{Cov}\left(\mathbf{X}^{(1)}, \mathbf{X}^{(2)}\right)$.

(e) $\text{Cov}\left(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)}\right)$.

6. Generate a random sample of $n = 100$ observations from the bivariate normal distribution with

$$\boldsymbol{\mu} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma} = \begin{bmatrix} 1.0 & 0.8 \\ 0.8 & 1.0 \end{bmatrix}$$

So that we all end up with the same numbers, first set your random seed to 101: `set.seed(101)`. Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means of the two components and

$$s_{11} = \frac{1}{n}\sum_{j=1}^{n}(x_{1j} - \bar{x}_1)^2, \quad s_{22} = \frac{1}{n}\sum_{j=1}^{n}(x_{2j} - \bar{x}_2)^2, \quad \text{and} \quad s_{12} = \frac{1}{n}\sum_{j=1}^{n}(x_{1j} - \bar{x}_1)(x_{2j} - \bar{x}_2)$$

be the sample variances and sample covariance, computed by dividing by n instead of $n-1$. Thus,

$$\mathbf{S}_n = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Also, let

$$r_{12} = \frac{s_{12}}{\sqrt{s_{11}}\sqrt{s_{22}}}$$

be the sample correlation between the two variables. Finally, with $\mathbf{y}_i$ the vector of n observations on variable i, let $\mathbf{d}_i = \mathbf{y}_i - \bar{x}_i\mathbf{1}$ be the ith deviation vector, and $\mathbf{D}$ be the $n \times 2$ matrix with columns equal to the $\mathbf{d}_i$, $i = 1, 2$. Verify the following relations:

(a) $s_{11} = \frac{1}{n}\mathbf{d}_1'\mathbf{d}_1$.

(b) $s_{22} = \frac{1}{n}\mathbf{d}_2'\mathbf{d}_2$.

(c) $s_{12} = \frac{1}{n}\mathbf{d}_1'\mathbf{d}_2$.

(d) $S_n = \frac{1}{n}\sum\limits_{j=1}^{n}(\mathbf{x}_j - \bar{\mathbf{x}})(\mathbf{x}_j - \bar{\mathbf{x}})'$.

(e) $S_n = \frac{1}{n}\mathbf{D}'\mathbf{D}$.

(f) $r_{12} = \cos(\theta)$, where θ is the angle between $\mathbf{d}_1$ and $\mathbf{d}_2$.