STAT 626 — Exam II
Summer 2016

Name: KEY

INSTRUCTIONS FOR THE STUDENT:

1. You have exactly 75 minutes to complete the exam.
2. There are 10 pages including this cover sheet, and 7 questions.
3. Point values are indicated in parentheses.
4. Please answer all questions and write your solutions only in the allocated space .
5. Show all your work on the test paper. **DO NOT separate the pages nor add extra pages to the exam.**
6. Do not discuss or provide any information to any one concerning any of the questions on this exam or your solutions until I post the solutions.
7. The only materials you may use are a calculator and one formula sheet. Do not use the textbook or class notes.

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1. Suppose a time series satisfies

$$x_t = w_t + 0.7w_{t-1} - 0.3w_{t-2}, \quad \text{for all } t,$$

where w is WN(0,1).

(a) (5) Compute its autocorrelation function and plot it.

(b) (5) Compute $\mathrm{Var}(x_t)$.

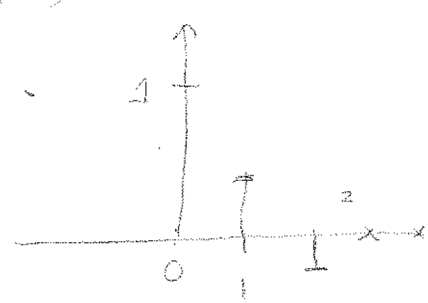
(b) (5) Is the model invertible? Why?

(c) (5) Write the predictor x^n_{n+2} and the prediction error variance P^n_{n+2}, based on the infinite past $x_t, t \le n$.

(a) $\gamma(h) = \mathrm{Cov}(x_{t+h}, x_t) = \mathrm{Cov}(w_{t+h} + 0.7w_{t+h-1} - 0.3w_{t+h-2},\ w_t + 0.7w_{t-1} - 0.3w_{t-2})$

$$= \begin{cases} (1+0.7^2+0.3^2)\sigma_w^2 = 1.58, & h=0, \\ (0.7-0.3\times0.7)\sigma_w^2 = 0.49, & h=1, \\ -0.3\sigma_w^2 = -0.3, & h=2, \\ 0 & h>2. \end{cases}$$

$$\rho(h) = \begin{cases} 1 & h=0, \\ 0.31 & h=1, \\ -0.2 & h=2, \\ 0 & h>2. \end{cases}$$



b) $Var(x_t) = \gamma(0)$ Problem 1 (cont.)

$$= 1.58.$$

(c) $x_t = (1 + 0.7B - 0.3B^2)\omega_t$,

$\theta(z) = 1 + 0.7z - 0.3z^2 = 0$, One root is $z = -1$, which is on the unit circle, so the model is <u>not invertible</u>.

(d) $x_{n+2} = \omega_{n+2} + 0.7\omega_{n+1} - 0.3\omega_n$

$$x^n_{n+2} = -0.3\omega_n$$

$$x_{n+2} - x^n_{n+2} = \omega_{n+2} + 0.7\omega_{n+1}.$$

$$P^n_{n+2} = Var(x_{n+2} - x^n_{n+2}) = Var(\omega_{n+2} + 0.7\omega_{n+1}) = 1.49.$$

2. (6 points) (a) Compute the mean and autocovariance functions of

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j} + w_{t+1}, \quad |\phi| < 1,$$

and decide if it is stationary.
(b) (4 points) Is the process causal? Why?

(a) $E(x_t) = E\left(\sum_{j=0}^{\infty} \phi^j \omega_{t-j} + \omega_{t+1} \right) = 0.$

$\gamma(h) = Cov(x_{t+h}, x_t) = Cov\left(\omega_{t+h+1} + \sum_{j=0}^{\infty} \phi^j \omega_{t+h-j},\ \omega_{t+1} + \sum_{j=0}^{\infty} \phi^j \omega_{t-j} \right)$

$$= \begin{cases} \sigma_\omega^2 + \sigma_\omega^2 \sum_{j=0}^{\infty} \phi^{2j} = \sigma_\omega^2 \left(1 + \frac{1}{1-\phi^2} \right), & h = 0 \\ \frac{\phi^h}{1-\phi^2}, & h = 1, 2, \cdots \end{cases}$$

$\{x_t\}$ is stationary.

(b) $\{x_t\}$ is not causal, because it depends on the future ~~wo~~ noise W_{t+1}.

3. Suppose the time series $\{x_t\}$ satisfies

$$x_t + 0.9x_{t-1} = w_t, \quad \text{for all } t,$$

where $\{w_t\}$ is $\mathrm{WN}(0, \sigma^2)$.

(a) (2) Is the time series causal? Why?

(b) (8) Write the moving average representation of $\{x_t\}$ and compute its autocovariance function.

(c) (3) Write the PACF of $\{x_t\}$.

(d) (2) Compute $\mathrm{Var}(x_t)$.

(e) (5) Write the predictor (normal) equations for $x_{n+1}^n = \phi_{21}x_n + \phi_{22}x_{n-1}$ based on x_{n-1} and x_n, and solve it.

(a) yes, because the root of $1+0.9z=0$ is $z=-\frac{1}{0.9}$ which is outside the unit circle.

(b) $x_t = \dfrac{1}{1-0.9B} w_t = \sum_{j=0}^{\infty} (-0.9)^j w_{t-j}$,

$\gamma(h) = \mathrm{Cov}(x_{t+h}, x_t) = \sigma_w^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+h} = \dfrac{\sigma_w^2}{1-\phi^2} \phi^h, \ h=0,1,\dots$

(c) For AR(1)

$\phi_{11} = \rho_1 = -0.9$

$\phi_{22} = 0, \quad \phi_{hh} = 0, \quad h > 2,$

(d) $\mathrm{Var}(x_t) = \gamma(0) = \dfrac{\sigma_w^2}{1-0.81} = \dfrac{\sigma_w^2}{0.91} = 5.263\,\sigma_w^2$

(e) $\Gamma_n \phi_n = \gamma_n \Rightarrow \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix} \begin{bmatrix} \phi_{21} \\ \phi_{22} \end{bmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$

For AR(1) with $\phi = -.9$, solving the above equ. gives $\phi_{21} = -0.9$, $\phi_{22} = 0$.

Problem 3 (cont.)

4. Consider the ARMA (1,1) model

$$x_t - 0.5x_{t-1} = w_t + \theta w_{t-1}.$$

(a) (5) Find the moving average coefficients (the ψ-weights) of $\{x_t\}$.

(b) (5) Use the MA coefficients from (a) and compute the autocovariance function of the process.

(c) (5) Write the predictor x_{n+2}^n and P_{n+2}^n based on the infinite past $x_t, t \leq n$.

(a) ψ_j's are the coefficients of

$$(1-0.5z)^{-1}(1+\theta z) = \sum_{j=0}^{\infty} \psi_j z^j \quad \text{or}$$

$$\frac{1+\theta z}{1-0.5z} = (1+\theta z)(1+0.5z+0.5^2z^2+\cdots$$

$$= 1 + 0.5z + 0.5^2 z^2 + \cdots$$
$$\theta z + 0.5\theta z^2 + \cdots,$$

so

$$\psi_j = (\theta + 0.5)\, 0.5^{j-1}, \quad j \geq 1, \quad \psi_0 = 1.$$

(b) $$\gamma(h) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \begin{cases} \dfrac{\theta^2+\theta+1}{1-0.5^2}\,\sigma_w^2, & h=0, \\ (0.5)^{h-1}\left[\theta + 0.5\,\dfrac{(\theta+0.5)^2}{0.5\times 0.25}\right]\sigma_w^2, & h \geq 1. \end{cases}$$

(c) $$x_{n+2} = 0.5\, x_{n+1} + w_{n+2} + \theta w_{n+1},$$

$$x_{n+2}^n = 0.5\, x_{n+1}^n = (0.5\, x_n + \theta w_n) \times 0.5$$

$$\mathrm{Var}\left[x_{n+2} - x_{n+2}^n\right] = \sigma_w^2(1+\psi_1^2) = \sigma_w^2\left[1+(\theta+0.5)^2\right].$$

5. Consider the model

$$x_t = \phi x_{t-1} + w_t + \Theta w_{t-6}.$$

(a) (5) Identify or rewrite the model using the notation SARIMA$(p,d,q) \times (P,D,Q)_s$.

(b) (5) Assuming that $|\Theta| < 1$, write the coefficients π_k's in the following representation in terms of ϕ and Θ,

$$w_t = \sum_{k=0}^{\infty} \pi_k x_{t-k}.$$

(c) (5) Assuming that $|\phi| < 1$, write the MA representation of the process and its coefficients in terms of ϕ and Θ.

(a) SARIMA $(1,0,0) \times (0,0,1)_6$.

(b) The π_k's are the coefficients of $\frac{1-\phi z}{1+\theta z^6} =$

$$(1-\phi z)(1 - \theta z^6 + \theta^2 z^{12} - \theta^3 z^{18} + \cdots)$$

$$= 1 - \theta z^6 + \theta^2 z^{12} - \theta^3 z^{18} + \cdots$$
$$- \phi z + \phi\theta z^7 - \phi\theta^2 z^{13} + \cdots$$

$\pi_0 = 1$, $\pi_1 = -\phi$ $\qquad$ $\pi_{6j} = (-\theta)^j$, $j = 0,1,2,\ldots$

$\pi_6 = -\theta$, $\pi_7 = \phi\theta$ $\qquad$ $\pi_{6j+1} = -\phi(-\theta)^j$, $j = 0,1,2,\ldots$;

$\pi_{12} = \theta^2$, $\pi_{13} = -\phi\theta^2$ $\qquad$ and the rest are zero.

$\vdots$

(c) The MA coefficients ψ_j's can be read off from

$$\frac{1+\Theta z^6}{1-\phi z} = (1 + \Theta z^6)(1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \cdots)$$

$$= 1 + \phi z + \phi^2 z^2 + \phi^3 z^3 + \cdots$$
$$+ \Theta z^6 + \Theta\phi z^7 + \Theta\phi^2 z^8 + \cdots, \text{ so}$$

$\psi_j = \phi^j$, $j = 0, 1, \ldots, 5$

$\psi_{j+6} = \phi^j(\phi^6 + \Theta)$, $j = 0,1,2,\ldots$.

Problem 5 (cont.)

6. Let $\{w_{t1}\}, \{w_{t2}\}, \{w_t\}$ be three independent WN(0, 1) series and define

$$y_{t1} = x_{t1} + \sum_{j=1}^{t} w_j, \quad y_{t2} = x_{t2} + 9\sum_{j=1}^{t} w_j,$$

where

$$x_{t1} = 0.7x_{t-1,1} + w_{t1}, \quad x_{t2} = 0.5x_{t-4,2} + w_{t2},$$

are two causal AR time series.

(a) (5) Compute the autocovariance function of $\{y_{t1}\}$. Is the time series $\{y_{t1}\}$ stationary?

(b) (5) Compute the cross-covariance function and cross-correlation function (CCF) between $\{y_{t1}\}$ and $\{y_{t2}\}$. Are the two time series $\{y_{t1}\}$ and $\{y_{t2}\}$ jointly stationary?

(c) (5) Is there a linear combination of $\{y_{t1}\}$ and $\{y_{t2}\}$ that is stationary? If so write it down, otherwise explain why there is none.

(a) $\mathrm{Cov}(y_{t+h,1}, y_{t,1}) = \mathrm{Cov}(x_{t+h,1}, x_t) +$ Cov of RW

$= ($Cov of AR(1) with $\phi = .7) +$ Cov of RW

$= \frac{1}{1-0.7^2}(0.7)^h + t$, depends on t, hence nonstationary.

(b) $\mathrm{Cov}(y_{t+h,1}, y_{t,2}) = 9 \times$ Cov of RW $= 9 \times t$, $h > 0$

depends on t, hence not jointly stationary.

(c) even though $\{y_{t1}\}$ and $\{y_{t2}\}$ are not stationary,

the linear combination

$$z_t = 9y_{t1} - y_{t2},$$

is stationary.

7. (5) It is felt that a time series follows either an AR(2) or an AR(3) model. A series of length 200 is observed, and the following results were obtained upon fitting an AR(3) model to the data:

i	1	2	3
$\hat{\phi}_i$	-0.930	0.250	-0.004
$SE(\hat{\phi}_i)$	0.071	0.095	0.071

Does the AR(3) model seem more plausible than the AR(2)? Test using a type I error probability of 0.05.

The estimated ϕ_3 has a small t-ratio and is not significant, so AR(2) is more plausible.