STATISTICS 630 - Solution to Test II **July 10, 2012**

1. A regional jet that can seat 36 passengers has a weight capacity of 7500 pounds. Suppose that the passengers that use this plane come from a population with a mean weight of 200 pounds and a standard deviation of 40 pounds. (Assume that this weight includes any baggage they may bring on board the plane.) Obtain the approximate probability that the weight of 36 randomly selected passengers from this population boarding the plane will exceed the weight capacity of the regional jet.

$$\begin{aligned} P\left[\sum_{i=1}^{36} X_i > 7500\right] &= P\left[\frac{\sum_{i=1}^{36} X_i - (36)(200)}{\sqrt{36(40^2)}} > \frac{7500-(36)(200)}{\sqrt{36(40^2)}}\right] \\ &= P[Z > 1.25] = 1 - \Phi(1.25) \end{aligned}$$

The next to last equality is an approximation justified by the Central Limit Theorem.

2. Suppose that (X, Y) have the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} x+y & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

and marginal probability density functions

$$f_X(x) = \begin{cases} \frac{1}{2}+x & 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2}+y & 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the values of $E(X)$, $E(Y)$, $\text{Var}(X)$, $\text{Var}(Y)$, and $\text{Cov}(X,Y)$.

$$E(X) = E(Y) = \int_0^1 x(1/2+x)dx = \frac{x^2}{(2)(2)} + \frac{x^3}{3}\bigg|_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

$$E(X^2) = E(Y^2) = \int_0^1 x^2(1/2+x)dx = \frac{x^3}{(2)(3)} + \frac{x^4}{4}\bigg|_0^1 = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}$$

Thus,

$$\text{Var}(X) = \text{Var}(Y) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

Next

$$\begin{aligned} E(XY) &= \int_0^1 \int_0^1 xy(x+y)dydx = \int_0^1 \frac{x^3y}{3} + \frac{x^2y^2}{2}\bigg|_{x=0}^{x=1} \\ &= \int_0^1 \frac{y}{3} + \frac{y^2}{2}dy = \frac{y^2}{6} + \frac{y^3}{6}\bigg|_0^1 = \frac{1}{3} \end{aligned}$$

Thus, $\text{Cov}(X,Y) = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}$.

3. The moment generating function of the Gamma(α, λ) distribution is

$$M(s) = \frac{\lambda^\alpha}{(\lambda - s)^\alpha}, s < \lambda.$$

Suppose that X and Y are independent random variables with the Gamma(α, λ) distribution. Obtain the moment generating functions of (i) $W = X + Y$ and (ii) $V = 5X$. Then use the moment generating function to identify the distributions of W and V.

The moment generating function of W is

$$M_W(s) = M_X(s) \times M_Y(s) = \frac{\lambda^{2\alpha}}{(\lambda - s)^{2\alpha}}.$$

Thus, W has a Gamma$(2\alpha, \lambda)$ distribution since this is the mgf of a Gamma$(2\alpha, \lambda)$ distribution.

Next,

$$M_V(s) = M_{5X}(s) = M_X(5s) = \frac{\lambda^\alpha}{(\lambda - 5s)^\alpha} = \frac{(\lambda/5)^\alpha}{((\lambda/5) - s)^\alpha}.$$

Thus, V has a Gamma$(\alpha, \lambda/5)$ distribution since this is the mgf of a Gamma$(\alpha, \lambda/5)$ distribution.

4. Suppose that Z_1, Z_2, Z_3, Z_4 are independent standard normal random variables. Let $U = Z_1^2$, $V = Z_2^2 + Z_3^2 + Z_4^2$, and $W = 3U/V$. Identify the distributions of U, V, and W. Then derive $E(W)$.

Using properties of standard normal rvs, $U \sim \chi^2(1)$ and $V \sim \chi^2(3)$. Since U and V are independent chi-square rvs, $W = 3U/V = (U/1)/(V/3)$ has an F distribution with (1,3) degrees of freedom.

Next using the pdf of the chi-squared distribution,

$$\begin{aligned} E(W) = E\left[\frac{3U}{V}\right] &= 3E[U]E[1/V] = 3\frac{(1/2)}{(1/2)} \int_0^\infty \frac{(1/2)^{3/2} v^{(3/2)-1} e^{-v/2}}{v\Gamma(3/2)} dv \\ &= 3\frac{(1/2)^{3/2}}{\Gamma(3/2)} \frac{\Gamma(1/2)}{(1/2)^{1/2}} \int_0^\infty \frac{(1/2)^{1/2}}{\Gamma(1/2)} v^{(1/2)-1} e^{-v/2} dv = 3 \end{aligned}$$

5. Suppose that $X_1, \ldots, X_n$ are a random sample from a distribution with pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} x^{(1/\theta)-1}, & 0 < x < 1,\ \theta > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and mean $E(X_i) = 1/(1+\theta)$. Find the maximum likelihood estimator estimator of θ and also the method of moments estimator of θ. Are they the same?

The likelihood function is

$$L(\theta|x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{\theta} x_i^{(1/\theta)-1} = \frac{1}{\theta^n} \prod_{i=1}^{n} x_i^{(1/\theta)-1}.$$

The log likelihood function is

$$\ell(\theta) = -n\log(\theta) + \left(\frac{1}{\theta} - 1\right) \sum_{i=1}^{n} \log(x_i).$$

Take the derivative and set equal to zero:

$$\frac{\partial \ell(\theta)}{\partial \theta} = -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \log(x_i) = 0.$$

The solution is

$$\hat{\theta} = -\frac{1}{n} \sum_{i=1}^{n} \log(x_i).$$

To verify that this is a maximum,

$$\frac{\partial^2 \ell(\hat{\theta})}{\partial \theta^2} = \frac{n}{\hat{\theta}^2} + \frac{2}{\hat{\theta}^3} \sum_{i=1}^{n} \log(x_i) = 0 = \frac{n}{\hat{\theta}^2}\left(1 - 2\frac{\hat{\theta}}{\hat{\theta}}\right) = -\frac{n}{\hat{\theta}^2} < 0.$$

To find the moment estimator, set $E(X) = \bar{X}$ and solve for θ:

$$\bar{X} = \frac{1}{1+\theta} \quad \Longrightarrow \quad \bar{X}(1+\theta) = 1 \quad \Longrightarrow \quad \theta = \frac{1}{\bar{X}} - 1.$$

Thus, the moment estimator is $\tilde{\theta} = \frac{1}{\bar{X}} - 1$, which is not the same as the mle.