STAT 630 Fall 2014
Homework 5 Solution

3.1.1

(a) $E(X) = \sum_i x_i P_X(x_i) = -4 \times 1/7 + 0 \times 2/7 + 3 \times 4/7 = 8/7$

(b) $E(X) = \sum_{i=0}^{+\infty} i \cdot 2^{-i-1} = 2^{-2} + 2 \times 2^{-3} + 3 \times 2^{-4} + \cdots n \times 2^{-n-1} + (n+1)2^{-n-2} + \cdots$. Also, we have $2E(X) = 2^{-1} + 2 \times 2^{-2} + 3 \times 2^{-3} + \cdots n \times 2^{-n} + (n+1)2^{-n-1} + \cdots$. Therefore, $2E(X) - E(X) = 2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-n} + \cdots = \frac{1/2-(1/2)^{+\infty}}{1-1/2} = 1$.

3.1.2

From the joint distribution of X and Y we can obtain: $P_X(x) = \begin{cases} \frac{3}{7} & if\ x = 5 \\ \frac{4}{7} & if\ x = 8 \\ 0 & otherwise. \end{cases}$ and

$P_Y(y) = \begin{cases} \frac{4}{7} & if\ y = 0 \\ \frac{1}{7} & if\ y = 3 \\ \frac{2}{7} & if\ y = 4 \\ 0 & otherwise. \end{cases}$

(a) $E(X) = 5 \times \frac{3}{7} + 8 \times \frac{4}{7} = \frac{47}{7}$.

(b) $E(Y) = 0 \times \frac{4}{7} + 3 \times \frac{1}{7} + 4 \times \frac{2}{7} = \frac{11}{7}$.

(c) $E(3X + 7Y) = 3E(X) + 7E(Y) = \frac{218}{7}$

(d) $E(X^2) = 5^2 \times \frac{3}{7} + 8^2 \times \frac{4}{7} = \frac{331}{7}$.

(f). $E(XY) = \frac{1}{7}(5 \times 0 + 5 \times 3 + 5 \times 4) + \frac{3}{7} \times 8 \times 0 + \frac{1}{7} \times 8 \times 4 = \frac{67}{7}$.

3.1.6

$$E(Y + Z) = E(Y) + E(Z) = 100 \times 0.3 + 7 = 37.$$

3.1.7

$$E(XY) = E(X)E(Y) = (80 \times \frac{1}{4})(\frac{3}{2}) = 30.$$

3.2.1

(a) First due to the definition of the probability density, we know $\int_5^9 C dx = 9C - 5C = 4C = 1$, thus $C = \frac{1}{4}$. Then $E(X) = \int_5^9 x\frac{1}{4}dx = \frac{1}{8}x^2|_5^9 = 7$.

(b) For f_X to be a valid pdf, we need $1 = \int_6^8 C(x+1)dx = \frac{C}{2}(x+1)^2|_6^8 = 16C$. Thus $C = \frac{1}{16}$ and $E(X) = \int_6^8 \frac{1}{16}x(x+1)dx = (\frac{1}{48}x^3 + \frac{1}{32}x^2)|_6^8 = 169/24 = 7.041667$.

3.2.2

(a) $f_X(x) = \int_0^1 (4x^2y + 2y^5)dy = (2x^2y^2 + \frac{y^6}{3})|_0^1 = 2x^2 + \frac{1}{3}$. Thus $E(X) = \int_0^1 (2x^3 + \frac{x}{3})dx = (x^4/2 + x^2/6)|_0^1 = \frac{2}{3}$.

(b) $f_Y(y) = \int_0^1 (4x^2y + 2y^5)dx = (3x^3y/4 + 2xy^5)|_0^1 = \frac{4}{3}y + 2y^5$.Thus $E(Y) = \int_0^1 (\frac{4}{3}y^2 + 2y^6)dy = (\frac{4}{9}y^3 + \frac{2}{7}y^7)|_0^1 = \frac{46}{63}$.

(d). $E(X^2) = \int_0^1 x^2(2x^2 + \frac{1}{3})dx = \frac{2}{5}x^5 + \frac{1}{9}x^3|_0^1 = \frac{23}{45}$.
(f). $E(XY) = \int_0^1 \int_0^1 xy(4x^2y + 2y^5)dxdy = \int_0^1 (x^4y^2 + x^2y^6)|_0^1 dy = \frac{1}{3} + \frac{1}{7} = \frac{10}{21}$.

3.2.5

$$E(-5X - 6Y) = -5E(X) - 6E(Y) = -5 \times \frac{3+7}{2} - 6 \times \frac{1}{9} = -\frac{77}{3} = -25.66667.$$

3.2.12

(a) $E(Z) = E(X+Y) = E(X) + E(Y) = 5 + 6 = 11$.

(b) Since X and Y are independent, $E(Z) = E(X)E(Y) = 5 \times 6 = 30$.

(c) $E(Z) = 2E(X) - 4E(Y) = 2 \times 5 - 4 \times 6 = -14$.

(d) $E(Z) = 6E(X) + 8E(XY) = 6 \times 5 + 8 \times 30 = 270$.

3.2.18

$E(X) = \int_0^\infty x\alpha x^{\alpha-1}e^{-x^\alpha}dx = \int_0^\infty \alpha x^\alpha e^{-x^\alpha}dx$. Let $u = x^\alpha$,$x = u^{\frac{1}{\alpha}}$,$du = \alpha x^{\alpha-1}dx$. Then $E(X) = \int_0^\infty u^{\frac{1}{\alpha}}e^{-u}du = \Gamma(\alpha^{-1} + 1)$

3.2.22

From the Question2.4.24 we know the pdf for beta distribution is $B(a,b)^{-1}x^{a-1}(1-x)^{b-1}$ where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$. Thus $E(X) = B(a,b)^{-1}\int_0^1 x^a(1-x)^{b-1}dx = B(a,b)^{-1} \times B(a+1,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \times \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}$. Since $\Gamma(a+1) = a\Gamma(a)$ and $\Gamma(a+b+1) = (a+b)\Gamma(a+b)$, thus $E(X) = \frac{a}{a+b}$.

3.3.2

From the Question 3.1.2 we know $E(X) = \frac{47}{7}, E(Y) = \frac{11}{7}$.
(b). $Cov(X,Y) = \frac{1}{7}((5 - E(X))(0 - E(Y)) + (5 - E(X))(3 - E(Y)) + (5 - E(X))(4 - E(Y))) + \frac{1}{7}(8 - E(X))(4 - E(Y)) + \frac{3}{7}(8 - E(X))(0 - E(Y)) = -\frac{48}{49} = -0.9796$
(c). $Var(X) = \frac{3}{7} \times (5 - E(X))^2 + \frac{4}{7} \times (8 - E(X))^2 = \frac{108}{49} = 2.2041$.
$Var(Y) = \frac{4}{7} \times (0 - E(Y))^2 + \frac{2}{7} \times (4 - E(Y))^2 + \frac{1}{7} \times (3 - E(Y))^2 = \frac{166}{49} = 3.3878$.
(d). $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = -0.3585$.

3.3.3

From 3.2.2 we know $E(X) = 2/3,\ E(Y) = 46/63$. $f_X(x) = 2x^2 + 1/3, f_Y(y) = 4y/3 + 2y^5$. Thus $Cov(X,Y) = \int_0^1 \int_0^1 (x - E(X))(y - E(y))(4x^2y + 2y^5)dxdy = -0.01058$, $E(X^2) = \int_0^1 x^2(2x^2 + 1/3)dx = 23/45$, $E(Y^2) = \int_0^1 y^2(4y/3 + 2y^5) = 7/12$. Thus, $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-0.01058}{\sqrt{(23/45-(2/3)^2)\times(7/12-(46/63)^2)}} = -0.18288$.

3.3.7

(a) Since X and Y are independent, $E(XY) = E(X)E(Y)$. Hence, $Cov(X,Z) = E(XZ) - E(X)E(Z) = E(X(X+Y)) - E(X)E(X+Y) = E(X^2) - E(X)^2 = Var(X) = \lambda^{-2} = \frac{1}{9}$.

(b) $Corr(X,Z) = \frac{Cov(X,Z)}{\sqrt{Var(X)Var(Z)}} = \sqrt{\frac{Var(X)}{Var(Z)}} = \sqrt{\frac{Var(X)}{Var(X)+Var(Y)}} = \sqrt{\frac{1/9}{1/9+5}} = \frac{1}{\sqrt{46}} = 0.147$.

3.3.14

Since $E(X) = 1/2, E(Y) = 0, Var(X) = 1/4$ and $Var(Y) = 1$, then $E(Z) = E(X+Y) = E(X) + E(Y) = 1/2$, $E(W) = E(X-Y) = E(X) - E(Y) = 1/2$, $Var(Z) = Var(X+Y) = Var(X) + Var(Y) = 5/4$, $Var(W) = Var(X-Y) = Var(X) + Var(Y) = 5/4$ and $E(ZW) = E(X^2 - Y^2) = 1/2 - 1 = -1/2$. Thus, $Cov(Z,W) = E(ZW) - E(Z)E(W) = -1/2 - (1/2)(1/2) = -3/4$ and $Corr(Z,W) = Cov(Z,W)/\sqrt{Var(Z)Var(W)} = -3/5$.

3.3.21

$E(X^2) = \int_0^\infty x^2 \alpha x^{\alpha-1} e^{-x^\alpha} dx = \int_0^\infty \alpha x^{\alpha+1} e^{-x^\alpha} dx$. Let $u = x^\alpha, x = u^{\frac{1}{\alpha}}, du = \alpha x^{\alpha-1}dx$. Then $E(X^2) = \int_0^\infty u^{\frac{2}{\alpha}} e^{-u} du = \Gamma(2\alpha^{-1} + 1)$ and $Var(X) = E(X^2) - (E(X))^2 = \Gamma(2\alpha^{-1} + 1) - \Gamma^2(\alpha^{-1} + 1)$.

3.3.24

$E(X^2) = \int_0^1 x^2 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 x^{a+1}(1-x)^{b-1}dx = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} = \frac{a(a+1)}{(a+b)(a+b+1)}$. Thus $Var(X) = E(X^2) - (E(X))^2 = \frac{a(a+1)}{(a+b)(a+b+1)} - (\frac{a}{a+b})^2 = \frac{ab}{(a+b)^2(a+b+1)}$.

3.4.5

$m_Y(s) = E(e^{sY}) = E(e^{3Xs+4s}) = E(e^{X\cdot 3s}) \times E(e^{4s}) = e^{4s}m_X(3s).$

3.4.8

(c). $m_X(s) = E(e^{sX}) = \frac{1}{2}e^{2s} + \frac{1}{3}e^{5s} + \frac{1}{6}e^{7s}.$
(d). $m'_X(s) = 2\frac{1}{2}e^{2s} + 5\frac{1}{3}e^{5s} + 7\frac{1}{6}e^{7s}, m''_X(s) = 2^2\frac{1}{2}e^{2s} + 5^2\frac{1}{3}e^{5s} + 7^2\frac{1}{6}e^{7s}$. Hence, $m'_X(0) = 2\frac{1}{2} + 5\frac{1}{3} + 7\frac{1}{6} = E(X), m''_X(0) = 2^2\frac{1}{2} + 5^2\frac{1}{3} + 7^2\frac{1}{6} = E(X^2).$

3.4.12

(a) $m_X(s) = \sum_{x=0,x\in N}^{\infty} e^{sx}(1-\theta)^x\theta = \theta \cdot \frac{1}{1-e^s(1-\theta)}$, if $|e^s(1-\theta)| < 1$.

(b) $E(X) = m'_X(0) = \theta(1-\theta)\cdot\frac{1}{(1-e^s(1-\theta))^2}e^s|_{s=0} = \frac{(1-\theta)}{\theta}$

(c) $E(X^2) = m''_X(0) = \theta(1-\theta)e^s\frac{1}{(1-e^s(1-\theta))^2} + 2\theta(1-\theta)\frac{e^{2s}}{(1-e^s(1-\theta))^3}|_{s=0} = \frac{(1-\theta)(2-\theta)}{\theta^2}$. So $Var(X) = E(X^2) - E(X)^2 = \frac{1-\theta}{\theta^2}$.

3.4.16

(a) $m_Y(s) = E(e^{sY}) = \int_{-\infty}^{\infty} e^{sy}\frac{1}{2}e^{-|y|}dy = \frac{1}{2}\{\int_0^{\infty} e^{sy-y}dy + \int_{-\infty}^{0} e^{sy+y}dy\} = \frac{1}{2}\frac{1}{1-s} + \frac{1}{1+s} = \frac{1}{1-s^2}$, provided $|s| < 1$.

(b) $m'_Y(s) = \frac{2s}{(1-s^2)^2}, E(Y) = m'_Y(0) = 0.$

(c) $m''_Y(s) = \frac{2+6s^2}{(1-s^2)^3}, E(Y^2) = m''_Y(0) = 2, Var(Y) = E(Y^2) - E(Y)^2 = 2.$

3.4.20

$$\begin{aligned}
m_X(t) &= \int_0^{\infty} e^{tx}\frac{\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)}e^{-\lambda x}dx \\
&= \int_0^{\infty} \frac{x^{\alpha-1}\lambda^\alpha e^{-(\lambda-t)x}}{\Gamma(\alpha)}dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^{\infty}((\lambda-t)x)^{\alpha-1}e^{-(\lambda-t)x}d(\lambda-t)x\cdot(\lambda-t)^{-\alpha} \\
&= \frac{\lambda^\alpha}{(\lambda-t)^\alpha}\times\Gamma(\alpha)^{-1}\times\Gamma(\alpha) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha}
\end{aligned}$$

The above holds for $\lambda - t > 0$, that is $t < \lambda$.For $t \geq \lambda$, $m_X(t)$ does not exist.

3.4.22

The mgf for Negative Binomial(r_i, θ) is $\frac{\theta^{r_i}}{(1-e^s(1-\theta))^{r_i}}$. Since $X_i's$ are independent, we have $m_Y(s) = \prod_{i=1}^n m_{X_i}(s) = \prod_{i=1}^n \frac{\theta^{r_i}}{(1-e^s(1-\theta))^{r_i}} = \frac{\theta^{\sum_{i=1}^n r_i}}{(1-e^s(1-\theta))^{\sum_{i=1}^n r_i}}$. By the uniqueness theorem, we know that Y has Negative Binomial(r, θ) distribution.

Additional Problem

$Cov(U,V) = Cov(X,Y) - Cov(X,Z) + Cov(Z,Y) - Var(Z)$. Since X, Y and Z are uncorrelated, thus $Cov(X,Y) = Cov(X,Z) = Cov(Z,Y) = 0$ and $Cov(U,V) = -Var(Z) = -\sigma_z^2$. Besides, $Var(U) = Var(X) + Var(Z) = \sigma_x^2 + \sigma_z^2$ and $Var(V) = Var(Y) + Var(Z) = \sigma_y^2 + \sigma_z^2$. So $\rho_{U,V} = \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} = \frac{-\sigma_z^2}{\sqrt{(\sigma_x^2+\sigma_z^2)(\sigma_y^2+\sigma_z^2)}}$.