STATISTICS 630 - Solution to Test I **October 3, 2012**

1. Suppose that a random variable X has the probability density function (pdf)

$$f_X(x) = \begin{cases} Cx(1-x) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of C that makes $f_X(x)$ a valid pdf for a continuous rv.

$$\int_0^1 Cx(1-x)dx = C\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{C}{6} = 1.$$

   Thus, $C = 6$.

   (b) Suppose $V = 1/X$. Obtain the pdf of V.
   For $v > 1$,

$$F_V(v) = P[V \le v] = P[1/X \le v] = P[X \ge 1/v] = 1 - \int_0^{1/v} 6x(1-x)dx$$
$$= 1 - 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^{1/v} = 1 - \frac{3}{v^2} + \frac{2}{v^3}.$$

   The pdf is

$$f_V(v) = \frac{dF_V(v)}{dv} = \frac{6}{v^3} - \frac{6}{v^4}, \quad v > 1.$$

2. Two litters of pet rats have been born, one with two brown-haired and one gray-haired rat (Litter 1 with 3 rats), and the other with three brown-haired and two gray-haired (Litter 2 with 5 rats). We select a litter at random (each litter with probability 1/2) and then select an offspring at random from the selected litter (each offspring in the litter has the same probability of being chosen).

   (a) What is the probability that the chosen animal is brown-haired?
   Let B be the event that the rat has brown hair and L_i be the event the rat comes from litter i, $i = 1, 2$. Then

$$P(B) = P(B|L_1)P(L_1) + P(B|L_2)P(L_2) = (2/3)(1/2) + (3/5)(1/2) = 19/30.$$

   (b) Given that a brown-haired offspring was selected, what is the probability that the rat was from Litter 1?

$$P(L_1|B) = \frac{P(B|L_1)P(B)}{P(B)} = \frac{(2/3)(1/2)}{(19/30)} = \frac{10}{19}$$

3. An absent-minded professor has five similar looking keys in his pocket. Two of the keys will unlock his office door while three of the keys will not. Find the probability that exactly one of the two keys tried will succeed in unlocking the office door in each of the following circumstances:

   (a) He selects two of the keys at random from his pocket. He then tries both of the keys on his office door and records how many of them can unlock the door.

   Let X = the number of keys that unlock the door. Then X has a hypergeometric distribution and
$$P(X=1) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{2\times 3}{10} = \frac{6}{10}.$$

   (b) He selects one of the keys at random. He tries the selected key on his office door and records whether it can unlock the door. He then places the key back in his pocket with the other keys. He again selects a key at random from his pocket. He tries it on his office door and records whether it can unlock the door.

   Let Y = the number of keys that unlock the door. Then Y has a binomial distri bution and
$$P(Y=1) = \binom{2}{1}\left(\frac{2}{5}\right)^1\left(\frac{3}{5}\right)^1 = \frac{12}{25}.$$

4. Suppose that (X,Y) have the joint probability density function
$$f_{X,Y}(x,y) = \begin{cases} 6xy^2 & \text{for } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal probability density functions of X and Y. Determine whether X and Y are independent.

   We first find the marginal pdfs of X and Y:
$$f_X(x) = \int_0^1 6xy^2 dy = 6x\frac{y^3}{3}\bigg|_0^1 = 2x,\ 0<x<1.$$
$$f_Y(y) = \int_0^1 6xy^2 dx = 6y^2\frac{x^3 2}{2}\bigg|_0^1 = 3y^2,\ 0<y<1.$$
   Since $f_{X,Y}(x,y) = 6xy^2 = (2x)(3y^2) = f_X(x)f_Y(y)$ for $0<x<1,\ 0<y<1$ and $=0$, elsewhere, X and Y are independent.

   (b) Compute $P(X>Y)$.
$$P(X>Y) = \int_0^1\int_0^x 6xy^2dydx = \int_0^1 6x\frac{y^3}{3}\bigg|_0^x dx = \int_0^1 2x^4dx = 2\frac{x^5}{5}\bigg|_0^1 = \frac{2}{5}.$$

5. Let the sample space $\mathcal{S}$ consist of the 6 permutations of the letters a, b, and c along with the three triples of each letter. Thus, $\mathcal{S} = \{aaa, bbb, ccc, abc, bca, cba, acb, bac, cab\}$. Suppose that each element of $\mathcal{S}$ is equally likely. Define the events $A_1 = \{aaa, abc, acb\}$, $A_2 = \{aaa, bac, cab\}$, and $A_3 = \{aaa, bca, cba\}$ with $P(A_1) = P(A_2) = P(A_3) = 1/3$.

   (a) Show that the events A_1, A_2, and A_3 are pairwise independent.

   $$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = P(\{aaa\}) = \frac{1}{9} = P(A_i) \times P(A_j).$$

   Thus, A_i and A_j are independent for $i \neq j$.

   (b) Show that the events A_1, A_2, and A_3 are not mutually independent.

   $$P(A_1 \cap A_2 \cap A_3) = P(\{aaa\}) = \frac{1}{9} \neq \frac{1}{27} = \left(\frac{1}{3}\right)^3 = P(A_1) \times P(A_2) \times P(A_3).$$

   Thus, the events A_1, A_2, and A_3 are not mutually independent.