STAT 630 Fall 2013
Homework 4 Solution

2.7.3

(a) From the joint distribution of x and y, we can see $p_X(2) = p_X(3) = p_X(-3) = p_X(-2) = p_X(17) = \frac{1}{5}$; $p_X = 0$ otherwise.

(b) Similarly, $p_Y(3) = p_Y(2) = p_Y(-2) = p_Y(-3) = p_Y(19) = \frac{1}{5}$; $p_Y = 0$ otherwise.

(c) $P(Y > X) = p(x = 2, y = 3) + p(x = -3, y = -2) + p(x = 17, y = 19) = \frac{3}{5}$.

(d) Since we can not find a pair of x and y satisfying $x = y$, thus $P(Y = X) = 0$.

(e) Similar to (d), $P(XY < 0) = 0$.

2.7.4

(a) From the definition of the density, we know $\int_0^1 \int_0^1 f_{X,Y}(x, y)dxdy = 1$. Therefore,

$$\int_0^1 \int_0^1 (2x^2y+Cy^5)dxdy = \int_0^1 \left(\frac{2}{3}yx^3 + Cy^5x\right)|_0^1 dy = \int_0^1 \left(\frac{2}{3}y + Cy^5\right) dy = \frac{1}{3}+\frac{C}{6} = 1$$

Thus $C = 4$. $f_X(x) = \int_0^1 (2x^2y + 4y^5)dy = x^2 + \frac{2}{3}$ and $f_X(x) = 0$ otherwise. $f_Y(y) = \frac{2}{3}y + 4y^5$ and $f_Y(y) = 0$ otherwise. $P(X \le 0.8, Y \le 0.6) = \int_0^{0.8} \int_0^{0.6} (2x^2y + 4y^5)dydx = 0.086323$.

(b) Let $\int_0^2 \int_0^1 (Cx^5y^5)dydx = 1$, then

$$\int_0^2 \int_0^1 (Cx^5y^5)dydx = \int_0^2 Cx^5dx \cdot y^6/6|_0^1 = C/6 \cdot x^6/6|_0^2 = C \times 2^6/36 = 1$$

So $C = \frac{9}{16}$. $f_X(x) = \int_0^1 \frac{9}{16}x^5y^5dy = \frac{3}{32}x^5$ for $0 \le x \le 2$ and $f_X(x) = 0$ otherwise; $f_Y(y) = \int_0^2 \frac{9}{16}x^5y^5dx = 6y^5$ for $0 \le y \le 1$ and $f_Y(y) = 0$ otherwise. $P(X \le 0.8, Y \le 0.6) = \int_0^{0.8} \int_0^{0.6} (\frac{9}{16}x^5y^5)dydx = \frac{1}{64}0.8^6 0.6^6 = 1.911 \times 10^{-4}$.

2.7.9

(a) $f_X(x) = \int_x^2 (x^2+y)/4dy = (x^2y + y^2/2)/4|_x^2 = \frac{3x^2-2x^3+4}{8}$ for $x \in (0,2)$ and 0 otherwise.

(b) $f_Y(y) = \int_0^y (x^2+y)/4dx = \frac{y^3+3y^2}{12}$ for $y \in (0,2)$ and 0 otherwise.

(c) $P(Y<1) = \int_0^1 (\frac{y^3+3y^2}{12})dy = \frac{5}{48}$.

2.7.10

(a) The marginal distribution of X is normal distribution with mean 3 and variance 4. To obtain this result, first do the transformation: Let $z_1 = (x-\mu_1)/\sigma_1$ and $z_2 = (y-\mu_2)/\sigma_2$. Therefore,

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y} f(x,y)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sqrt{1-\rho^2}} e^{-\frac{z_1^2+z_2^2-2\rho z_1 z_2}{2(1-\rho^2)}} dz_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{z_1^2}{2(1-\rho^2)}} \cdot e^{\frac{\rho^2 z_1^2}{2(1-\rho^2)}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(z_2-\rho z_1)^2}{2(1-\rho^2)}} dz_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{z_1^2}{2}}
\end{aligned}$$

we can see X has a normal distribution with mean 3 and variance 4.

(b) Change X to Y and use the same method we can see the marginal distribution of Y is normal distribution with mean 5 and variance 16.

(c) Since the correlation coefficient $\rho = 0.5$, thus X and Y are not independent.

2.7.16

(a) $\int_0^\infty \int_0^y f_{X,Y} f(x,y)dxdy = \int_0^\infty \int_0^y Ce^{-(x+y)}dxdy = C\int_0^\infty -e^{-(x+y)}|_y^0 dy = C\int_0^\infty e^{-y}(1-e^{-y})dy = C(1-\int_0^\infty e^{-2y}dy) = C(1-1/2) = C/2$. So C=2

(b) $f_x(x) = \int_x^\infty 2e^{-(x+y)}dy = 2e^{-x}\int_x^\infty e^{-y}dy = 2e^{-2x}$
$f_y(y) = \int_0^y 2e^{-(x+y)}dx = 2e^{-y}\int_0^y e^{-x}dx = 2e^{-y}(1-e^{-y})$ for $y > 0$

2.8.1

(a) $p_X(-2) = p_{X,Y}(-2,3) + p_{X,Y}(-2,5) = 1/6 + 1/12 = 1/4$;$p_X(9) = p_{X,Y}(9,3) + p_{X,Y}(9,5) = 1/6 + 1/12 = 1/4$;$p_X(13) = p_{X,Y}(13,3) + p_{X,Y}(13,5) = 1/3 + 1/6 = 1/2$; Otherwise, $p_X(x) = 0$.

(b) $p_Y(3) = p_{X,Y}(-2,3) + p_{X,Y}(9,3) + p_{X,Y}(13,3) = 1/6 + 1/6 + 1/3 = 2/3$;$p_Y(5) = p_{X,Y}(-2,5) + p_{X,Y}(9,5) + p_{X,Y}(13,5) = 1/12 + 1/12 + 1/6 = 1/3$;Otherwise, $p_Y(y) = 0$.

(c) Yes, since $p_X(x)p_Y(y) = p_{X,Y}(x,y)$ for all x and y.

2.8.2

(a) $p_X(-2) = p_{X,Y}(-2,3) + p_{X,Y}(-2,5) = 1/16 + 1/4 = 5/16$, $p_X(9) = P_{X,Y}(9,3) + p_{X,Y}(9,5) = 1/2 + 1/16 = 9/16$, $p_X(13) = P_{X,Y}(13,3) + p_{X,Y}(13,5) = 1/16 + 1/16 = 1/8$. $p_X(x) = 0$ otherwise.

(b) $p_Y(3) = p_{X,Y}(-2,3) + p_{X,Y}(9,3) + p_{X,Y}(13,3) = 1/16 + 1/2 + 1/16 = 5/8$; $p_Y(5) = p_{X,Y}(-2,5) + p_{X,Y}(9,5) + p_{X,Y}(13,5) = 1/12 + 1/12 + 1/6 = 3/8$; $p_Y(y) = 0$ otherwise.

(c) No, for example you can see $p_{X,Y}(-2,3) \neq p_X(-2) * p_Y(3)$

2.8.5

We can obtain $p_X(-4) = 1/9$, $p_X(5) = 2/9$ and $p_X(9) = 3/9 + 2/9 + 1/9 = 2/3$; $p_Y(-2) = 1/9 + 2/9 + 3/9 = 6/9 = 2/3$, $p_Y(0) = 2/9$ and $p_Y(4) = 1/9$.

(a) $P(Y = 4|X = 9) = p_{X,Y}(9,4)/p_X(9) = 1/9/(2/3) = 1/6$.

(b) $P(Y = -2|X = 9) = p_{X,Y}(9,-2)/p_X(9) = 3/9/(2/3) = 1/2$.

(c) $P(Y = 0|X = -4) = p_{X,Y}(-4,0)/p_X(-4) = 0/(1/9) = 0$.

2.8.7

(a) From 2.7.4(a), we know $C = 4$ and $f_X(x) = x^2 + \frac{2}{3}$ for $0 \le x \le 1$ and $f_X(x) = 0$ otherwise. $f_Y(y) = \frac{2}{3}y + 4y^5$ for $0 \le y \le 1$ and $f_Y(y) = 0$ otherwise. $f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x) = x^2 + \frac{2}{3} = \frac{2x^2y+4y^5}{x^2+\frac{2}{3}} \neq f_Y(y)$. Thus X and Y are not independent.
(d) From 2.7.4(b), we know $C = \frac{9}{16}$, $f_X(x) = \frac{3}{32}x^5$ and $f_Y(y) = 6y^5$. Thus $f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x) = \frac{9}{16}x^5y^5/(\frac{3}{32}x^5) = 6y^5 = f_Y(y)$. Thus X, Y are independent.

2.8.10

To prove the independence, we need to show

$$P(X = x, Y = y) = P(X = x)P(Y = y) \text{ for } x = 0,1 \text{ and } y = 0,1.$$

When $x = y = 1$, it immediately holds. When $x = 0, y = 1$, $P(X = 0, Y = 1) = P(Y = 1) - P(X = 1, Y = 1) = P(Y = 1) - P(X = 1)P(Y = 1) = \Phi - \theta\Phi = (1-\theta)\Phi = P(X = 0)P(Y = 1)$. The other two situations can be showed similarly.

2.8.15

In Exercise 2.7.9, we already showed that
$f_X(x) = \frac{4+3x^2-2x^3}{8}$ for $0 < x < 2$ and otherwise $f_X(x) = 0$
$f_Y(y) = \frac{y^3+3y^2}{12}$ for $0 < y < 2$ and otherwise$f_Y(y) = 0$

(a) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x^2+y)/4}{(4+3x^2-2x^3)/8} = \frac{2(x^2+y)}{4+3x^2-2x^3}$ for $x < y < 2$, otherwise $f_{Y|X}(y|x) = 0$.

(b) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{(x^2+y)/4}{(y^3+3y^2)/12} = \frac{3(x^2+y)}{y^3+3y^2}$ for $0 < x < y$, otherwise $f_{X|Y}(x|y) = 0$.

(c) No, because the marginal density and conditional density are different.

2.8.23

We fisrt get the joint distribution of (X_1, X_2). It's obvious that when $X_1 = x_1, X_2 = x_2$, then $X_3 = n - x_1 - x_2$ since they add up to n. Hence

$$\begin{aligned} P(X_1 = x_1, X_2 = x_2) &= P(X_1 = x_1, X_2 = x_2, X_3 = n - x_1 - x_2) \\ &= \binom{n}{x_1 x_2 (n - x_1 - x_2)} \theta_1^{x_1} \theta_2^{x_2} \theta_3^{n-x_1-x_2}, \end{aligned}$$

for $x_1, x_2 \geq 0, x_1 + x_2 \leq n$. From 2.8.22 we know that $X_1 \sim$ Binomial(n, θ_1),i.e., $P(X_1 = x_1) = \binom{n}{x_1}\theta_1^{x_1}(1-\theta_1)^{(n-x_1)}$ for $0 \leq x_1 \leq n$. Hence, the conditional distribution of X_2 given $X_1 = x_1$ is

$$\begin{aligned} P(X_2 = x_2 \mid X_1 = x_1) &= P(X_1 = x_1, X_2 = x_2)/P(X_1 = x_1) \\ &= \binom{n - x_1}{x_2} \left(\frac{\theta_2}{1-\theta_1}\right)^{x_2} \left(\frac{\theta_3}{1-\theta_1}\right)^{(n-x_1-x_2)}, \end{aligned}$$

for $0 \leq x_2 \leq n - x_1$. That is $X_2 \mid X_1 = x_1 \sim$ Binomial$\left(n - x_1, \frac{\theta_2}{1-\theta_1}\right)$.

2.8.24

First $f(x_i) = \lambda e^{-\lambda x_i}$ for $x_i > 0$ and $f(x_i) = 0$ otherwise; $P(x_i \leq x) = \int_0^x \lambda e^{-\lambda x_i} dx_i = 1 - e^{-\lambda x}$ and $P(x_i > x) = e^{-\lambda x}$ for $i = 1, 2, \cdots, n$.

(a) $P(X_{(1)} > x) = P(all\ X_i > x) = \prod_{i=1}^{n} e^{-\lambda x} = e^{-n\lambda x}$. So $f_{X_{(1)}}(x) = -\frac{dP(X_{(1)}>x)}{dx} = n\lambda e^{-n\lambda x}$.

(b) $P(X_{(n)} < x) = P(all\ X_i < x) = \prod_{i=1}^{n}(1 - e^{-\lambda x}) = (1 - e^{-\lambda x})^n$. So $f_{X_{(n)}}(x) = \frac{dP(X_{(n)}<x)}{dx} = n\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{n-1}$.

2.9.7

$$p_{X,Y}(x,y) = \begin{cases} 1/18 & x = 0, y = 2 \\ 1/36 & x = 0, y = 5 \\ 1/4 & x = 0, y = 9 \\ 1/12 & x = 2, y = 2 \\ 1/24 & x = 2, y = 5 \\ 3/8 & x = 2, y = 9 \\ 1/36 & x = 3, y = 2 \\ 1/72 & x = 3, y = 5 \\ 1/8 & x = 3, y = 9 \end{cases}$$

Therefore, $p_Z(2) = 1/18$, $p_Z(4) = 1/12$, $p_Z(5) = 1/36 + 1/36 = 1/18$, $P_Z(7) = 1/24$, $p_Z(8) = 1/72$, $p_Z(9) = 1/4$, $p_Z(11) = 3/8$, $p_Z(12) = 1/8$. $p_Z(z) = 0$ otherwise.

2.9.14

Let Y=Z-X, then,

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(z-w-\mu_1)^2}{2\sigma_1^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(w-\mu_2)^2}{2\sigma_2^2}} dw \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu_1)^2}{2\sigma_1^2}} e^{-\frac{\mu_2}{2\sigma_2^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1\sigma_2} e^{-\frac{(\sigma_1^2+\sigma_2^2)w^2-2w((z-\mu_1)\sigma_2^2+\mu_2\sigma_1^2)}{2\sigma_1^2\sigma_2^2}} dw \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu_1)^2}{2\sigma_1^2}} \cdot e^{-\frac{\mu_2}{2\sigma_2^2}} \cdot e^{\frac{((z-\mu_1)\sigma_2^2+\mu_2\sigma_1^2)^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1\sigma_2} e^{-\frac{(\sigma_1^2+\sigma_2^2)\left(w-\frac{(z-\mu_1)\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)}{\sigma_1^2\sigma_2^2}} dw \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\mu_1)^2}{2\sigma_1^2}} \cdot e^{-\frac{\mu_2}{2\sigma_2^2}} \cdot e^{\frac{((z-\mu_1)\sigma_2^2+\mu_2\sigma_1^2)^2}{2\sigma_1^2\sigma_2^2(\sigma_1^2+\sigma_2^2)}} \cdot \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2}} \\
&= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{(z-\mu_1-\mu_2)^2}{\sigma_1^2+\sigma_2^2}}
\end{aligned}
$$

The integral

$$
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1\sigma_2} e^{-\frac{(\sigma_1^2+\sigma_2^2)\left(w-\frac{(z-\mu_1)\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)}{\sigma_1^2\sigma_2^2}} dw = \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2}}
$$

this is because we can see $\frac{(z-\mu_1)\sigma_2^2+\mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} = C$ is a constant and the formula inside the integral is the density of the normal distribution with mean C and variance $(\sigma_1^2\sigma_2^2)/(\sigma_1^2+\sigma_2^2)$.