STATISTICS 630 - Solution to Test I **October 2, 2013**

1. After travelling to the Joint Statistical Meetings in Montreal, I arrived back in College Station with 3 U. S. quarters and 2 Canadian quarters in my pocket. I randomly chose two quarters without replacement from my pocket.

   (a) Obtain the probability mass function of the number of U. S. quarters among the two that I chose from my pocket. Evaluate this probability mass function using fractions (e.g., P(X=1)=3/4).

$$P(X=0) = \frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \frac{(1)(1)}{10} = \frac{1}{10},\ P(X=1) = \frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{(3)(2)}{10} = \frac{6}{10},$$
$$P(X=2) = \frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \frac{(3)(1)}{10} = \frac{3}{10},\ P(X=x) = 0,\ \text{for all other } x.$$

   (b) Obtain the conditional probability that both the quarters that I randomly selected were U. S. quarters given that at least one of the two quarters was a U. S. quarter.

$$P(X=2|X\geq 1) = \frac{P(X=2)}{P(X\geq 1)} = \frac{\frac{3}{10}}{\frac{6}{10}+\frac{3}{10}} = \frac{3}{9}.$$

2. A continuous random variable X has the probability density function (pdf)

$$f_X(x) = \begin{cases} cx^4 & \text{for } 0<x<2, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Obtain the value of c that makes $f_X(x)$ a valid pdf for a continuous rv.

   We require that
$$\int_0^2 cx^4 dx = c\frac{x^5}{5}\bigg|_0^2 = \frac{c32}{5} = 1.$$
   Thus, $c = 1/(32/5) = 5/32$.

   (b) Find the 50^{th} percentile of the distribution of X.

   We want to find $x_{.5}$ such that
$$0.5 = P[X\leq x_{.5}] = \int_0^{x_{.5}} \frac{5t^4}{32}dt = \frac{5x^5}{32\times 5}\bigg|_0^{x_{.5}} = \frac{x_{.5}^5}{32}.$$

   Thus, $x_{.5} = 16^{1/5}$.

3. Suppose that the random variables (X, Y) have joint probability density function

$$f(x,y) = \begin{cases} 15x^2y, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Obtain the marginal probability density functions of X and Y.

$$f_X(x) = \int_x^1 15x^2y\,dy = 15x^2 \frac{y^2}{2}\bigg|_x^1 = \frac{15}{2}x^2(1-x^2),\ 0 \le x \le 1.$$

$$f_Y(y) = \int_0^y 15x^2y\,dx = 15y\frac{x^3}{3}\bigg|_0^y = 5y^4,\ 0 \le y \le 1.$$

(b) Obtain the conditional probability density function of X given $Y = y$ and use this function to determine whether X and Y are independent.

The conditional pdf of X given $Y = y$ for $0 < y < 1$ is given by

$$f_{X|Y}(x|y) = \frac{15x^2y}{5y^4} = \frac{3x^2}{y^3},\ 0 \le x \le y.$$

Since the marginal pdf $f_X(x) = \frac{15}{2}x^2(1-x^2)$ does not equal the conditional pdf $f_{X|Y}(x|y) = \frac{3x^2}{y^3}$ for all $0 < x < 1$, X and Y are not independent.

4. A tennis player with a two-handed backhand often has elbow problems. Let A be the event that her right elbow is sore on a given morning and B be the event that her left elbow is sore on that morning. Suppose that $P(A) = 0.3$ and $P(B) = 0.2$. What the probability that at least one of her elbows is sore on the given morning under each of the following assumptions?

(a) A and B are mutually exclusive.
$P(A \cup B) = P(A) + P(B) + 0 = 0.3 + 0.2 = 0.5.$

(b) A and B are independent.
$P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.3 + 0.2 - (0.3)(0.2) = 0.5 - 0.06 = 0.44.$

(c) $B \subset A$.
$P(A \cup B) = P(A) + P(B) - P(B) = 0.3 + 0.2 - 0.2 = 0.3.$

(d) The probability that both her elbows are sore on the given morning equals 0.1.
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.2 - 0.1 = 0.4.$

5. Suppose that X is an exponential random variable with probability density function

$$f(x) = \begin{cases} 2e^{-2x}, & 0 < x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Obtain the cumulative distribution function of X.
For $x > 0$

$$F_X(x) = \int_0^x 2e^{-2t} dt = 2 \frac{e^{-2t}}{-2} \bigg|_0^x = (-1)(e^{-2x} - 1) = 1 - e^{-2x}.$$

For $x \le 0$, $F_X(x) = \int_{-\infty}^x 0 \, dt = 0$.

(b) Obtain the probability density function of $Y = e^X$.
For $y > 1$, the cdf of Y is

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[e^X \le y] = P[X \le \log(y)] = F_X(\log(y)) \\ &= 1 - \exp(-2\log(y)) = 1 - 1/y^2, \end{aligned}$$

and for $y \le 1$, $F_Y(y) = 0$. Thus, the pdf of Y is

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{d}{dy}(1 - 1/y^2) = (-2)(-1)/y^3 = \frac{2}{y^3},\ y > 1.$$

For $y \le 1$, $f_Y(y) = 0$.