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STAT 630-720
HW 03

1) 2.4.2: let W ~ Uniform[1, 4]. Compute each of the following

a) $P(W \geq 5)$: since 5 is outside of L and R, $P(W \geq 5) = 0$
b) $P(W \geq 2)$: $\frac{1}{4-1} = \frac{1}{3}$
c) $P(W^2 \leq 9)$: Then $P(W \leq 3) = \frac{1}{3}$

2) 2.4.4: Establish for which constants c the following functions are densities

a) $f(x) = cx$ on (0,1) and 0 otherwise: To solve for c:

$$c\int_0^1 x\,dx = c\frac{1^2}{2} - \frac{0^2}{2} = \frac{c}{2} = 1$$
$$c = 2$$

b) $f(x) = cx^n$ on (0,1) and 0 otherwise, for n a nonnegative interger

$$c\int_0^1 x^n\,dx = c\frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = c\frac{1}{n+1} = 1$$
$$c = n+1$$

c) $f(x) = cx^{\frac{1}{2}}$ on (0,2) and 0 otherwise

$$c\int_0^1 x^. 5\,dx = c(1.52^1.5 - 0) = c(1.5 * \sqrt{8}) = 1$$
$$c = \frac{3}{2\sqrt{8}}$$

3) 2.4.6: Let X ~ Exponential(3). Compute each of the following.

b) $P(0 < X < 3)$

$$\begin{aligned}\int_0^3 \lambda e^{-\lambda x}\,dx &= (-e^{-\lambda 3}) - (-e^{-\lambda 0}) \\ &= (-e^{-(3)3}) - (-e^{-(3)0}) \\ &= 1 - e^{-9} \\ &= \ .9998\end{aligned}$$

c) $P(2 < X < 10)$

$$\begin{aligned}\int_2^{10} \lambda e^{-\lambda x}\,dx &= (-e^{-\lambda 10}) - (-e^{-\lambda 2}) \\ &= (-e^{-(3)10}) - (-e^{-(3)2}) \\ &= e^{-6} - e^{-30} \\ &= .002\end{aligned}$$

4) 2.4.19: (Weibuill(α) distribution) Consider, for $\alpha > 0$ fixed, the function given by $f(x) = \alpha(1+x)^{\alpha-1}e^{-x^{\alpha}}$ for $0 < x < \infty$ and 0 otherwise. Prove that f is a density function. Plug in any positive integer for α, this case I will use $\alpha = 1$. $1\alpha^{1-1}e^{-x^1} = e^{-x}$ so now we need to integrate e^{-x} between 0 and infinity

$$\begin{aligned}\int_0^{\infty} e^{-x}\,dx &= \lim_{t\to\infty}\int_0^t e^{-x}\,dx \\ &= \lim_{t\to\infty}(-e^{-t}) - (-e^{-0}) \\ &= \lim_{t\to\infty}(1-e^{-t}) \\ &= 1\end{aligned}$$

5) 2.4.22: (Laplace distribution) Consider the function given by $f(x) = \frac{e^{-|x|}}{2}$ for $-\infty < x < \infty$ and 0 otherwise. Prove that f is a density function.

$$\begin{aligned}f(x) &= \frac{e^{-|x|}}{2}, for(-\infty < x < \infty) \\ &= \frac{e^{-x}}{2}, for(x \geq 0) + \frac{e^{-|x|}}{2}, for(x < 0) \\ &= e^{-x}, for(x \geq 0)\end{aligned}$$

6) 2.5.3: For each of the following functions F, determine whether or not F is a valid cumulative distribution function satisfying properties a-d of theorem 2.5.2

a) F(x) = x for all $x \in R^1$: Yes, R -> [0, 1] so all x's are between [0,1]
b) Yes, all x's between [0,1] for x^2 are less than or equal to 1
c) No, when x is > 1 f(x) is > 1 so it does not satify the cdf requirement
d) Yes, if you plug in the max $3^2/9 = 1$ so this satisfies the cdf requirement
e) No, as x increases from -1 to 0, F(x) decreases to the cdf requirement is not satisfied

7) 2.5.5 (use R): Let Y ~ N(-8, 4). Compute each of the following in terms of the function Φ of definition 2.5.2

a) $P(Y \leq -5)$:

```
pnorm(q = -5, mean = -8, sd = 4)
```

```
## [1] 0.7734
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b) $P(-2 \leq Y \leq 7)$

```
d2 = pnorm(q = -2, mean = -8, sd = 4)
d7 = pnorm(q = 7, mean = -8, sd = 4)
d7 - d2
```

```
## [1] 0.06672
```

c) $P(Y \geq 3)$

```r
1 - pnorm(q = 3, mean = -8, sd = 4)
```

```
## [1] 0.00298
```

d) Obtain the 35th and 84th percentiles of the distribution of Y

```r
qnorm(p = c(.35, .84), mean = -8, sd = 4)
```

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## [1] -9.541 -4.022
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8) 2.5.7: Suppose $F_x(x) = x^2$ for $0 \leq x \leq 1$. Compute the following.

a) $P(X < \frac{1}{3})$: $\frac{1}{3}^2 = \frac{1}{9}$
b) $P(\frac{1}{4} < X < \frac{1}{2})$: $\frac{1}{2}^2 - \frac{1}{4}^2 = \frac{3}{16}$
c) $P(X < -1)$: Outside the bounds of [0,1] so 0
d) $P(X < 3)$: 1
e) $P(X = \frac{3}{7})$: 0, the probability of a single value in a continuous distribution is 0.
f) Obtain the 40th and 72nd percentiles of the distribution of X:
40th percentile: $.4^2 = .16$
72th percentile: $.72^2 = .518$

9) 2.5.8: Suppose $F_y(y) = y^3$ for $0 \leq y < \frac{1}{2}$, and $F_y(y) = 1 - y^3$ for $\frac{-1}{2} \leq y \leq 1$. Comput the following a)

$$
\begin{aligned}
P(\frac{1}{3} < Y < \frac{3}{4}) = & \\
&= P(\frac{2}{6} \leq Y < \frac{3}{6}) = \frac{3}{6}^3 - \frac{2}{6}^3 \\
&= \frac{27}{216} - \frac{8}{216} \\
&= .087 \\
P(\frac{2}{4} \leq Y \leq \frac{3}{4}) &= (1 - (1 - \frac{3}{4})^3) - (1 - (1 - \frac{2}{4})^3 \\
&= (1 - \frac{1}{64}) - (1 - \frac{1}{8}) \\
&= .109
\end{aligned}
$$

$$.087 + .109 = .196$$

b) $P(Y = \frac{1}{3})$: $P(\frac{1}{3}) = \frac{1}{3}^3 = \frac{1}{27} = .037$

c) $P(Y = \frac{1}{2})$: $P(\frac{1}{2}) = \frac{1}{2}^3 = \frac{1}{8} = .125$

10) 2.5.21: Using problem 4

a) Determine the distribution function For the Weibuill(α) distribution.

$$
\begin{aligned}
F(x) &= \int_0^\infty \alpha x^{\alpha-1} e^{x^\alpha}\, dx \\
x^\alpha &\to u \\
\alpha x^{\alpha-1} &\to du \\
\int_0^\infty &\to \int_0^x \\
&= \int_0^x e^- u\, du \\
&= -e^{-u} - 1 \\
&= 1 - e^{-u}
\end{aligned}
$$

b) Derive the quantile function for the Weibuill (α) distribution

$$
\begin{aligned}
y &= 1 - e^{-u} \\
y - 1 &= -e^{-u} \\
1 - y &= e^{-u} \\
log(1-y) &= -u \\
-log(1-y) &= u
\end{aligned}
$$

The inverse of y is: $y = -log(1-u)$

11) 2.5.24: Determine the distribution function for the LaPlace distribution in problem 5

$$f(x) = \frac{e^{-x}}{2}$$

$$
\begin{aligned}
\int_0^\infty \frac{e^{-x}}{2}\, dx &= \lim_{x\to\infty} \frac{-e^{-x}}{2} - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty \frac{e^{-|x|}}{2}\, dx &= \lim_{x\to-\infty} \frac{-e^{-|x|}}{2} - \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

12) 2.6.1: Let X ~ Uniform[L, R]. Let $Y = cX + d$, where c > 0. Prove that Y ~ Uniform[cL + d, cR + d]. Look at 2.6.4

$$y = cX + d, y' = x, y^{-1} = \frac{y-d}{c}$$

$$\begin{aligned}\frac{\frac{y-d}{c}}{x\frac{y-d}{c}} &= \frac{y-d}{c}\frac{xc}{xy-xd} \\ &= \frac{1}{x}\end{aligned}$$

13) 2.6.5: Let X ~ Exponential(λ). Let $Y = X^3$. Compute the density of f_Y of Y.

$$Y = X^3, Y^{-1} = X^{\frac{1}{3}}, Y' = 3x^2$$

$$ExponentialFunction : \lambda e^{-\lambda x}$$

$$\frac{fx(y^{-1})}{y'(y^{-1}(x))} = \frac{\lambda e^{-\lambda x^{1/3}}}{3x^2 x^{1/3} x^3}$$

14) 2.6.9: Let X have the density function $f_Y(x) = \frac{x^3}{4}$ for $0 < x < 2$, otherwise $f_X(x) = 0$.

a) Let $Y = X^2$. Computer the density function $f_Y(y)$ for Y.

$$Y = x^2, Y' = 2x, Y^{-1} = \sqrt{x}$$

$$\frac{fx(\sqrt{x})}{2x\sqrt{x}} = \frac{x^3}{42x} = \frac{x^2}{8} = \frac{Y}{8}$$

b) Let $Z = \sqrt{X}$. Compute the density function $f_Z(z)$ for Z.

$$Z = \sqrt{x}, Z' = \frac{1}{\sqrt{x}}, Z^{-1} = x^2$$

$$\frac{fx(x^2)}{x^{\frac{-1}{2}}x^2} = \frac{x^3}{4x^{\frac{-1}{2}}} = \frac{x^{3.5}}{4}$$

15) 2.6.12: Let X have density function $f_X(x) = \frac{1}{x^2}$ for x > 1, otherwise $f_X(x) = 0$. Let $Y = X^{\frac{1}{3}}$. Compute the density function $f_Y(y)$ for Y.

$$Y = x^{\frac{1}{3}}, Y' = \frac{1}{3x^{\frac{2}{3}}}, Y^{-1} = x^3$$

$$\frac{fx(x^3)}{\frac{1}{3x^{\frac{2}{3}}}x^3} = \frac{3x^{\frac{2}{3}}}{x^2} = \frac{3}{x^{1\frac{1}{3}}}$$

16) 2.6.18 (assume $\beta > 0$): Suppose that X ~ Weibull(α) (see problem 4), determine the distribution function of $Y = (1+X)^\beta$.

$$Y = X^B, Y' = Bx^{B-1}, Y^{-1} = \frac{log(Y)}{log(B)}$$

$$\begin{aligned}
\frac{fx(\frac{log(Y)}{log(B)})}{Bx^{B-1}\frac{log(Y)}{log(B)}} &= \\
&= \frac{\alpha(1+x)^{\alpha-1}e^{-x^\alpha}}{Bx^{B-1}} \\
&= \frac{\alpha(1+\frac{log(Y)}{log(B)})^{\alpha-1}e^{-\frac{log(Y)}{log(B)}^\alpha}}{B\frac{log(Y)}{log(B)}^{B-1}}
\end{aligned}$$