STATISTICS 630 - Solution to Test I **June 19, 2012**

1. A box in a supply room contains four 60W bulbs, six 75W bulbs, and five 100W bulbs. Suppose that three bulbs are randomly selected (without replacement).

   (a) Find the probability that all three bulbs have the same wattage.
   Define the events A = the event that all 3 bulbs are 60W, B = the event that all 3 bulbs are 75W, and C = the event that all 3 bulbs are 100W. Then

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{\binom{4}{3}}{\binom{15}{3}} + \frac{\binom{6}{3}}{\binom{15}{3}} + \frac{\binom{5}{3}}{\binom{15}{3}} = \frac{\binom{4}{3} + \binom{6}{3} + \binom{5}{3}}{\binom{15}{3}} = \frac{34}{455}$$

   (b) Conditional on all three bulbs having the same wattage, obtain the probability that all three bulbs are 100W bulbs.

   $$P(C|A \cup B \cup C) = \frac{\frac{\binom{5}{3}}{\binom{15}{3}}}{\frac{\binom{4}{3}+\binom{6}{3}+\binom{5}{3}}{\binom{15}{3}}} = \frac{\binom{5}{3}}{\binom{4}{3} + \binom{6}{3} + \binom{5}{3}} = \frac{10}{34}$$

2. Suppose that X is a random variable with cumulative distribution function

   $$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x^2, & 0 < x < 0.4 \\ x, & 0.4 \leq x < 1 \\ 1, & 1 \leq x. \end{cases}$$

   Compute each of the following:

   (a) $P(X = 0.5)$
   $P(X = 0.5) = F_X(0.5) - F_X(0.5-) = 0.5 - 0.5 = 0$

   (b) $P(X = 0.4)$
   $P(X = 0.4) = F_X(0.4) - F_X(0.4-) = 0.4 - 0.4^2 = 0.24$

   (c) $P(0.3 \leq X \leq 0.6)$
   $P(0.3 \leq X \leq 0.6) = F_X(0.6) - F_X(0.3-) = 0.6 - 0.3^2 = 0.51$

   (d) $P(0.4 \leq X \leq 0.6)$
   $P(0.4 \leq X \leq 0.6) = F_X(0.6) - F_X(0.4-) = 0.6 - 0.4^2 = 0.44$

3. Suppose that V is a random variable with the cumulative distribution function

$$F_V(v) = \begin{cases} 0 & v \le -1 \\ \frac{(1+v)^2}{4}, & -1 < v < 1 \\ 1, & v \ge 1 \end{cases}$$

Obtain the probability density function of $W = V^2$.

Let $0 < w < 1$. Then

$$\begin{aligned} P[W \le w] &= P[-\sqrt{w} \le V \le \sqrt{w}] = F_V(\sqrt{w}) - F_V(\sqrt{-w}) \\ &= \frac{1 + 2\sqrt{w} + \sqrt{w}^2}{4} - \frac{1 + 2(-\sqrt{w}) + (-\sqrt{w})^2}{4} = \sqrt{w} \end{aligned}$$

Since $\frac{d\sqrt{w}}{dw} = \frac{1}{2\sqrt{w}}$,

$$f_W(w) = \begin{cases} \frac{1}{2\sqrt{w}}, & 0 < w < 1 \\ 0, & \text{otherwise.} \end{cases}$$

4. Consider rolling a fair six-sided die and independently tossing two fair two-sided coins. Let $Z =$ the number showing on the die, and let $W =$ the number of heads showing in the two tosses. Find the probability distributions of (i) W and (ii) $U = WZ$.

   (i) The sample space is $\{HH, HT, TH, TT\}$ with probability 1/4 for each outcome. Thus, $P(W = 0) = P[\{TT\}] = 1/4$, $P(W = 1) = P[\{HT, TH\}] = 1/2$, and $P(W = 2) = P[\{HH\}] = 1/4$.

   (ii) Enumerate the possible values of wz where $w = 0, 1, 2$ and $z = 1, 2, ..., 6$. Then compute the probabilities. $P[U = 0] = P[W = 0] = 1/4$, $P[U = 1] = P[W = 1, Z = 1] = P[W = 1] \times P[Z = 1] = 1/2 \times 1/6 = 1/12$, $P[U = 2] = P(W = 1, Z = 2) + P(W = 2, Z = 1) = 1/2 \times 1/6 + 1/4 \times 1/6 = 1/8$, etc. The pmf of U is

u	0	1	2	3	4	5	6	8	10	12
$f_U(u)$	1/4	1/12	1/8	1/12	1/8	1/12	1/8	1/24	1/24	1/24

5. Suppose that (X, Y) have the joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} x + Cy & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find the value of C that makes $f_{X,Y}(x,y)$ a valid probability density function. Then determine whether X and Y are independent. Justify your answer.

$$\int_0^1 \int_0^1 (x + Cy)dxdy = \int_0^1 (1/2 + Cy)dy = 1/2 + C/2 = 1.$$

Thus, $C = 1$. The marginal pdfs are $f_X(x) = 1/2 + x,\ 0 \le x \le 1$ and $f_Y(y) = 1/2 + y,\ 0 \le y \le 1$. Since $(1/2 + x) \times (1/2 + y) \ne x + y$ for $0 \le x \le 1,\ 0 \le y \le 1$, X and Y are not independent.