STAT 630–Formulas for Test 2

Cumulative Distribution Function

The cdf of a random variable X is a function $F_X(x) = P(X \leq x)$ for each x.

Relationship of CDF and PDF for a Continuous RV $f_X(x) = \frac{d}{dx} F_X(x)$.

Function of a Discrete RV

Let $Y = h(X)$ where X is a discrete rv with pmf $p_X(x)$. Then the pmf of Y is $p_Y(y) = \sum_{\{x:h(x)=y\}} p_X(x)$.

Function of a Continuous RV

Let $Y = h(X)$ where X is a continuous rv with pdf $f_X(x)$. Then the cdf of Y is

$$F_Y(y) = P[h(X) \leq y] = \int_{\{x:h(x)\leq y\}} f_X(x)dx$$

If h is differentiable and strictly monotonic on some interval I which includes the range of X, the pdf of Y equals

$$f_Y(y) = f_X(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|.$$

Joint Probability Mass Function $p(x, y) = P(X = x, Y = y)$

Joint Probability Density Function A joint pdf f is a nonnegative function such that

$$P((X,Y) \in A) = \int_A \int f(x,y)\, dxdy.$$

Bivariate Distribution Function This is the function F such that $F(x, y) = P(X \leq x, Y \leq y)$.

Obtaining Joint PDF from CDF If X and Y are continuous rvs, $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$

Marginal Distributions When X and Y have joint pmf p or joint pdf f, the marginal pmf or pdf of X is

$$p_X(x) = \sum_y p(x,y), \text{ for } X, Y \text{ discrete} \qquad f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy, \text{for } X, Y \text{ continuous}.$$

Conditional PMF or PDF When X and Y have joint pmf p or pdf f and X has marginal pmf p_X or pdf f_X, the conditional pmf or pdf of Y given that $X = x$ is

$$p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)}, \qquad \text{or} \qquad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}.$$

Independent RVs When X and Y have joint pmf p or pdf f, they are independent iff

$$p(x,y) = p_X(x)p_Y(y) \qquad \text{or} \qquad f(x,y) = f_X(x)f_Y(y), \text{ all } x, y.$$

Convolutions When X and Y are independent continuous rvs with pdfs f_X and f_Y, $Z = X + Y$ has cdf and pdf

$$F_Z(z) = \int_{-\infty}^{\infty} f_Y(y)F_X(z-y)dy \qquad f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$

When X and Y are independent discrete rvs with pmfs p_X and p_Y, the pmf of $Z = X + Y$ is

$$p_Z(z) = \sum_{\{(x,y):x+y=z\}} p_X(x)p_Y(y).$$

Maxima and Minima When $X_1, \ldots, X_n$ are continuous rvs with the same cdf F, the cdfs of the maximum U and minimum V are, respectively, $F_U(u) = [F(u)]^n$ and $F_V(v) = 1 - (1 - F(v))^n$.

Expectations Let p or f be the pmf or pdf of a random variable X. The expectation of $h(X)$ is

$$E[h(X)] = \begin{cases} \int_{-\infty}^{\infty} h(x)f(x)\,dx, & X \text{ continuous}, \\ \sum_x h(x)p(x), & X \text{ discrete}. \end{cases}$$

Expectation of a Linear Function: $E(aX + b) = aE(X) + b$

Expectation of a Sum: $E(b + a_1X_1 + \cdots + a_nX_n) = b + a_1E(X_1) + \cdots + a_nE(X_n)$

Expectation of a Product of Independent RVs: Suppose $X_1, \ldots, X_n$ are independent rvs. Then

$$E\left(\prod_{i=1}^{n} h(X_i)\right) = \prod_{i=1}^{n} E(h(X_i))$$

Variance Let $\mu = E(X)$. The variance of X is $\text{Var}(X) = E[(X - \mu)^2] = E(X^2) - (E(X))^2$.

Covariance and correlation

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - E(X)E(Y), \qquad \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Covariance of Sums and Variance of a Sum

$$\text{Cov}(\sum_{i=1}^{n} a + b_iX_i, c + \sum_{j=1}^{m} d_jY_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} b_id_j\text{Cov}(X_i, Y_j), \ \text{Var}\left(\sum_{i=1}^{n} a_iX_i + b\right) = \sum_{i=1}^{n} a_i^2\text{Var}(X_i) + \sum_{i=1}^{n}\sum_{j \neq i} a_ia_j\text{Cov}(X_i, X_j)$$

Moments The moments of X are $E(X^k),\ k = 1, 2, \ldots$.

Moment generating function The mgf of X is $m_X(s) = E(e^{sX})$.

$$E(X^k) = \frac{d^k m_X(s)}{ds^k}\bigg|_{s=0}, \ m_{aX+b}(s) = e^{bs}m_X(as), \ m_{X_1+\cdots+X_n}(s) = \prod_{i=1}^{n} m_{X_i}(s), \text{ if } X_1, \ldots, X_n \text{ are independent}$$

Conditional Mean The conditional mean of Y given $X = x$ is

$$E(Y|x) = \int_{-\infty}^{\infty} yf_{Y|X}(y|x)dy \quad \text{or} \quad E(Y|x) = \sum_y yp_{Y|X}(y|x)$$

$$E[E[Y|X]] = E[Y]$$

Conditional Variance The conditional variance of Y given $X = x$ is

$$\text{Var}(Y|X = x) = E[(Y - E(Y|x))^2|X = x]$$

$$\text{Var}(Y) = \text{Var}[E(Y|X)] + E[\text{Var}(Y|X)]$$

Markov Inequality For any nonnegative rv X and $a > 0$, $P[X \geq a] \leq \frac{E(X)}{a}$

Chebyshev Inequality For all $a > 0$, $P(|X - \mu_X| \geq a) \leq \frac{\sigma^2}{a^2}$

Convergence in Probability The sequence $Z_1, Z_2, \ldots$ converges in probability to the constant b if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|Z_n - b| \geq \varepsilon) = 0.$$

Law of Large Numbers If $X_1, X_2, \ldots$ is a sequence of independent rvs with mean μ and variance σ^2, then $\bar{X}_n$ converges in probability to μ.

Convergence in Distribution The sequence of rvs $X_1, X_2, \ldots$ converges in distribution to the rv X if for all x where $F_X(x)$ is continuous, $\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x) = F_X(x)$.

Central Limit Theorem If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with mean μ and variance σ^2, $S_n = \sum_{i=1}^{n} X_i$, and $\bar{X}_n = S_n/n$, then

$$\lim_{n\to\infty} P\left(\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \leq z\right) = \lim_{n\to\infty} P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq z\right) = \Phi(z).$$

Chi-Squared Distribution: The chi-squared distribution with n degrees of freedom (df) is a gamma distribution with $\alpha = n/2$ and $\lambda = 1/2$. If $Z \sim N(0,1)$, then $Z^2 \sim \chi^2(1)$. If $X_1, \ldots, X_k$ are independent chi-squared rvs with dfs $n_1, \ldots, n_k$, respectively, then $X_1 + \cdots + X_k \sim \chi^2(n_1 + \cdots + n_k)$.

F **Distribution:** If X and Y are independent chi-squared rvs with m and n degrees of freedom, respectively, the rv

$$F = \frac{X/m}{Y/n}, \qquad \text{with pdf} \quad f(w) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)} \left(\frac{m}{n}\right)^{m/2} w^{m/2-1} \left(1 + \frac{m}{n}w\right)^{-(m+n)/2}, \ w \geq 0$$

has an F distribution with m, n degrees of freedom.

t **Distribution:** If $Z \sim N(0,1)$ and $Y \sim \chi^2(n)$ are independent rvs, then the rv

$$T = \frac{Z}{\left(\frac{Y}{n}\right)^{1/2}} \quad \text{with pdf} \quad f(t) = \frac{\Gamma[(n+1)/2]}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}, \ -\infty < t < \infty$$

has a t distribution with n df.

Distribution of a Sum of Independent Normal RVs: Let $X_1, \ldots, X_n$ be independent random variables such that $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \ldots, n$. Then $Y = b + a_1 X_1 + \cdots + a_n X_n \sim N\left(b + \sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$.

Sampling from the Normal Distribution: For a random sample from a $N(\mu, \sigma^2)$ distribution, the sample mean $\bar{X}$ and the sample variance $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ are independent random variables satisfying

$$\bar{X} \sim N(\mu, \sigma^2/n) \quad \text{and} \quad \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2(n-1).$$

Maximum Likelihood Estimation: The likelihood function for a random sample $X_1, \ldots, X_n$ from a distribution with pmf/pf $f_\theta(x)$ is

$$L(\theta|x_1, \ldots, x_n) = f_\theta(x_1) \cdot f_\theta(x_2) \cdots f_\theta(x_n).$$

The maximum likelihood estimation $\hat{\theta}$ maximizes the likelihood, or equivalently, maximizes the log-likelihood by solving the score equation:

$$\frac{\partial}{\partial\theta} \log\left[L(\theta|x_1, \ldots, x_n)\right] = 0 \ \text{ for } \theta.$$

Invariance Property of MLE: If $\hat{\theta}$ is the mle of θ and $\psi(\theta)$ is a one-to-one function, then $\psi(\hat{\theta})$ is the mle of $\psi(\theta)$.

Method of Moment Estimators: Equate the kth population moment $E_{\theta_1, \ldots, \theta_m}[X^k]$ with the kth sample moment $m_k = \frac{1}{n}\sum x_i^k$, for $k = 1, \ldots, m$ and solve for $\theta_1, \ldots, \theta_m$.

Bias: The bias of an estimator T of $\psi(\theta)$ is $\text{Bias}_\theta(T) = E_\theta(T) - \psi(\theta)$.

Standard Error: The standard error of an estimator T is $\sqrt{\text{Var}_\theta(T)}$.

Mean Squared Error: The MSE of an estimator T of $\psi(\theta)$ is $\text{MSE}_\theta(T) = E_\theta[(T - \psi(\theta))^2] = \text{Var}_\theta(T) + \text{Bias}_\theta(T)^2$.

Binomial Distribution

$$p(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, \quad x = 0, 1, \ldots, n, \quad \text{Mean} = n\theta, \quad \text{Variance} = n\theta(1-\theta), \quad m(s) = (1 - \theta + \theta e^s)^n$$

Binomial Sum

$$\sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} = (a+b)^n$$

Hypergeometric Distribution

$$p(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, \quad \max\{0, n-(N-M)\} \leq x \leq \min(n, M), \text{ Mean} = n\left(\frac{M}{N}\right)$$

Negative Binomial Distribution

$$p(x) = \binom{r-1+x}{x}\theta^r(1-\theta)^x, \ x = 0, 1, 2, \dots \ \text{Mean} = \frac{r(1-\theta)}{\theta}, \ \text{Var} = \frac{r(1-\theta)}{\theta^2}, \ m(s) = \left(\frac{\theta}{1-(1-\theta)e^s}\right)^r$$

Geometric Distribution (Negative binomial with $r = 1$)

$$p(x) = p(1-p)^x, \quad x = 0, 1, 2, \dots \qquad \text{Mean} = \frac{1-\theta}{\theta}$$

Geometric Sum

$$\sum_{x=0}^{\infty} \alpha^x = \frac{1}{1-\alpha}, \text{ for } 0 < \alpha < 1$$

Poisson Distribution

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \dots, \quad \lambda > 0, \qquad \text{Mean} = \lambda \qquad \text{Variance} = \lambda, \quad m(s) = e^{\lambda(e^s - 1)}$$

Normal Distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < \mu < \infty, \ \sigma > 0$$

$$\text{Mean} = \mu \qquad \text{Variance} = \sigma^2, \qquad m(s) = \exp(\mu s + \sigma^2 s^2/2)$$

Gamma Distribution

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x), \text{ where } \alpha > 0 \text{ and } \lambda > 0.$$

$$\text{Mean} = \frac{\alpha}{\lambda}, \qquad \text{Variance} = \frac{\alpha}{\lambda^2}, \qquad m(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha, \text{ for } t < \lambda.$$

Exponential Distribution (Gamma with $\alpha = 1$)

$$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x), \quad \lambda > 0, \qquad \text{Mean} = \frac{1}{\lambda}, \qquad \text{Variance} = \frac{1}{\lambda^2}$$

Uniform Distribution

$$f(x) = \frac{1}{(R-L)} \cdot I_{(L,R)}(x), \quad -\infty < L < R < \infty, \qquad \text{Mean} = \frac{(L+R)}{2} \qquad \text{Variance} = \frac{(R-L)^2}{12}$$

Beta Distribution

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \ 0 \leq x \leq 1, \quad \text{Mean} = \frac{a}{a+b}, \quad \text{Variance} = \frac{ab}{(a+b)^2(a+b+1)}.$$

A Few Indefinite Integrals and One Definite Integral

$$\int x^n dx = \frac{x^{n+1}}{n+1} \qquad \text{except for } n = -1$$

$$\int \frac{1}{x} dx = \log_e(x)$$

$$\int u\,dv = uv - \int v\,du \qquad \text{integration by parts}$$

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx, \qquad a > 0, \qquad \Gamma(a+1) = a\Gamma(a), \qquad \Gamma(1/2) = \sqrt{\pi}$$