STAT 630 Fall 2014
Homework 10 Solution

7.1.3

For the prior distribution of μ, since it is a normal distribution with mean zero, thus the probability for $\mu > 0$ is 0.5. For the posterior distribution, from example 7.1.2, we known it is a normal distribution with mean $\left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2} + \frac{n\bar{x}}{\sigma_0^2}\right)$ and variance $\left(\frac{1}{\tau_0^2} + \frac{n}{\sigma_0^2}\right)^{-1}$. Then we plug in the values given in the question and known the posterior distribution obeys normal distribution with mean $\frac{10}{10.1}$ and variance $\frac{1}{10.1}$. Therefore, $P(\mu > 0|n, \bar{x}) = 1-\Phi\left(\frac{0-10/10.1}{\sqrt{1/10.1}}\right) = 0.99917$.

7.1.4

(a) We have $L(\lambda|x_1, \cdots, x_n) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$ and the prior density $\frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta\lambda}$. Thus the posterior distribution

$$\begin{aligned} P(\lambda|x_1, \cdots, x_n) &\propto L(\lambda|x_1, \cdots, x_n) \cdot \pi(\lambda) \\ &= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \cdot \frac{\beta^\alpha \lambda^{\alpha-1}}{\Gamma(\alpha)} e^{-\beta\lambda} \\ &\propto e^{-(n+\beta)\lambda} \cdot \lambda^{n\bar{x}+\alpha-1} \end{aligned}$$

The last term is proportional to Gamma$(n\bar{x}+\alpha, n+\beta)$ except for a normalizing constant. Thus the posterior distribution of λ has a Gamma$(n\bar{x}+\alpha, n+\beta)$ distribution.

(b) According to the property of Gamma distribution, we have the posterior mean is $\frac{n\bar{x}+\alpha}{n+\beta}$, the posterior variance is $\frac{n\bar{x}+\alpha}{(n+\beta)^2}$ and the posterior mode is $\frac{n\bar{x}+\alpha-1}{n+\beta}$.

7.1.9

(a)

$$\begin{aligned} P(\text{nheads}) &= \int_0^1 P(\text{nheads}|\theta)\cdot\pi(\theta)d\theta \\ &= \int_0^1 \theta^n(1-\theta)^0\cdot 5I_{[0.4,0.6](\theta)}d\theta \\ &= \int_{0.4}^{0.6} 5\theta^n d\theta = \frac{5(0.6^{n+1}-0.4^{n+1})}{n+1} \end{aligned}$$

Then we can obtain the posterior distribution of θ:

$$\begin{aligned} P(\theta|\text{nheads}) &= \frac{P(\text{nheads}|\theta)\cdot\pi(\theta)}{P(\text{nheads})} \\ &= \frac{(n+1)\theta^n I_{[0.4,0.6]}(\theta)}{(0.6^{n+1}-0.4^{n+1})} \end{aligned}$$

(b) No. Because for any $\epsilon\in(0,0.01)$, we have

$$P(\theta\in[0.99-\epsilon,0.99+\epsilon]|n) = \int_{0.99-\epsilon}^{0.99+\epsilon}(n+1)\theta^n I_{[0.4,0.6]}(\theta)/(0.6^{n+1}-0.4^{n+1})d\theta = 0$$

Thus the posterior distribution will not put any probability mass for θ around 0.99.

(c) We can see if you set a prior which exclude a parameter value, then no matter what data is obtained, the posterior will not be positive for that parameter value. Thus we should set prior to be greater than zero for any possible parameter values.

7.1.14

The posterior density of μ is

$$N\left(\left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}\right),\left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\right).$$

When $\mu_0=2,\tau_0^2=1,\sigma_0^2=1,n=20,\bar{x}=8.2$, $\mu|x_1,\cdots,x_n\sim N(166/21,1/21)$. Here is the R code I use. In my simulation, I got the posterior mean to be 0.1266 and variance 1.254e-05.

```
n=10^4
mean=166/21
sd=sqrt(1/21)
mu=rnorm(n,mean,sd)
mean(1/mu)
var(1/mu)
```

7.2.1

From example 7.1.1, we know the posterior distribution for θ obeys Beta$(n\bar{x}+\alpha, n(1-\bar{x})+\beta)$. Therefore,

$$\begin{aligned}
E(\theta^m|x_1,\cdots,x_n) &= \int_0^1 B^{-1}(n\bar{x}+\alpha, n(1-\bar{x})+\beta)\cdot\theta^{m+n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1}d\theta \\
&= \frac{B(n\bar{x}+\alpha+m, n(1-\bar{x})+\beta)}{B(n\bar{x}+\alpha, n(1-\bar{x})+\beta)}\int_0^1 B^{-1}(n\bar{x}+\alpha+m, n(1-\bar{x})+\beta) \\
&\quad \cdot\theta^{m+n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1}d\theta \\
&= \frac{B(n\bar{x}+\alpha+m, n(1-\bar{x})+\beta)}{B(n\bar{x}+\alpha, n(1-\bar{x})+\beta)} \\
&= \frac{\Gamma(n\bar{x}+\alpha+m)\Gamma(n+\alpha+\beta)}{\Gamma(n\bar{x}+\alpha)\Gamma(n+m+\alpha+\beta)}
\end{aligned}$$

7.2.2

For the model discussed in Example 7.1.2, the posterior distribution of μ is

$$N((\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2})^{-1}(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}), (\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2})^{-1})$$

Hence, the posterior distribution of the third quartile Ψ is

$$N((\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2})^{-1}(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x})+\sigma_0 z_{0.75}, (\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2})^{-1})$$

The mode is identical to its mean, which is given by

$$\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2})^{-1}(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x})+\sigma_0 z_{0.75}$$

7.2.10

(a) The likelihood function $L(\lambda|x_1,\cdots,x_n)=\prod_{i=1}^n \lambda e^{-\lambda x_i}=\lambda^n e^{-n\bar{x}\lambda}$ and the prior is $\frac{\beta_0^{\alpha_0}\lambda^{\alpha_0-1}}{\Gamma(\alpha_0)}e^{-\beta_0\lambda}$. Thus the posterior

$$P(\lambda|x_1,\cdots,x_n)\propto\lambda^n e^{-n\bar{x}\lambda}\cdot\lambda^{\alpha_0-1}e^{-\beta_0\lambda}=\lambda^{n+\alpha_0-1}e^{-(\beta_0+n\bar{x})\lambda}$$

Compared with the density of the gamma distribution, we can find the posterior distribution of λ obeys Gamma$(n+\alpha_0, \beta_0+n\bar{x})$. So the posterior mean is $\frac{n+\alpha_0}{\beta_0+n\bar{x}}$ and the posterior variance is $\frac{n+\alpha_0}{(\beta_0+n\bar{x})^2}$. To obtain the mode of the posterior distribution, we need to find the λ which maximizes the objective function:$\log(\lambda^{n+\alpha_0-1}e^{-(\beta_0+n\bar{x})\lambda}) = (n+\alpha_0-1)\log(\lambda)-(\beta_0+n\bar{x})\lambda$. So take derivative with respect to λ for the objective function and let it be zero, we can obtain $\hat{\lambda}=\frac{n+\alpha_0-1}{\beta_0+n\bar{x}}$. You can easily verify the second derivative of the objective function is negative, so the mode is $\frac{n+\alpha_0-1}{\beta_0+n\bar{x}}$.

(b) When $n = 20$, $\bar{x} = 5.1$

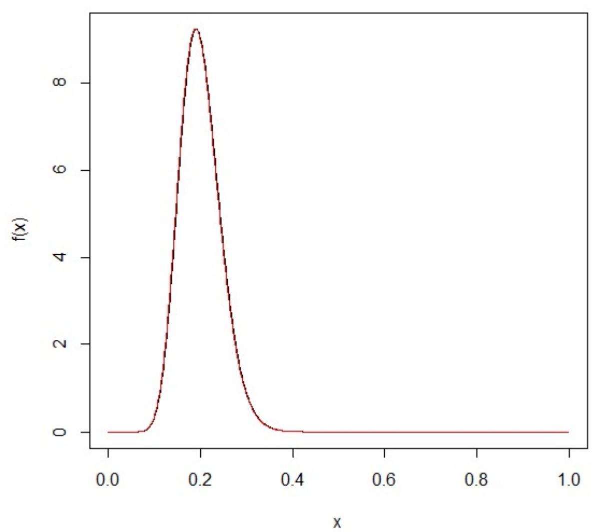
(i) If the prior is with mean 0.5 and the standard deviation 1. That is, $\frac{\alpha_0}{\beta_0} = 0.5$ and $\frac{\sqrt{\alpha_0}}{\beta_0} = 1$. Thus $\alpha_0 = 0.25$ and $\beta_0 = 0.5$ The posterior distribution is $Gamma(20 + 0.25, 20 \cdot 5.1 + 0.5)$=$Gamma(20.25, 102.5)$. The posterior mean=0.1976.

(ii) If the prior is with mean 10 and the standard deviation 20. That is, $\frac{\alpha_0}{\beta_0} = 10$ and $\frac{\sqrt{\alpha_0}}{\beta_0} = 20$. Thus $\alpha_0 = 0.25$ and $\beta_0 = 0.025$ The posterior distribution is $Gamma(20 + 0.25, 20 \cdot 5.1 + 0.025)$=$Gamma(20.25, 102.025)$. The posterior mean=0.1985.

R code:

```
x=seq(0,1,length=1001)
plot(x,dgamma(x,20.5,102.5),type="l",ylab="f(x)")
lines(x,dgamma(x,20.25,102.025),col=2)
```

The posteriors are plotted below. The posteriors are nearly identical.



6.3.27

The power of a test is the probability we reject the null hypothesis H_0. From the question we know if the p-value $1-\Phi\left(\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}\right)<\alpha$, we reject H_0. Thus the power function is

$$\begin{aligned}
\beta(\mu) &= P_\mu\left(1-\Phi\left(\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}\right)<\alpha\right) \\
&= P_\mu\left(\Phi\left(\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}\right)>1-\alpha\right) \\
&= P_\mu\left(\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}>Z_{1-\alpha}\right) \\
&= P_\mu\left(\frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}}>\frac{\mu_0-\mu}{\sigma_0/\sqrt{n}}+Z_{1-\alpha}\right)
\end{aligned}$$

Again, since $\frac{\bar{X}-\mu}{\sigma_0/\sqrt{n}}\sim \mathrm{N}(0,1)$, from the last equality above we know $\beta(\mu)=1-\Phi\left(\frac{\mu_0-\mu}{\sigma_0/\sqrt{n}}+Z_{1-\alpha}\right)$.

8.2.5

(a) Since $P_{\theta\in H_0}(\text{reject}H_0)=P_{\theta\in H_0}(x>\theta)=0$, the size of this test is 0.

(b) If $\theta>1$,

$$\begin{aligned}
\beta(\theta) &= P_\theta(x>1) \\
&= \int_1^\theta \frac{1}{\theta}dx \\
&= 1-\frac{1}{\theta}
\end{aligned}$$

otherwise it is zero.

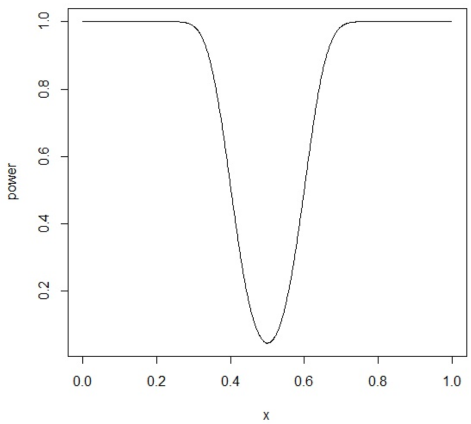
Addtional Problem: A

(a) To rejects $H_0$when $|X-50|>10$ is to reject H_0 when $X>60$ or $X<40$. $\alpha=P(Reject\ H_0|H_0)=P(X>60\ or\ X<40|H_0)=1-P(X<60|H_0)+P(X<40|H_0)$ $\approx\ 1-\Phi(\frac{60-50}{\sqrt{25}})+\Phi(\frac{40-50}{\sqrt{25}})=1-\Phi(2)+\Phi(-2)=1-0.9772499+0.02275013=0.04550026$.

(b) The power function is $\beta(\theta)=P(X>60\ or\ X<40)$ $\approx\ 1-\Phi(\frac{60-100\theta}{\sqrt{100\theta(1-\theta)}})+\Phi(\frac{40-100\theta}{\sqrt{100\theta(1-\theta)}})$.

R code:

```
x=seq(0,1,by=0.001)
y=1-pnorm((60-100*x)/sqrt(100*x*(1-x)))+pnorm((40-100*x)/sqrt(100*x*(1-x)))
plot(x,y,ylab="power",type="l")
```



Addtional Problem: B

(a)

x	1	2	3	4
$f_0(x)$	0.2	0.3	0.3	0.2
$f_1(x)$	0.1	0.4	0.1	0.4
LR	1/2	4/3	1/3	2

where $LR = \frac{P(X=x_i|H_1)}{P(X=x_i|H_0)}$. Thus $LR(3) < LR(1) < LR(2) < LR(4)$.

(b) Likelihood ratio test:
When $\alpha = 0.2$, we reject H_0 if $LR \geq 2$, i.e., $X = X_4$.
When $\alpha = 0.5$, we reject H_0 if $LR \geq 4/3$, i.e., $X = X_2$ or X_4.

(c) When $\alpha = 0.2$, the power$= 0.4$.
When $\alpha = 0.5$, the power$= 0.4 + 0.4 = 0.8$.

(d) When $P(H_0) = P(H_a)$, $\frac{P(H_0|x)}{P(H_a|x)} = \frac{\frac{P(x|H_0)P(H_0)}{P(x)}}{\frac{P(x|H_a)P(H_a)}{P(x)}} = \frac{P(x|H_0)}{P(x|H_a)}$.
For $X = x_1$, we have $\frac{P(H_0|x_1)}{P(H_a|x_1)} = 1/2 < 1$. For $X = x_3$, we have $\frac{P(H_0|x_3)}{P(H_a|x_3)} = 1/3 < 1$. Hence $\{x_1, x_3\}$ favor H_0.