STAT 638: Solution for Homework #6

7.3 (a)

$$\begin{aligned}
p(\boldsymbol{\theta}) &\propto |\Lambda_0|^{-1/2} \exp\{-\frac{1}{2}(\boldsymbol{\theta}-\mu_0)^T \Lambda_0^{-1}(\boldsymbol{\theta}-\mu_0)\} \\
&\propto |\Lambda_0|^{-1/2} \exp\{-\frac{1}{2}\boldsymbol{\theta}^T \Lambda_0^{-1}\boldsymbol{\theta} + \boldsymbol{\theta}^T \Lambda_0^{-1}\mu_0\} \\
&= \exp\{-\frac{1}{2}\boldsymbol{\theta}^T A_0 \boldsymbol{\theta} + \boldsymbol{\theta}^T b_0\} \text{ where } A_0 = \Lambda_0^{-1} \text{ and } b_0 = \Lambda_0^{-1}\mu_0.
\end{aligned}$$

$$\begin{aligned}
p(\boldsymbol{y}_1, ..., \boldsymbol{y}_n|\boldsymbol{\theta}, \Sigma) &\propto |\Sigma|^{-1/2} \exp\{-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{y}_i - \bar{\boldsymbol{y}})^T \Sigma^{-1}(\boldsymbol{y}_i - \bar{\boldsymbol{y}})\} \\
&\propto \exp\{-\frac{1}{2}\boldsymbol{\theta}^T A_1 \boldsymbol{\theta} + \boldsymbol{\theta}^T b_1\} \text{ where } A_1 = n\Sigma^{-1} \text{ and } b_1 = n\Sigma^{-1}\bar{\boldsymbol{y}}
\end{aligned}$$

Thus,

$$\begin{aligned}
p(\boldsymbol{\theta}|\Sigma, \boldsymbol{y}_1, ..., \boldsymbol{y}_n) &\propto \exp\{-\frac{1}{2}\boldsymbol{\theta}^T A_0 \boldsymbol{\theta} + \boldsymbol{\theta}^T b_0\} \exp\{-\frac{1}{2}\boldsymbol{\theta}^T A_1 \boldsymbol{\theta} + \boldsymbol{\theta}^T b_1\} \\
&= \exp\{-\frac{1}{2}\boldsymbol{\theta}^T A_n \boldsymbol{\theta} + \boldsymbol{\theta}^T b_n\} \text{ where } A_n = \Lambda_0^{-1} + n\Sigma^{-1} \text{ and } b_n = \Lambda_0^{-1}\mu_0 + n\Sigma^{-1}\bar{\boldsymbol{y}}
\end{aligned}$$

Hence, this distribution is multivariate normal$(A_n^{-1}b_n, A_n^{-1})$.

$$\begin{aligned}
p(\Sigma|\boldsymbol{y}_1, ..., \boldsymbol{y}_n, \boldsymbol{\theta}) &\propto |\Sigma|^{-(\nu_0+p+1)/2} \exp\{-\frac{1}{2}tr[\boldsymbol{S}_0\Sigma^{-1}]\}|\Sigma|^{-n/2} \exp\{-\frac{1}{2}tr(\boldsymbol{S}_\theta\Sigma^{-1}\} \\
&= |\Sigma|^{-(\nu_0+p+n+1)/2} \exp\{-\frac{1}{2}tr[(\boldsymbol{S}_0 + \boldsymbol{S}_\theta)\Sigma^{-1}]\} \text{ where } \boldsymbol{S}_\theta = \sum_{i=1}^{n}(\boldsymbol{y}_i - \boldsymbol{\theta})(\boldsymbol{y}_i - \boldsymbol{\theta})^T
\end{aligned}$$

Thus, this distribution is inverse-Wishart$(\nu_0 + n, (\boldsymbol{S}_0 + \boldsymbol{S}_\theta)^{-1})$.

(b) The plot shows that body depth and rear width of orange crabs are larger on average than those of blue crabs.

(c) $P(\rho_{\text{blue}} < \rho_{\text{orange}}|\boldsymbol{y}_{\text{blue}}, \boldsymbol{y}_{\text{orange}}) = 0.989$. This probability strongly suggests that blue crabs have a lower correlation between body depth and rear width than do orange crabs.

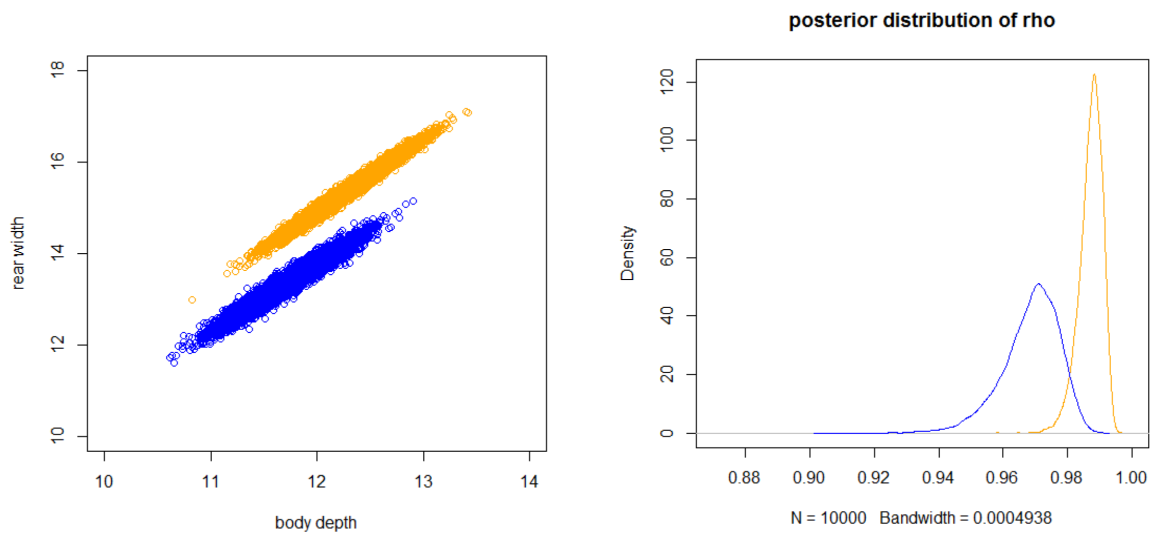
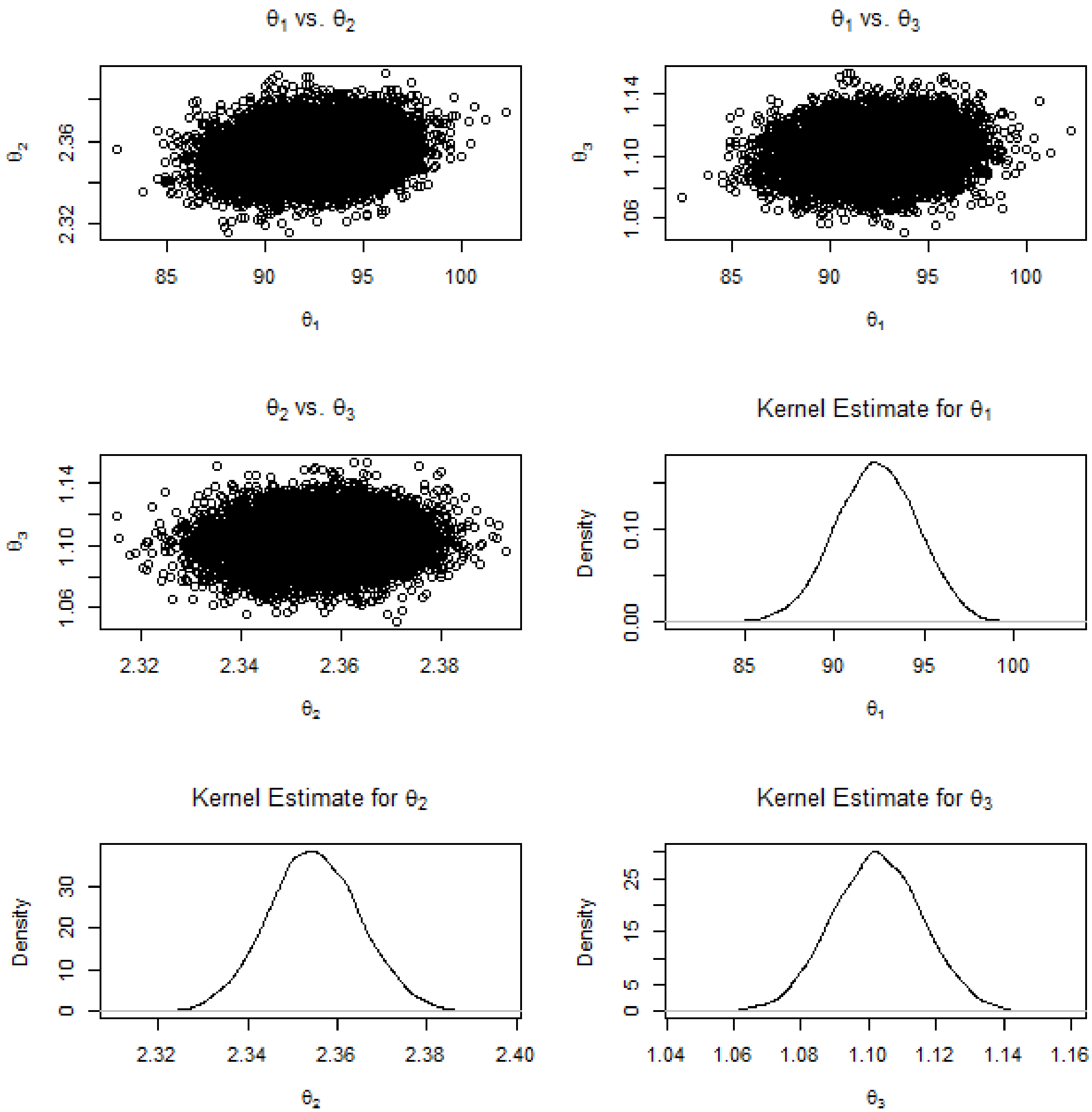
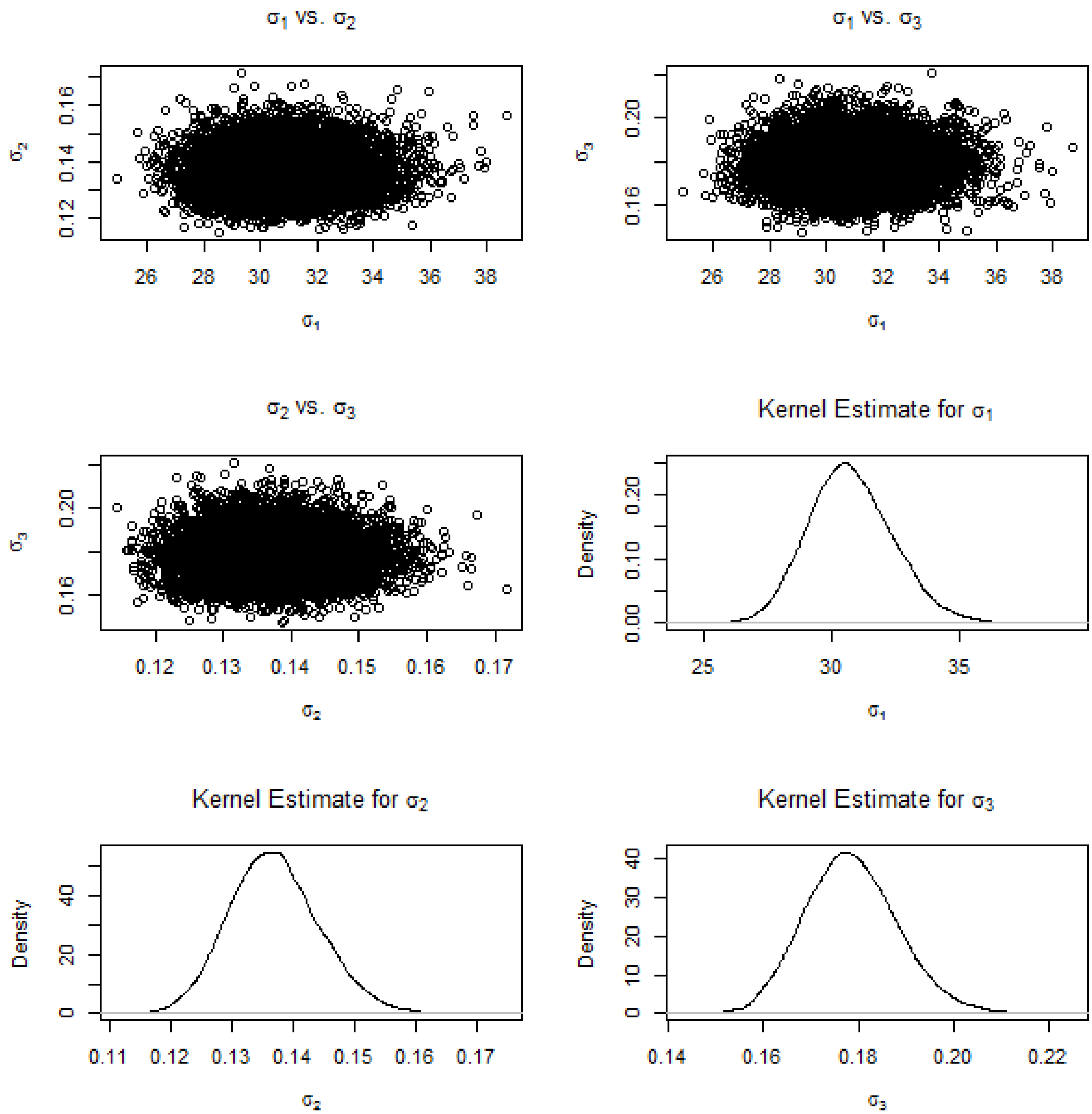


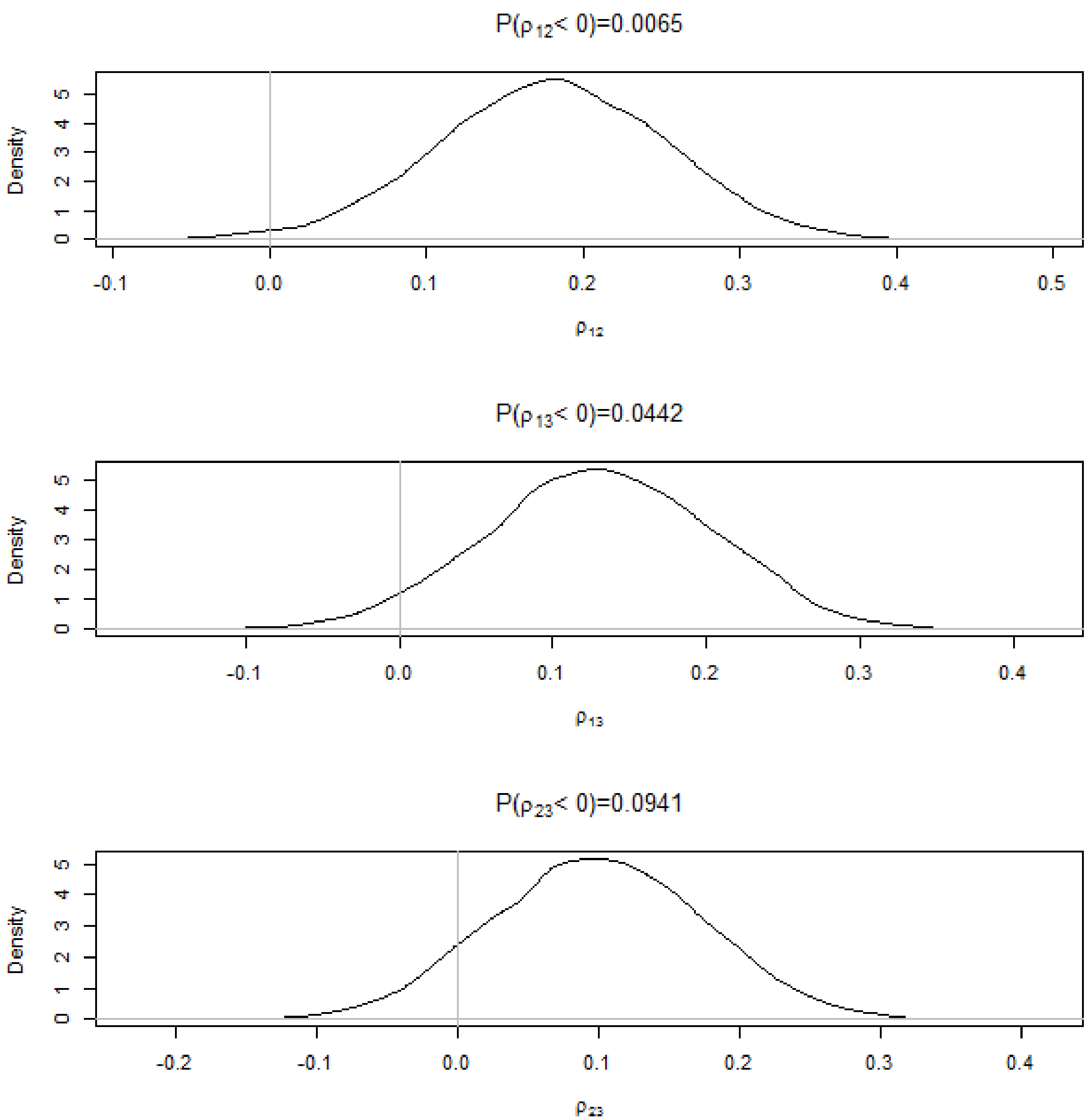
Figure 1: The left figure shows posterior samples of $\boldsymbol{\theta}$, and the right figure is an approximation of the posterior density of ρ. In both figures orange corresponds to orange crabs and blue to blue crabs.

Missing data problem





σ1 vs. σ2
σ1 vs. σ3
σ2 vs. σ3
Kernel Estimate for σ1
Kernel Estimate for σ2
Kernel Estimate for σ3
Density



P(ρ12< 0)=0.0065
Density
ρ12
P(ρ13< 0)=0.0442
Density
ρ13
P(ρ23< 0)=0.0941
Density
ρ23

	lower limit (mean +/- 1.96(sd))	upper limit (mean +/- 1.96(sd))	lower limit (percentile)	upper limit (percentile)
theta1	87.84511	96.90088	87.77296	96.82194
theta2	2.334672	2.375126	2.335148	2.375500
theta3	1.076107	1.128975	1.076581	1.128931
sigma1	27.51728	34.02	27.76954	34.27099
sigma2	0.1225796	0.1512929	0.1235071	0.1521936
sigma3	0.1595768	0.1974373	0.160743	0.1986015