# AMS526: Numerical Analysis I (Numerical Linear Algebra)

## Review Session

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# Announcement: Final Exam

- Tuesday **12/11** between **8:30pm** and **11:00pm** in classroom Physics P122
- Final exam will be **accumulative** covering **all** the material from the semester
- About 50–60% will be on material after Test 2 (i.e., eigenvalue problems and iterative methods, which are closely connected with earlier materials, especially QR and SVD)
- As usual, you can have a **single-sided, one-page**, **letter-size** (8.5inx11in) cheat sheet

# Topics Covered in The Course

- Fundamental concepts: norms, orthogonality, conditioning, stability
- Least squares problems using direct method (QR factorization)
- Singular value decomposition, properties, and relationship with eigenvalue problems
- Eigenvalue problems, properties, and algorithms (QR algorithm and Lanczos iterations)
- Solving linear systems using direct (Gaussian elimination) and iterative (Krylov subspace) methods
- Conditioning of problems, stability and backward stability of algorithms
- Efficiency of algorithms, convergence rate of iterative methods

# Matrix Properties and Transformations

- Properties
  - Hermitian (symmetric), skew symmetry, positive definite
  - unitary (orthogonal), normal, (orthogonal and oblique) projection matrix
  - singular/nonsingular, defective/nondefective
  - triangular, Hessenberg, tridiagonal, diagonal, Jordan-form, sparse
- Transformations
  - orthogonalization (Gram-Schmidt)
  - triangularization (Gaussian elimination, Cholesky factorization, Householder QR)
  - reduction to Hessenberg or tridiagonal form
  - similarity transformation and unitary similarity transformation (Schur factorization)
  - congruence transformation (preserves symmetry and inertia)

# Fundamental Algorithms

- QR factorization using classical and modified Gram-Schmidt
- QR factorization using Householder triangularization
- Gaussian elimination with partial pivoting and Cholesky factorization
- Reduction to Hessenberg/tridiagonal form for eigenvalue problems
- QR algorithm with or without shifts for eigenvalue problems
- Lanczos iterations and conjugate gradients
- Do not need to memorize the details of the algorithms
- Understand when they work, how they work, why they work, and how well they work
- Understand relationships among each other: how one transforms into another, and to make an intelligent choice

# Eigenvalue Problem

- *Eigenvalue problem* of $m \times m$ matrix $\boldsymbol{A}$ is $\boldsymbol{Ax} = \lambda \boldsymbol{x}$
- *Characteristic polynomial* is $\det(\boldsymbol{A} - \lambda \boldsymbol{I})$
- *Eigenvalue decomposition* of $\boldsymbol{A}$ is $\boldsymbol{A} = \boldsymbol{X \Lambda X}^{-1}$ (does not always exist)
- *Geometric multiplicity* of $\lambda$ is $\dim(\text{null}(\boldsymbol{A} - \lambda \boldsymbol{I}))$, and *algebraic multiplicity* of $\lambda$ is its multiplicity as a root of $p_{\boldsymbol{A}}$, where algebraic multiplicity $\geq$ geometric multiplicity
- *Similar* matrices have the same eigenvalues, and algebraic and geometric multiplicities
- *Schur factorization* $\boldsymbol{A} = \boldsymbol{QTQ}^*$ uses unitary similarity transformations

# Eigenvalue Algorithms

- Underlying concepts: power iterations, Rayleigh quotient, inverse iterations, convergence rate
- *Schur factorization* is typically done in two steps
  - Reduction to Hessenberg form for nonhermitian matrices or reduction to tridiagonal form for hermitian matrices by unitary similarity transformation
  - Finding eigenvalues of Hessenberg or **tridiagonal** form
- Finding eigenvalue of tridiagonal forms
  - QR algorithm with shifts, and their interpretations as (inverse) simultaneous iterations
  - Others: Bisection and divide-and-conquer
- Alternative method is Jacobi method for symmetric matrices using Jacobi rotations

# Relationship between SVD and Eigenvalue Decomposition

- SVD works for all matrices (even rectangular matrices), but eigenvalue decomposition (i.e., diagonalization) works only for nondefective square matrices
- Singular vectors are always orthonormal and singular values are always real, while eigenvectors may not be orthogonal and eigenvalues may be complex numbers
- For normal matrices, singular values and eigenvalue are particularly closely related, which make them particularly powerful analytical tools

# Iterative Methods

- Advantages and disadvantages of iterative methods vs. direct methods
- We focus on Krylov subspace methods for symmetric matrices
- Given $\boldsymbol{A}$ and $\boldsymbol{b}$, *Krylov subspace* is $\{\boldsymbol{b}, \boldsymbol{A}\boldsymbol{b}, \boldsymbol{A}^2\boldsymbol{b}, \cdots \boldsymbol{A}^k\boldsymbol{b}\}$
- Key observation: QR factorization of leading vectors of Krylov subspace leads to Hessenberg form for nonsymmetric matrices and tridiagonal form for symmetric matrices
- Lanczos iterations takes advantage of the tridiagonal form to get three-term recurrence version of Arnoldi iterations
- **Conjugate gradient methods** for solving SPD linear systems: solution as quadratic optimization problem, finite-termination properties with exact arithmetic, and convergence with floating-point arithmetic
- GMRES, Bi-CG, Bi-CGSTAB for nonsymmetric matrices
- Concepts of preconditioners, and multigrid method