Formula Sheet for Linear Algebra Final

6.3 Orthonormal bases; Gram-Schmidt process; QR-decomposition

- Apply the Gram-Schmidt process to a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ of vectors to obtain an orthonormal set $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_r\}$ of vectors with the same span as the original set. If the original set was a basis, then the final set will be an orthonormal basis. To find the vector $\mathbf{q}_i$, subtract from $\mathbf{v}_i$ its projections onto the vectors $\mathbf{q}_1, \ldots, \mathbf{q}_{i-1}$ (to get $\mathbf{w}_i$) and then normalize this vector to get $\mathbf{q}_i$:

$$\mathbf{w}_i = \mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{q}_1 \rangle \mathbf{q}_1 - \langle \mathbf{v}_i, \mathbf{q}_2 \rangle \mathbf{q}_2 - \cdots - \langle \mathbf{v}_i, \mathbf{q}_{i-1} \rangle \mathbf{q}_{i-1},$$

$$\mathbf{q}_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}.$$

- Use the results of Gram-Schmidt orthogonalization to write a matrix A with linearly independent columns as a product

$$A = QR,$$

where Q has orthonormal columns and R is an invertible upper triangular matrix. With

$$A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n],$$

we set

$$Q = [\mathbf{q}_1 \ \mathbf{q}_2 \ \cdots \ \mathbf{q}_n]$$

and

$$R = \begin{bmatrix} \|\mathbf{w}_1\| & \langle \mathbf{v}_2, \mathbf{q}_1 \rangle & \langle \mathbf{v}_3, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{v}_n, \mathbf{q}_1 \rangle \\ 0 & \|\mathbf{w}_2\| & \langle \mathbf{v}_3, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{v}_n, \mathbf{q}_2 \rangle \\ 0 & 0 & \|\mathbf{w}_3\| & \cdots & \langle \mathbf{v}_n, \mathbf{q}_3 \rangle \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \|\mathbf{w}_n\| \end{bmatrix}.$$

Note that $\|\mathbf{w}_i\| = \langle \mathbf{w}_i, \mathbf{q}_i \rangle = \langle \mathbf{v}_i, \mathbf{q}_i \rangle$.

6.4 Least squares approximation

- If W is a finite-dimensional subspace of an inner product space V, and if $\mathbf{v}$ is a vector in V, then the vector in W closest to $\mathbf{v}$ is $\operatorname{proj}_W \mathbf{v}$.

- Given a matrix $A \in M_{m \times n}$, the set of all vectors of the form $A\mathbf{x}$ is the span of the column vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of A. Thus $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ if and only if

$$\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

- If $\mathbf{b} \notin \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $A\mathbf{x} = \mathbf{b}$ does not have any solutions $\mathbf{x}$. However, the associated *normal system*

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

always has a solution $\mathbf{x}$. A solution $\mathbf{x}$ of the normal system satisfies $\operatorname{proj}_W \mathbf{b} = A\mathbf{x}$, so that $A\mathbf{x}$ is as close to $\mathbf{b}$ as possible. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are linearly independent, then $A^T A$ will be invertible, so we have

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}.$$

- It follows that
$$\operatorname{proj}_W \mathbf{b} = A\mathbf{x} = A(A^TA)^{-1}A^T\mathbf{b},$$
so the standard matrix for the linear transformation $\operatorname{proj}_W : \mathbb{R}^m \to \mathbb{R}^m$ is
$$[\operatorname{proj}_W] = A(A^TA)^{-1}A^T.$$

6.5 Change of basis (for vectors)

- $$\mathbf{x} = P_B[\mathbf{x}]_B$$
Solving for $[\mathbf{x}]_B$, we have
$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x}.$$
- To change directly from a nonstandard basis B to a nonstandard basis B', we have the formula
$$[\mathbf{x}]_{B'} = P_{B'}^{-1}P_B[\mathbf{x}]_B.$$
- The same formulas hold in a general vector space V, as long as we set $P_B = \Big[[\mathbf{v}_1]_E \cdots [\mathbf{v}_n]_E\Big]$ and replace $\mathbf{x}$ by $[\mathbf{x}]_E$ for some basis E of V (usually chosen to be "standard").

8.4 Matrices of general linear transformations

- The matrix $[T]_{B'}^B$ can be computed as
$$[T]_{B'}^B = \Big[[T(\mathbf{v}_1)]_{B'} \quad [T(\mathbf{v}_2)]_{B'} \quad \cdots \quad [T(\mathbf{v}_n)]_{B'}\Big],$$
where $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
- $$[T(\mathbf{x})]_{B'} = [T]_{B'}^B[\mathbf{x}]_B$$
- $$[T_2 \circ T_1]_{B''}^B = [T_2]_{B''}^{B'}[T_1]_{B'}^B$$

8.5 Similarity (change of basis for linear operators)

- $$[T] = P_B[T]_B^BP_B^{-1}$$