Exam Cheat Sheet

Binomial Coefficient: For $r \leq n$,

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

represents the number of possible combinations of n objects taken m at a time.

Multinomial Coefficient: For $n_1 + n_2 + \cdots + n_r = n$,

$$\binom{n}{n_1, n_2, ..., n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

represents the number of possible ways n objects can be divided into r groups with $n_1, n_2, ..., n_r$ objects in each, respectively.

Axioms for Probability Measures: The following 3 axioms define a probability measure:

1. For all events E, $0 \leq P(E) \leq 1$.
2. If S is the sample space, $P(S) = 1$.
3. For any sequence of mutually exclusive events $E_1, E_2, ..., E_n$, $P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$.

Useful Formulas for Probability Measures:

$$\begin{aligned}
P(E^c) &= 1 - P(E) \\
P(E) &\leq P(F) \text{ if } E \subseteq F \\
P(E) &= P(EF) \text{ if } E \subseteq F \\
P(E \cup F) &= P(E) + P(F) - P(EF) \\
P(E) &= P(EF) + P(EF^c) \\
P(E \cup F \cup G) &= P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG)
\end{aligned}$$

Conditional Probability: The probability of event E, given that we observe F, is:

$$P(E \mid F) = \frac{P(EF)}{P(F)}$$

Also, $P(E \mid F)$ defines a new probability measure with F being the restriced sample space.

Independence: Two items are independent if $P(EF) = P(E)P(F)$. If $P(E \mid F) = P(E)$, E and F are independent; when $P(F) \neq 0$, the two definitions are exactly equivalent.

Useful Formulas for Contitional Probability:

$$\begin{aligned}
P(EF) &= P(E \mid F)P(F) = P(F \mid E)P(E) \\
P(E) &= P(E \mid F)P(F) + P(E \mid F^c)P(F^c) \\
P(E_1 E_2 \cdots E_n) &= P(E_1)P(E_2 \mid E_1)P(E_3 \mid E_2 E_1) \cdots P(E_n \mid E_{n-1} E_{n-2} \cdots E_1) \\
&= P(E_n)P(E_{n-1} \mid E_n)P(E_{n-2} \mid E_{n-1} E_n) \cdots P(E_1 \mid E_2 E_3 \cdots E_n) \\
P(E \mid F) &= \frac{P(F \mid E)P(E)}{P(F)}
\end{aligned}$$

Expectation – Discrete: Suppose $p(x) = \mathrm{P}(X = x)$ is the probability mass function of a R.V. X. Then

$$\mathrm{E}\, X = \sum_x xp(x)$$
$$\mathrm{E}\,[g(X)] = \sum_x g(x)p(x)$$
$$\mathrm{Var}\,[X] = \mathrm{E}\left[(X-\mu)^2\right] = \mathrm{E}\left[X^2\right] - (\mathrm{E}\, X)^2, \quad \text{where } \mu = \mathrm{E}\, X$$

Discrete Distributions:

If $X \sim \mathit{Bernoulli}(x;p)$, where p is the probability of success, then

$$\mathrm{P}(X=x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \mathrm{E}\, X = p, \qquad \mathrm{Var}\,[X] = p(1-p)$$

If $X \sim \mathit{Binomial}(x;n,p)$, where p is the prob. of success and n is the number of tries, then

$$\mathrm{P}(X=x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x \in \{0,1,...,n\} \\ 0 & \text{otherwise} \end{cases}, \qquad \mathrm{E}\, X = np, \quad \mathrm{Var}\,[X] = np(1-p)$$

If $X \sim \mathit{Geometric}(x;p)$, then

$$\mathrm{P}(X=x) = \begin{cases} (1-p)^{x-1} p & x \in \{1,...\} \\ 0 & \text{otherwise} \end{cases}, \qquad \mathrm{E}\, X = 1/p \quad \mathrm{Var}\,[X] = \frac{1-p}{p^2}$$

If $X \sim \mathit{NegativeBinomial}(x;r,p)$, where p is the prob. of success, and r is the number of successes required,

$$\mathrm{P}(X=x) = \begin{cases} \binom{x-1}{r-1} p^r (1-p)^{x-r} & x \in \{r, r+1, ...\} \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{E}\, X = \frac{r}{p}$$

If $X \sim \mathit{Poisson}(x;\lambda)$, where λ is the rate parameter, then

$$\mathrm{P}(X=x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x \in \{0,1,2,...\} \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{E}\, X = \lambda \qquad \mathrm{Var}\,[X] = \lambda$$

If $X \sim \mathit{HyperGeometric}(x;n,N,m)$, then X represents the number of red balls in a sample of n balls drawn from an urn with N balls, m of which are red. Then

$$\mathrm{P}(X=x) = \begin{cases} \left[\binom{m}{x}\binom{N-m}{n-x}\right]\left[\binom{N}{n}^{-1}\right] & x \in \{0,1,...,n\} \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{E}\, X = \frac{nm}{N}$$

Expectation: Continuous case. Suppose $f_X(x)$ is the probability density function of X. Then

$$\begin{aligned}
\mathrm{E}\, X &= \int_{-\infty}^{\infty} x f_X(x)\mathrm{d}x \\
\mathrm{E}\left[g(X)\right] &= \int_{-\infty}^{\infty} g(x) f_X(x)\mathrm{d}x \\
\mathrm{Var}\left[X\right] &= \mathrm{E}\left[(X-\mu)^2\right] = \mathrm{E}\left[X^2\right] - (\mathrm{E}\, X)^2, \quad \text{where } \mu = \mathrm{E}\, X
\end{aligned}$$

Continuous distributions:

If $X \sim \mathit{Uniform}(x; a, b)$, with $a < b$, then

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{E}\, X = \frac{b+a}{2} \qquad \mathrm{Var}\left[X\right] = \frac{(b-a)^2}{12}$$

If $X \sim \mathit{Exponential}(x; \lambda)$, with λ being the rate, then

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{E}\, X = \frac{1}{\lambda} \qquad \mathrm{Var}\left[X\right] = \frac{1}{\lambda^2}$$

If $X \sim \mathcal{N}(x; \mu, \sigma^2)$, where μ is the expectation and σ^2 is the variance, then

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad \mathrm{E}\, X = \mu \qquad \mathrm{Var}\left[X\right] = \sigma^2$$

If $X \sim \mathit{Gamma}(x; \alpha, \lambda)$, where α is the shape parameter and λ is the rate, then

$$f_X(x) = \begin{cases} (\lambda x)^{\alpha-1}\left(\lambda e^{-\lambda x}\right)/\Gamma(\alpha) & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \mathrm{E}\, X = \frac{\alpha}{\lambda} \qquad \mathrm{Var}\left[X\right] = \frac{\alpha}{\lambda^2}$$

If $X \sim \mathit{Beta}(x; \alpha, \beta)$, then

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{E}\, X = \frac{\alpha}{\alpha+\beta}$$

Standard Normal CDF: If $Z \sim \mathcal{N}(z; 0, 1)$, then define $\Phi(z) = P(Z \leq z)$.

Poisson Approximation Theorem: Let $X \sim \mathit{Binomial}(x; n, p)$. For np small relative to n, $\mathrm{P}(X = x) \simeq \mathrm{P}(Y = x)$, where $Y \sim \mathit{Poisson}(y; \lambda = np)$.

DeMoivre-Laplace Approximation Theorem: Let $X \sim \mathit{Binomial}(x; n, p)$. For $np(1-p)$ sufficiently large,

$$\mathrm{P}\left(\frac{X - np}{\sqrt{np(1-p)}} \leq z\right) \simeq \mathrm{P}(Z \leq z) = \Phi(z)$$

where $Z \sim \mathcal{N}(z; 0, 1)$.