Design and Analysis of Data Structures for Dynamic Trees

Renato Werneck

Advisor: Robert Tarjan

Outline

$\Rightarrow$ The Dynamic Trees problem

- Existing data structures
- A new worst-case data structure
- A new amortized data structure
- Experimental results
- Final remarks

The Dynamic Trees Problem

- Dynamic trees:
  - Goal: maintain an n-vertex forest that changes over time.
    - link(v, w): adds edge between v and w.
    - cut(v, w): deletes edge (v, w).
  - Application-specific data associated with vertices/edges:
    - updates/queries can happen in bulk (entire paths or trees at once).

- Concrete examples:
  - Find minimum-weight edge on the path between v and w.
  - Add a value to every edge on the path between v and w.
  - Find total weight of all vertices in a tree.

- Goal: $O(\log n)$ time per operation.

Applications

- Subroutine of network flow algorithms
  - maximum flow
  - minimum cost flow
- Subroutine of dynamic graph algorithms
  - dynamic biconnected components
  - dynamic minimum spanning trees
  - dynamic minimum cut
- Subroutine of standard graph algorithms
  - multiple-source shortest paths in planar graphs
  - online minimum spanning trees
- $\cdots$

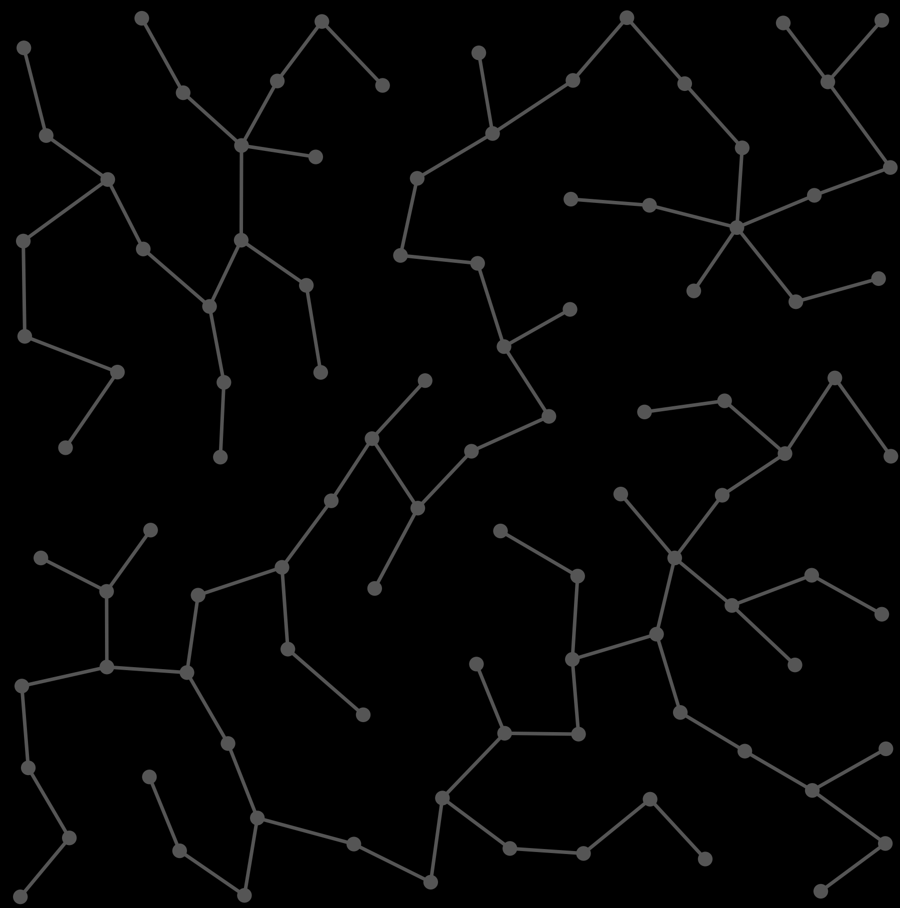
Application: Online Minimum Spanning Trees

- Problem:
  - Graph on n vertices, with new edges "arriving" one at a time.
  - Goal: maintain the minimum spanning forest (MSF) of G.

- Algorithm:
  - Edge $e = (v, w)$ with length $\ell(e)$ arrives:
    1. If v and w in different components: insert e;
    2. Otherwise, find longest edge f on the path $v \cdots w$:
       * $\ell(e) < \ell(f)$: remove f and insert e.
       * $\ell(e) \geq \ell(f)$: discard e.

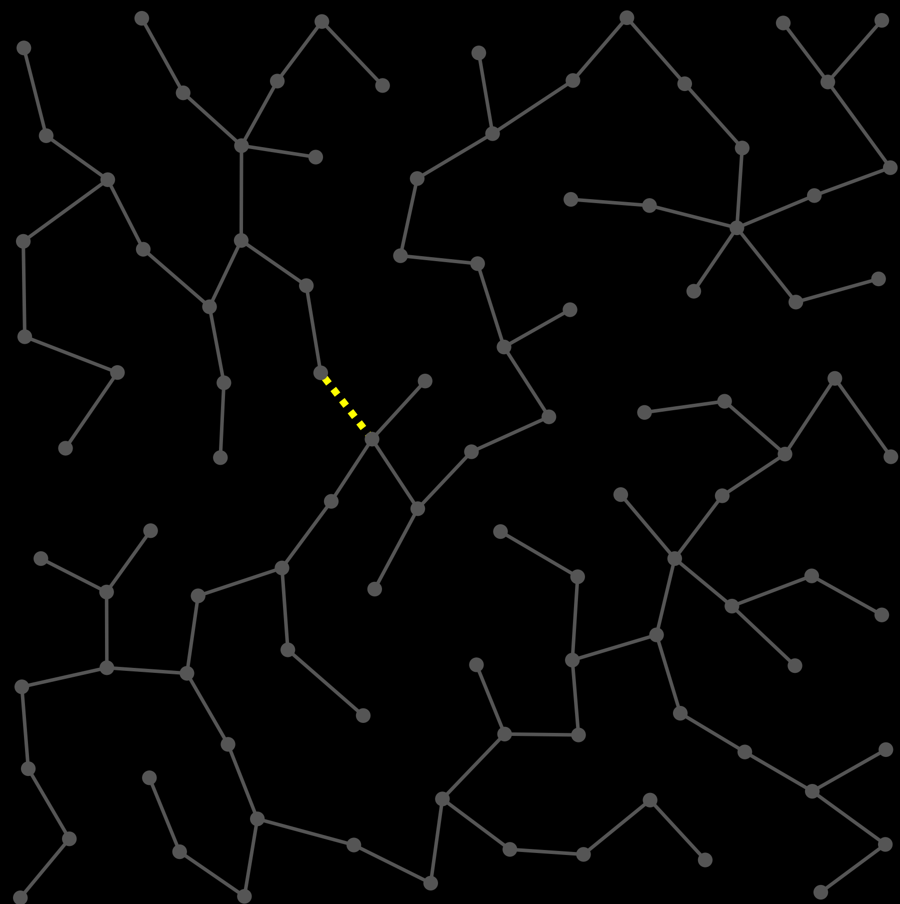
Example: Online Minimum Spanning Trees

- Current minimum spanning forest.



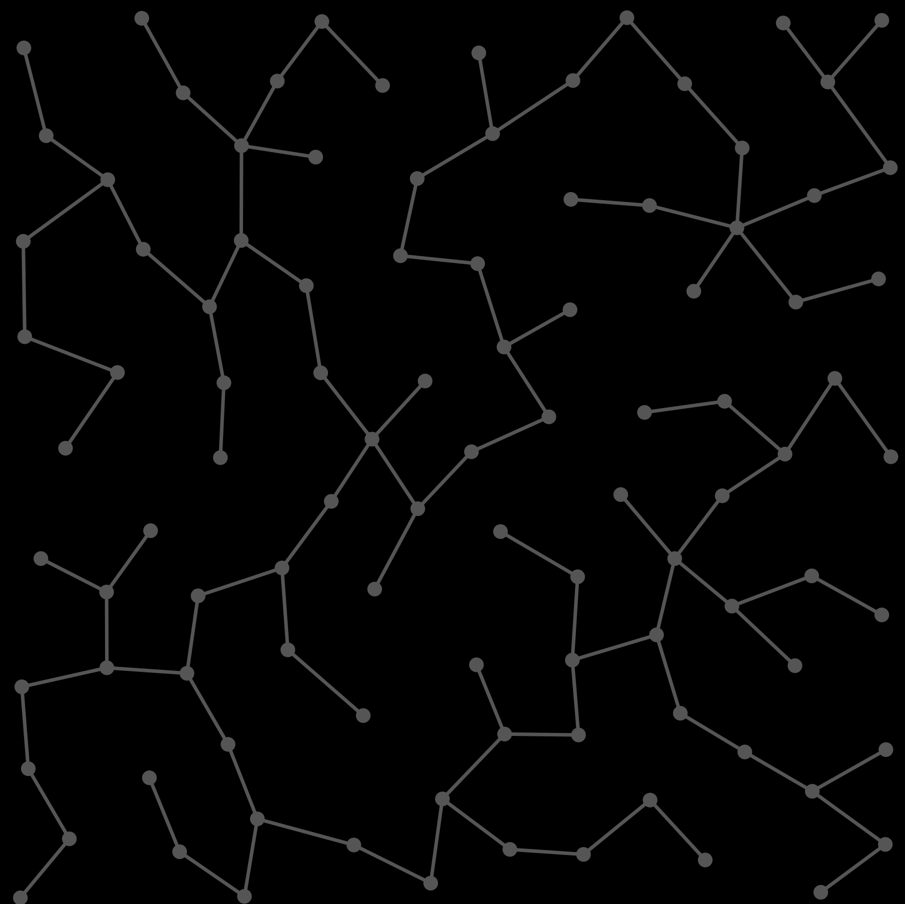
Example: Online Minimum Spanning Trees

- Edge between different components arrives.



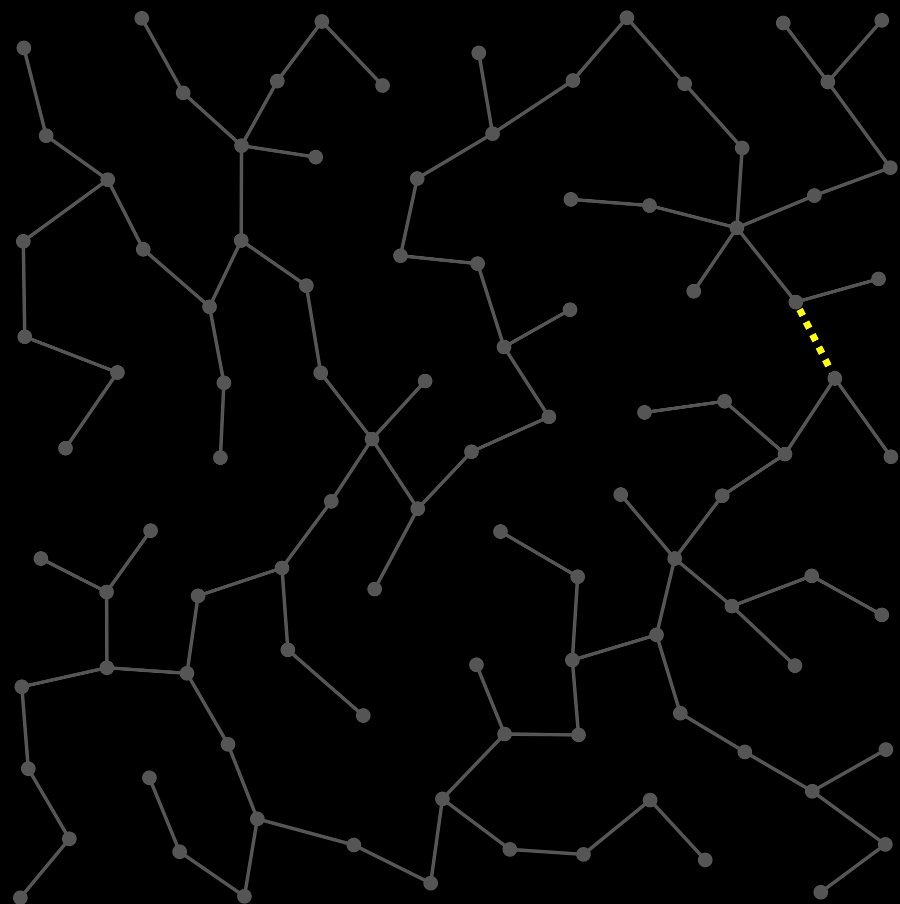
Example: Online Minimum Spanning Trees

- Edge between different components arrives:
  - add it to the forest.



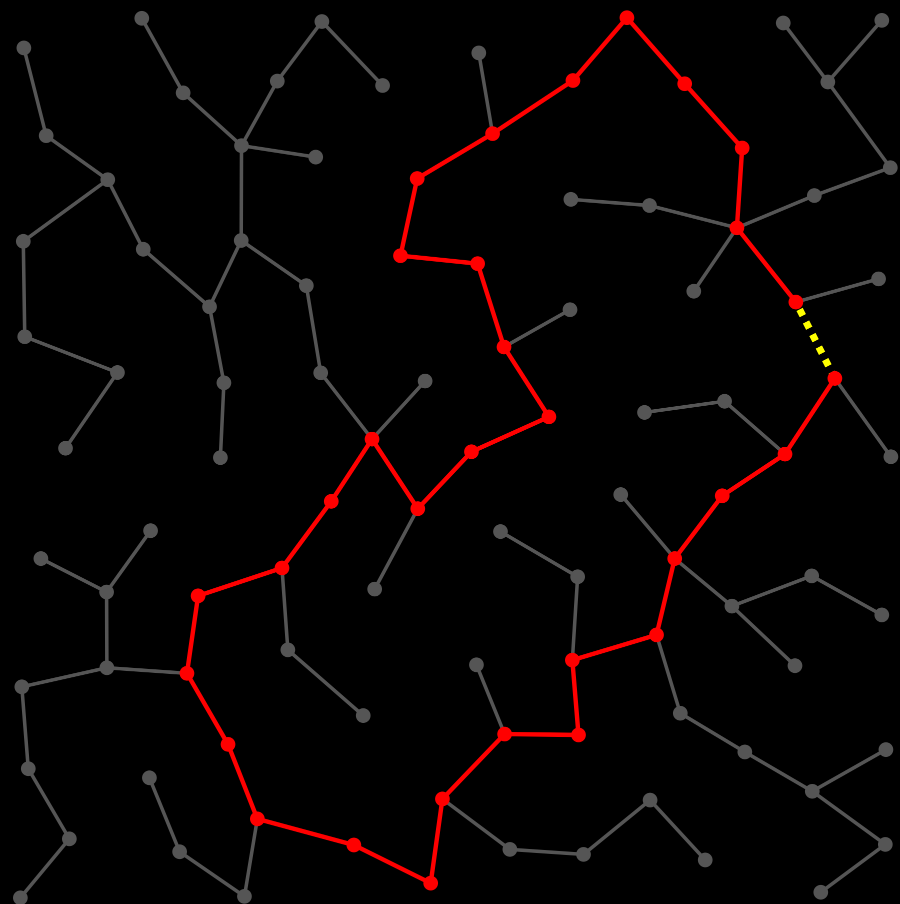
Example: Online Minimum Spanning Trees

- Edge within single component arrives.



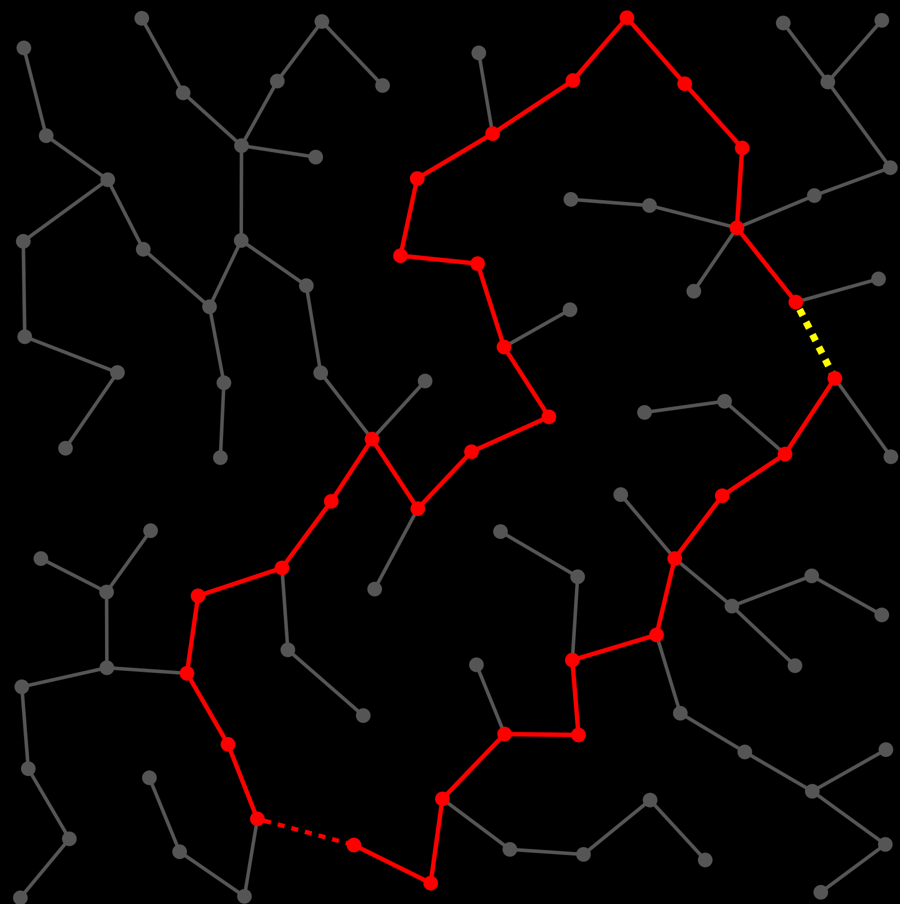
Example: Online Minimum Spanning Trees

- Edge within single component arrives:
  - determine path between its endpoints.



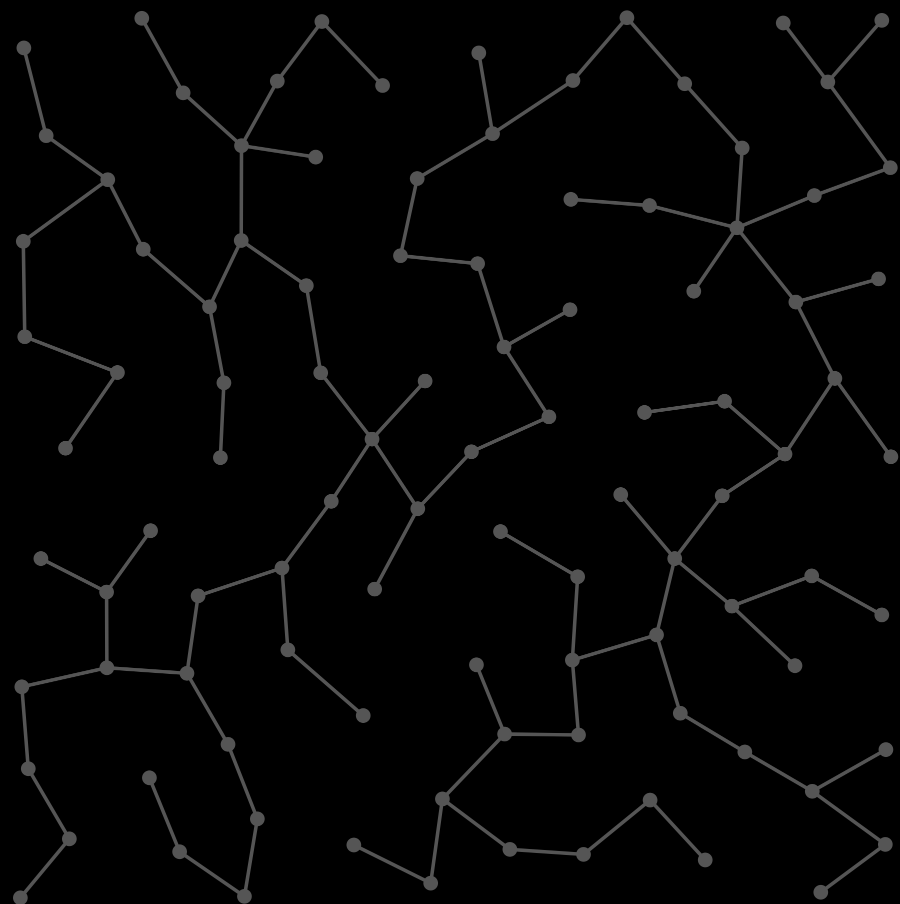
Example: Online Minimum Spanning Trees

- Edge within single component arrives:
  - find longest edge on the path between its endpoints.



Example: Online Minimum Spanning Trees

- Edge within single component arrives:
  - if longest edge on the path longer than new edge, swap them.



Online Minimum Spanning Trees

- Data structure must support the following operations:
  - add an edge;
  - remove an edge;
  - decide whether two vertices belong to the same component;
  - find the longest edge on a specific path.

- Each in $O(\log n)$ time:
  - basic strategy: map arbitrary tree onto balanced tree.

Data Structures for Dynamic Trees

- Different applications require different operations.

- Desirable features of a data structure for Dynamic Trees:
  - low overhead (fast);
  - simple to implement;
  - intuitive interface (easy to adapt, specialize, modify);
  - general:
    - path and tree queries;
    - no degree constraints;
    - rooted and unrooted trees.

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Main Strategies

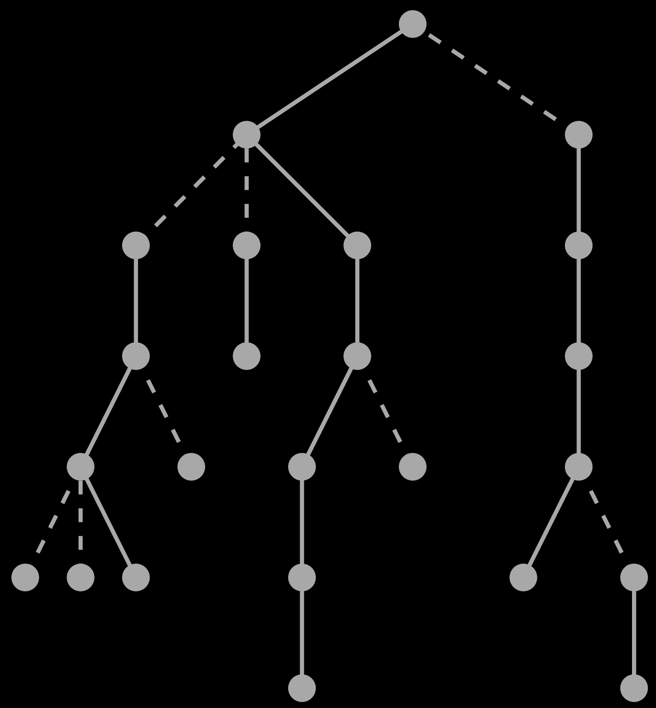
- Path decomposition:
  - ST-trees [Sleator and Tarjan 83, 85];
    - also known as link-cut trees or dynamic trees.

- Tree contraction:
  - Topology trees [Frederickson 85];
  - Top trees [Alstrup, Holm, de Lichtenberg, and Thorup 97];
  - RC-trees [Acar, Blelloch, Harper, Vittes, and Woo 04].

- Linearization:
  - ET-trees [Henzinger and King 95, Tarjan 97];
  - Less general: cannot handle path queries.

Path Decomposition

- ST-trees [Sleator and Tarjan 83, 85]:
  - partition the tree into vertex-disjoint paths;
  - represent each path as a binary tree;
  - "glue" the binary trees appropriately.



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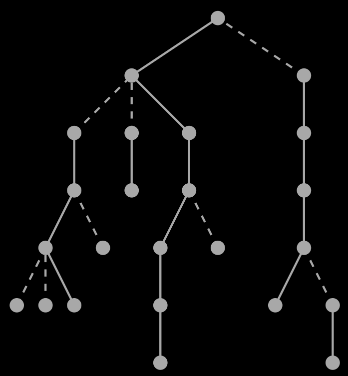
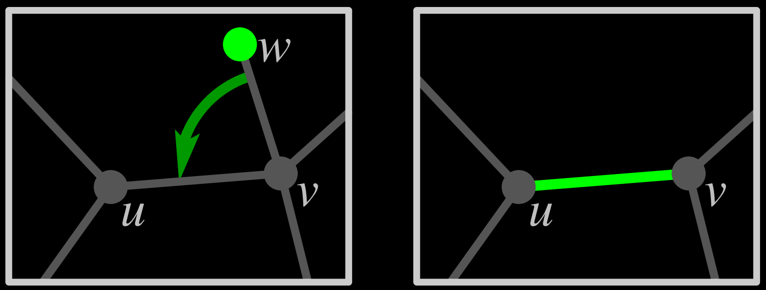


- Complexity:
  - with ordinary balanced trees: $O(\log^2 n)$ worst-case;
  - with globally biased trees: $O(\log n)$ worst-case;
  - with splay trees: $O(\log n)$ amortized, but much simpler.

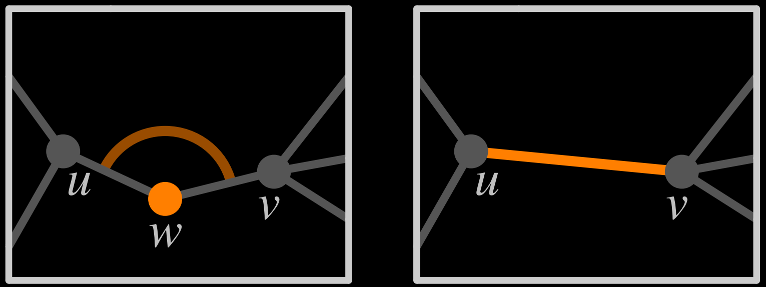
- Main features:
  - relatively low overhead (one node per vertex/edge);
  - adapting to different applications requires knowledge of inner workings;
  - tree-related queries require bounded degrees (or ternarization).

Contractions: Rake and Compress

- Proposed by Miller and Reif [1985] (parallel setting).
- Rake:
  - eliminates a degree-one vertex;
  - edge combined with successor:
    * assumes circular order.



- Compress:
  - eliminates a degree-two vertex;
  - combines two edges into one.



- Original and resulting edges are clusters:
  - cluster represents both a path and a subtree;
  - user defines what to store in the new cluster.

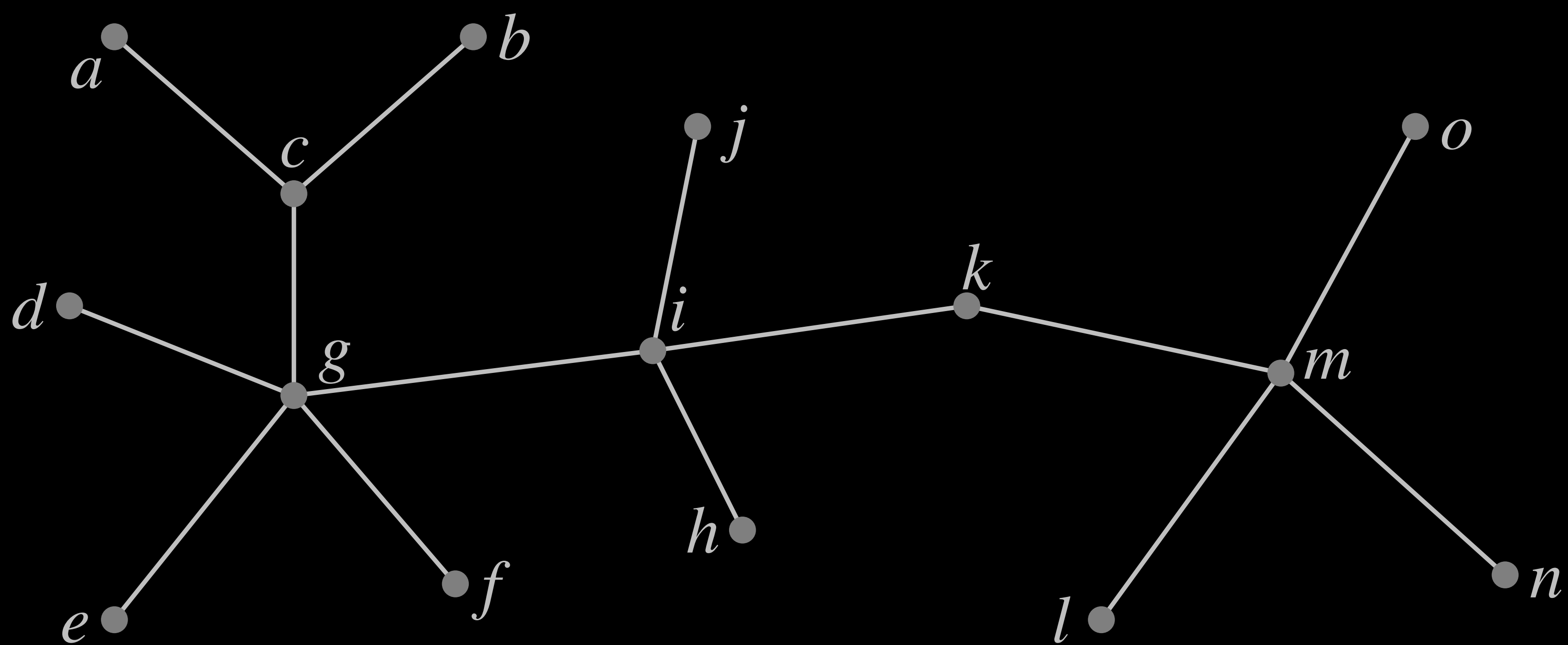
Contractions

- Contraction:
  - series of rakes and compresses;
  - reduces tree to a single cluster (edge).

- Any order of rakes and compresses is “right”:
  - final cluster will have the correct information;
  - data structure decides which moves to make:
    - just “asks” the user how to update values after each move.

- Example:
  - work in rounds;
  - perform a maximal set of independent moves in each round.

Contractions: Example



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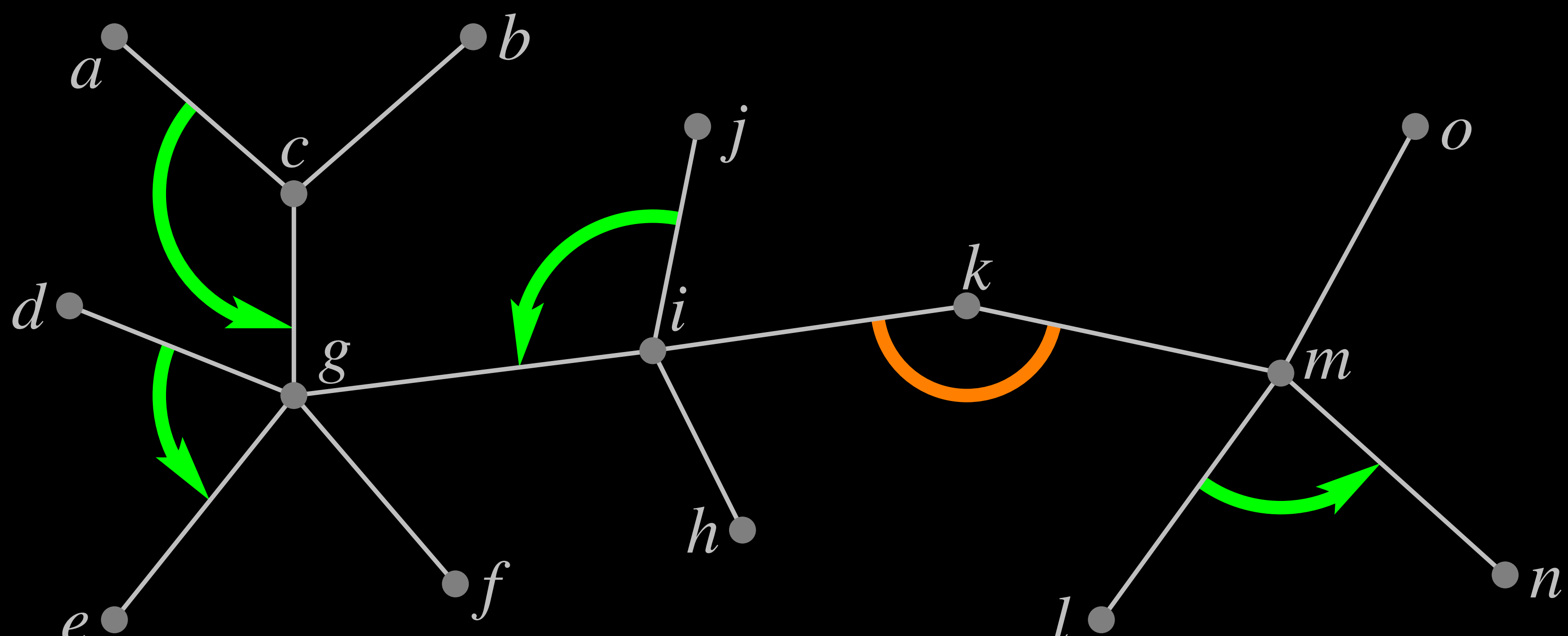

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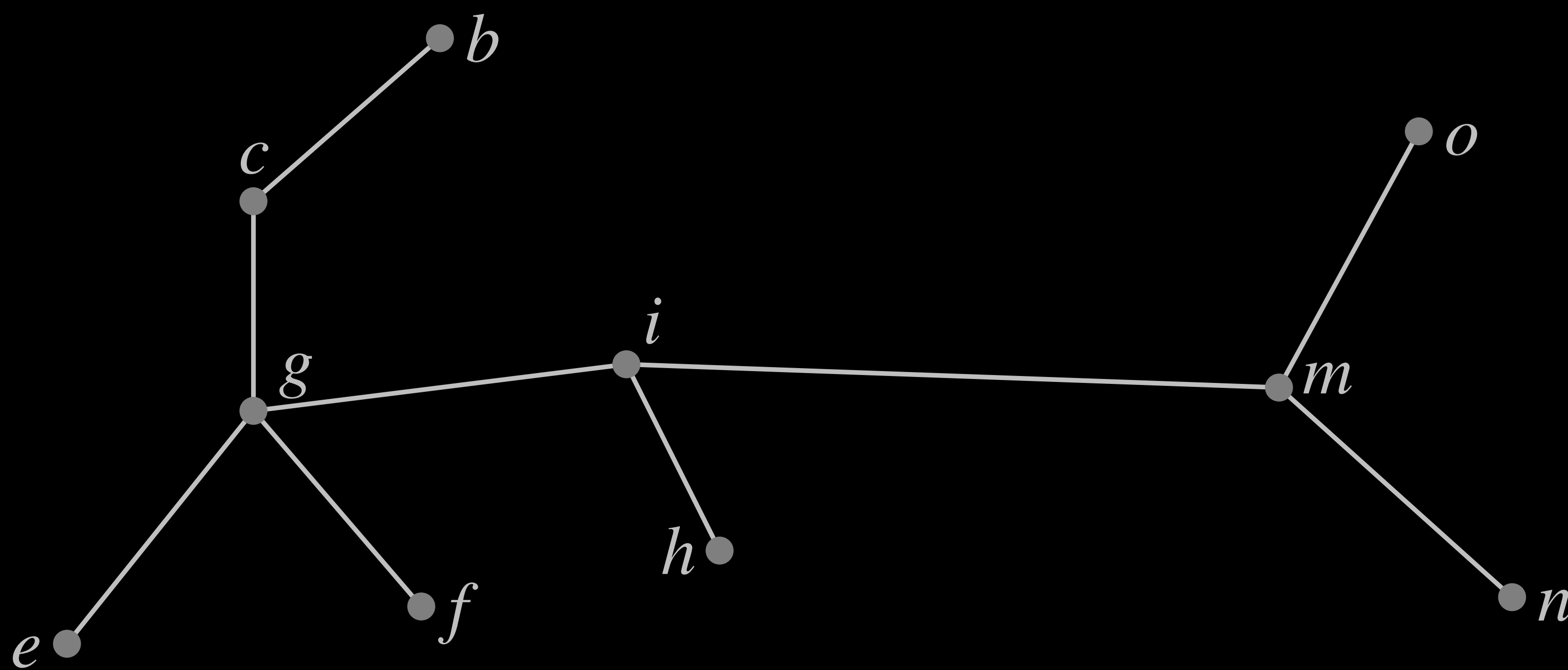

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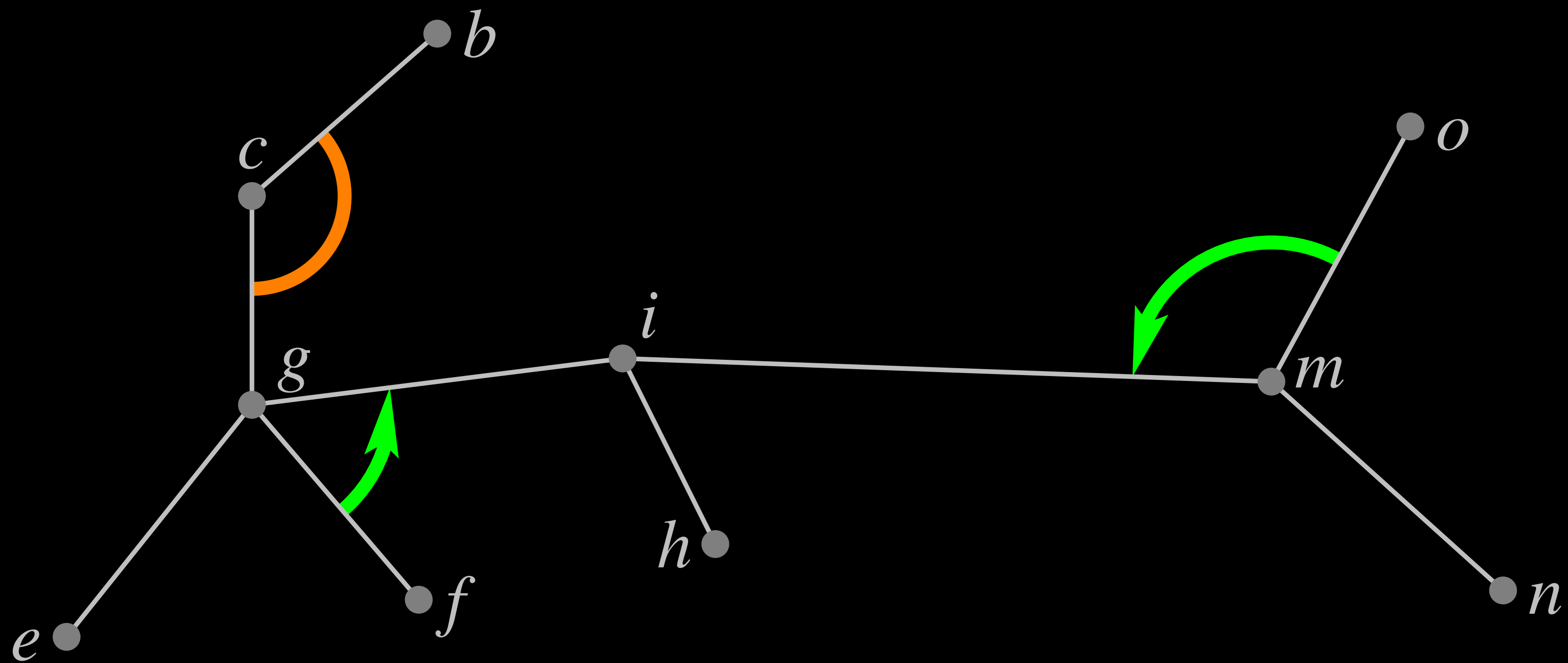

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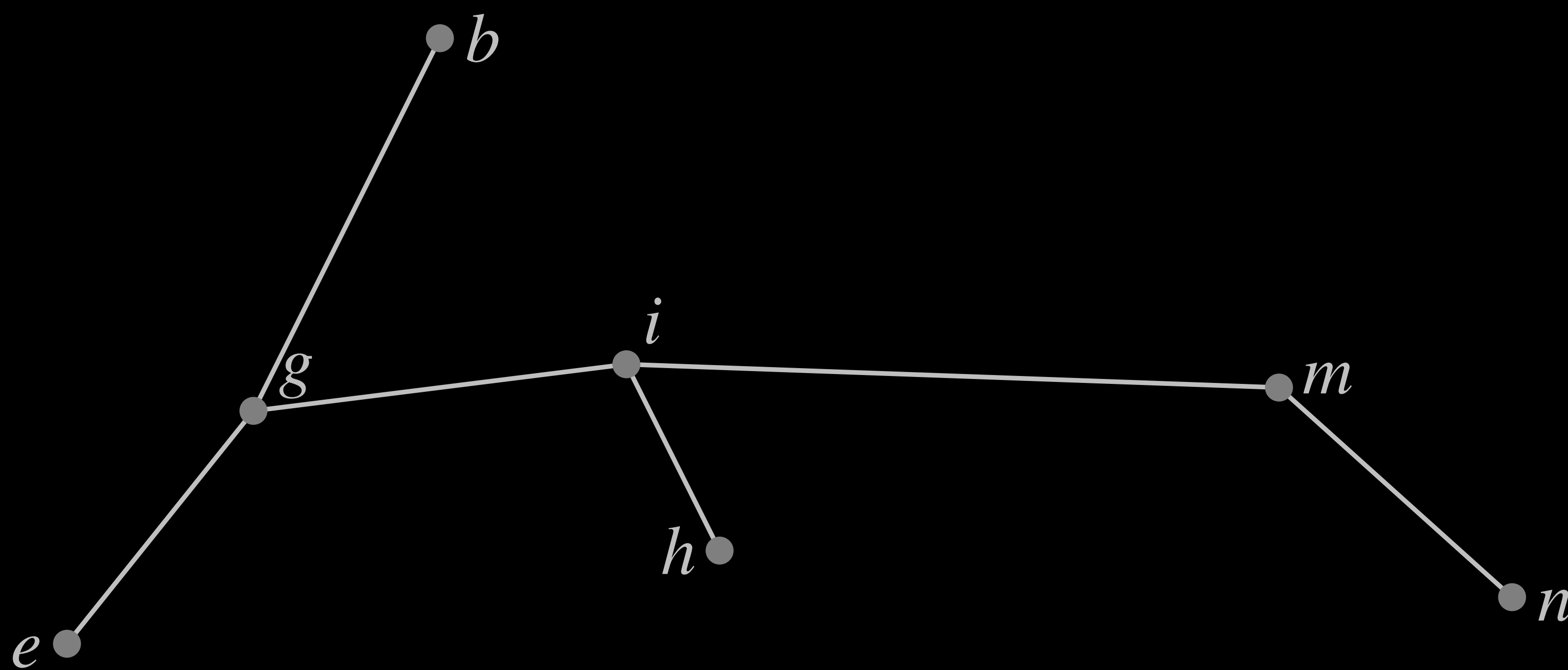

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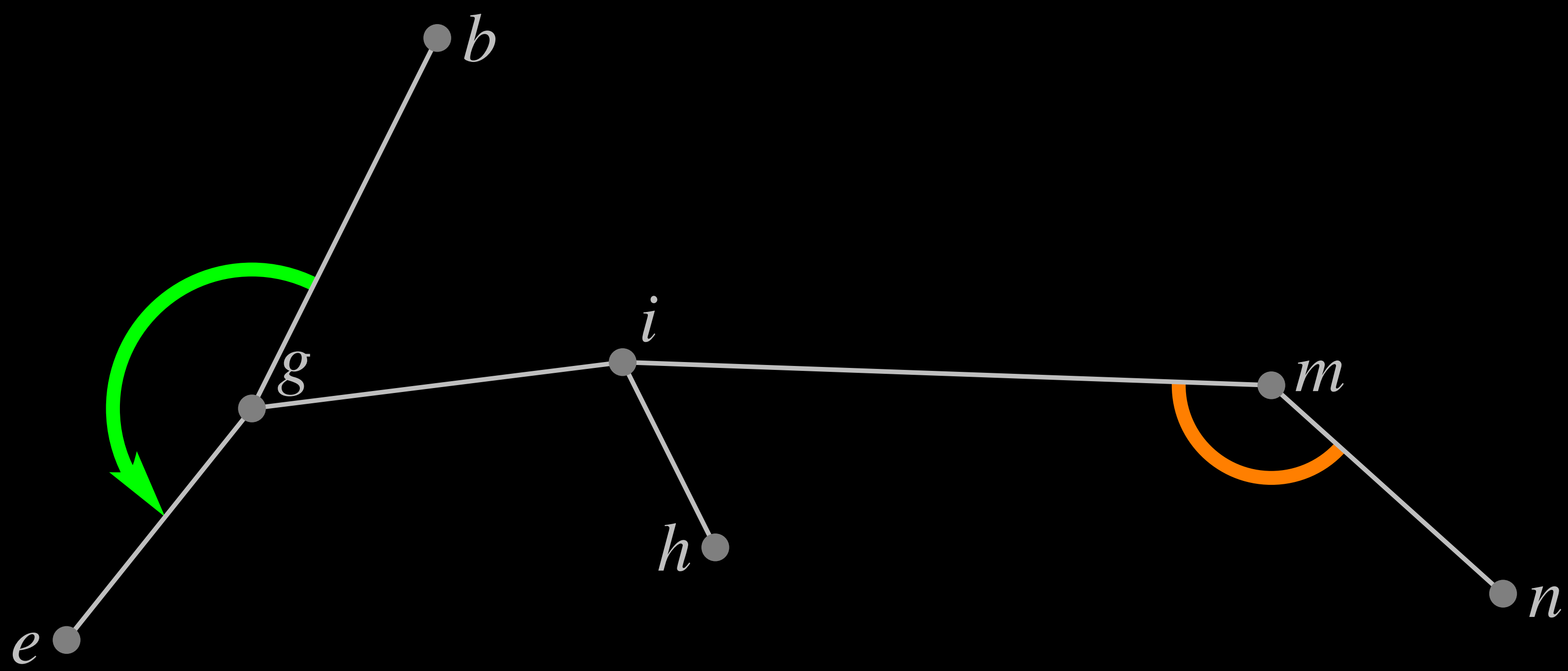

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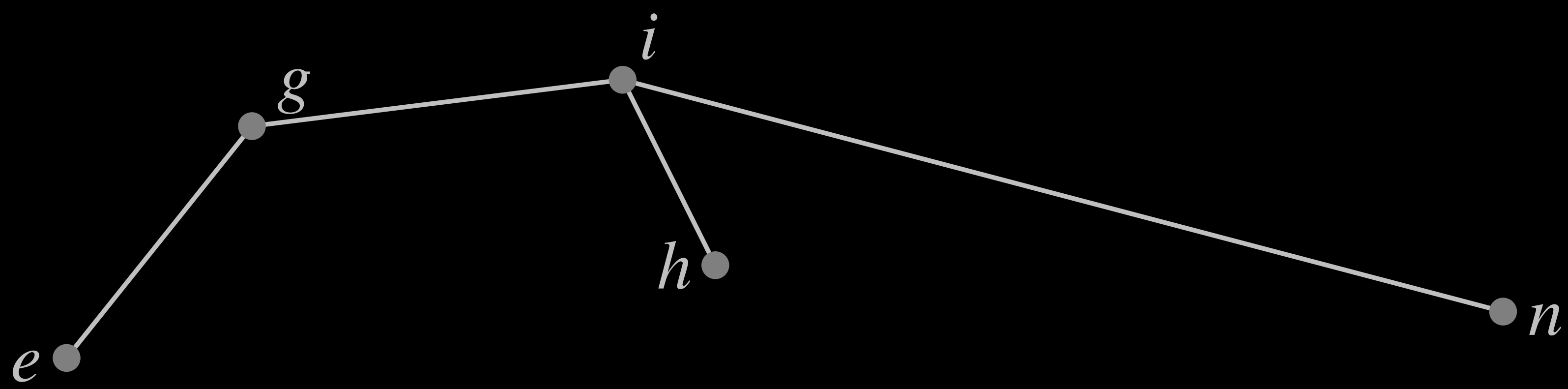

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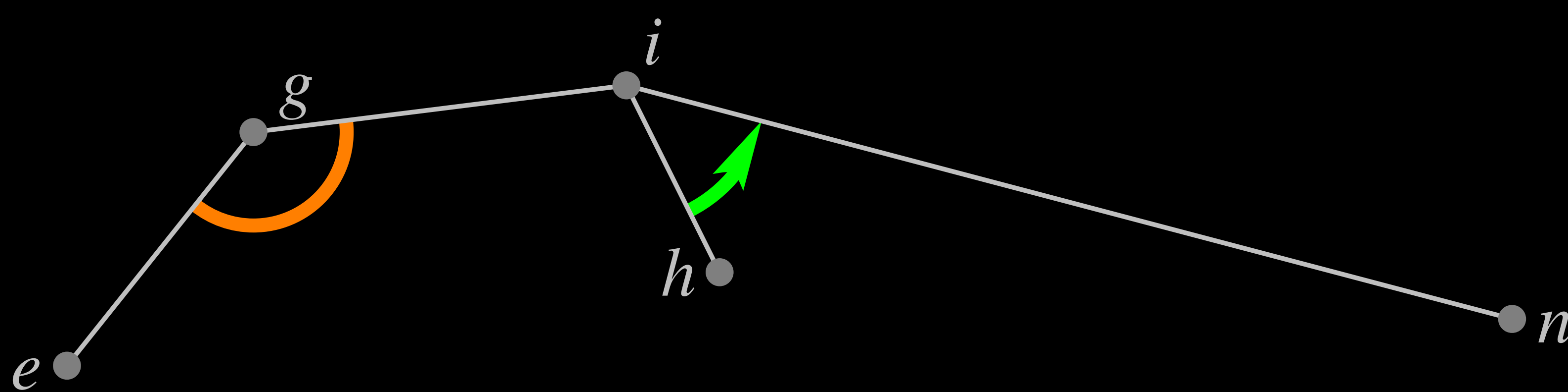

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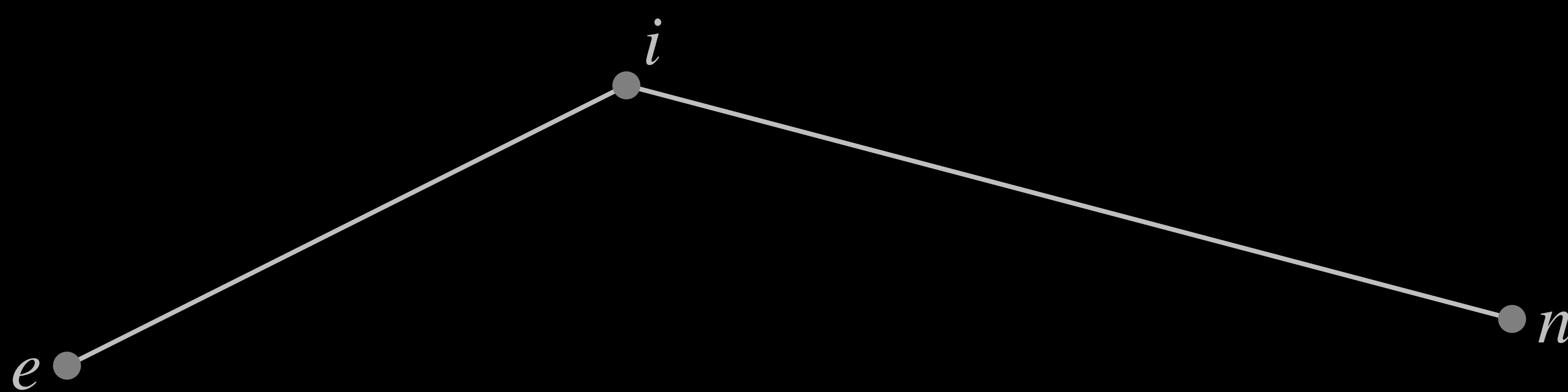

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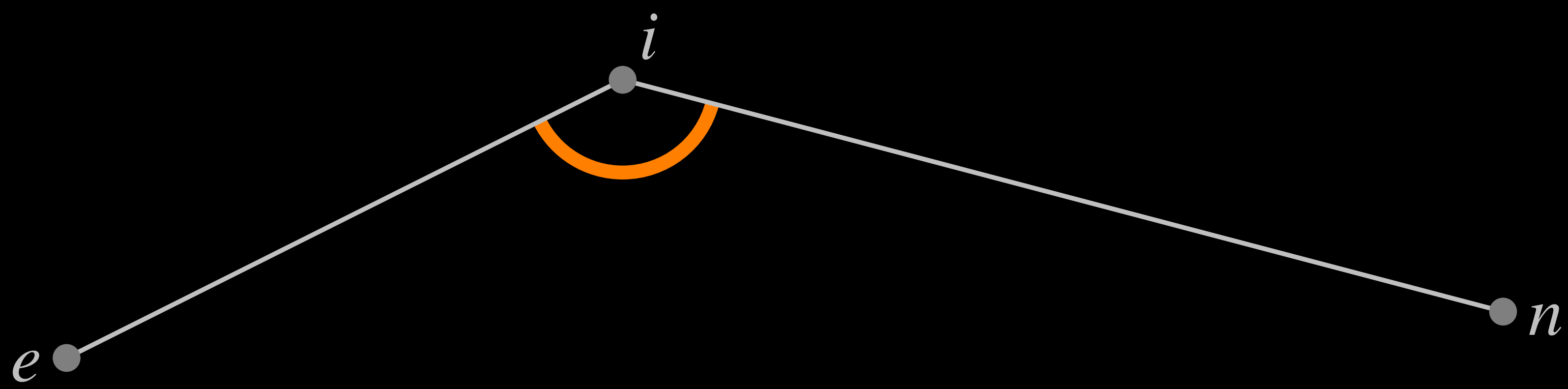

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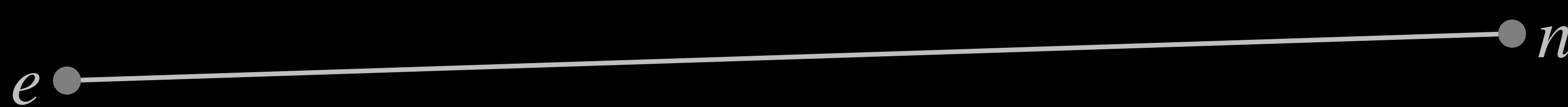
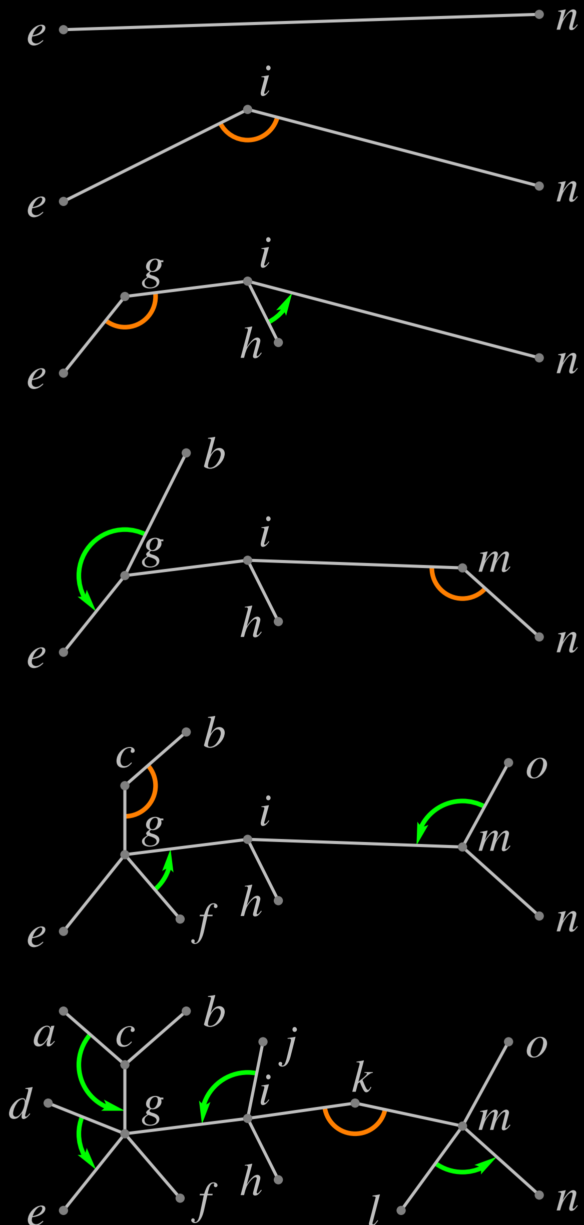


Top Trees: Example



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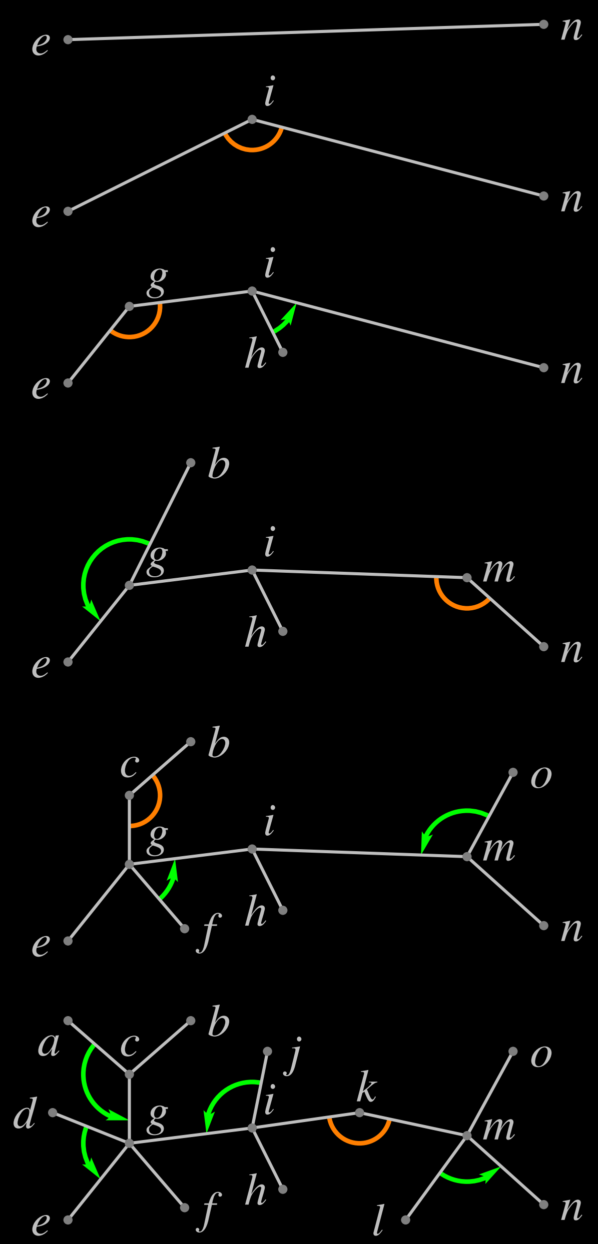

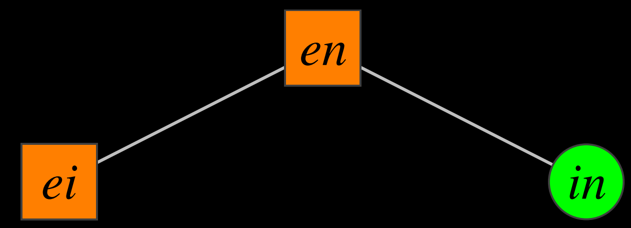

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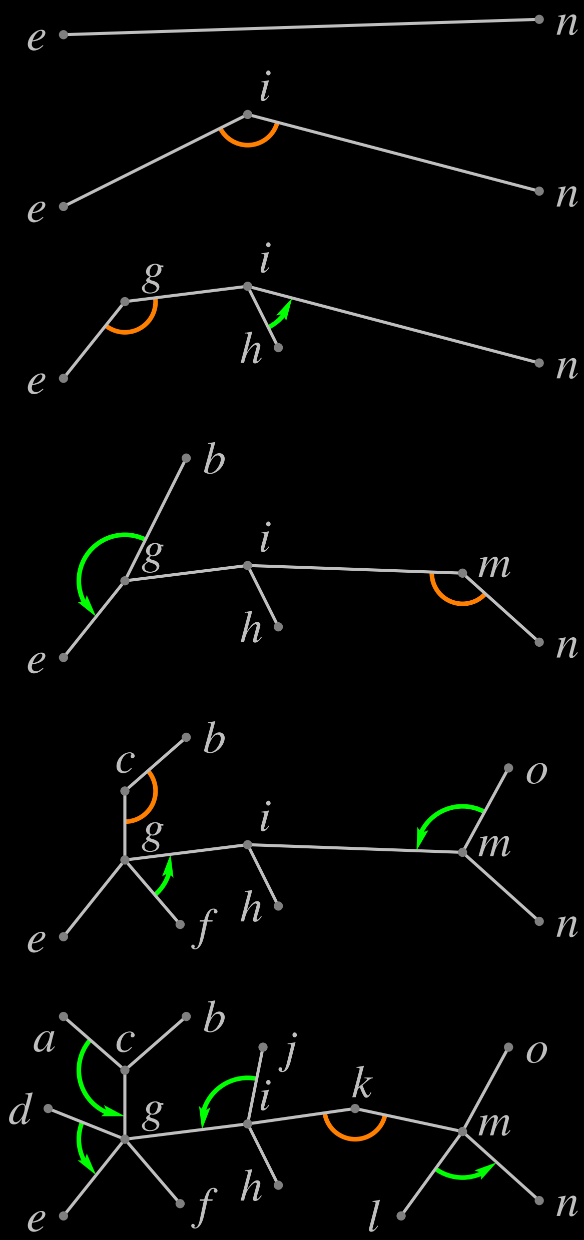

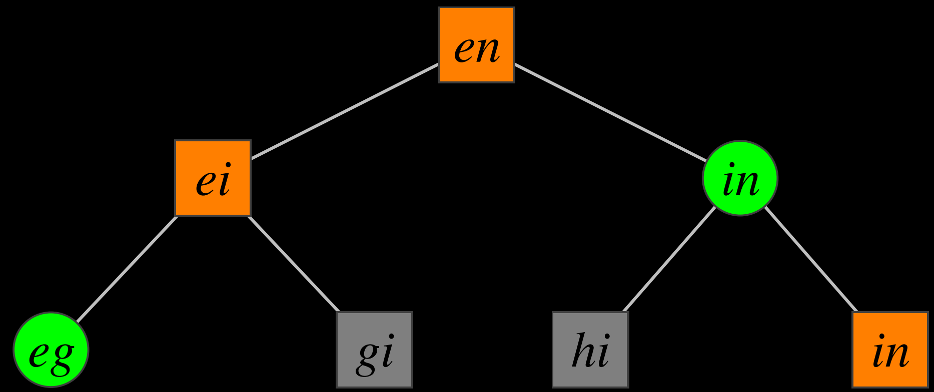

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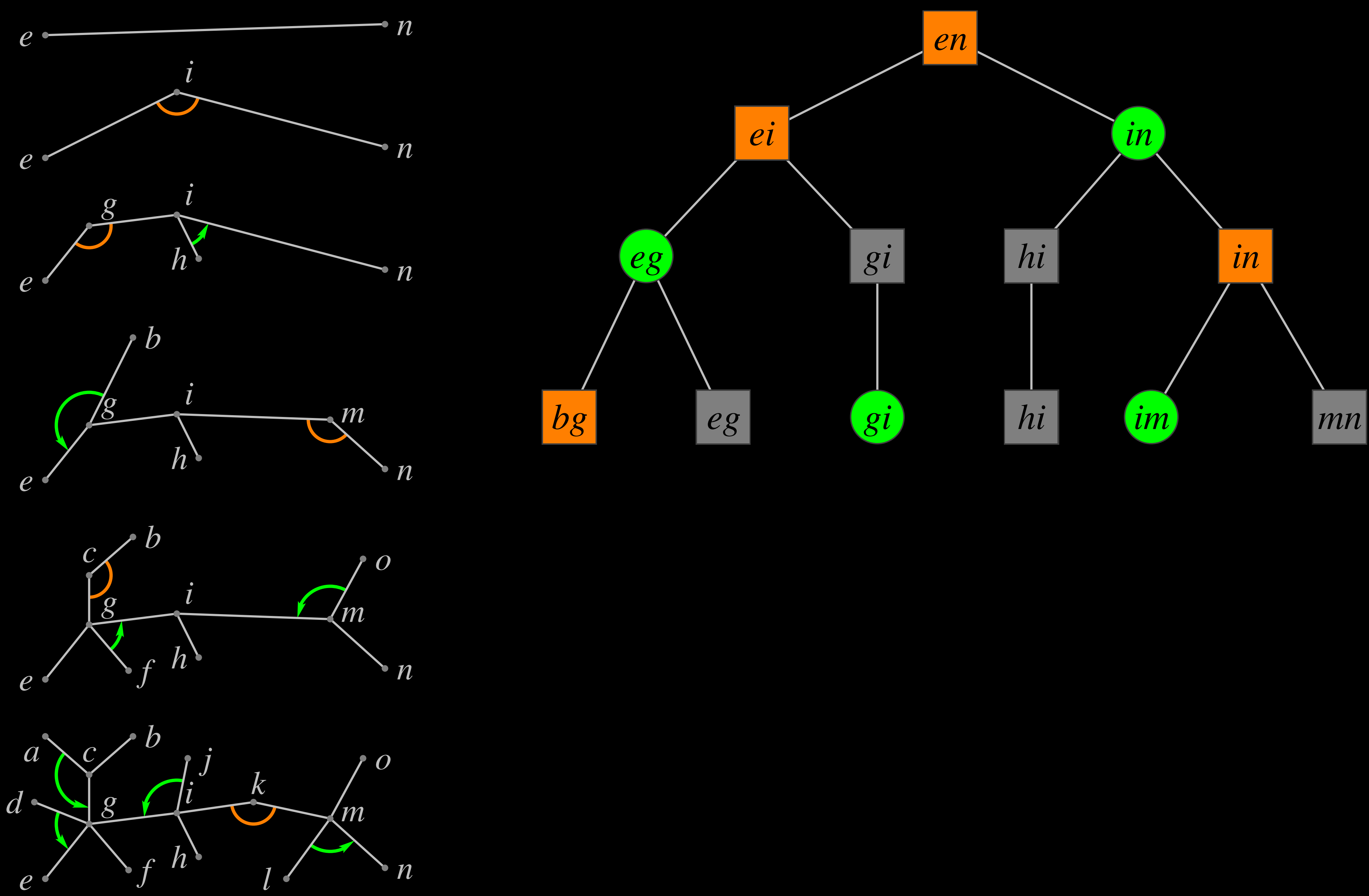

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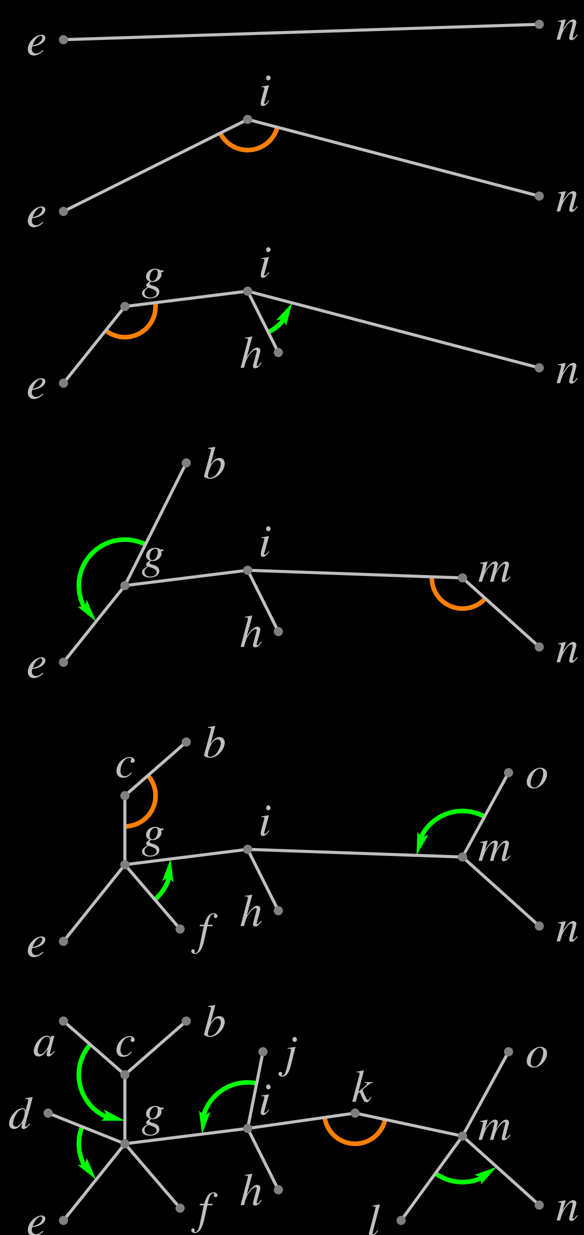

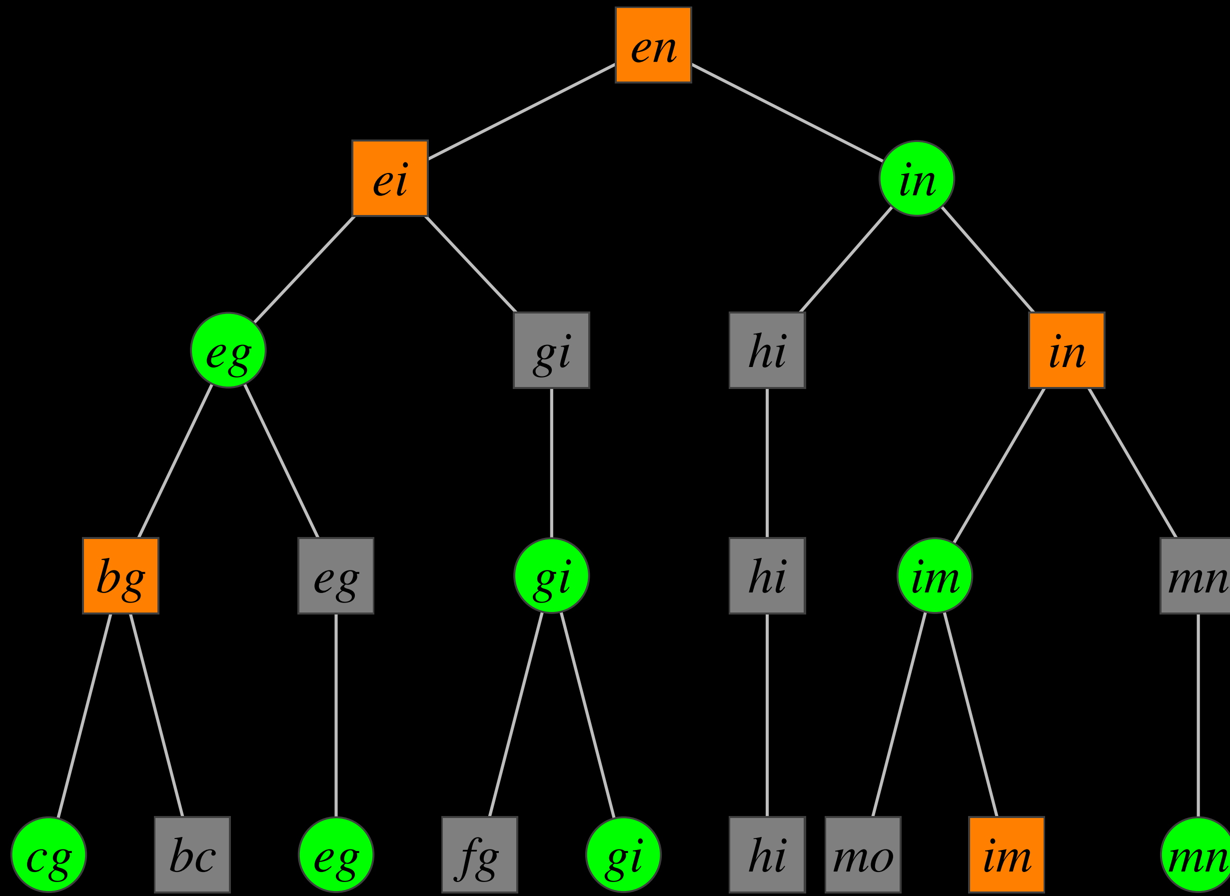

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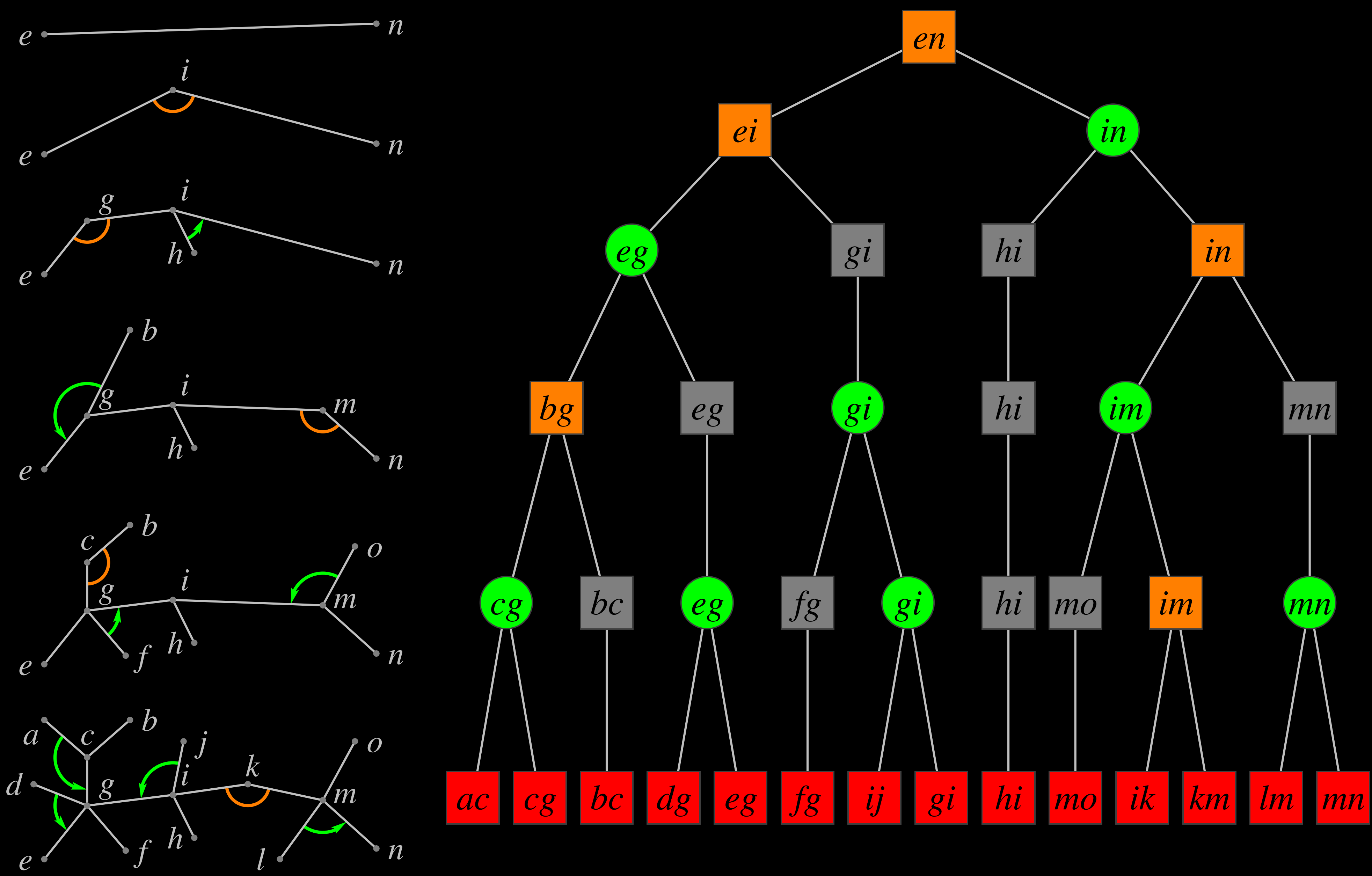
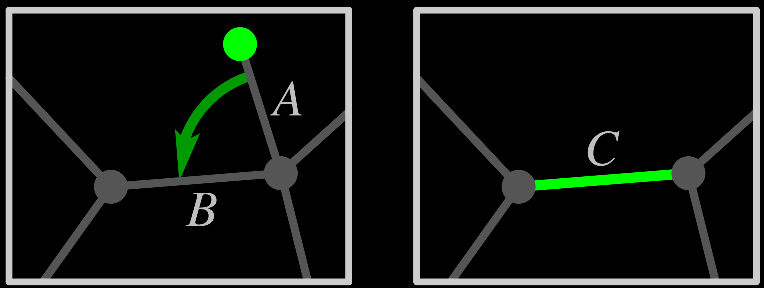
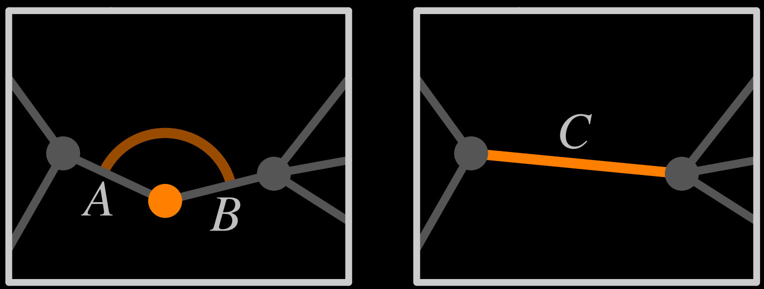


Contractions: Updating Values

- To find the minimum-cost edge on the tree:
  - rake: $C \leftarrow \min\{A, B\}$
  - compress: $C \leftarrow \min\{A, B\}$

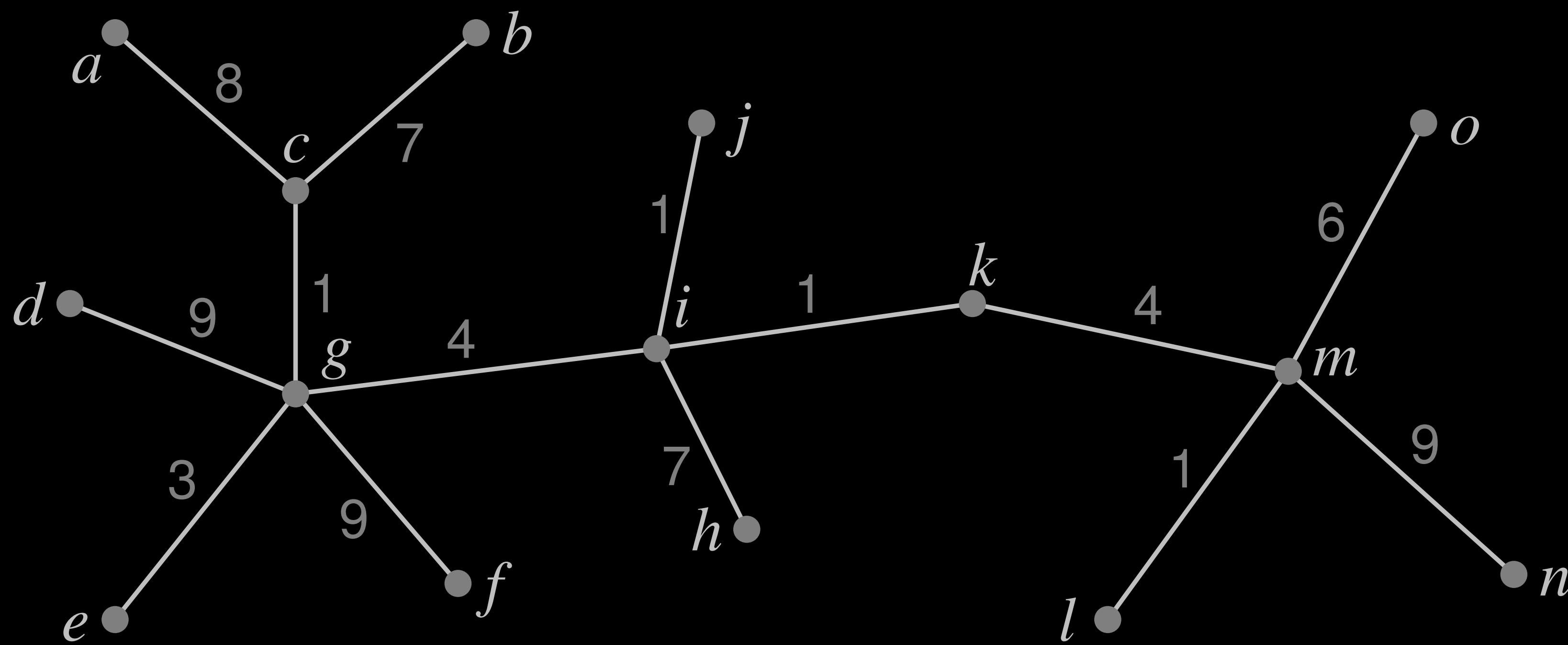


- To find the maximum edge on a path:
  - rake: $C \leftarrow B$.
  - compress: $C \leftarrow \max\{A, B\}$

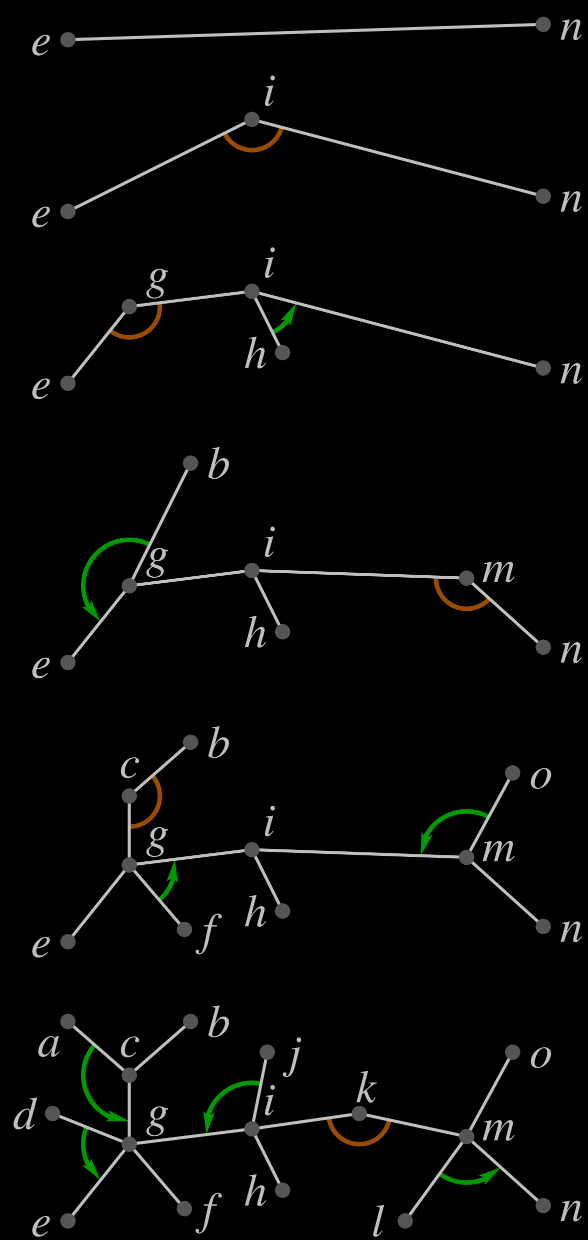


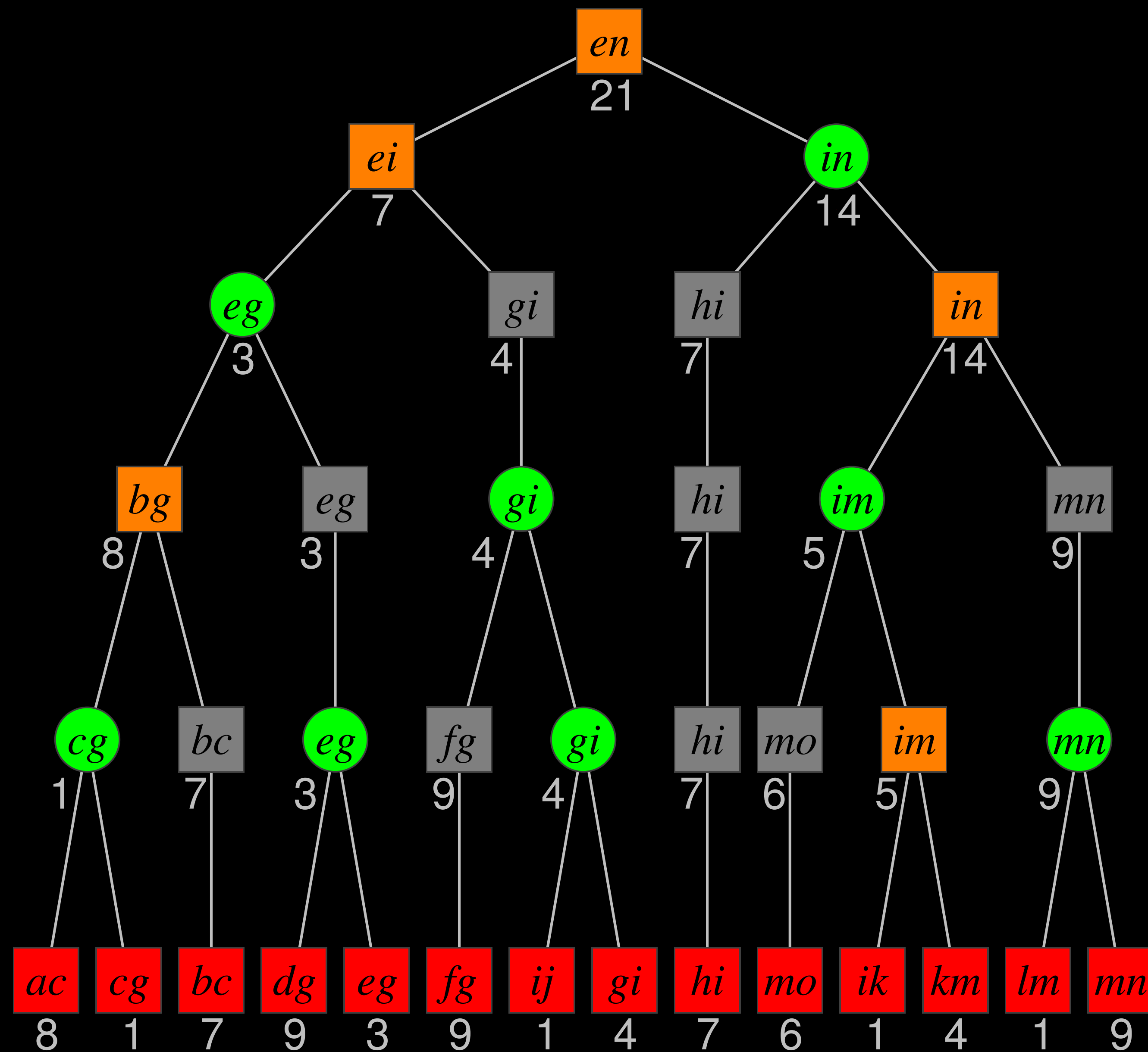
- To find the length of a path:
  - rake: $C \leftarrow B$
  - compress: $C \leftarrow A + B$

Contractions: Example with Values



Top Trees: Path Lengths





Top Trees

- Top tree embodies a contraction:
  - top tree root represents the entire original tree;
  - expose(v, w) ensures that path $v \cdots w$ is represented at the root;
  - interface only allows direct access to the root.

- Other operations:
  - link: joins two top trees;
  - cut: splits a top tree in two;
  - Goal: make link, cut, and expose run in $O(\log n)$ time.

- Known implementation [Alstrup et al. 97]:
  - interface to topology trees:
    * variant where clusters are vertices;
    * degree must be bounded.

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⇒ A new worst-case data structure

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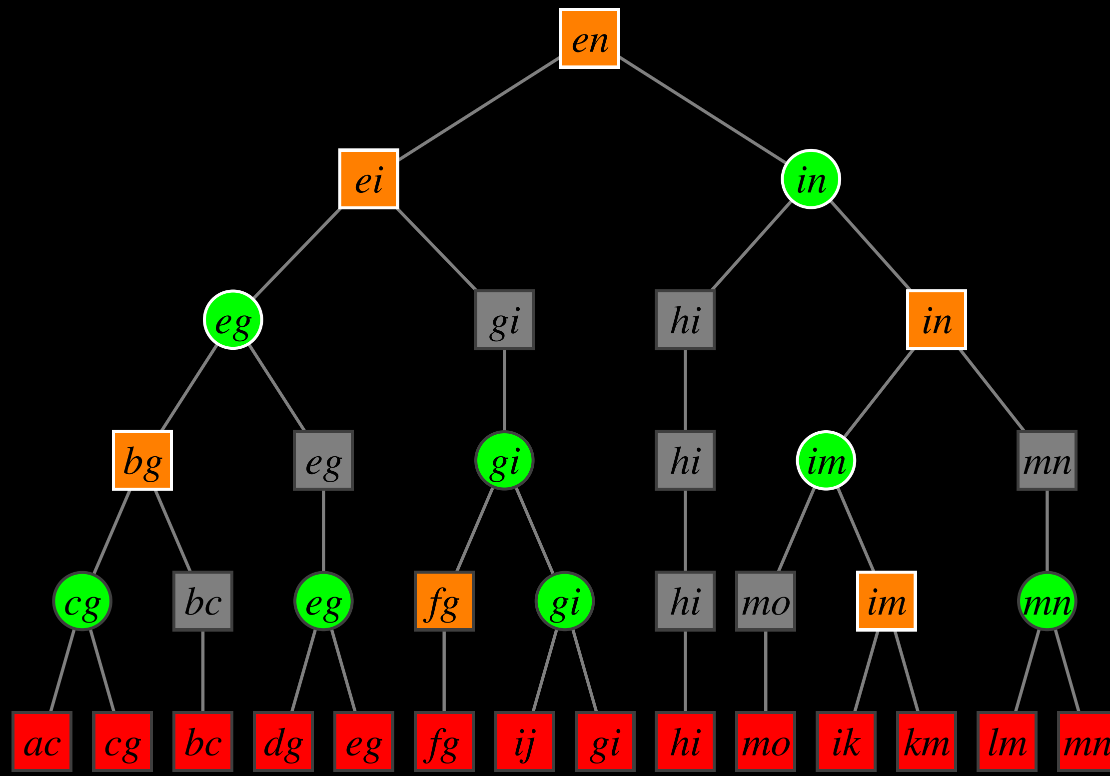
Tree Contraction

- Contraction scheme:
  - Work in rounds, each with a maximal set of independent moves.

- Lemma: there will be at most $O(\log n)$ levels.
  - At least half the vertices have degree 1 or 2.
  - At least one third of those will disappear:
    - a move blocks at most 2 others.
  - No more than 5/6 of the original vertices will remain.

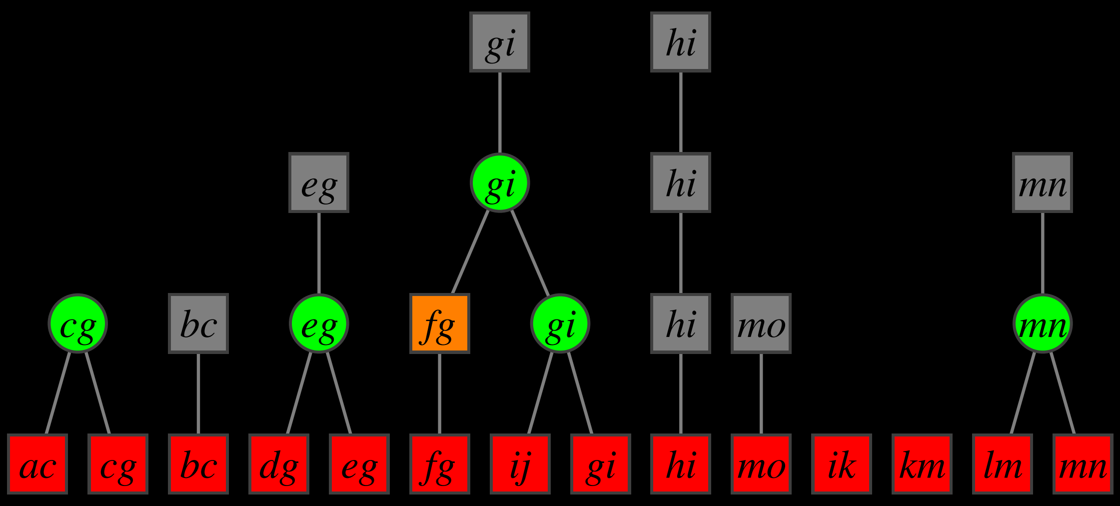
Online MSF on Augmented Stars

- Corollary: expose(v, w) can be implemented in $O(\log n)$ time.
  - Temporarily eliminate all clusters with v or w as internal vertices:
    - $*$ at most two per level: $O(\log n)$.



Online MSF on Augmented Stars

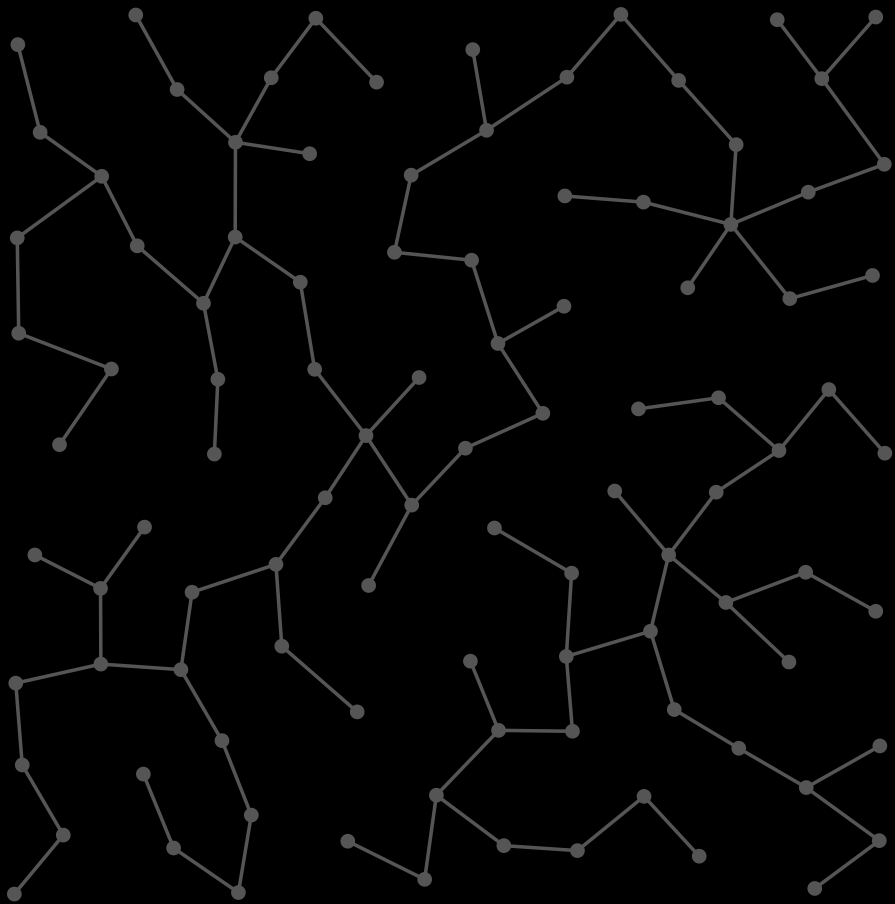
- Corollary: expose(v, w) can be implemented in $O(\log n)$ time.
  - Temporarily eliminate all clusters with v or w as internal vertices:
    - at most two per level: $O(\log n)$.
  - Build a temporary top tree on the remaining root clusters.
  - Restore original tree.

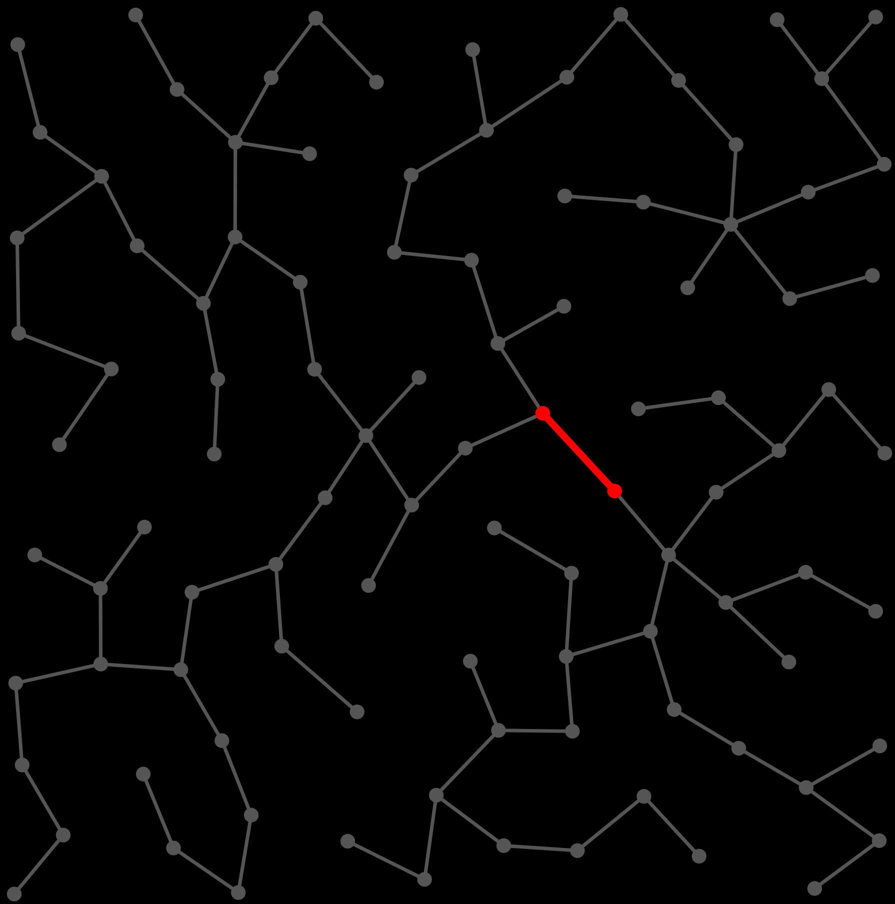


Tree Contraction

- Update scheme:
  - Goal: minimize “damage” to original contraction after link or cut.
  - For each level (bottom-up), execute two steps:
    1. replicate as many original moves as possible;
    2. perform new moves until maximality is achieved.
  - Step 1 is implicit.

Updates: Example





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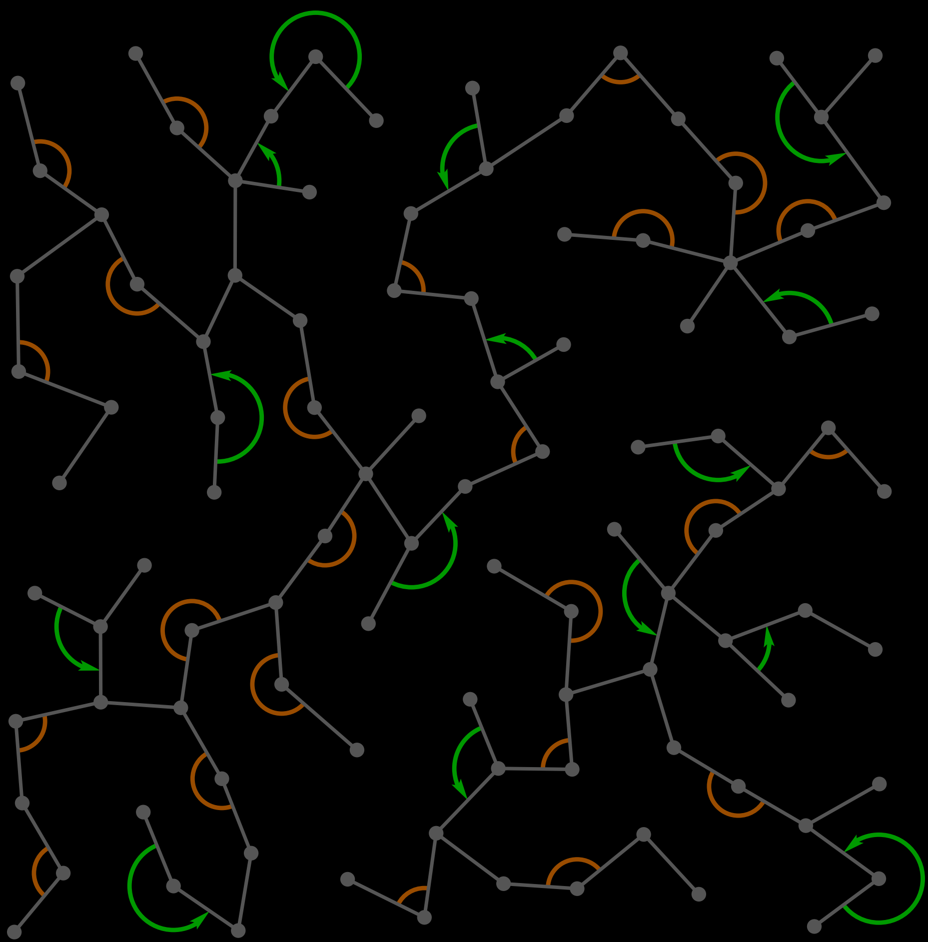

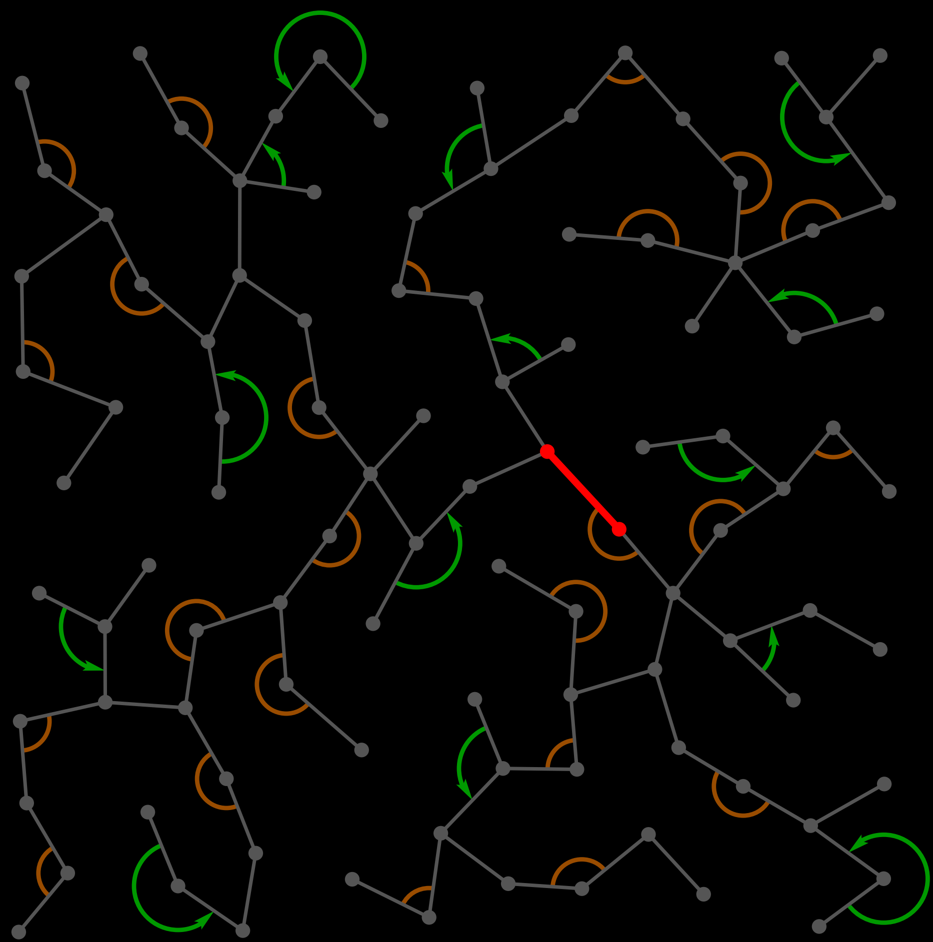

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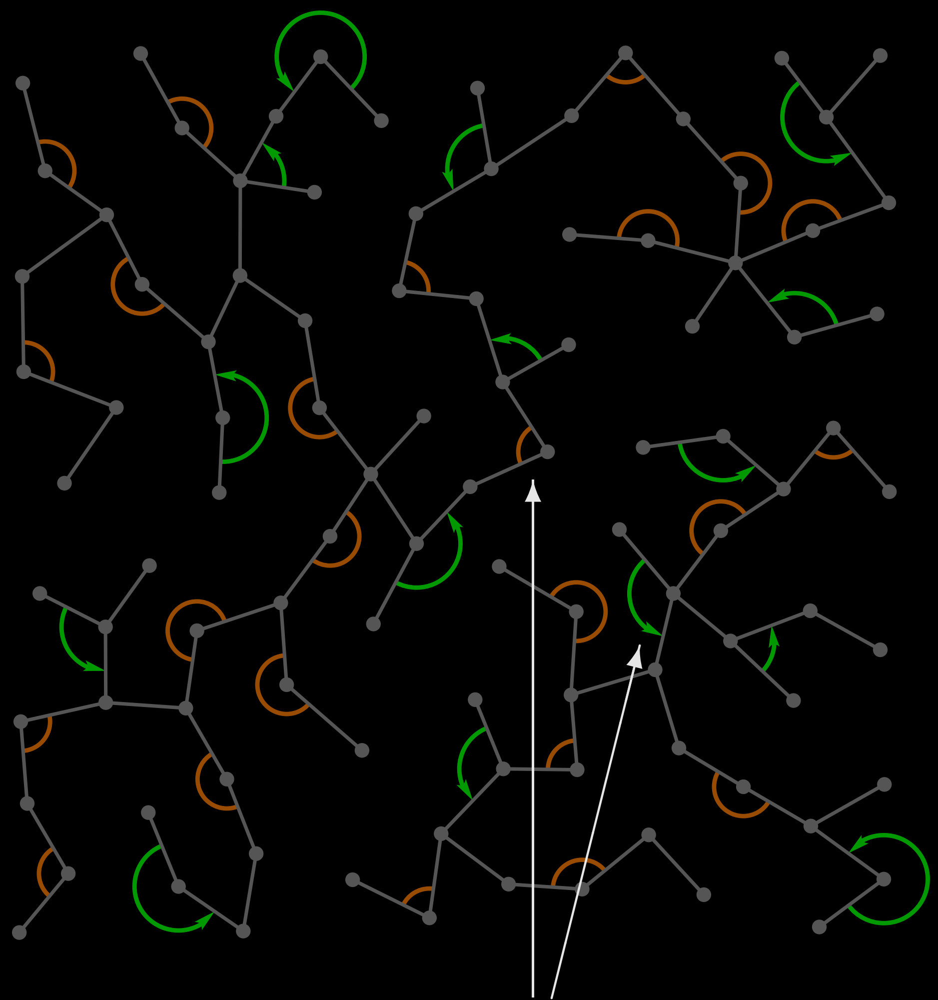
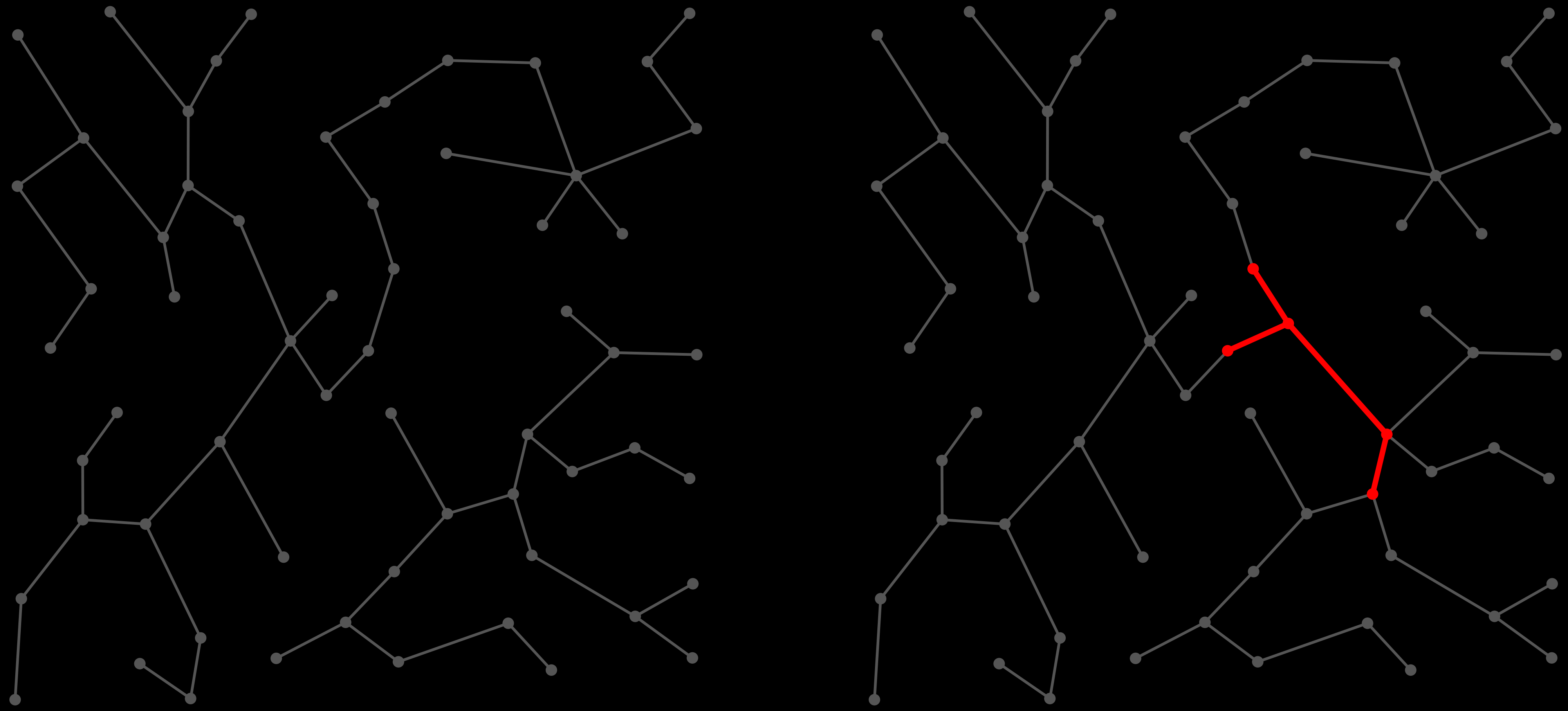


these moves cannot be replicated

new move

Updates: Example



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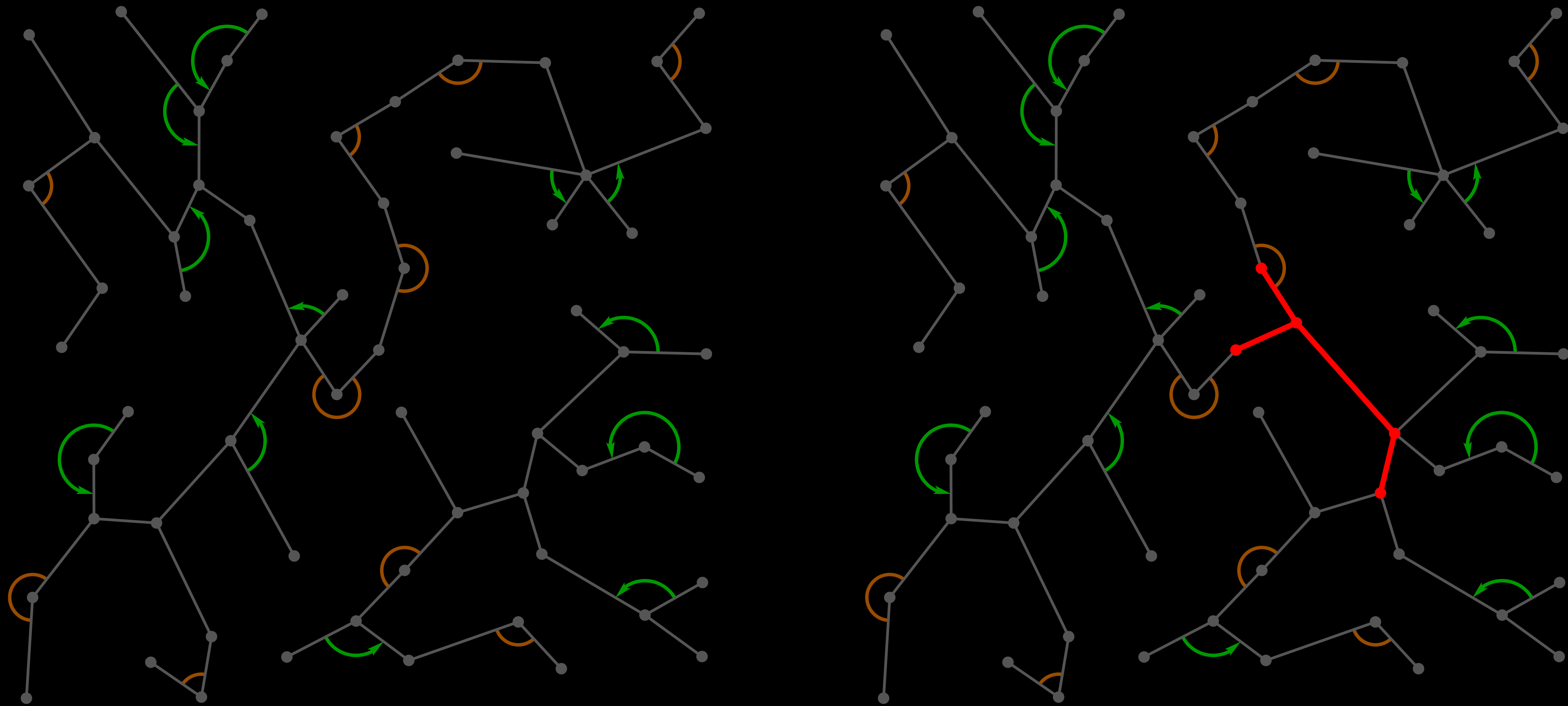

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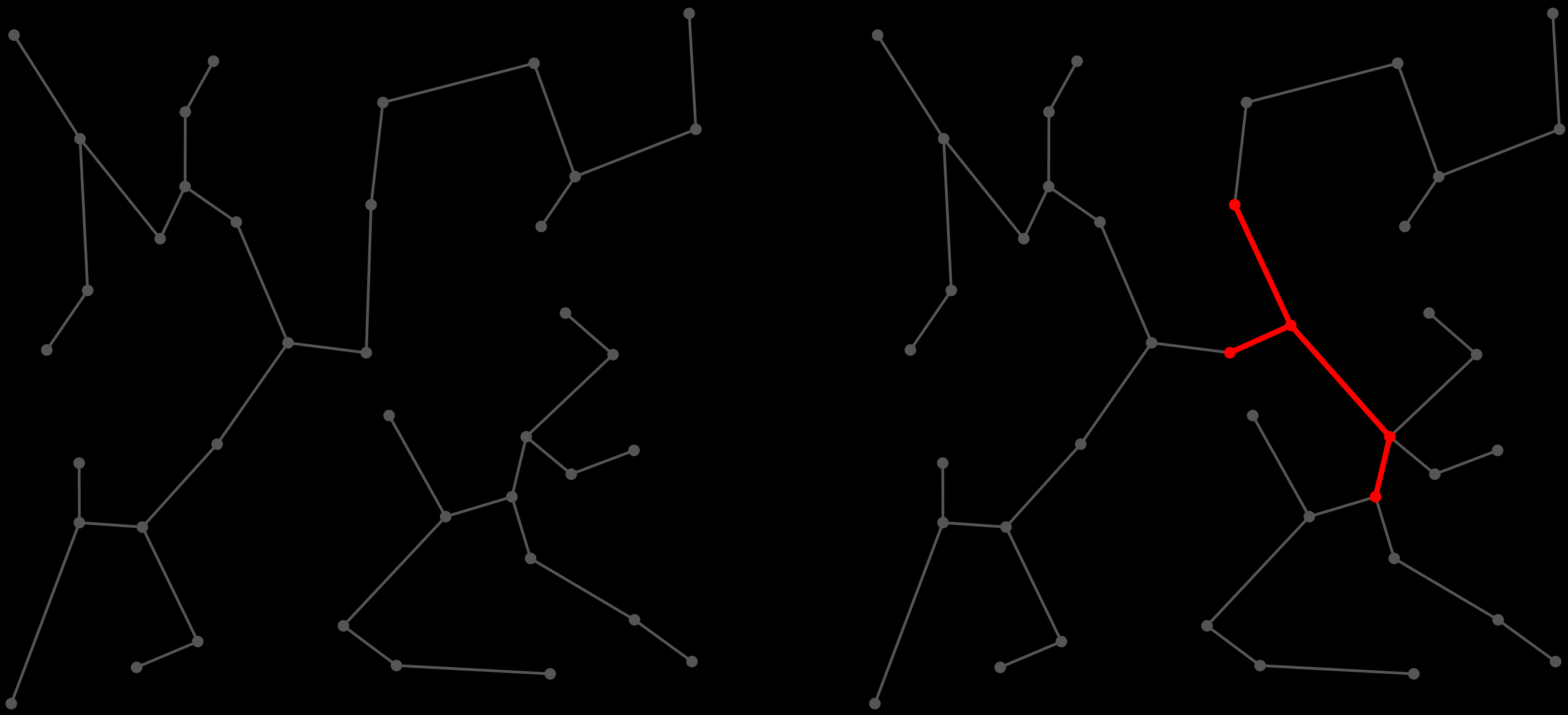

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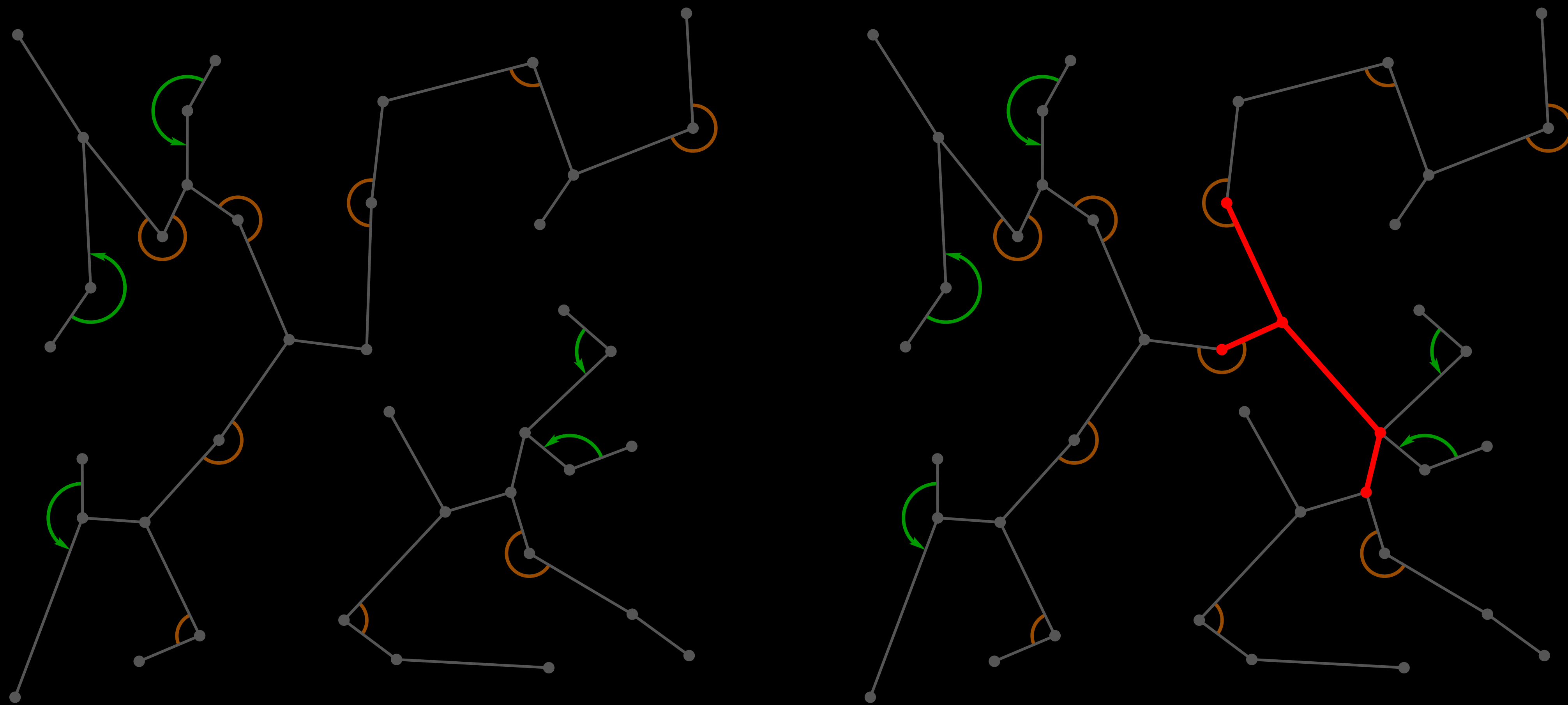

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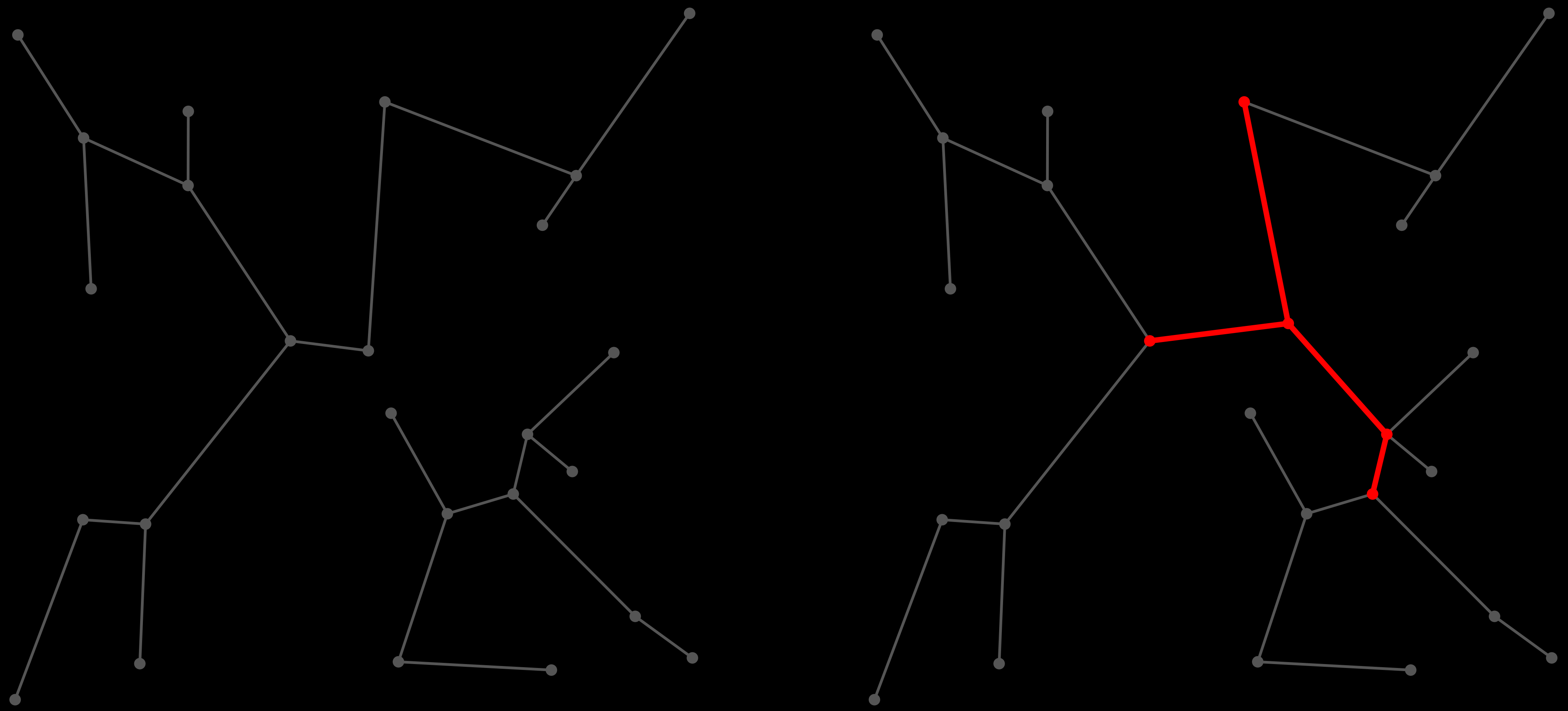

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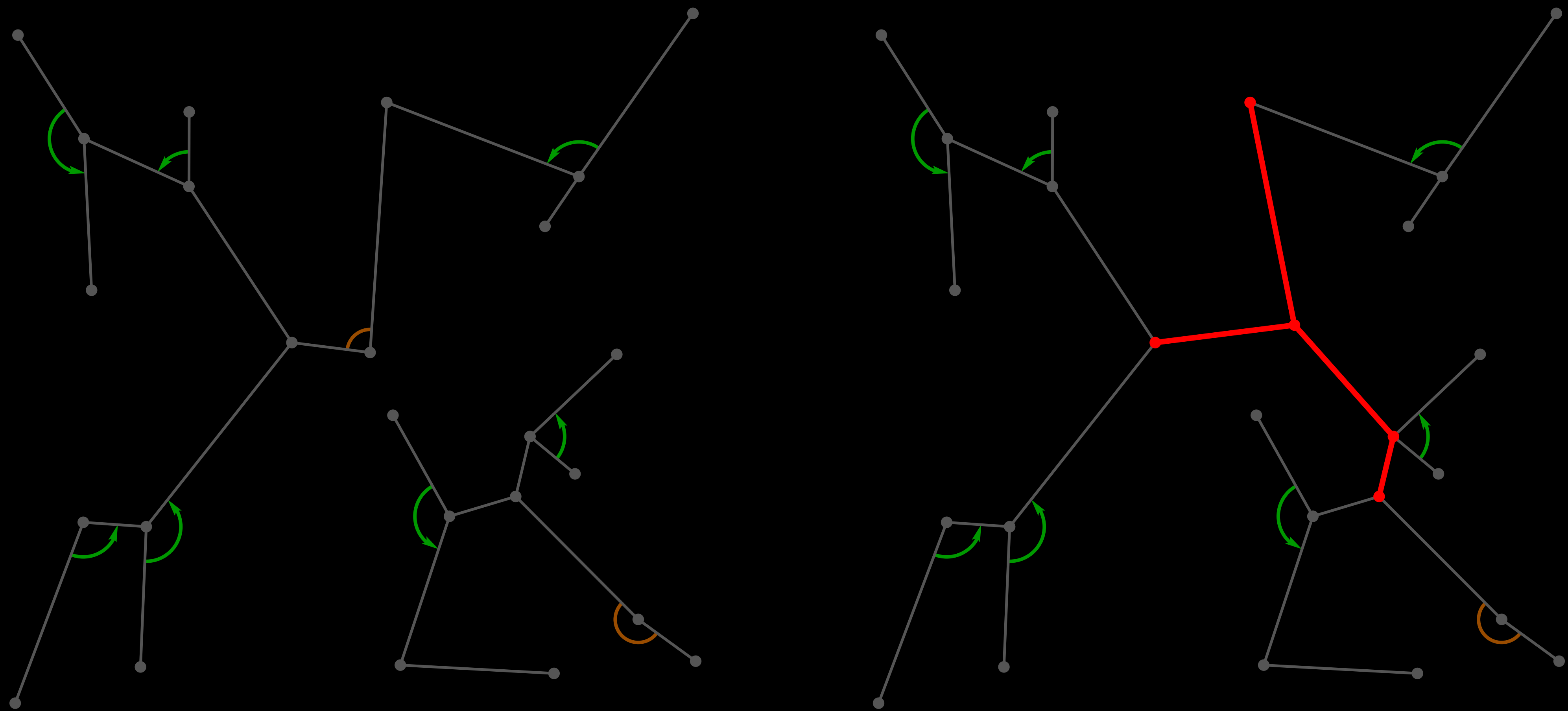

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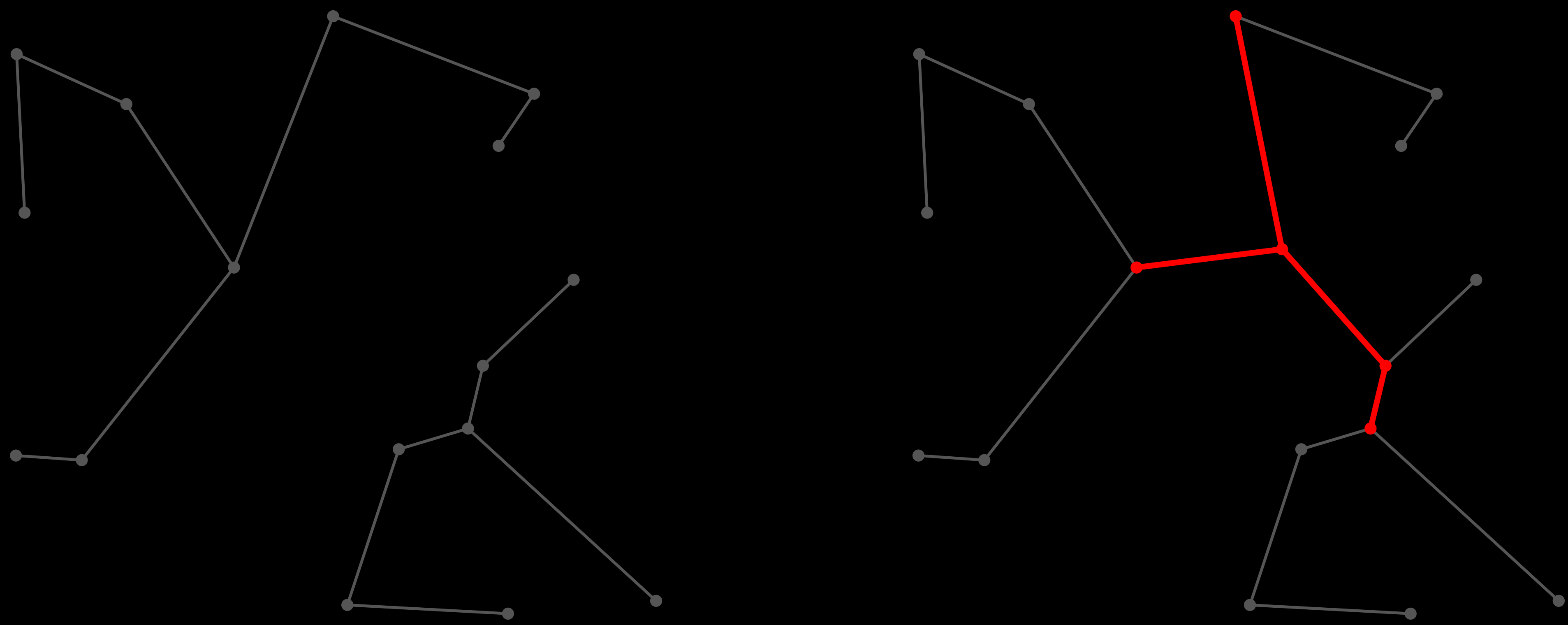

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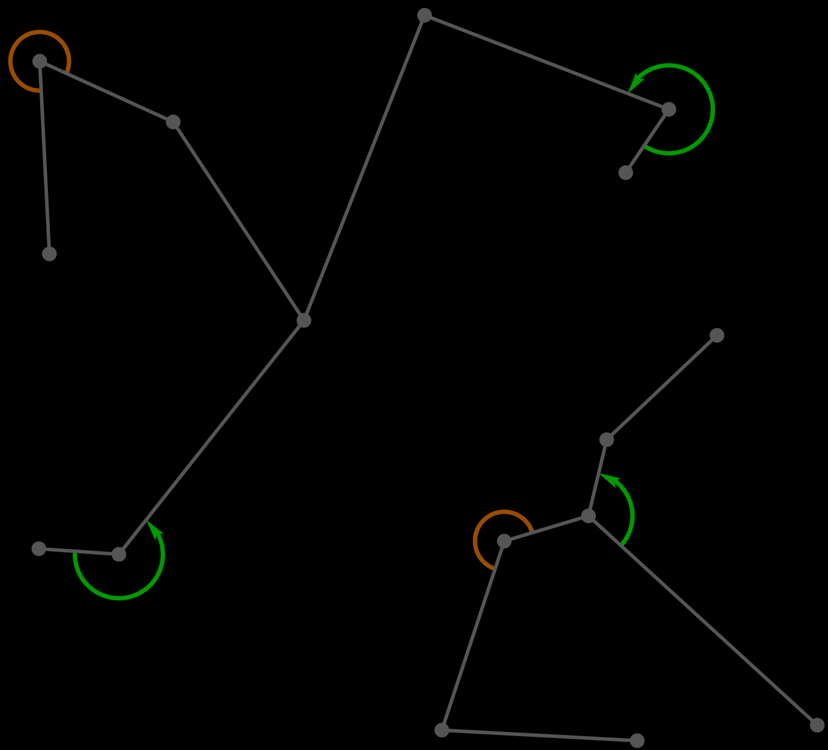

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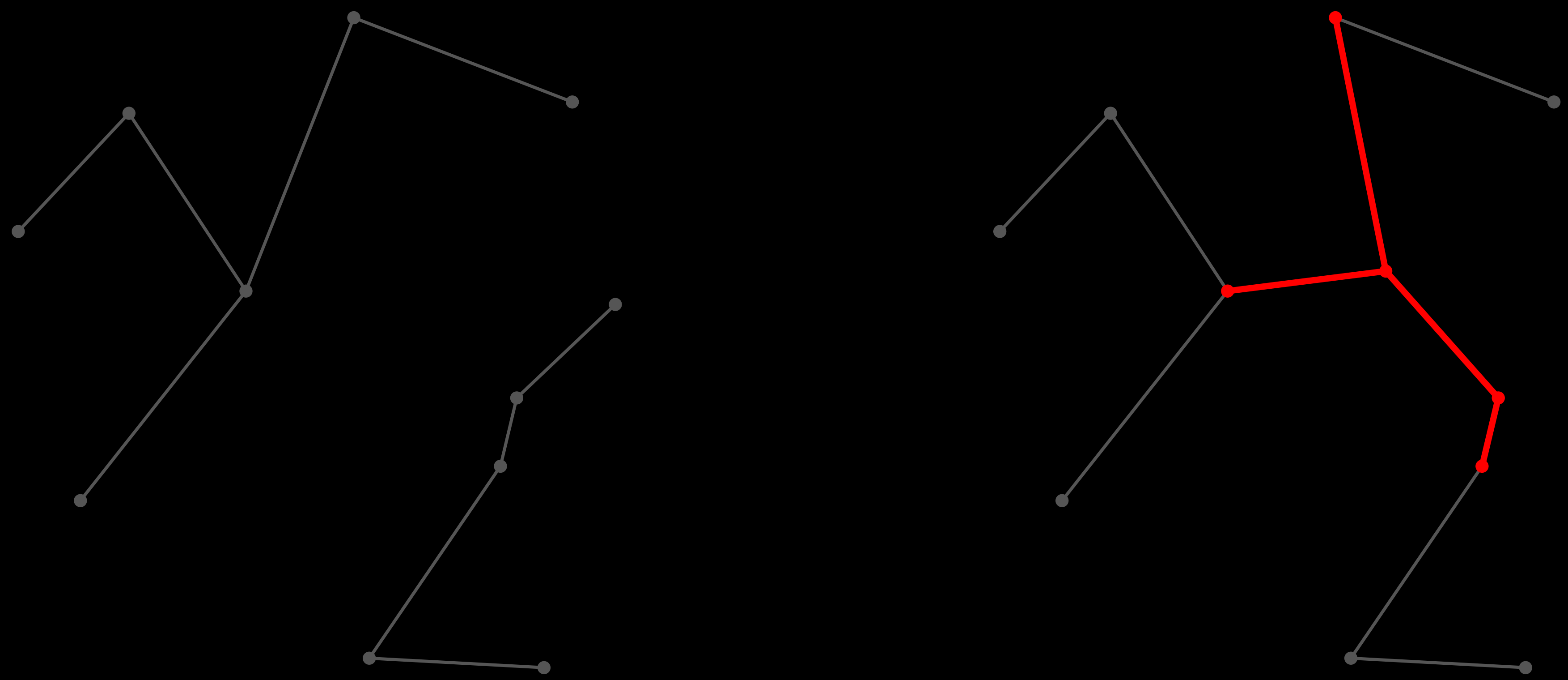

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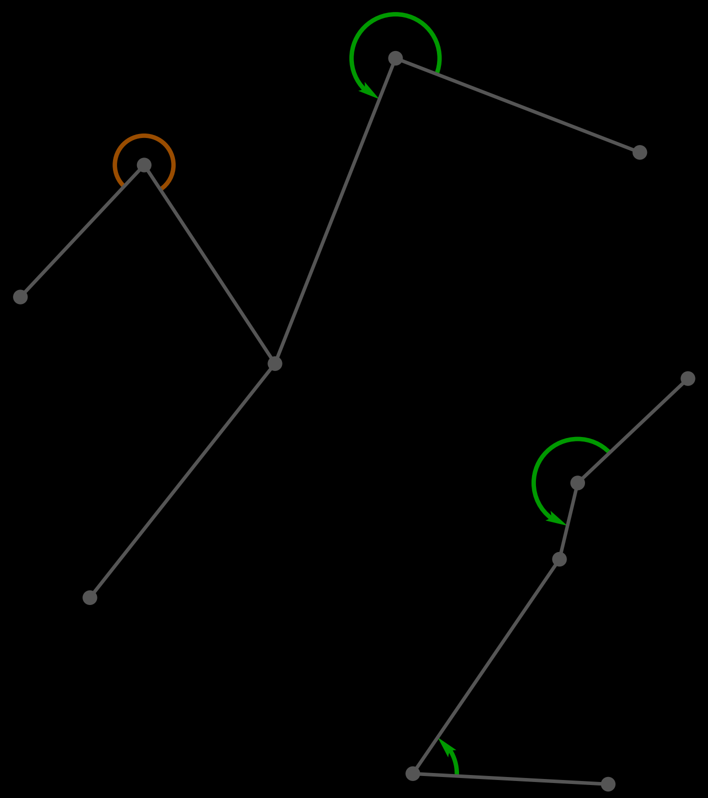
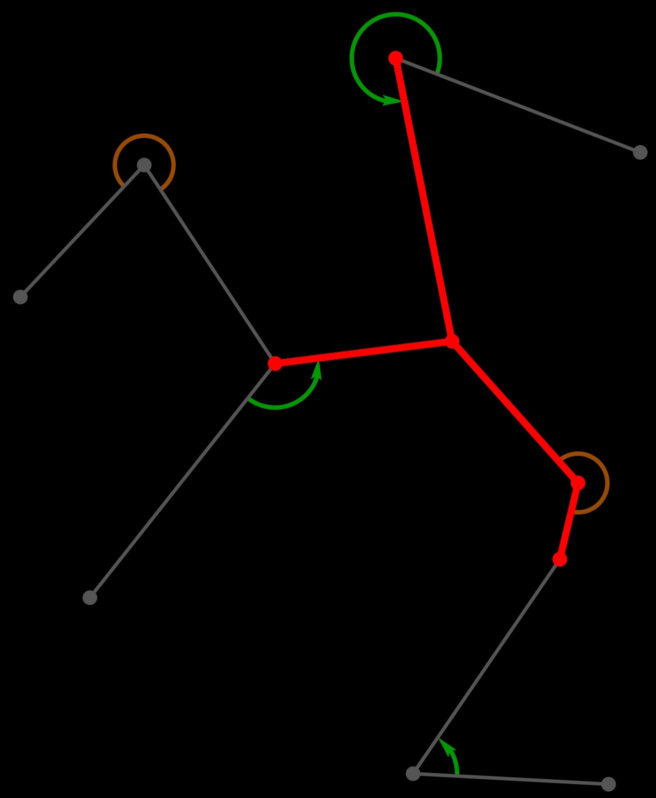
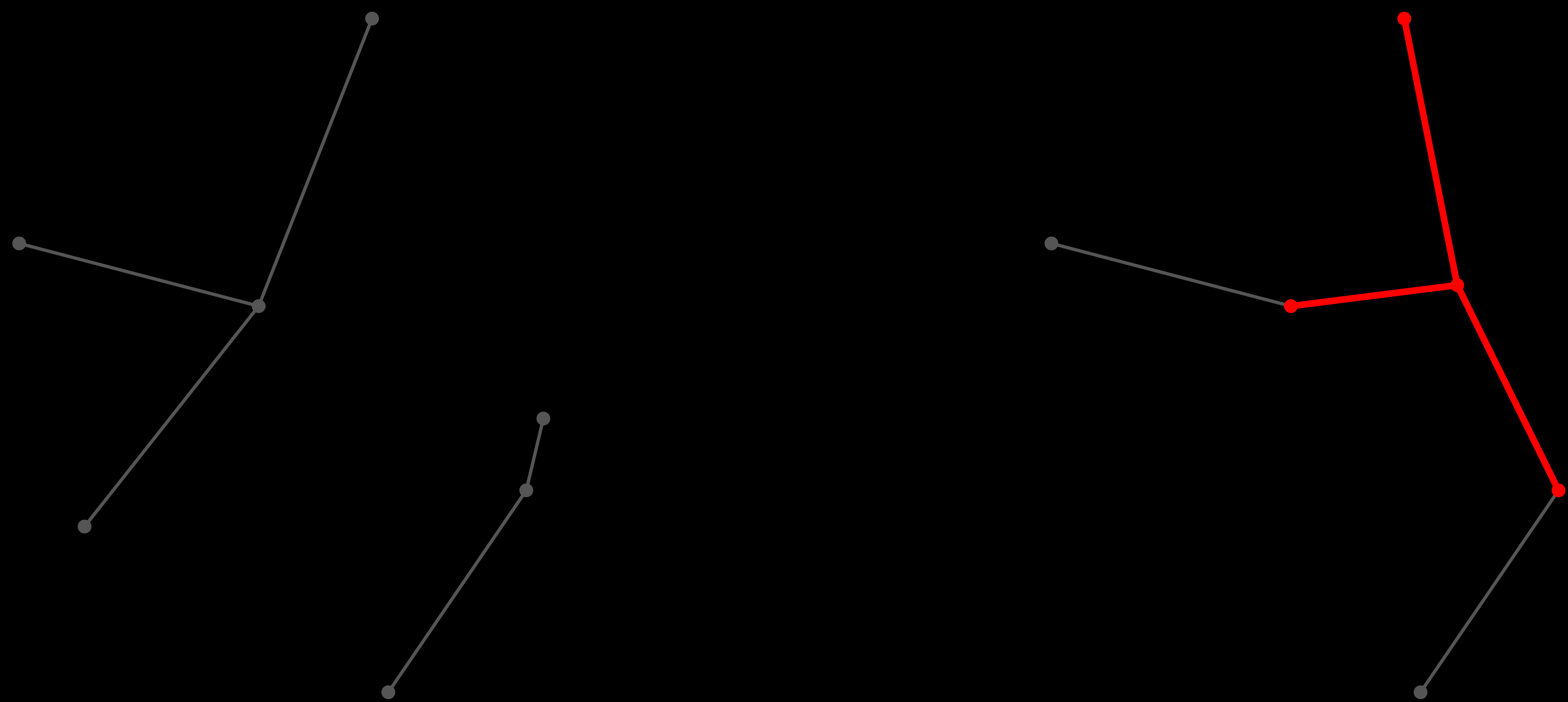




Updates: Example



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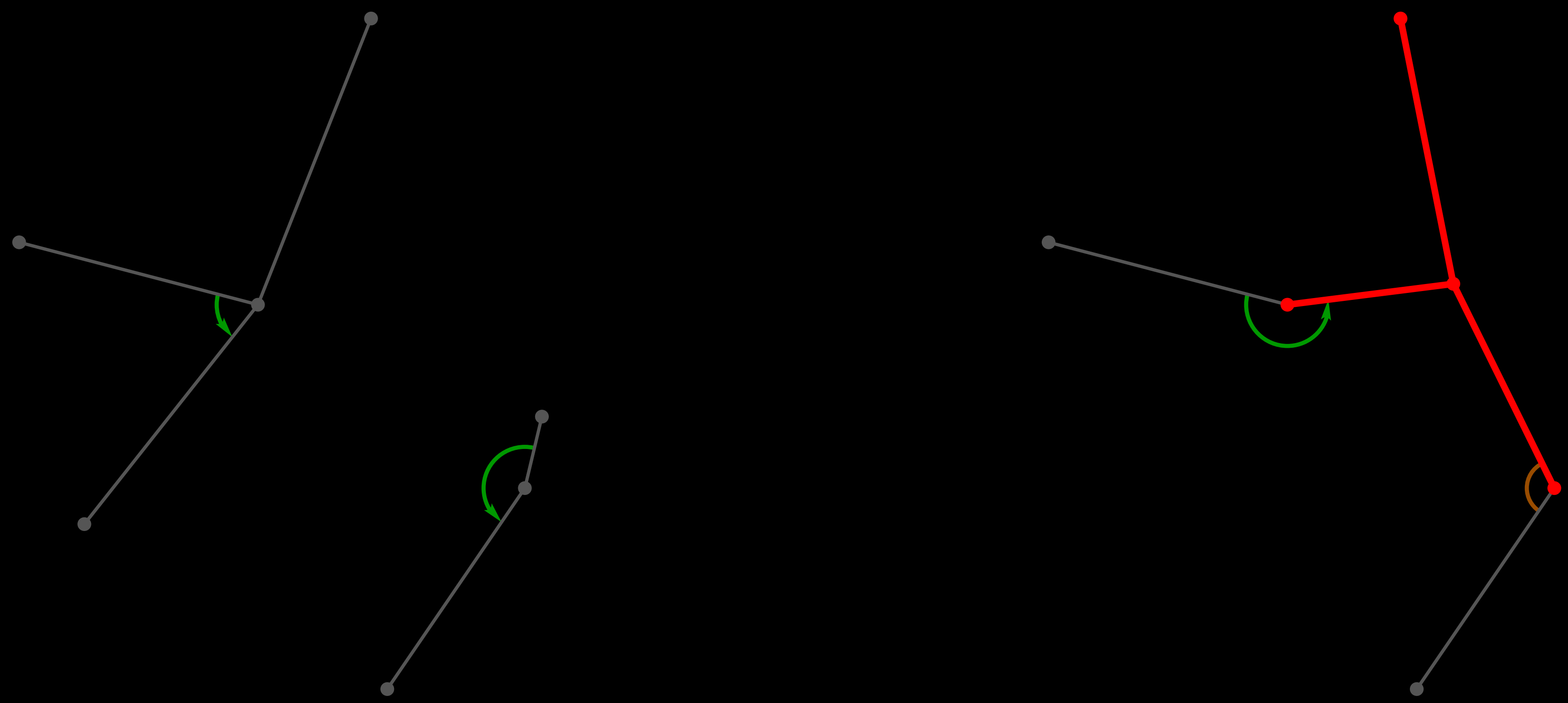

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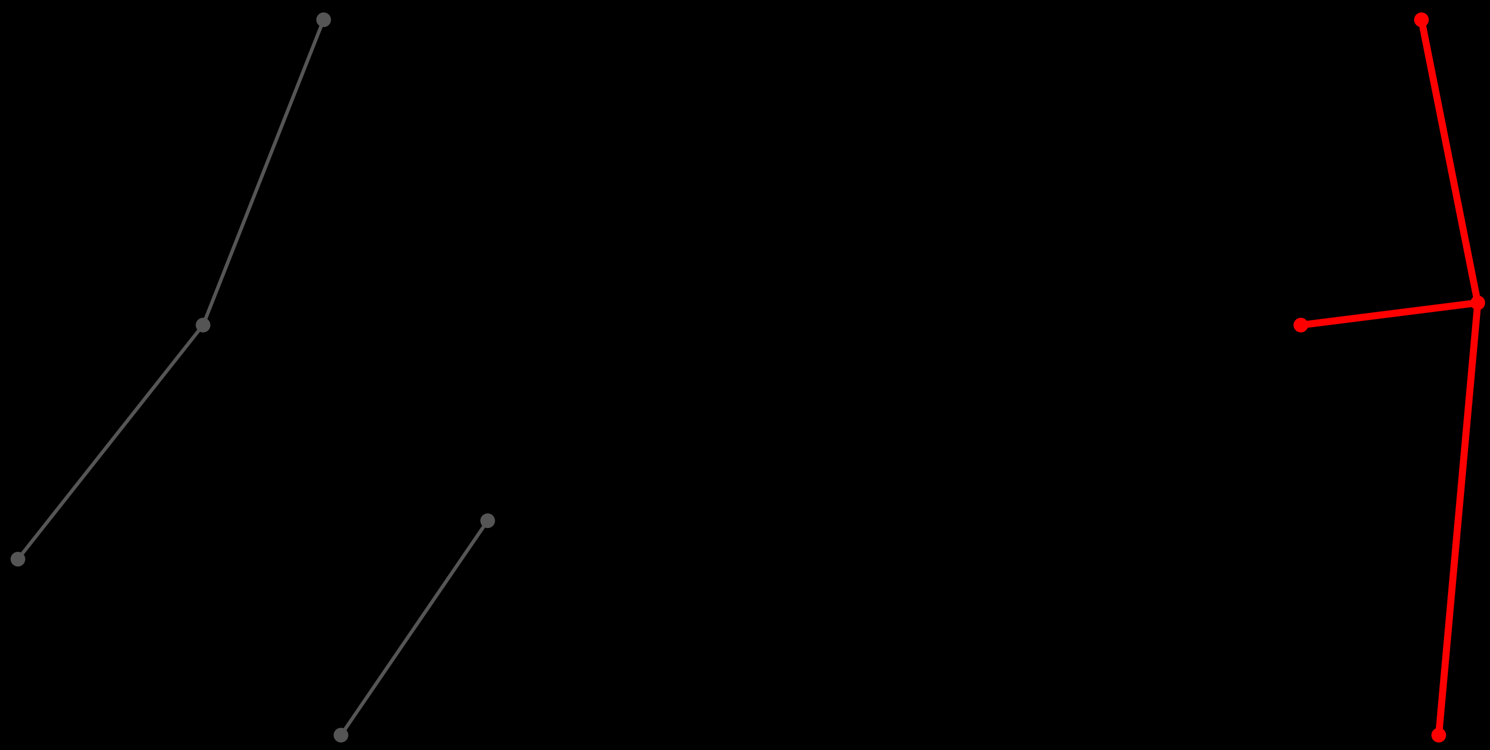

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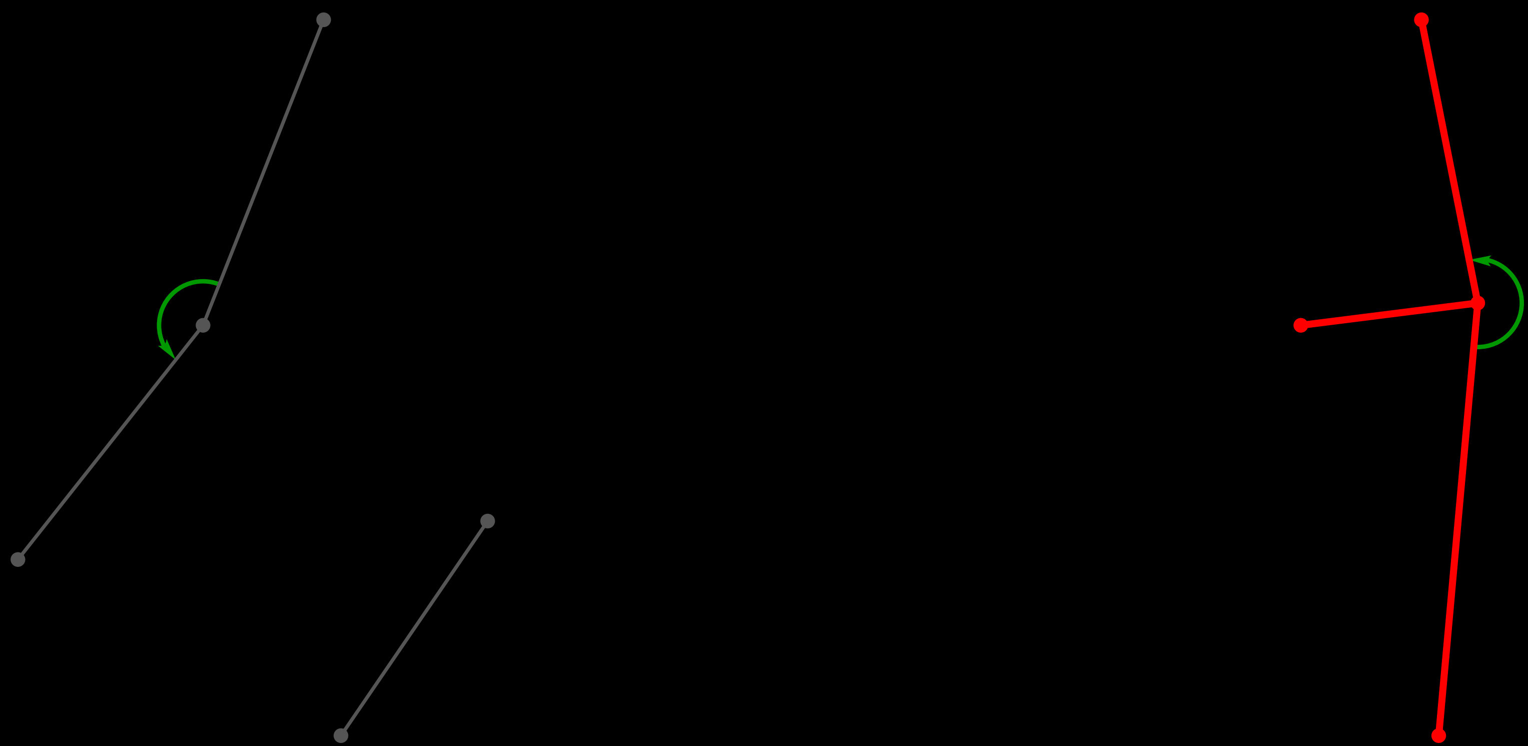

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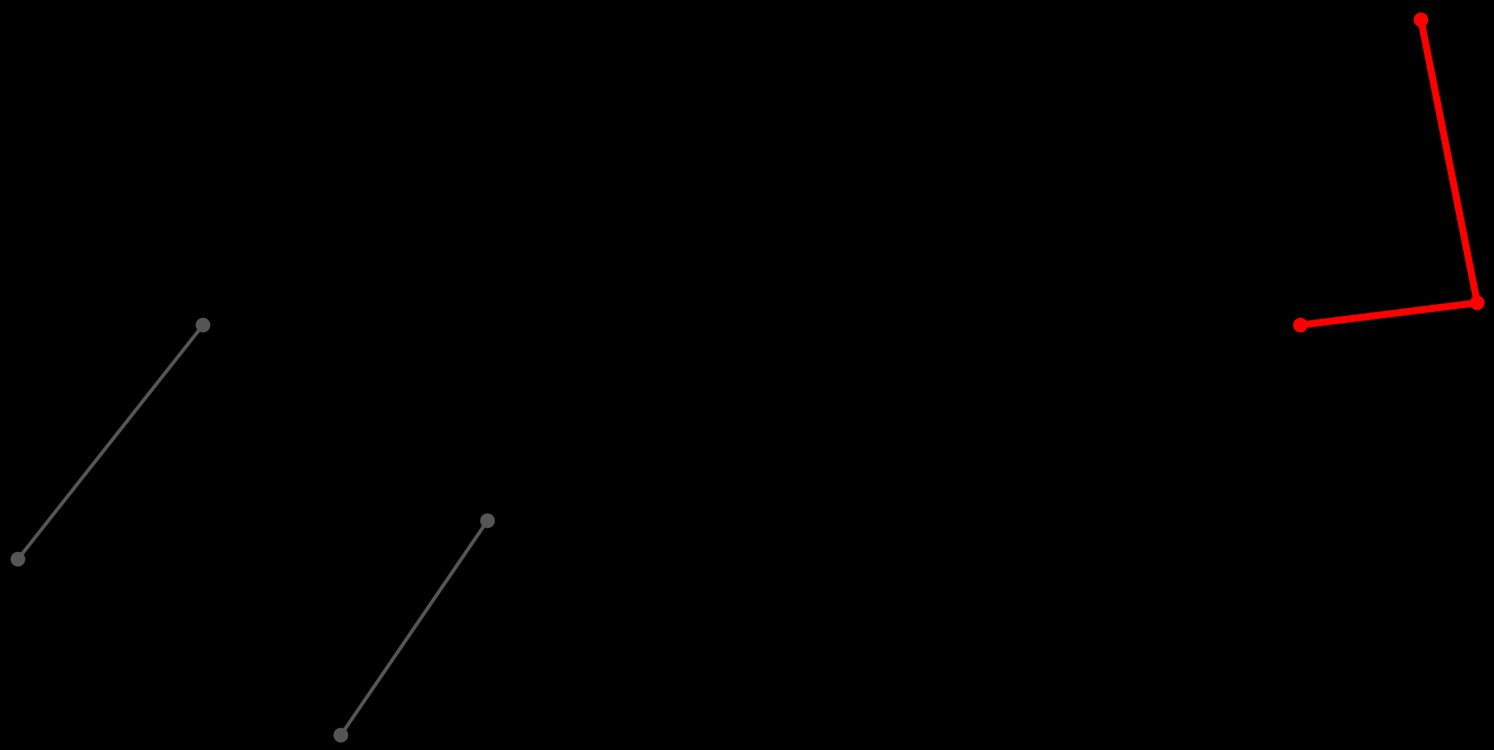

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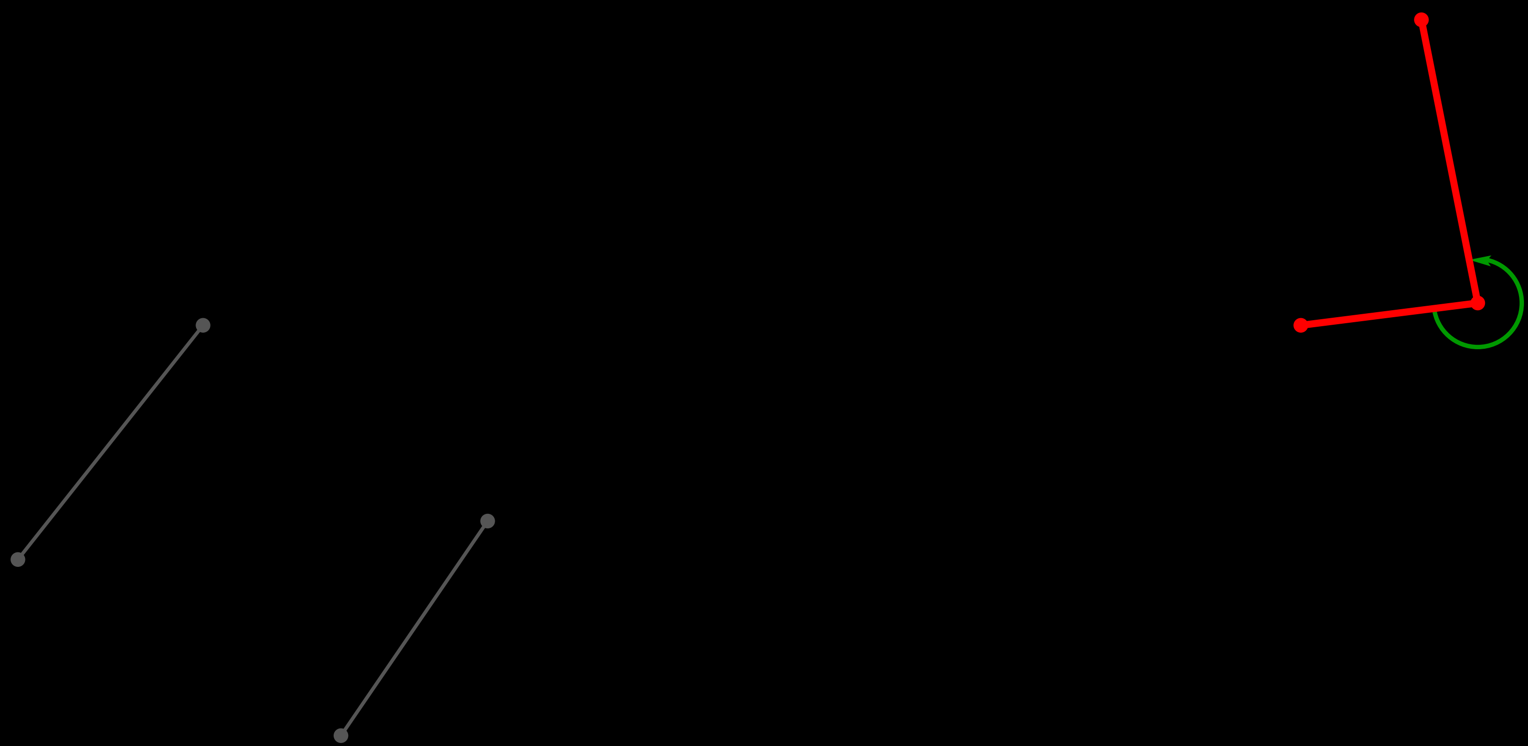

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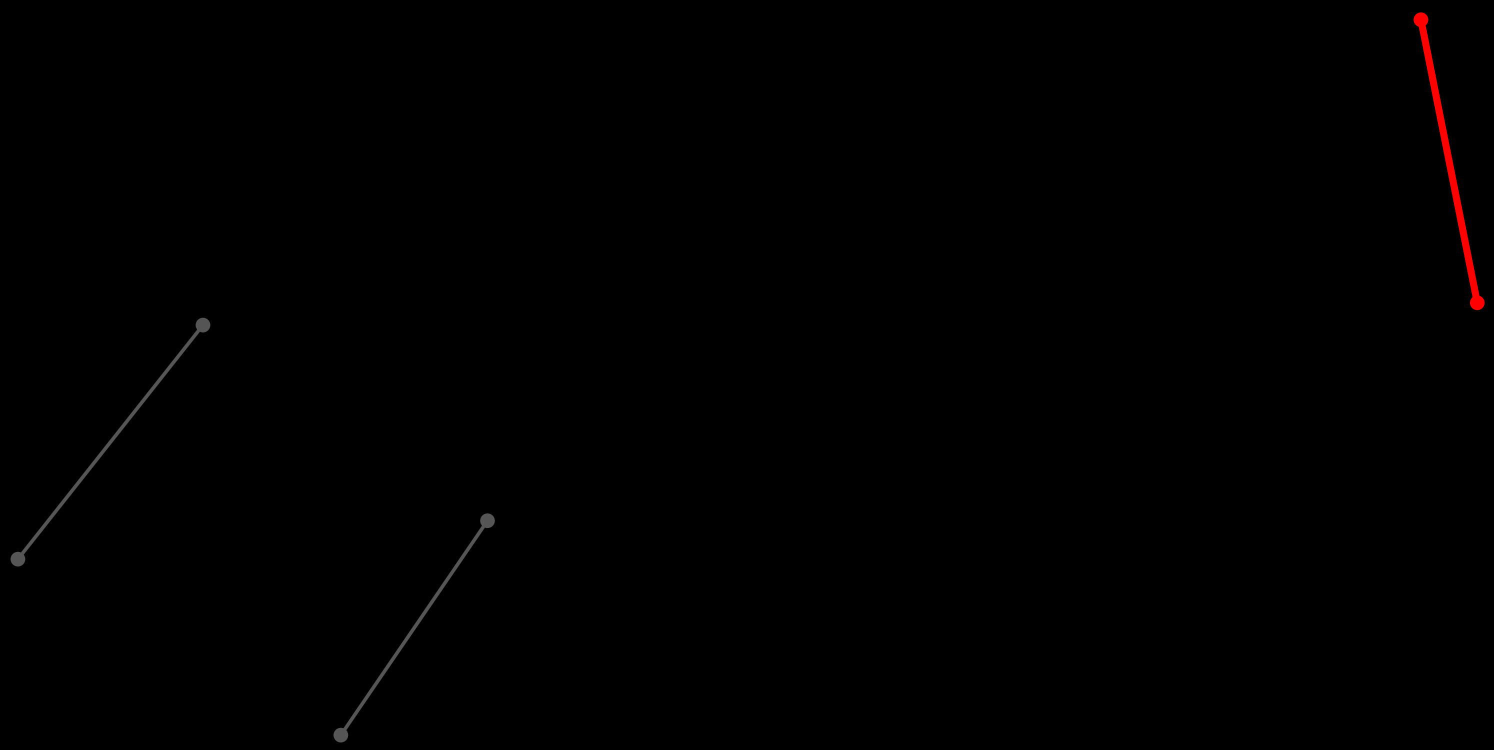
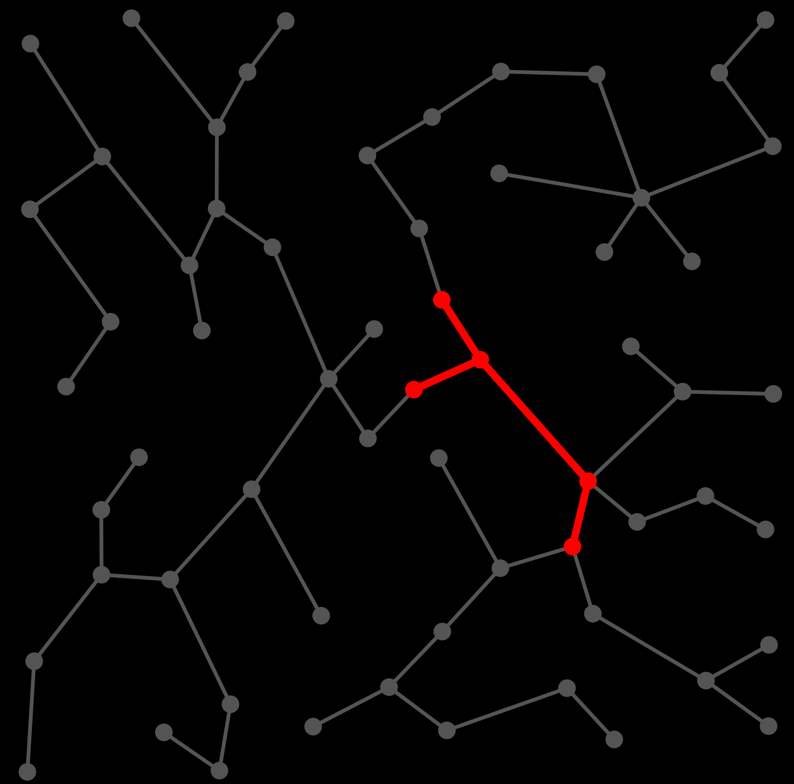


Running Time

- Theorem: each level can be updated in $O(1)$ time.

- Only need to worry about the core:
  - connected subgraph induced by the new (active) clusters;
  - remaining clusters: inactive subgraph.



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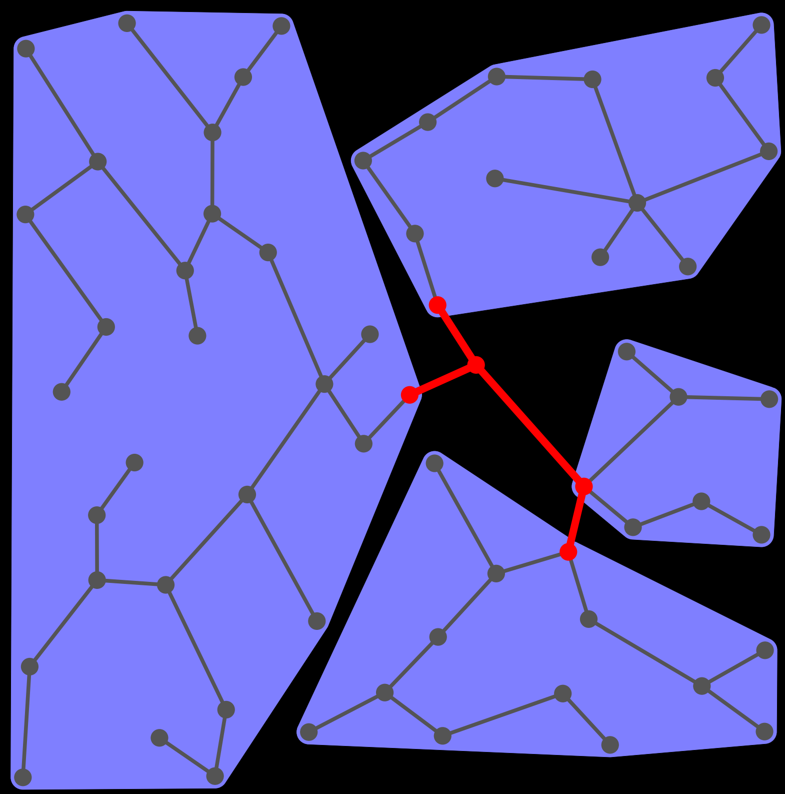


Running Time

- Theorem: each level can be updated in $O(1)$ time.

- Only need to worry about the core:
  - connected subgraph induced by the new (active) clusters;
  - remaining clusters: inactive subgraph.

- Suffices to prove that core size s is bounded by a constant.
  - If core were a free tree, would shrink by $1/6$ between rounds.
  - Each point of contact will add $O(1)$ clusters to the core.
  - Claim: there are at most four points of contact.
  - Constant initial size + multiplicative decrease + additive increase:

  $\Rightarrow$ constant size.

Contraction-based Data Structure

- Positive aspects:
  - general (top tree interface);
  - conceptually simple algorithm;
  - direct implementation (does not use topology trees);
  - $O(\log n)$ worst case.

- Potential overheads:
  - pointers within and among levels:
    - need Euler tour of each level;
  - unmatched clusters are repeated (dummy nodes).

- Joint work with J. Holm, R. Tarjan, and M. Thorup.

Outline

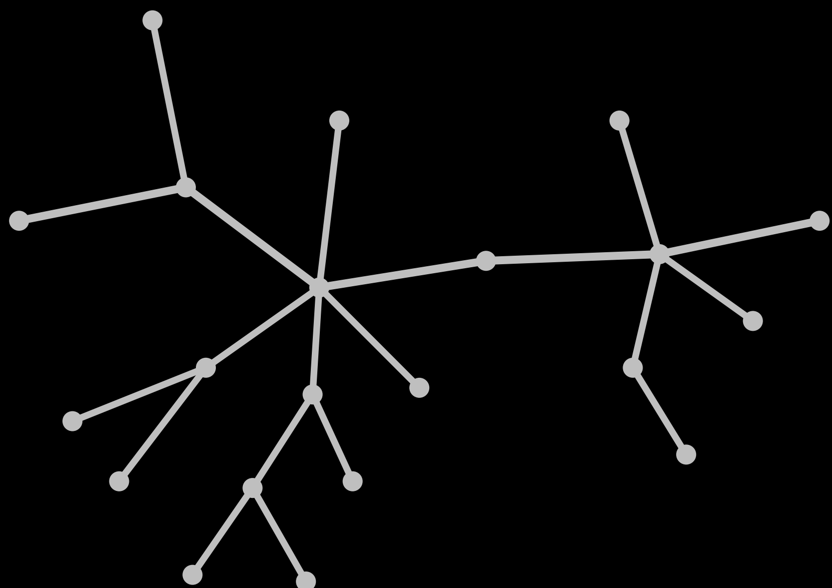
- The Dynamic Trees problem
- Existing data structures
- A new worst-case data structure

⇒ A new amortized data structure

- Experimental results
- Final remarks

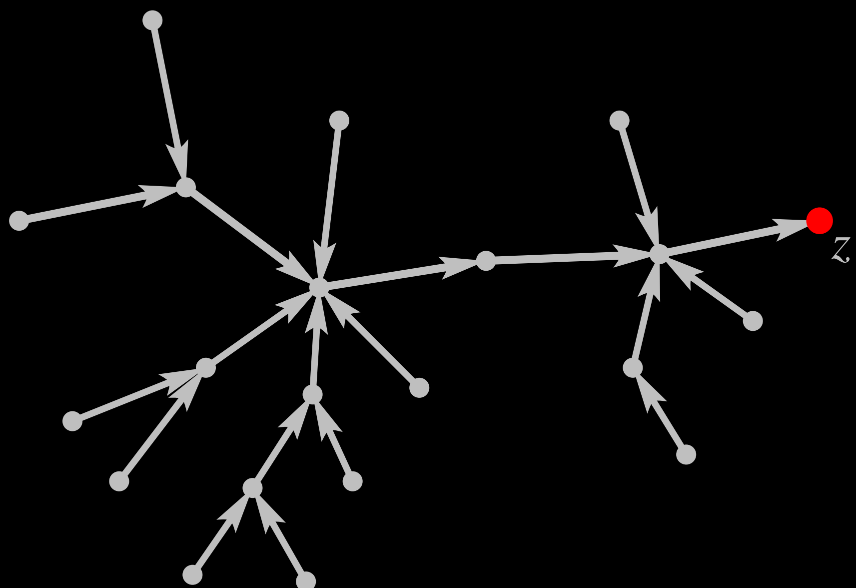
Representation

- Consider some unrooted tree:



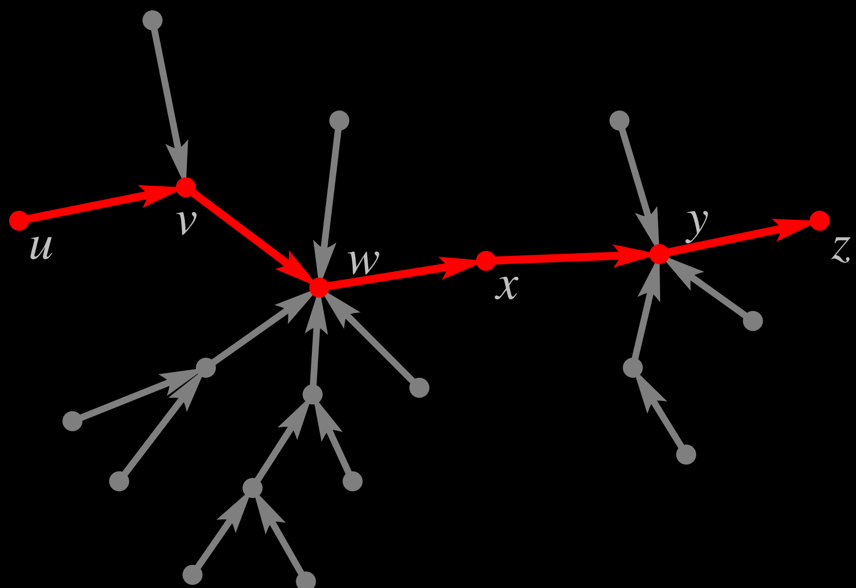
Representation

- Make it a unit tree:
  - directed tree with degree-one root.



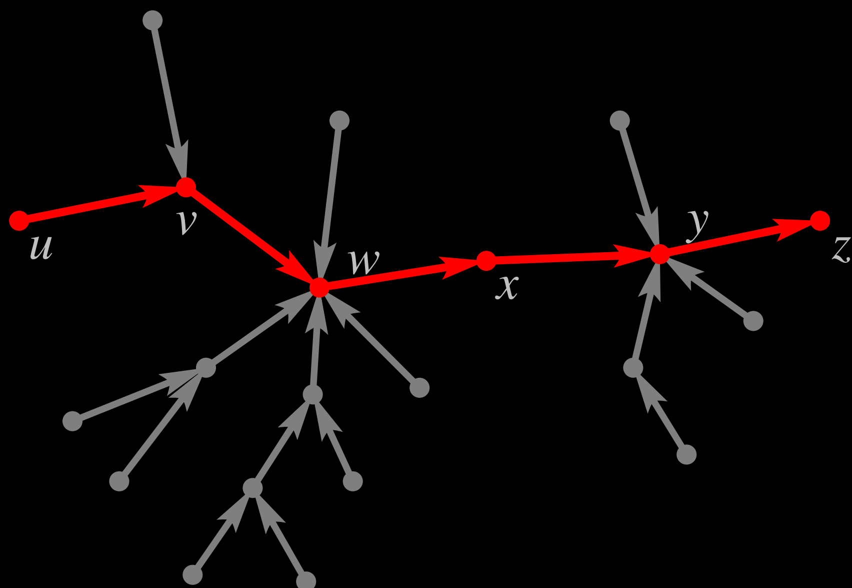
Representation

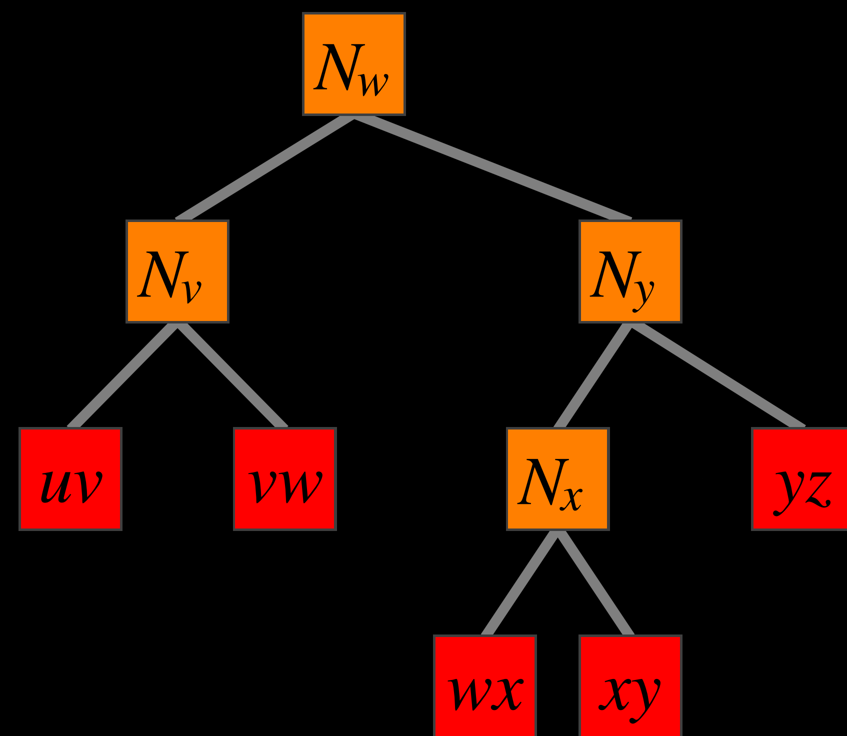
- Pick a root path:
  - starts at a leaf;
  - ends at the root.



Representation

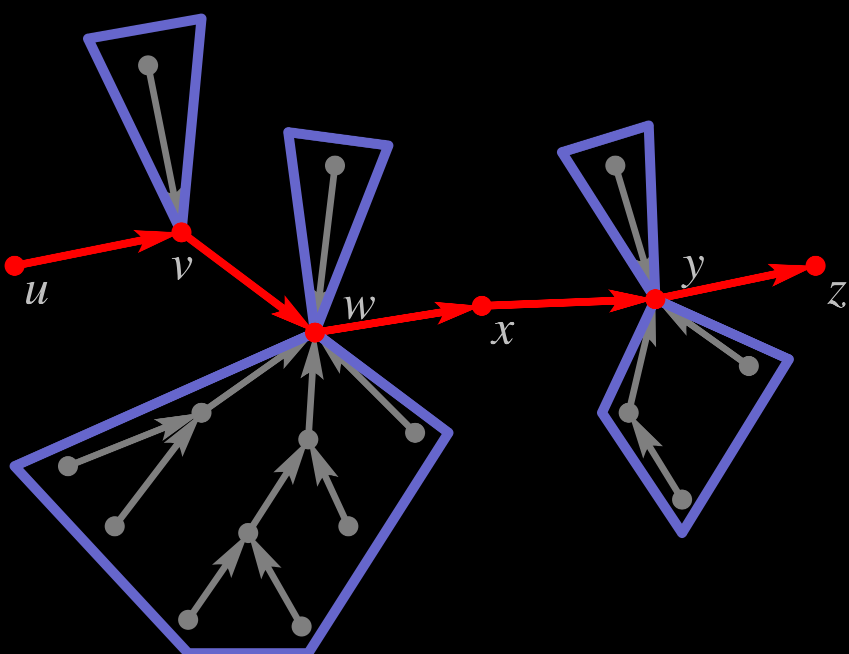
- Represent the root path as a binary tree:
  - leaves: base clusters (original edges);
  - internal nodes: compress clusters.

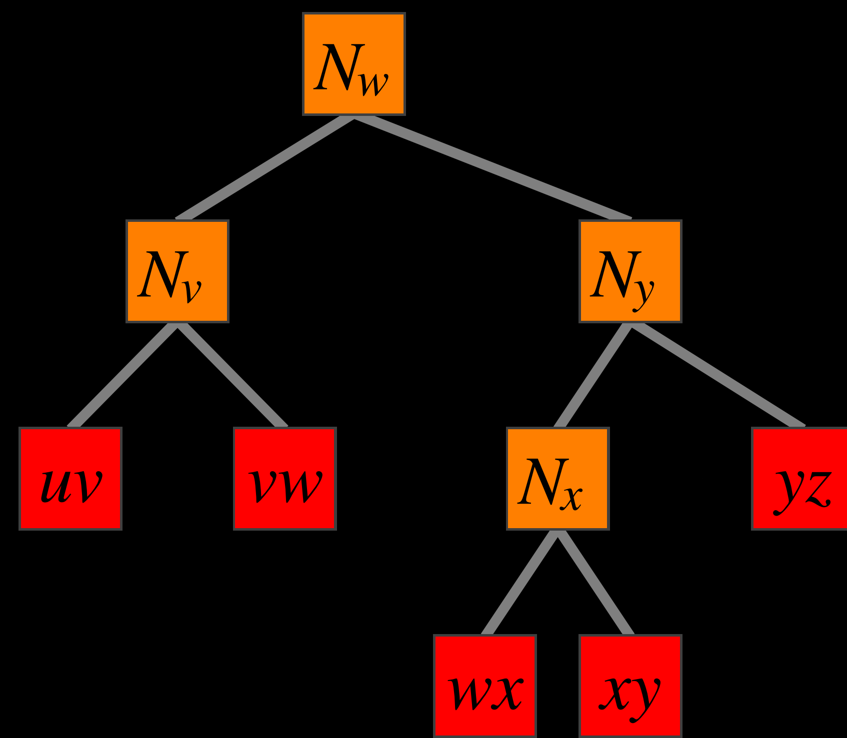




Representation

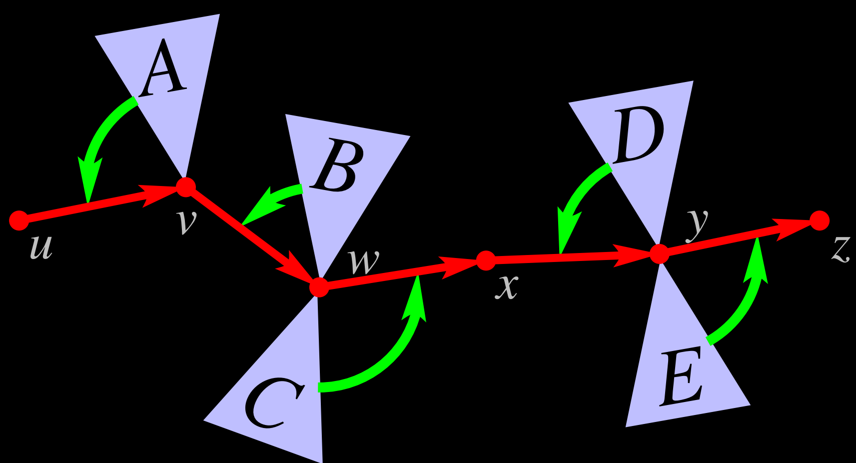
- What if a vertex has degree greater than two?
  - Recursively represent each subtree rooted at the vertex (at most two, because of circular order).

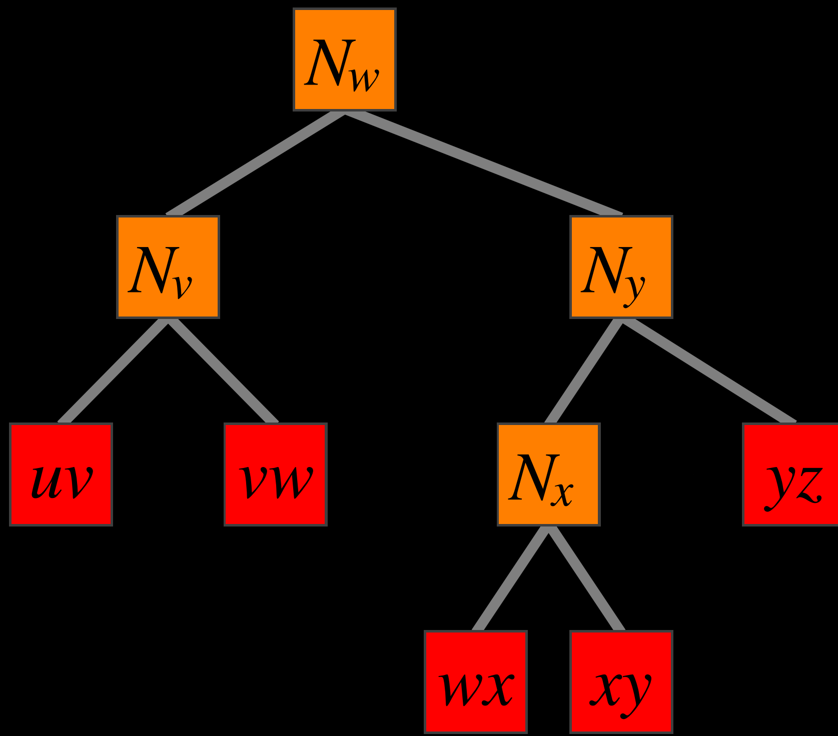




Representation

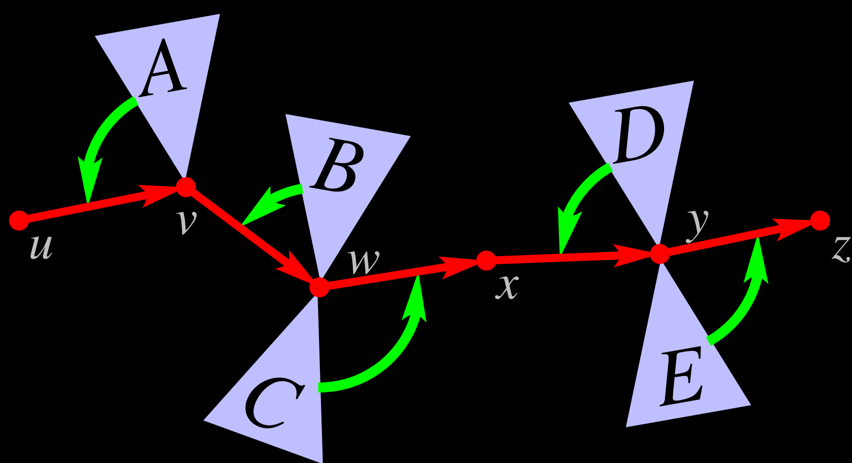
- What if a vertex has degree greater than two?
  - Recursively represent each subtree rooted at the vertex.
  - Before vertex is compressed, rake subtrees onto adjacent cluster.

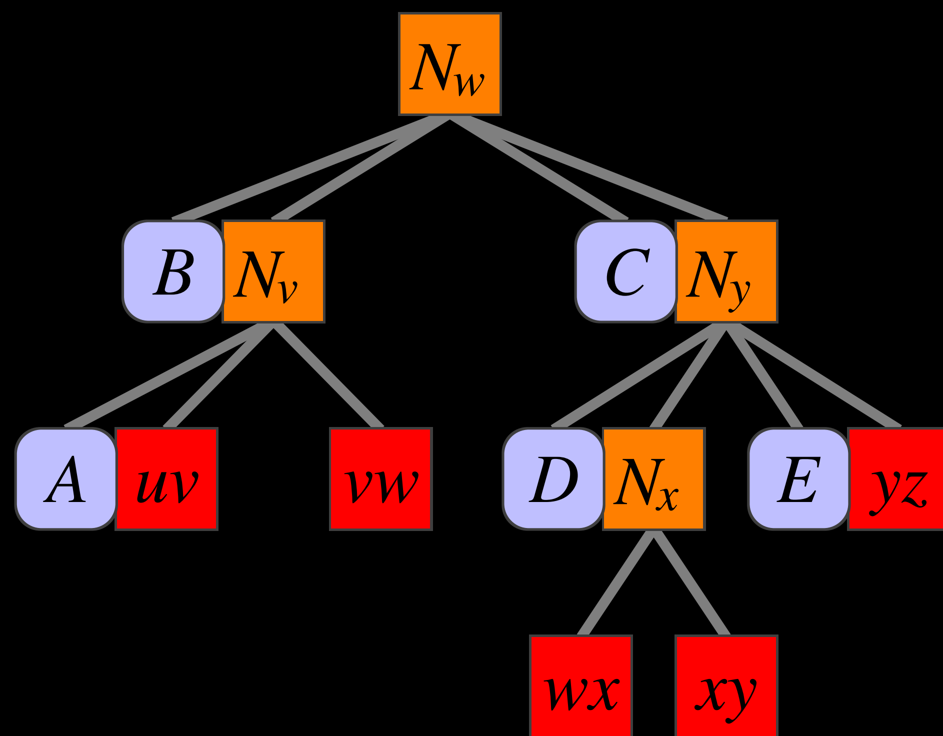




Representation

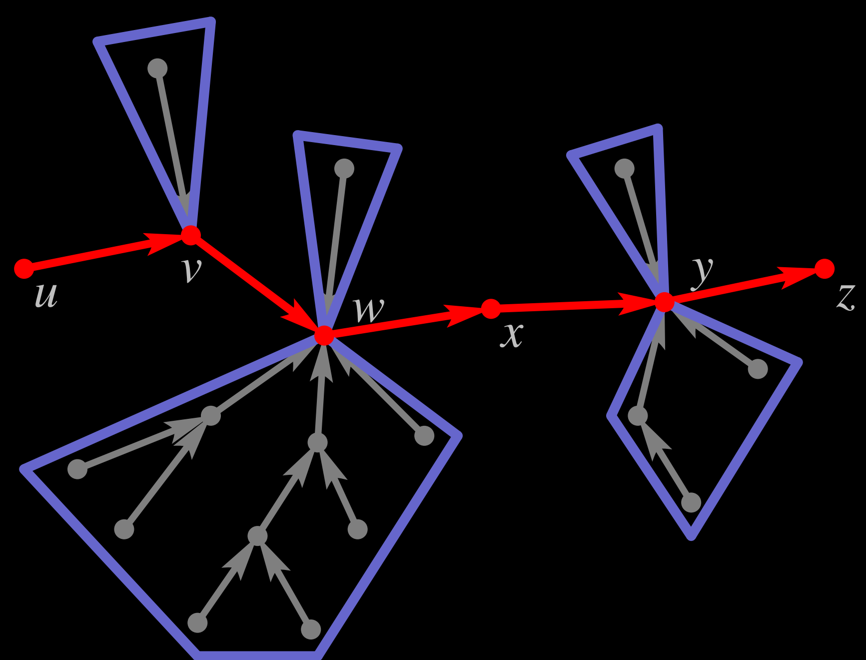
- Representation:
  - Up to four children per node (two proper ones, two foster ones)
  - Meaning: up to two rakes followed by a compress.
  - Example: $N_y = \text{compress}(\text{rake}(D, N_x), \text{rake}(E, yz)) = wz$.





Representation

- How does the recursive representation work?
  - Must represent subtrees rooted at the root path.



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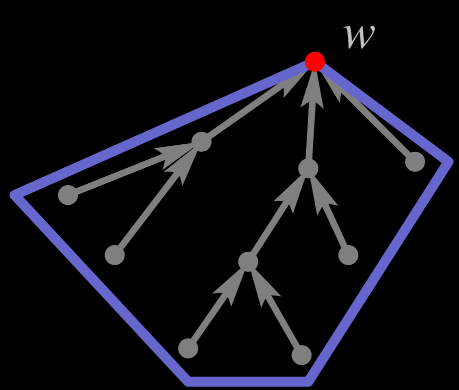

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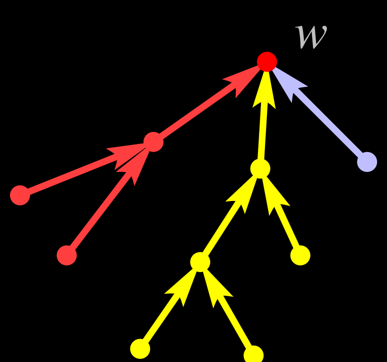
  - Each subtree is a sequence of unit trees with a common root.



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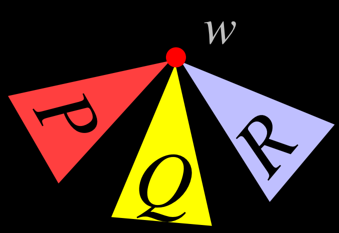
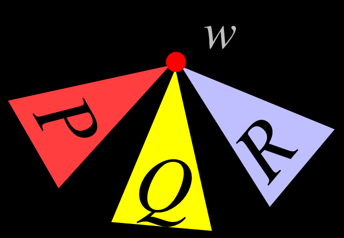
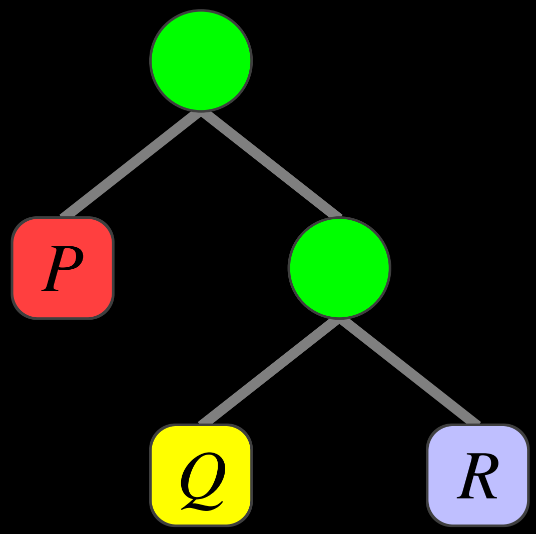


Representation

- How does the recursive representation work?
  - Must represent subtrees rooted at the root path.
  - Each subtree is a sequence of unit trees with a common root.
  - Represent each recursively.
  - Build a binary tree of rakes.



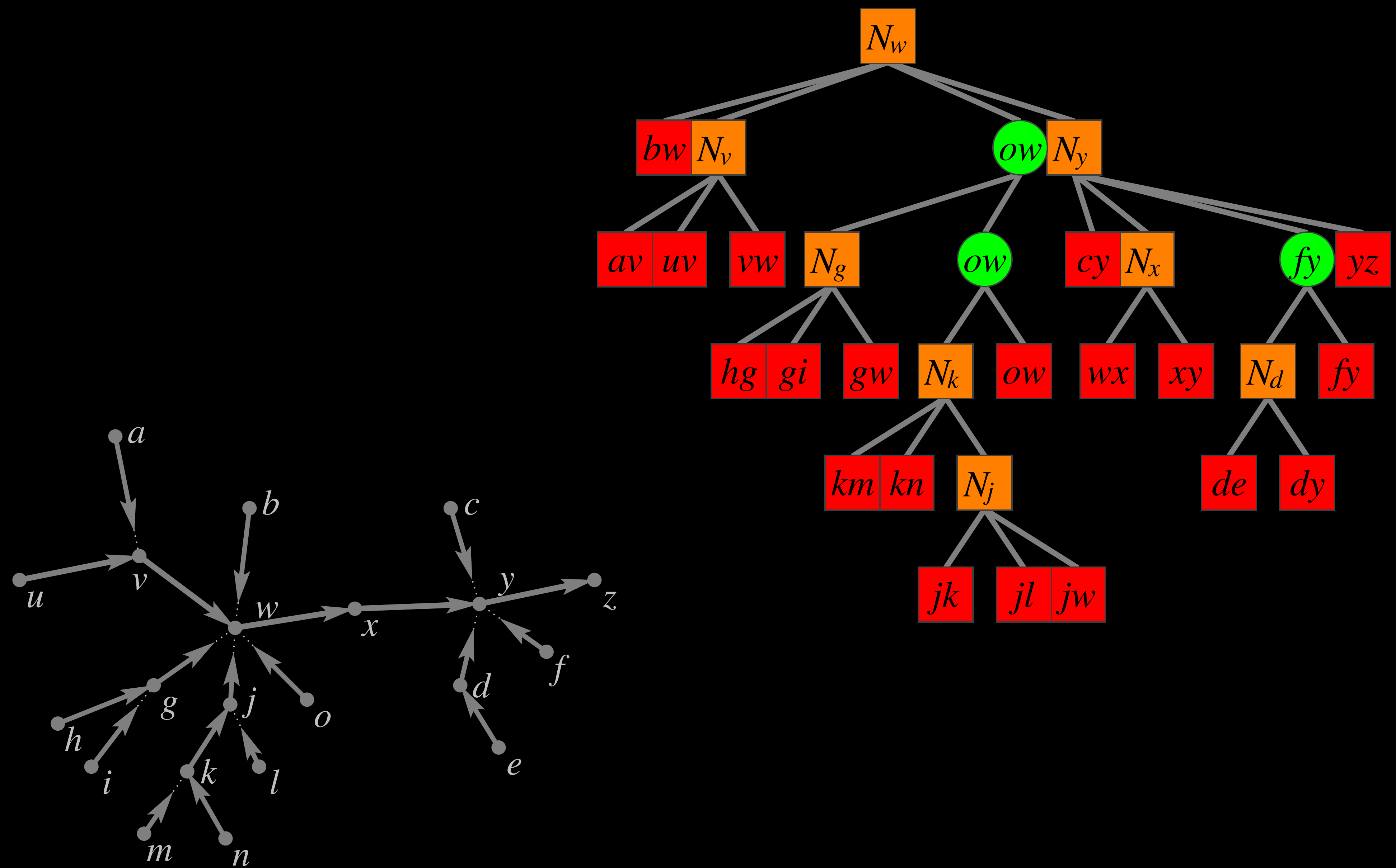


Representation

- Two different views:
  - User interface: tree contraction.
    * sequence of rakes and compresses;
    * a single tree (a top tree).
  - Implementation: path decomposition.
    * maximal edge-disjoint paths;
    * hierarchy of binary trees (rake trees/compress trees);
    * similar to ST-trees.

Representation

- Full example:



Self-Adjusting Top Trees

- Topmost compress tree represents the root path:
  - determined by expose(v, w);
  - implementation similar to ST-trees;
  - basic operations:
    * splay: rebalances each binary tree;
    * splice: changes the partition into paths.

- Main result: expose takes $O(\log n)$ amortized time.

- Operations link and cut use expose as the main subroutine.

- Joint work with R. Tarjan [SODA'05].

Outline

- The Dynamic Trees problem
- Existing data structures
- A new worst-case data structure
- A new amortized data structure

$\Rightarrow$ Experimental results

- Final remarks

Experimental Results

- Data structures implemented (in `C++`):
  - TOP-W: worst-case top trees;
  - TOP-S: self-adjusting top trees;
  - ET-S: self-adjusting ET-trees;
  - ST-V/ST-E: self-adjusting ST-trees;
  - LIN-V/LIN-E: "obvious" linear-time data structure.

- Machine: 2.0 GHz Pentium IV with 1 GB of RAM.

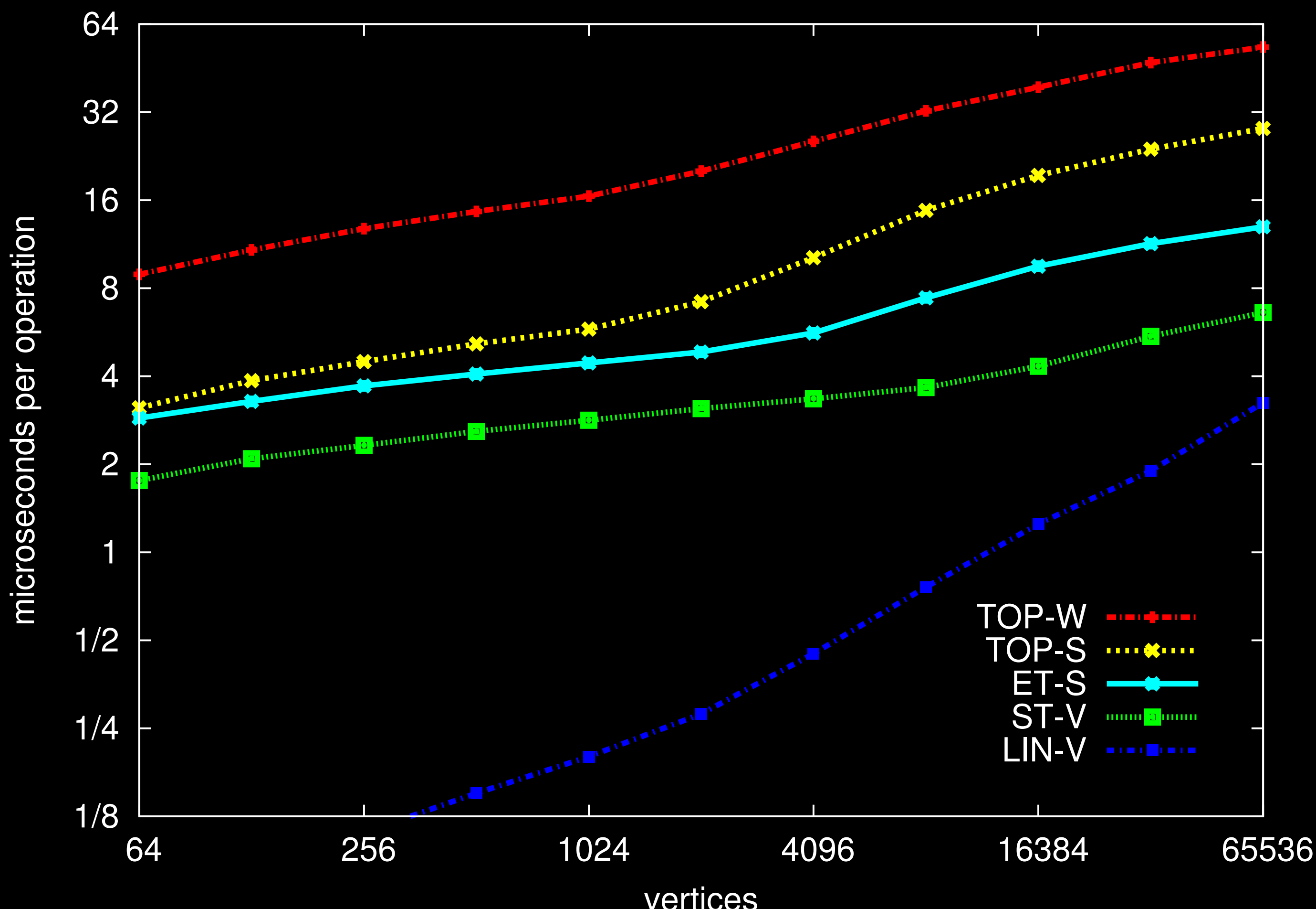
- Applications:
  - random operations;
  - maximum flows;
  - online minimum spanning trees;
  - shortest paths.

Experimental Results: Executive Summary

- ST-V: fastest $O(\log n)$ algorithm;
  - ST-E and ET-S: up to twice as slow.
- TOP-S is 2–4 times as slow as ST-V.
- TOP-W vs. TOP-S:
  - TOP-S faster for links and cuts;
  - TOP-W faster for expose;
  - TOP-S benefits from correlated operations (splaying).
- LIN-V/LIN-E: fastest for paths with up to $\sim$1000 vertices.

Experiment: Random Operations

- Setup:
  - Start with empty graph on n vertices;
  - use $n-1$ links to create a random spanning tree;
  - alternate cuts and links until $\#\text{ops} = 10n$.



Experiment: Maximum Flow

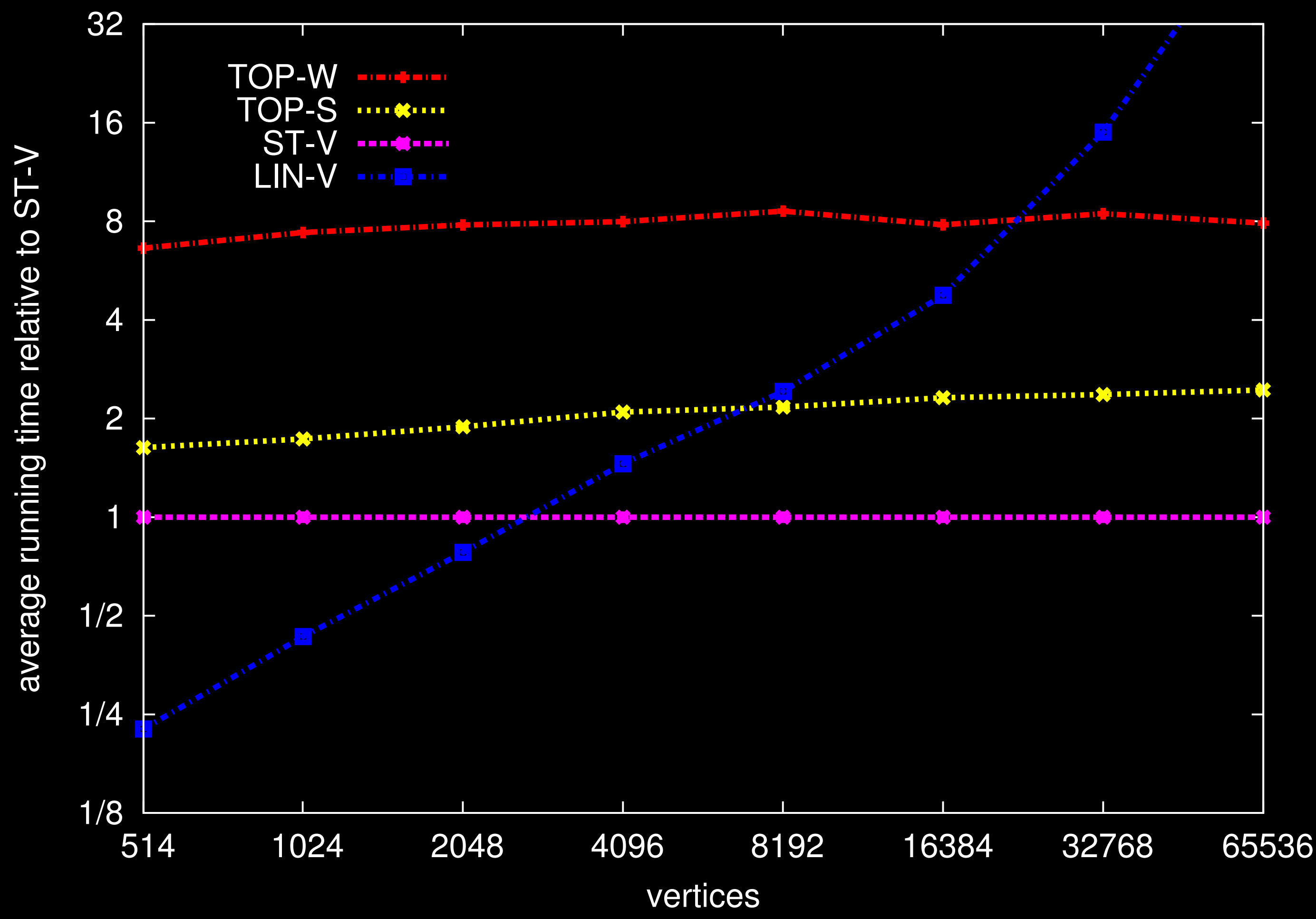
- Maximum flow:
  - Given: directed graph with n vertices and m edges, source s, sink t;
  - Goal: find maximum flow from s to t.

- Algorithm:
  - Find shortest path from s to t with positive residual capacity;
  - send as much flow as possible, update residual capacities, repeat;
  - running time: $O(mn^2)$.

- With dynamic trees:
  - Keep "partial paths" between iterations as a forest;
  - allows augmentations in $O(\log n)$ time;
  - running time: $O(mn \log n)$.

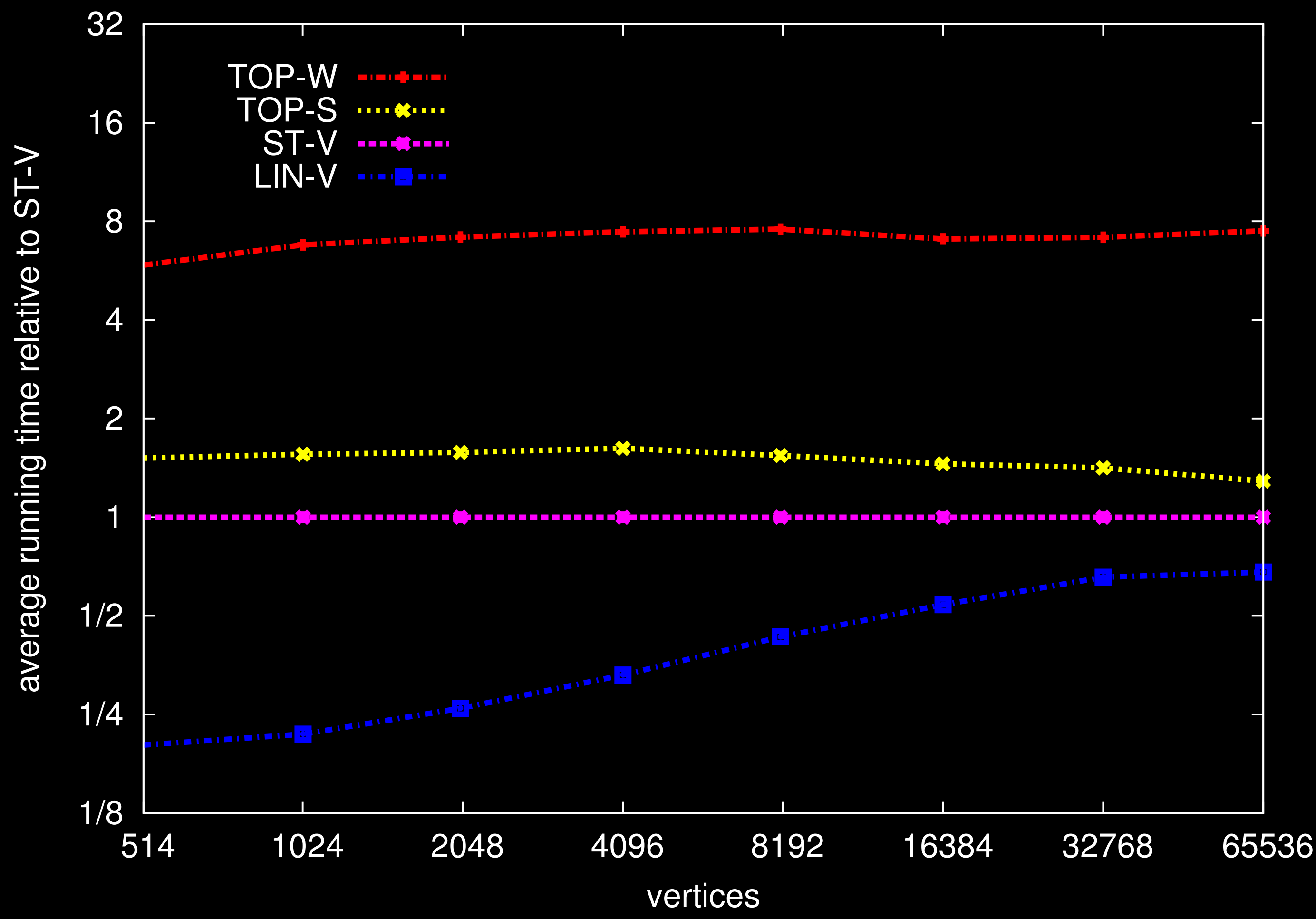
Maximum Flow on Layered Networks

- Layered networks:
  - 4 rows, $\lfloor n/4 \rfloor$ columns between s and t;
  - each vertex connected to 3 random vertices in the next column;
  - augmenting paths have $\Omega(n)$ arcs.



Maximum Flow on Square Meshes

- Square meshes:
  - full $\lfloor\sqrt{n}\rfloor \times \lfloor\sqrt{n}\rfloor$ grid between s and t;
  - each grid vertex connected to all 4 neighbors;
  - augmenting paths have $\Omega(\sqrt{n})$ arcs.



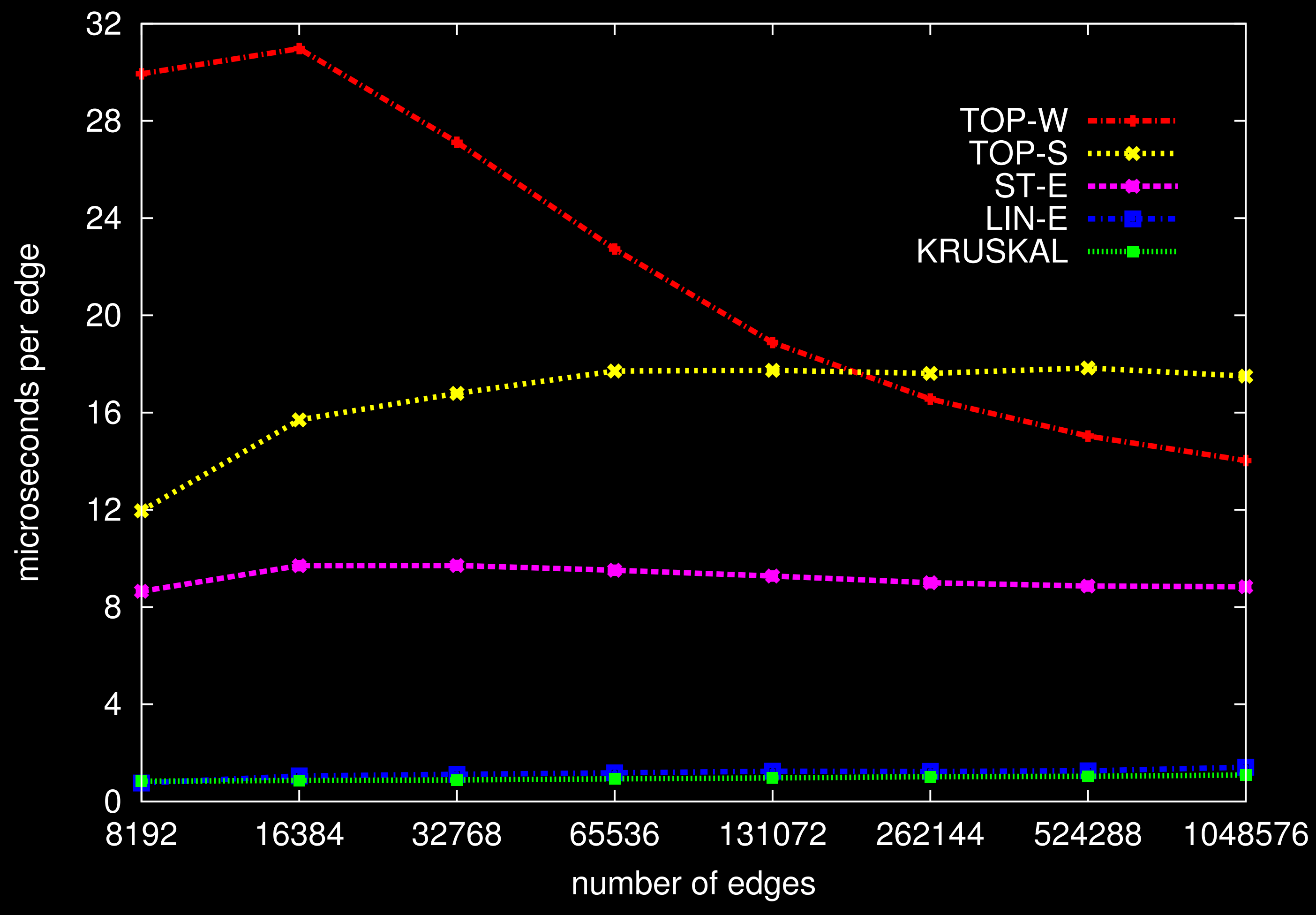
Experiment: Online Minimum Spanning Forest

- Online minimum spanning forest:
  - Edges processed one at a time;
  - Edge (v, w) inserted if
    - v and w in different components; or
    - (v, w) is shorter than longest edge on path from v to w:
      - longest edge is removed.
  - $O(\log n)$ time per edge.

- Kruskal as reference algorithm:
  1. Quicksort all m edges (offline);
  2. Insert edge iff its endpoints are in different components:
     - use union-find data structure.

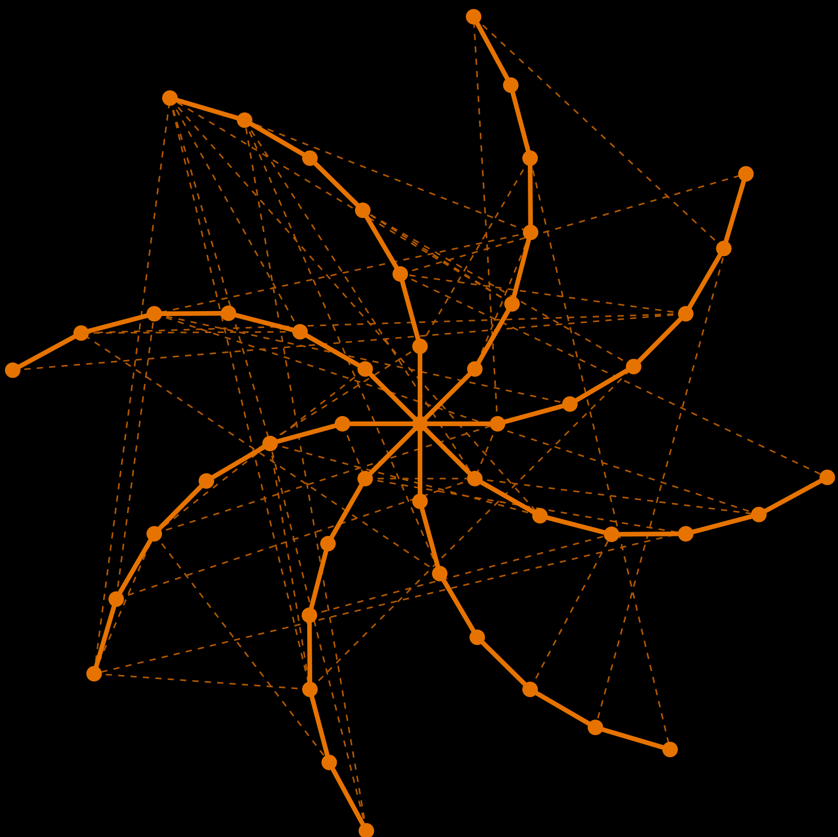
Online MSF on Random Graphs

- Random multigraphs:
  - $n = 4096$, edges with random endpoints and weights;
  - expected diameter: $O(\log n)$;
  - more edges $\Rightarrow$ greater percentage of exposes.



Online MSF on Augmented Stars

- Augmented stars:
  - $n = 65537$, varying number (and length) of spokes;
  - diameter: $O(65537/\#\text{spokes})$;
  - spoke edges, with length 1, processed first;
  - other edges (random), with length 2, processed later;
  - $10n$ edges in total.



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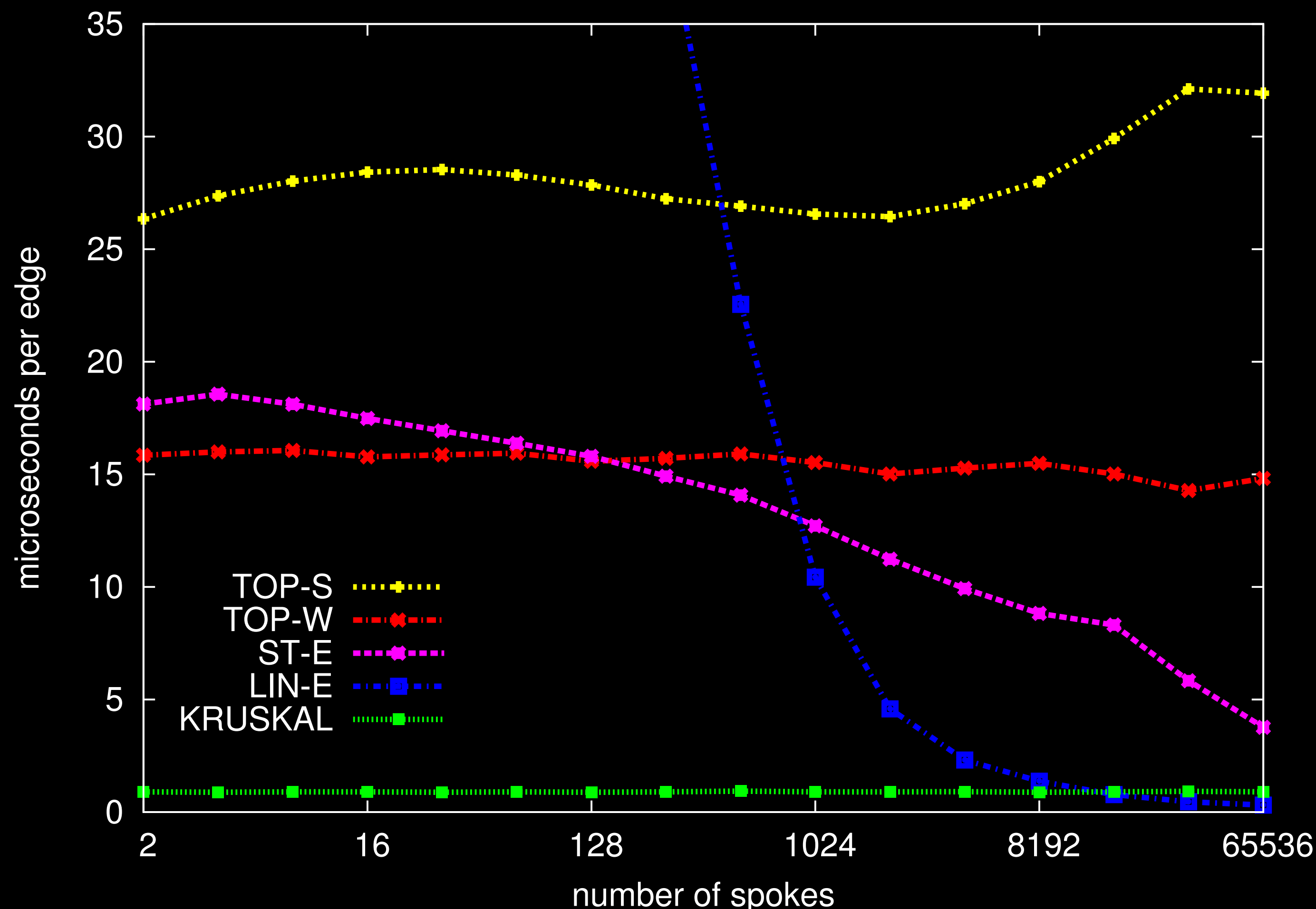
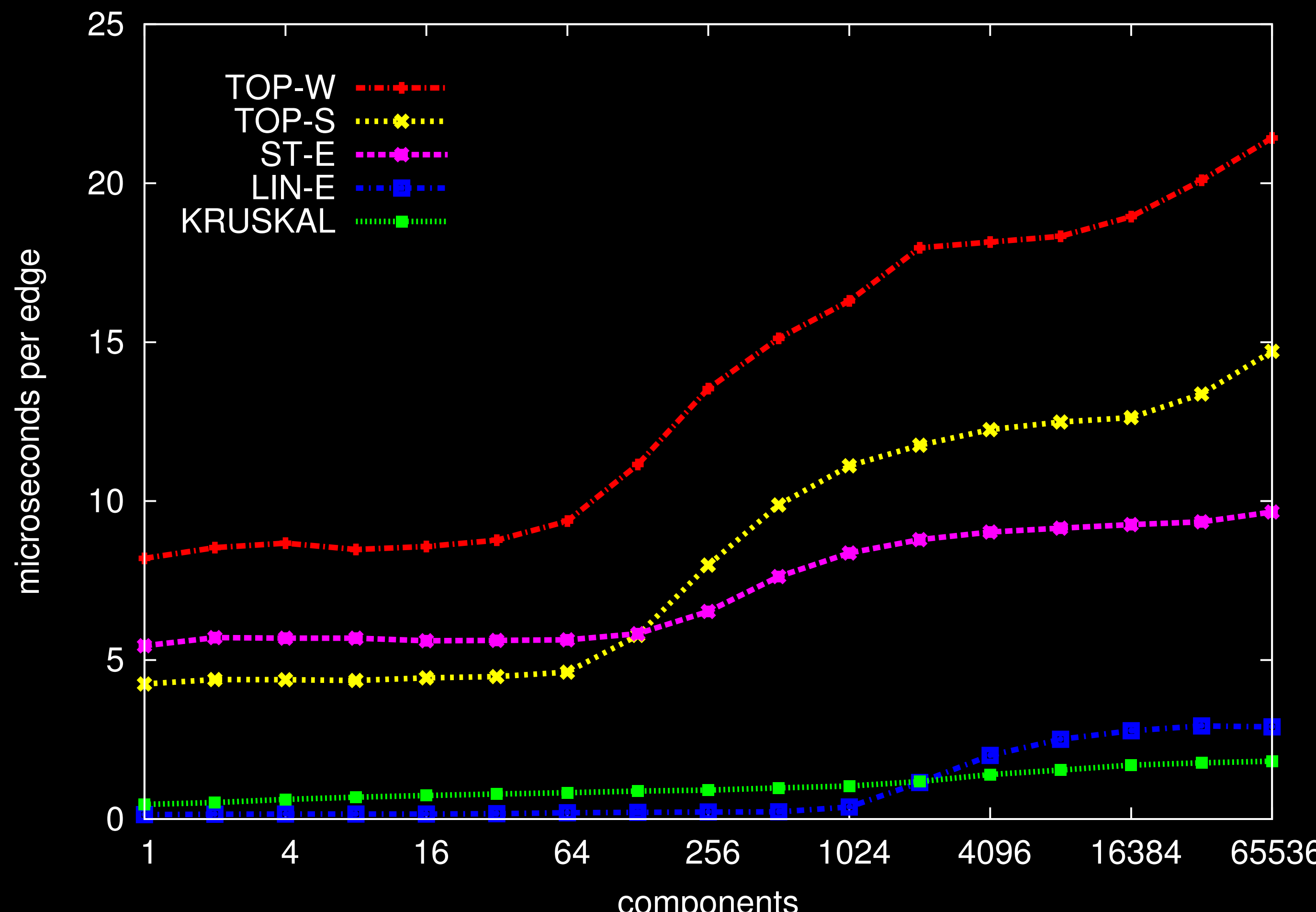


Online MSF and Cache Effects

- Procedure:
  - Partition vertices at random into $n/32$ components of size 32;
  - Random edges: pick random component, then random pair within it.
  - Total number of edges: $4n$.



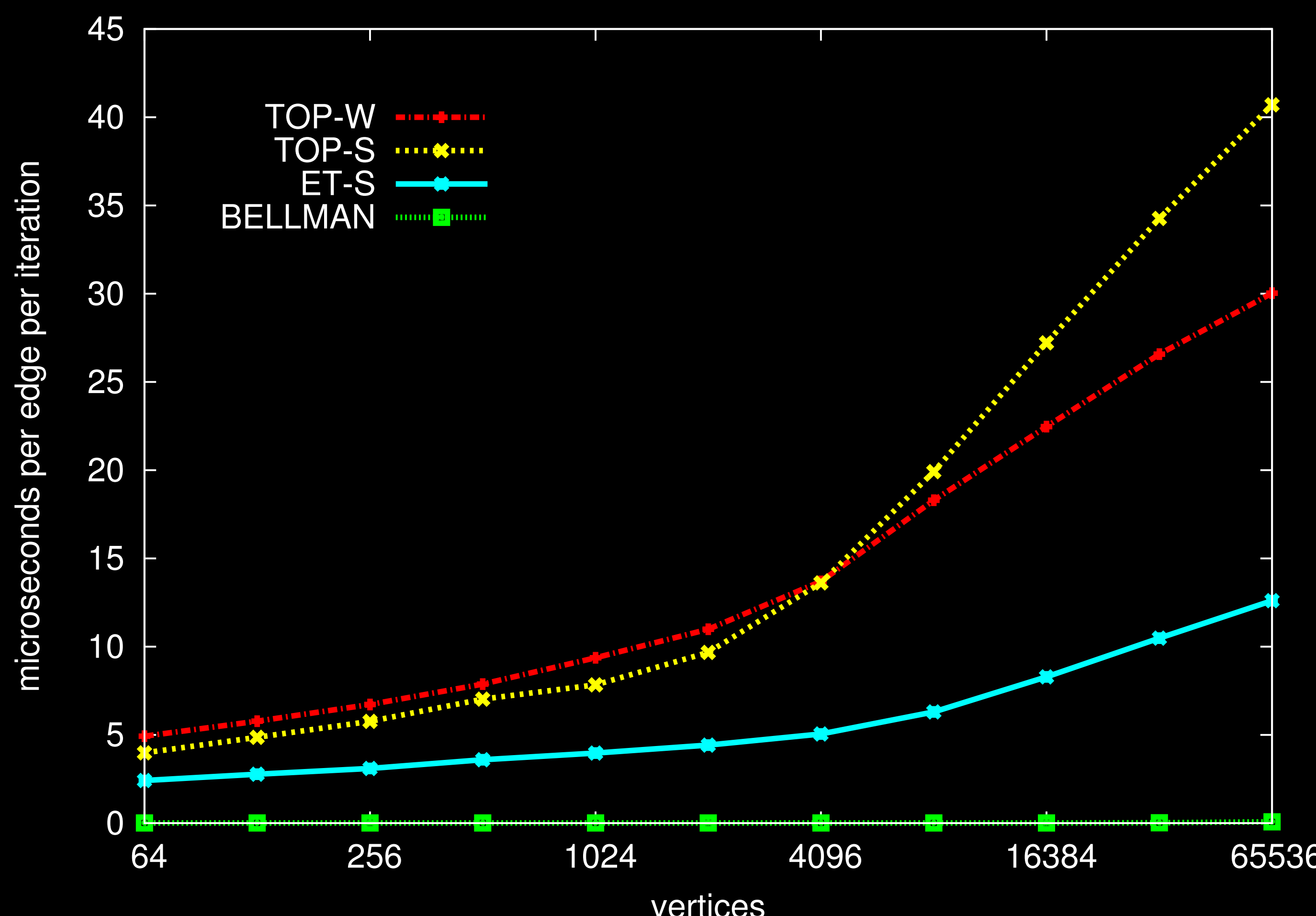
Experiment: Shortest Paths

- Bellman's single source shortest path algorithm:
  - Assign distance label to each vertex.
    * initially, $d(s) \leftarrow 0$ (source) and $d(v) \leftarrow \infty$ (remaining vertices).
  - In each iteration, process all arcs (v, w) in fixed order:
    * if $d(w) < d(v) + \ell(v, w)$, relax (v, w):
    * decrement $d(w)$ by $\Delta = d(v) + \ell(v, w) - d(w)$.
  - Stop when an iteration does not relax any arc.
  - Running time: $O(mn)$.

- Dynamic trees:
  - Maintain the current shortest path tree and current $d(\cdot)$;
  - When relaxing (v, w), decrement $d(\cdot)$ for all descendants of w;
  - $O(mn \log n)$, but may require fewer iterations.

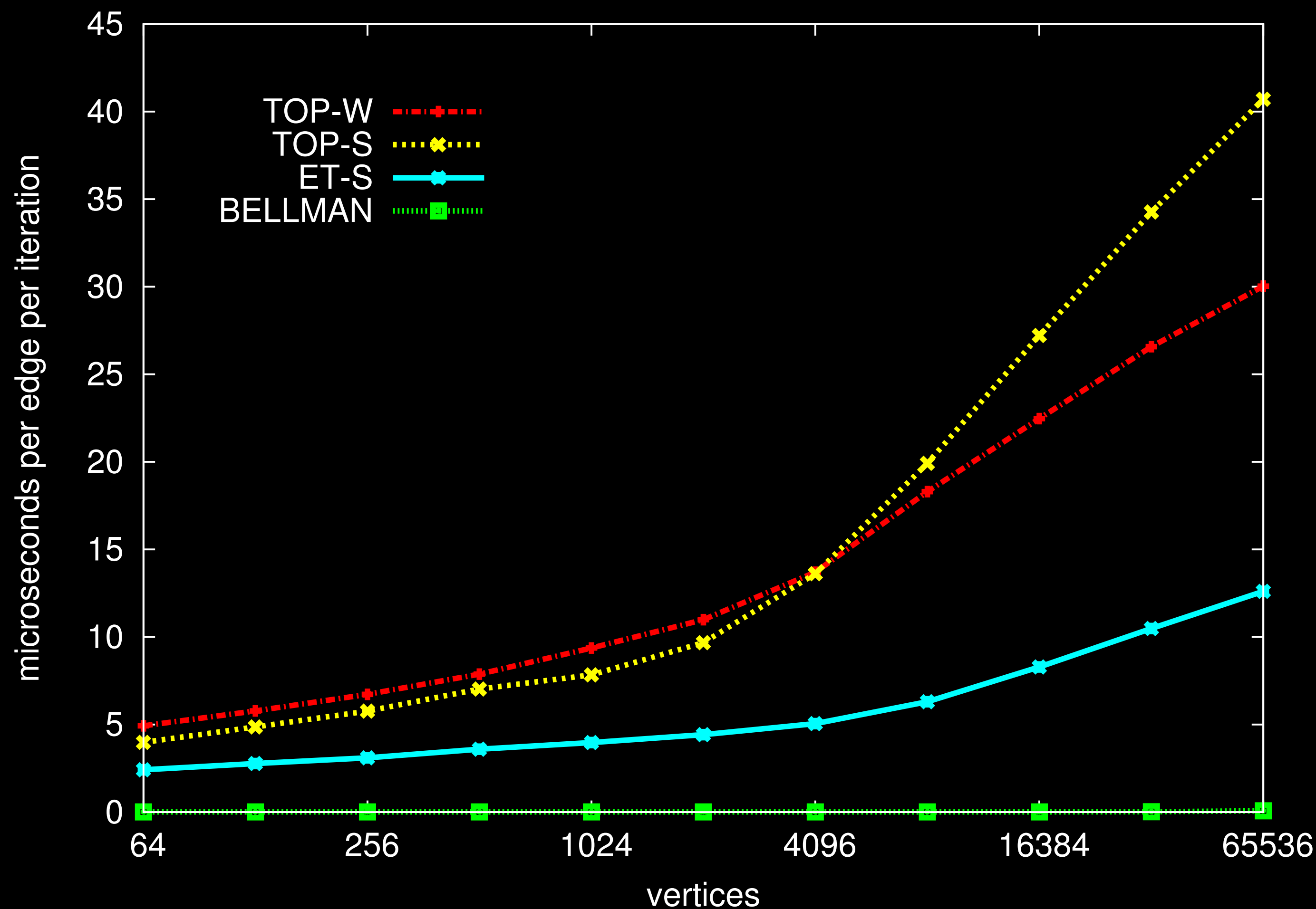
Experiment: Shortest Paths

- Random graph with Hamiltonian cycle:
  - n arcs on Hamiltonian cycle with weight in [1;10].
  - $3n$ random arcs with weight [1;1000].
  - Edges processed in random order.



Experiment: Shortest Paths

- Random graph with Hamiltonian cycle:
  - Dynamic trees reduce #iterations by 60% to 75%...
  - ...but increases running times by a factor of at least 50.



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Summary

- Main contributions:
  - new worst-case data structure;
  - new self-adjusting data structure:
    - uses contraction and path decomposition.
  - experimental analysis.

- Future work and open problems:
  - Worst-case data structure: do we really need Euler tours?
  - Self-adjusting data structure: can we make it worst-case?
  - Hybrid data structure?
  - Extend top trees?
  - Generalize top trees (grids, planar graphs, ...)?

Thank You

Main Approaches

	Top Trees	ST-trees
principle	tree contraction	path decomposition
general interface	**YES**	**NO**
unrestricted trees	**YES**	**NO**
representation	level-based	recursive
overhead	**HIGH**	**LOW**er