Parameterization of 2D and 3D Cusp–Core Simulations

1 Introduction

This document summarizes the mathematical formulation and parameter choices used in the 2D and 3D particle-on-grid simulations designed to study cusp–core formation. Both frameworks share the same physical principles but differ in dimensional scaling, smoothing kernels, and threshold criteria.

2 Governing Equations

The general curvature field $C(\mathbf{r}, t)$ evolves under particle contributions, smoothing, and repulsion terms:

$$C(\mathbf{r}, t+\Delta t) = C(\mathbf{r}, t) + \Delta t \left[\sum_i K_{\text{well}} f(|\mathbf{r} - \mathbf{r}_i|) - K_{\text{rep}} g(C) - K_{\text{decay}} C \right]. \tag{1}$$

Here:

- K_{well}: well strength coefficient.
- K_{rep}: repulsion coefficient.
- K_{decay}: damping/decay constant.
- $f(|\mathbf{r} - \mathbf{r}_i|)$: smoothing kernel centered on particle i.
- $g(C)$: nonlinear repulsion function.

2.1 2D Kernel

In two dimensions, the smoothing kernel is typically Gaussian-like:

$$f_{2D}(r) = \exp\left(-\frac{r^2}{2\sigma^2}\right), \tag{2}$$

with normalization

$$\int f_{2D}(r)\, d^2 r = 1. \tag{3}$$

2.2 3D Kernel

In three dimensions, the kernel generalizes to:

$$f_{3D}(r) = \exp\left(-\frac{r^2}{2\sigma^2}\right), \tag{4}$$

with normalization

$$\int f_{3D}(r)\, d^3r = 1. \tag{5}$$

Although the functional form matches, the scaling of normalization differs between 2D and 3D.

2.3 Units and Nondimensionalization

For consistency across simulations, the curvature field C is reported in dimensionless form. Introducing characteristic scales for length L_0, density ρ_0, and time T_0, we define

$$C_0 = \frac{\alpha\, \rho_0\, L_0^2}{D_C}, \qquad \tilde{C} = \frac{C}{C_0}, \qquad \tilde{\rho} = \frac{\rho}{\rho_0}. \tag{6}$$

The governing equation in dimensionless form becomes

$$\frac{\partial \tilde{C}}{\partial \tilde{t}} = \nabla^2_{\tilde{x}} \tilde{C} - \left(\frac{\lambda L_0^2}{D_C}\right) \tilde{C} + \tilde{\rho}, \tag{7}$$

so that the diffusion coefficient is unity. Maximum curvature is reported as $\tilde{C}_{\max}$, the nondimensional peak relative to C_0.

3 Thresholds for Well Formation

A central diagnostic of cusp–core behavior is the detection of curvature wells above a threshold C_{th}.

3.1 2D Criterion

A well is registered when

$$C_{\text{max, 2D}} > C_{\text{th, 2D}} \approx 0.8\, C_{\text{sat, 2D}}, \tag{8}$$

where $C_{\text{sat, 2D}}$ is the saturation curvature amplitude from repeated particle accumulation.

3.2 3D Criterion

Similarly, in 3D:

$$C_{\text{max, 3D}} > C_{\text{th, 3D}} \approx 0.8\, C_{\text{sat, 3D}}. \tag{9}$$

The threshold value differs numerically due to dimensional scaling.

4 Stochastic Smoothing

Both 2D and 3D formulations include a stochastic smoothing term to mimic thermal fluctuations:

$$C_{\text{smoothed}} = (1-\eta)C + \eta\,\xi, \tag{10}$$

where η is the noise strength and ξ is Gaussian white noise. This term prevents artificial sharp wells and encourages realistic diffusion of curvature.

5 Parameter Comparison

Parameter	2D Simulation	3D Simulation
Grid size	256×256	$128 \times 128 \times 128$
Particles	$N_{2D} \sim 10^4$	$N_{3D} \sim 10^5$
Kernel width σ	2–3 grid units	1–2 grid units
Well strength K_{well}	0.1–0.5	0.05–0.2
Repulsion K_{rep}	0.8–1.2	0.6–1.0
Decay K_{decay}	0.01–0.05	0.01–0.03
Threshold C_{th}	$0.8\ C_{\text{sat, 2D}}$	$0.8\ C_{\text{sat, 3D}}$
Noise strength η	0.02–0.05	0.01–0.03

6 Discussion

The 2D and 3D models reproduce the same qualitative cusp–core transition, despite differences in parameter scaling and kernel normalization. Both display saturation at $\sim 0.8C_{\text{sat}}$ and a flattening of central curvature profiles, consistent with observational interpretations of dwarf galaxy cores.