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Assignment Sheet Nr. 5

Exercise 1

1.1)

The system is given by:

$$\tau \dot{u}_1(t) = -u_1(t) + \frac{s_1}{1 + u_2(t)},$$
$$\tau \dot{u}_2(t) = -u_2(t) + \frac{s_2}{1 + u_1(t)}.$$

Here, $\tau > 0$ is a time constant, and $s_1, s_2 \geq 0$ are non-negative inputs. Let us analyze each equation separately.

For Case: $u_1(t) = 0$

If $u_1 = 0$, the first equation becomes:

$$\tau \dot{u}_1 = -0 + \frac{s_1}{1 + u_2}.$$

Since $s_1 \geq 0$ and $1 + u_2 > 0$, the right-hand side is non-negative:

$$\dot{u}_1 = \frac{s_1}{\tau(1 + u_2)} \geq 0.$$

This implies that if u_1 reaches 0, it will not decrease further; instead, u_1 will either remain at 0 or increase.

For Case: $u_2(t) = 0$

If $u_2 = 0$, the second equation becomes:

$$\tau \dot{u}_2 = -0 + \frac{s_2}{1 + u_1}.$$

Similarly, since $s_2 \geq 0$ and $1 + u_1 > 0$, the right-hand side is non-negative:

$$\dot{u}_2 = \frac{s_2}{\tau(1 + u_1)} \geq 0.$$

This implies that if u_2 reaches 0, it will not decrease further; instead, u_2 will either remain at 0 or increase.

From the above cases, we see that the dynamics prevent u_1 or u_2 from becoming negative:

- If $u_1 = 0$, $\dot{u}_1 \geq 0$, so u_1 cannot decrease below 0.

- If $u_2 = 0$, $\dot{u}_2 \geq 0$, so u_2 cannot decrease below 0.

Thus, starting from $u_1(0) \geq 0$ and $u_2(0) \geq 0$, the trajectories $u_1(t)$ and $u_2(t)$ remain non-negative for all $t \geq 0$.

Given that the positive quadrant $u_1 \geq 0$, $u_2 \geq 0$ is invariant under the dynamics of the system. Therefore, if $u_1(0) \geq 0$ and $u_2(0) \geq 0$, then $u_1(t) \geq 0$ and $u_2(t) \geq 0$ for all $t \geq 0$.

1.2)

To compute the fixed points of the system for $s_1 = s_2 = s \geq 0$, we solve for the steady-state conditions where $\dot{u}_1 = 0$ and $\dot{u}_2 = 0$. The system is given by:

$$\tau \dot{u}_1 = -u_1 + \frac{s}{1+u_2}, \quad \tau \dot{u}_2 = -u_2 + \frac{s}{1+u_1}.$$

At a fixed point, $\dot{u}_1 = 0$ and $\dot{u}_2 = 0$, so:

$$u_1 = \frac{s}{1+u_2}, \quad u_2 = \frac{s}{1+u_1}.$$

Substituting $u_1 = \frac{s}{1+u_2}$ into $u_2 = \frac{s}{1+u_1}$, we get:

$$u_2 = \frac{s}{1 + \frac{s}{1+u_2}}.$$

Simplifying the denominator:

$$u_2 = \frac{s}{\frac{1+u_2+s}{1+u_2}} = \frac{s(1+u_2)}{1+u_2+s}.$$

Rearranging:

$$u_2(1+u_2+s) = s(1+u_2),$$
$$u_2 + u_2^2 + su_2 = s + su_2.$$

Canceling su_2 from both sides:

$$u_2^2 + u_2 = s.$$

We solve for u_2:

$$u_2 = \frac{-1 \pm \sqrt{1+4s}}{2}.$$

Since $u_2 \geq 0$, we take the positive root:

$$u_2 = \frac{-1 + \sqrt{1+4s}}{2}.$$

Substituting $u_2 = \frac{-1+\sqrt{1+4s}}{2}$ into $u_1 = \frac{s}{1+u_2}$, we find:

$$u_1 = \frac{s}{1 + \frac{-1+\sqrt{1+4s}}{2}} = \frac{s}{\frac{1+\sqrt{1+4s}}{2}} = \frac{2s}{1+\sqrt{1+4s}}.$$

Thus, the fixed points are:

$$u_1 = \frac{2s}{1+\sqrt{1+4s}}, \quad u_2 = \frac{-1+\sqrt{1+4s}}{2}.$$

Now computing the two fixed points:

For Case: $s = 0$

For $s = 0$, substituting into the fixed point equations:

$$u_2 = \frac{-1 + \sqrt{1 + 4(0)}}{2} = \frac{-1 + 1}{2} = 0,$$

$$u_1 = \frac{2(0)}{1 + \sqrt{1 + 4(0)}} = 0.$$

The fixed point is:

$$(u_1, u_2) = (0, 0).$$

For Case: $s = \frac{3}{4}$

For $s = \frac{3}{4}$, substituting into the fixed point equations:

$$u_2 = \frac{-1 + \sqrt{1 + 4 \cdot \frac{3}{4}}}{2} = \frac{-1 + \sqrt{1 + 3}}{2} = \frac{-1 + 2}{2} = \frac{1}{2},$$

$$u_1 = \frac{2 \cdot \frac{3}{4}}{1 + \sqrt{1 + 4 \cdot \frac{3}{4}}} = \frac{\frac{3}{2}}{1 + 2} = \frac{\frac{3}{2}}{3} = \frac{1}{2}.$$

The fixed point is:

$$(u_1, u_2) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

1.3)

To analyze the stability of the system, we linearize it around the fixed points and compute the eigenvalues of the Jacobian matrix.

The system is given by:

$$\tau \dot{u}_1 = -u_1 + \frac{s}{1 + u_2}, \quad \tau \dot{u}_2 = -u_2 + \frac{s}{1 + u_1}.$$

Define:

$$f_1(u_1, u_2) = -u_1 + \frac{s}{1 + u_2}, \quad f_2(u_1, u_2) = -u_2 + \frac{s}{1 + u_1}.$$

The Jacobian matrix is:

$$J(u_1, u_2) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}.$$

The partial derivatives of f_1 and f_2 are:

$$\frac{\partial f_1}{\partial u_1} = -1, \quad \frac{\partial f_1}{\partial u_2} = -\frac{s}{(1 + u_2)^2},$$

$$\frac{\partial f_2}{\partial u_1} = -\frac{s}{(1 + u_1)^2}, \quad \frac{\partial f_2}{\partial u_2} = -1.$$

Thus, the Jacobian matrix becomes:

$$J(u_1, u_2) = \begin{bmatrix} -1 & -\frac{s}{(1+u_2)^2} \\ -\frac{s}{(1+u_1)^2} & -1 \end{bmatrix}.$$

For Case: $s = 0$, Fixed Point $(u_1, u_2) = (0, 0)$

For $s = 0$, substitute $u_1 = 0$ and $u_2 = 0$ into the Jacobian matrix:

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of this matrix are:

$$\lambda_1 = -1, \quad \lambda_2 = -1.$$

Since both eigenvalues are negative, the fixed point $(u_1, u_2) = (0, 0)$ is **stable**.

For Case: $s = \frac{3}{4}$, Fixed Point $\left(\frac{1}{2}, \frac{1}{2}\right)$

For $s = \frac{3}{4}$, substitute $u_1 = \frac{1}{2}$ and $u_2 = \frac{1}{2}$ into the Jacobian matrix:

$$J\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{bmatrix} -1 & -\frac{\frac{3}{4}}{(1+\frac{1}{2})^2} \\ -\frac{\frac{3}{4}}{(1+\frac{1}{2})^2} & -1 \end{bmatrix}.$$

Simplify:

$$1 + \frac{1}{2} = \frac{3}{2}, \quad \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

$$J\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{bmatrix} -1 & -\frac{\frac{3}{4}}{\frac{9}{4}} \\ -\frac{\frac{3}{4}}{\frac{9}{4}} & -1 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{3} \\ -\frac{1}{3} & -1 \end{bmatrix}.$$

The characteristic equation of this matrix is:

$$\det\left(\begin{bmatrix} -1-\lambda & -\frac{1}{3} \\ -\frac{1}{3} & -1-\lambda \end{bmatrix}\right) = 0.$$

Expanding the determinant:

$$(-1-\lambda)^2 - \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = 0,$$

$$(-1-\lambda)^2 - \frac{1}{9} = 0,$$

$$1 + 2\lambda + \lambda^2 - \frac{1}{9} = 0,$$

$$\lambda^2 + 2\lambda + \frac{8}{9} = 0.$$

Solving using the quadratic formula:

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot \frac{8}{9}}}{2},$$

$$\lambda = \frac{-2 \pm \sqrt{4 - \frac{32}{9}}}{2},$$

$$\lambda = \frac{-2 \pm \sqrt{\frac{36}{9} - \frac{32}{9}}}{2},$$
$$\lambda = \frac{-2 \pm \sqrt{\frac{4}{9}}}{2},$$
$$\lambda = \frac{-2 \pm \frac{2}{3}}{2}.$$

The eigenvalues are:

$$\lambda_1 = \frac{-2 + \frac{2}{3}}{2} = -\frac{4}{3}, \quad \lambda_2 = \frac{-2 - \frac{2}{3}}{2} = -\frac{8}{3}.$$

Since both eigenvalues are negative, the fixed point $\left(\frac{1}{2}, \frac{1}{2}\right)$ is **stable**.

1.4)

No math calculation is required for this part. Please, see the plot in the notebook.

1.5)

Assuming the following Lyapunov function:

$$V(u_1, u_2) = \frac{1}{2}\left(u_1^2 + u_2^2\right),$$

which is positive definite and zero at the fixed points.

Now we analyze its stability by computing the time derivative of $V(u_1, u_2)$:

$$\dot{V}(u_1, u_2) = \frac{\partial V}{\partial u_1}\dot{u}_1 + \frac{\partial V}{\partial u_2}\dot{u}_2.$$

Since:

$$\frac{\partial V}{\partial u_1} = u_1, \quad \frac{\partial V}{\partial u_2} = u_2,$$

we have:

$$\dot{V}(u_1, u_2) = u_1\dot{u}_1 + u_2\dot{u}_2.$$

Substituting the system equations:

$$\dot{u}_1 = -\frac{u_1}{\tau} + \frac{s}{\tau(1+u_2)}, \quad \dot{u}_2 = -\frac{u_2}{\tau} + \frac{s}{\tau(1+u_1)}.$$

Thus:

$$\dot{V}(u_1, u_2) = u_1\left(-\frac{u_1}{\tau} + \frac{s}{\tau(1+u_2)}\right) + u_2\left(-\frac{u_2}{\tau} + \frac{s}{\tau(1+u_1)}\right).$$

Simplifying, we get:

$$\dot{V}(u_1, u_2) = -\frac{1}{\tau}(u_1^2 + u_2^2) + \frac{s}{\tau}\left(\frac{u_1}{1+u_2} + \frac{u_2}{1+u_1}\right).$$

So for $\dot{V}$

- The term $-\frac{1}{\tau}(u_1^2 + u_2^2)$ is always non-positive.
- The term $\frac{s}{\tau}\left(\frac{u_1}{1+u_2} + \frac{u_2}{1+u_1}\right)$ is non-negative because $u_1, u_2 \geq 0$.

Here the sign of $\dot{V}$ is determined by the balance between these two terms.

Where $\dot{V}$ is Negative

The function $\dot{V}$ is negative definite if:

$$\frac{s}{\tau}\left(\frac{u_1}{1+u_2} + \frac{u_2}{1+u_1}\right) < \frac{1}{\tau}(u_1^2 + u_2^2).$$

This inequality holds for small values of u_1 and u_2, as the quadratic term $u_1^2 + u_2^2$ dominates the linear term $u_1 + u_2$ near the origin.

Therefore, we can conclude the following regrading Stability:

1. When $\dot{V} < 0$, the system's energy decreases, indicating movement toward the fixed point.
2. By **LaSalle's Invariance Principle**:
   - The system asymptotically approaches the set where $\dot{V} = 0$.
   - In this case, the set where $\dot{V} = 0$ corresponds to the fixed points (u_1, u_2).

Thus, the Lyapunov function confirms that the fixed points are **globally asymptotically stable** for $u_1, u_2 \geq 0$.

Exercise 2

2.1)

For a stationary solution, we assume $\dot{u}(x,t) = 0$, so the equation reduces to:

$$0 = -u(x) + \int_{-\infty}^{\infty} w(x - x')u(x')\,\mathrm{d}x' + s(x),$$

or equivalently:

$$u(x) = \int_{-\infty}^{\infty} w(x - x')u(x')\,\mathrm{d}x' + s(x).$$

Since the input signal is given as:

$$s(x) = \frac{1}{2\sqrt{\pi}d}\exp\left(-\frac{x^2}{4d^2}\right).$$

Substituting this into the equation, we get:

$$u(x) = \int_{-\infty}^{\infty} w(x - x')u(x')\,\mathrm{d}x' + \frac{1}{2\sqrt{\pi}d}\exp\left(-\frac{x^2}{4d^2}\right).$$

Given the interaction kernel:

$$w(x) = \frac{1}{\sqrt{\pi}b}a\left(\exp\left(-\frac{x^2}{4b^2}\right)\cos(k_0x)\right).$$

Thus, $w(x - x')$ becomes:

$$w(x - x') = \frac{1}{\sqrt{\pi}b}a\left(\exp\left(-\frac{(x - x')^2}{4b^2}\right)\cos(k_0(x - x'))\right).$$

Substituting $w(x - x')$ into the equation, we have:

$$u(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}b}a\left(\exp\left(-\frac{(x - x')^2}{4b^2}\right)\cos(k_0(x - x'))\right)u(x')\,\mathrm{d}x' + \frac{1}{2\sqrt{\pi}d}\exp\left(-\frac{x^2}{4d^2}\right).$$

Therefore, the stationary solution can be expressed as:

$$u(x) = \frac{a}{\sqrt{\pi}b}\int_{-\infty}^{\infty} \exp\left(-\frac{(x - x')^2}{4b^2}\right)\cos(k_0(x - x'))u(x')\,\mathrm{d}x' + \frac{1}{2\sqrt{\pi}d}\exp\left(-\frac{x^2}{4d^2}\right).$$

We observe here that:

- The stationary solution $u(x)$ depends on both the interaction kernel $w(x - x')$ and the input signal $s(x)$.
- Solving this equation explicitly may require numerical methods or additional assumptions about $u(x)$.

2.2)

The stationary equation from **2.1** is:

$$u(x) = \int_{-\infty}^{\infty} w(x - x')u(x')\, \mathrm{d}x' + s(x),$$

where:

$$s(x) = \frac{1}{2\sqrt{\pi}d} \exp\left(-\frac{x^2}{4d^2}\right),$$

and

$$w(x) = \frac{1}{\sqrt{\pi}b} a \exp\left(-\frac{x^2}{4b^2}\right) \cos(k_0 x).$$

Taking the Fourier transform $\mathcal{F}$ of both sides:

$$\tilde{u}(k) = \mathcal{F}[w(x) * u(x)](k) + \tilde{s}(k),$$

where $*$ denotes convolution. Using the convolution theorem:

$$\mathcal{F}[w(x) * u(x)](k) = \tilde{w}(k)\tilde{u}(k).$$

Thus, the Fourier transform of the stationary equation becomes:

$$\tilde{u}(k) = \tilde{w}(k)\tilde{u}(k) + \tilde{s}(k).$$

Rearranging:

$$\tilde{u}(k) = \frac{\tilde{s}(k)}{1 - \tilde{w}(k)}.$$

Fourier Transform of $s(x)$

The input signal $s(x)$ is:

$$s(x) = \frac{1}{2\sqrt{\pi}d} \exp\left(-\frac{x^2}{4d^2}\right).$$

This matches the Gaussian form:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

with $\sigma = \sqrt{2}d$. The Fourier transform of $f(x)$ is:

$$\tilde{f}(k) = \exp\left(-\frac{\sigma^2 k^2}{2}\right).$$

Thus:

$$\tilde{s}(k) = \exp\left(-d^2 k^2\right).$$

Fourier Transform of $w(x)$

The interaction kernel is:

$$w(x) = \frac{1}{\sqrt{\pi}b} a \exp\left(-\frac{x^2}{4b^2}\right) \cos(k_0 x).$$

Using $\cos(k_0 x) = \frac{1}{2}\left(e^{ik_0 x} + e^{-ik_0 x}\right)$, write:

$$w(x) = \frac{1}{\sqrt{\pi}b} a \exp\left(-\frac{x^2}{4b^2}\right) \cdot \frac{1}{2}\left(e^{ik_0 x} + e^{-ik_0 x}\right).$$

The Fourier transform of $w(x)$ is then:

$$\tilde{w}(k) = \frac{a}{2\sqrt{\pi}b} \mathcal{F}\left[\exp\left(-\frac{x^2}{4b^2}\right)\right] \cdot \left(\mathcal{F}\left[e^{ik_0 x}\right] + \mathcal{F}\left[e^{-ik_0 x}\right]\right).$$

(a) Fourier Transform of the Gaussian Term For $\exp(-x^2/(4b^2))$, set $\sigma = \sqrt{2}b$, so its Fourier transform is:

$$\mathcal{F}\left[\exp\left(-\frac{x^2}{4b^2}\right)\right] = \exp\left(-b^2 k^2\right).$$

(b) Fourier Transform of the Exponential Term Using the time-shift property:

$$\mathcal{F}\left[e^{ik_0 x} f(x)\right] = \tilde{f}(k - k_0),$$

the Fourier transform of $\exp(-x^2/(4b^2))\cos(k_0 x)$ becomes:

$$\mathcal{F}\left[\exp\left(-\frac{x^2}{4b^2}\right)\cos(k_0 x)\right] = \frac{1}{2}\left(\exp(-b^2(k-k_0)^2) + \exp(-b^2(k+k_0)^2)\right).$$

Thus:

$$\tilde{w}(k) = \frac{a}{\sqrt{\pi}b} \cdot \frac{1}{2}\left(\exp(-b^2(k-k_0)^2) + \exp(-b^2(k+k_0)^2)\right).$$

Therefore, the Fourier transform of the stationary solution is:

$$\tilde{u}(k) = \frac{\tilde{s}(k)}{1 - \tilde{w}(k)}.$$

Substituting $\tilde{s}(k)$ and $\tilde{w}(k)$, we get the solution in the frequency domain:

$$\tilde{u}(k) = \frac{\exp(-d^2 k^2)}{1 - \frac{a}{2\sqrt{\pi}b}\left(\exp(-b^2(k-k_0)^2) + \exp(-b^2(k+k_0)^2)\right)}.$$

To obtain $u^*(x)$ in the spatial domain, take the inverse Fourier transform of $\tilde{u}(k)$.

2.3)

The Fourier transform of the stationary solution is:

$$\tilde{u}(k) = \frac{\exp(-d^2k^2)}{1 - \frac{a}{2\sqrt{\pi}b}\left(\exp(-b^2(k-k_0)^2) + \exp(-b^2(k+k_0)^2)\right)}.$$

As $a \to 1^-$, the denominator approaches zero in certain regions of k-space, leading to resonances that dominate the behavior of $u(x)$. We analyze these contributions in the following cases.

For Case: $b \gg d$

When $b \gg d$, the interaction kernel $w(x)$ is much broader than the input signal $s(x)$ in the x-domain. The behavior in k-space is as follows:

(a) Small k Region ($k \approx 0$)

For small k, the Gaussian terms $\exp(-b^2(k-k_0)^2)$ and $\exp(-b^2(k+k_0)^2)$ in the denominator are approximately 1 because b^2 is large. Thus, the denominator becomes:

$$1 - \frac{a}{\sqrt{\pi}b}.$$

As $a \to 1^-$, this denominator approaches zero, causing a significant resonance at $k = 0$.

(b) Near $k = \pm k_0$

Around $k = \pm k_0$, the Gaussian terms $\exp(-b^2(k-k_0)^2)$ or $\exp(-b^2(k+k_0)^2)$ dominate, while the term for $k = 0$ becomes negligible. The denominator becomes:

$$1 - \frac{a}{2\sqrt{\pi}b} \cdot \exp(-b^2(k-k_0)^2),$$

or similarly for $k + k_0$. As $a \to 1^-$, this denominator also approaches zero, leading to strong resonances near $k = \pm k_0$.

Prediction for $u(x)$ in x-Domain

- The $k = 0$ resonance contributes a broad, slowly varying component to $u(x)$.
- The resonances near $k = \pm k_0$ introduce oscillatory components with wavelength $2\pi/k_0$, resulting in a pattern of regularly spaced peaks.
- Since $b \gg d$, the oscillatory term dominates due to the broad interaction kernel $w(x)$.

For Case: $d \gg b$

When $d \gg b$, the input signal $s(x)$ is much broader than the interaction kernel $w(x)$ in the x-domain. The behavior in k-space is as follows:

(a) Small k Region ($k \approx 0$)

For small k, as in the previous case, the Gaussian terms $\exp(-b^2(k-k_0)^2)$ and $\exp(-b^2(k+k_0)^2)$ are approximately 1. The denominator becomes:

$$1 - \frac{a}{\sqrt{\pi}b}.$$

As $a \to 1^-$, a resonance forms at $k = 0$, leading to a broad, smooth contribution in $u(x)$.

(b) Near $k = \pm k_0$

Around $k = \pm k_0$, the Gaussian terms $\exp(-b^2(k - k_0)^2)$ or $\exp(-b^2(k + k_0)^2)$ dominate. However, since b is small, these Gaussian terms decay very quickly for k away from k_0. This suppresses the resonance around $k = \pm k_0$, reducing the contribution of oscillatory components.

Prediction for $u(x)$ in x-Domain

- The $k = 0$ resonance dominates, producing a broad, slowly varying profile in $u(x)$ that follows the shape of $s(x)$.
- The oscillatory contributions are minimal because the narrow kernel $w(x)$ suppresses the influence of $k = \pm k_0$.

2.4)

No math calculation is required for this part. Please, see the plot in the notebook.

Exercise 3

3.1)

No math calculation is required for this part. Please, see the plot in the notebook.

3.2)

In the case with no excited region, we assume that initially $u(x) = h$ (the resting potential, which is $h = -1$) for all x. This corresponds to the assumption that no region of the field is excited, so $u(x') = h$ for all x'. Therefore, $1(u(x')) = 0$ for all x', since $u(x') = h = -1$ is below the threshold for activation (the threshold is typically 0, so $1(u(x')) = 0$ for $u(x') \leq 0$).

This simplifies the equation to:

$$u(x) = s(x) + h.$$

Since the input $s(x)$ is a piecewise linear function defined as:

$$s(x) = \begin{cases} C\left(1 - \frac{|x|}{d}\right), & \text{for } |x| \leq d, \\ 0, & \text{otherwise,} \end{cases}$$

we have:

$$u(x) = \begin{cases} C\left(1 - \frac{|x|}{d}\right) - 1, & \text{for } |x| \leq d, \\ -1, & \text{otherwise.} \end{cases}$$

This is the stationary solution when there is no excited region, and it shows a linear variation in $u(x)$ over the region where $|x| \leq d$, and a constant value of -1 elsewhere.

Why Does No Peak Arise Here?

A peak would arise if $u(x)$ exceeds a certain threshold (typically 0) in some region, activating that region and creating a localized high value. However, in this case, since the input $s(x)$ only has a maximum value of $C(1-\frac{|x|}{d})$, and the resting potential $h = -1$, the function $u(x)$ does not exceed the threshold anywhere. Specifically, for $|x| \leq d$, $u(x) = C(1-\frac{|x|}{d})-1$, and for this to exceed the threshold (which is 0), we need:

$$C\left(1 - \frac{|x|}{d}\right) - 1 > 0.$$

This condition does not hold for all x unless C is large enough to push $u(x)$ above 0 in some region.

Threshold Value for the Input Amplitude C

To find the threshold value of C needed for an activated region to emerge, we require $u(x) > 0$ for some x. The critical condition is:

$$C\left(1 - \frac{|x|}{d}\right) - 1 > 0.$$

Solving this inequality:

$$C\left(1 - \frac{|x|}{d}\right) > 1,$$

$$C > \frac{1}{1 - \frac{|x|}{d}}.$$

At the boundary of the excited region, when $|x| = 0$, we have:

$$C > 1.$$

Thus, for $C > 1$, $u(x)$ will exceed 0 at the center of the field (at $x = 0$), and the region around $x = 0$ will be activated. This will give rise to a peak in the field. If $C \leq 1$, the solution will remain at the resting potential $u(x) = -1$, and no activated region will emerge.

3.3)

No math calculation is required for this part. Please, see the plot in the notebook.

End of Solutions