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Assignment Sheet Nr. 1

Exercise 1

1.1)

Since at equilibrium, there is no net flux of ions ($J_{\mathrm{A}} = 0$). Thus, we set the Nernst-Planck equation to zero:

$$0 = -D_{\mathrm{A}} \left(\frac{\mathrm{d}[\mathrm{A}]}{\mathrm{d}x} + \frac{z_{\mathrm{A}} F}{RT} [\mathrm{A}] \frac{\mathrm{d}V}{\mathrm{d}x} \right) \tag{1}$$

Dividing both sides by $-D_{\mathrm{A}}$ gives:

$$\frac{\mathrm{d}[\mathrm{A}]}{\mathrm{d}x} = -\frac{z_{\mathrm{A}} F}{RT} [\mathrm{A}] \frac{\mathrm{d}V}{\mathrm{d}x} \tag{2}$$

Rearranging the equation to separate variables:

$$\frac{1}{[\mathrm{A}]} \frac{\mathrm{d}[\mathrm{A}]}{\mathrm{d}x} = -\frac{z_{\mathrm{A}} F}{RT} \frac{\mathrm{d}V}{\mathrm{d}x} \tag{3}$$

Now integrating both sides with respect to x across the membrane, from the inside ($x = 0$) to the outside ($x =$ membrane thickness):

$$\int_{[\mathrm{A}]_{\mathrm{in}}}^{[\mathrm{A}]_{\mathrm{out}}} \frac{1}{[\mathrm{A}]} \, \mathrm{d}[\mathrm{A}] = -\frac{z_{\mathrm{A}} F}{RT} \int_{V_{\mathrm{in}}}^{V_{\mathrm{out}}} \mathrm{d}V \tag{4}$$

The left integral with respect to [A] is:

$$\int_{[\mathrm{A}]_{\mathrm{in}}}^{[\mathrm{A}]_{\mathrm{out}}} \frac{1}{[\mathrm{A}]} \, \mathrm{d}[\mathrm{A}] = \ln \left(\frac{[\mathrm{A}]_{\mathrm{out}}}{[\mathrm{A}]_{\mathrm{in}}} \right) \tag{5}$$

The right integral with respect to V is:

$$\int_{V_{\mathrm{in}}}^{V_{\mathrm{out}}} \mathrm{d}V = \underline{V_{\mathrm{out}} - V_{\mathrm{in}}} = E_m \tag{6}$$

where E_m is the membrane potential difference. $V_{in} - V_{out} = E_m$

Thus, we have:

$$\ln \left(\frac{[\mathrm{A}]_{\mathrm{out}}}{[\mathrm{A}]_{\mathrm{in}}} \right) = -\frac{z_{\mathrm{A}} F}{RT} E_m \tag{7}$$

Now multiplying both sides by $-\frac{RT}{z_{\mathrm{A}} F}$ to isolate E_m, we get:

$$E_m = \frac{RT}{z_{\mathrm{A}} F} \ln \left(\frac{[\mathrm{A}]_{\mathrm{out}}}{[\mathrm{A}]_{\mathrm{in}}} \right) \tag{8}$$

which is the Nernst equation:

$$E_m = \frac{RT}{z_{\mathrm{A}} F} \log \left(\frac{[\mathrm{A}]_{\mathrm{out}}}{[\mathrm{A}]_{\mathrm{in}}} \right) \tag{2}$$

1.2)

In the case where the membrane is selectively permeable only to one type of monovalent anion A^-, with permeability P_A, and all other permeabilities are zero, the GHK equation simplifies as follows:

- For cations C_i^+, the permeability terms $P_{C_i^+}$ are zero, so their contributions vanish.
- For anions A_j^-, only the permeability P_A for A^- survives.

Thus, the GHK equation becomes:

$$E_m = \frac{RT}{F} \log\left(\frac{P_A[A^-]_{in}}{P_A[A^-]_{out}}\right)$$

Since the permeability P_A is the same on both sides of the membrane, it cancels out. Therefore, the equation simplifies to:

$$E_m = \frac{RT}{F} \log\left(\frac{[A^-]_{in}}{[A^-]_{out}}\right)$$

Since the Nernst equation for an ion X, whether a cation or anion, is typically written as:

$$E_m = \frac{RT}{zF} \log\left(\frac{[X]_{out}}{[X]_{in}}\right)$$

where z is the valency of the ion.

Since, for a monovalent anion A^-, the valency $z = -1$. Therefore, we modify the equation as follows:

$$E_m = -\frac{RT}{F} \log\left(\frac{[A^-]_{out}}{[A^-]_{in}}\right)$$

Exercise 2

2.1)

Since the total current across the membrane, I_{total}, is the sum of the currents carried by each ion:

$$I_{\text{total}} = I_{\text{K}^+} + I_{\text{Na}^+}.$$

And since each ionic current I_{ion} can be expressed according to Ohm's law as:

$$I_{\text{ion}} = g_{\text{ion}}(E_m - E_{\text{ion}}),$$

where:

- g_{ion} is the conductance of the ion,
- E_{ion} is the reversal potential of the ion, and
- E_m is the membrane potential.

Therefore, for two ions, K^+ and Na^+, we have:

$$I_{\text{K}^+} = g_{\text{K}^+}(E_m - E_{\text{K}^+}),$$

$$I_{\text{Na}^+} = g_{\text{Na}^+}(E_m - E_{\text{Na}^+}).$$

Now, given that at steady state (when E_m is stable), there is no net current across the membrane, so $I_{\text{total}} = 0$. Thus,

$$I_{\text{K}^+} + I_{\text{Na}^+} = 0.$$

Substituting the expressions for I_{K^+} and I_{Na^+}:

$$g_{\text{K}^+}(E_m - E_{\text{K}^+}) + g_{\text{Na}^+}(E_m - E_{\text{Na}^+}) = 0.$$

Expanding and grouping terms involving E_m gives:

$$g_{\text{K}^+}E_m - g_{\text{K}^+}E_{\text{K}^+} + g_{\text{Na}^+}E_m - g_{\text{Na}^+}E_{\text{Na}^+} = 0.$$

Combining terms with E_m on the left side:

$$E_m(g_{\text{K}^+} + g_{\text{Na}^+}) = g_{\text{K}^+}E_{\text{K}^+} + g_{\text{Na}^+}E_{\text{Na}^+}.$$

Solving for E_m, we get:

$$E_m = \frac{g_{\text{K}^+}E_{\text{K}^+} + g_{\text{Na}^+}E_{\text{Na}^+}}{g_{\text{K}^+} + g_{\text{Na}^+}}.$$

2.2)

Given the membrane equation,

$$I_e(t) = C_m \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V - E_m}{R_m},$$

where:

- $I_e(t)$ is the input current,
- C_m is the membrane capacitance,
- R_m is the membrane resistance,
- E_m is the resting membrane potential.

and given that the current $I_e(t)$ is:

$$I_e(t) = \begin{cases} 0 & t \le t_0, \\ I_0 & t > t_0. \end{cases}$$

$\boxed{\textbf{For } \mathrm{t} \le \mathrm{t}_0}$

i.e., before t_0, there is no applied current, and the membrane potential is at rest: $V(t) = E_m$.

$\boxed{\textbf{For } \mathrm{t} > \mathrm{t}_0}$

when $I_e(t) = I_0$, the equation becomes:

$$C_m \frac{\mathrm{d}V}{\mathrm{d}t} + \frac{V}{R_m} = \frac{E_m}{R_m} + I_0.$$

If the current I_0 were applied indefinitely, $V(t)$ would eventually reach a constant value V_∞ such that $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$. Setting $\frac{\mathrm{d}V}{\mathrm{d}t} = 0$, we find:

$$\frac{V_\infty}{R_m} = \frac{E_m}{R_m} + I_0.$$

Solving for V_∞:

$$V_\infty = E_m + I_0 R_m.$$

Now rewruiting the equation in terms of V_∞:

$$C_m \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{V_\infty - V}{R_m}.$$

Dividing by C_m:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{R_m C_m}(V - V_\infty).$$

Since membrane time constant τ_m is defined as:

$$\tau_m = R_m C_m.$$

Therefore, the differentual equation equation becomes:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{\tau_m}(V - V_\infty).$$

which has a general solution:

$$V(t) = V_\infty + (V(t_0) - V_\infty)e^{-\frac{(t-t_0)}{\tau_m}}.$$

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Given that $V(t_0) = E_m$, by substituting $V(t_0) = E_m$ and $V_\infty = E_m + I_0 R_m$ we obtain:

$$V(t) = E_m + I_0 R_m \left(1 - e^{-\frac{(t-t_0)}{\tau_m}}\right) \quad \text{for } t > t_0.$$

2.3)

Because the network is time-invariant, the solution to the input current $I_e(t)$ can be obtained by applying the previous result in a piecewise fashion.

$\boxed{\textbf{For} \ \ \mathrm{t} < 0}$

Before the current is applied (i.e., when $t < 0$), there is no external current ($I_e(t) = 0$). The membrane potential is just the resting potential:

$$V(t) = E_m \quad \text{for } t < 0.$$

$\boxed{\textbf{For } 0 \leq t \leq t_e}$

Here, the current $I_e(t) = I_0$ is applied. We can use the previous result, but we need to adjust it for the fact that the current starts at $t = 0$ (i.e., $t_0 = 0$) and lasts for a finite duration t_e. The membrane potential starts to change from the resting potential at $t = 0$, and the response follows:

$$V(t) = E_m + I_0 R_m \left(1 - e^{-\frac{t}{\tau_m}}\right) \quad \text{for } 0 \leq t \leq t_e.$$

$\boxed{\textbf{For } \mathrm{t} > \mathrm{t}_e}$

After $t = t_e$, the external current is turned off ($I_e(t) = 0$ for $t > t_e$). The membrane potential will start to decay from the value it reached at $t = t_e$ according to the time constant τ_m, and will eventually return to the resting potential E_m.

At $t = t_e$, the membrane potential is:

$$V(t_e) = E_m + I_0 R_m \left(1 - e^{-\frac{t_e}{\tau_m}}\right).$$

And the potential will decay from this value according to the equation for a decaying exponential response, starting at $t = t_e$:

$$V(t) = V(t_e) \cdot e^{-\frac{(t - t_e)}{\tau_m}} + E_m.$$

Substituting the expression for $V(t_e)$, we get:

$$V(t) = \left[E_m + I_0 R_m \left(1 - e^{-\frac{t_e}{\tau_m}}\right)\right] e^{-\frac{(t - t_e)}{\tau_m}} + E_m \quad \text{for } t > t_e.$$

Therefore, the complete response for $V(t)$ to the input current $I_e(t)$ is:

$$V(t) = \begin{cases} E_m & t < 0, \\ E_m + I_0 R_m \left(1 - e^{-\frac{t}{\tau_m}}\right) & 0 \leq t \leq t_e, \\ \left[E_m + I_0 R_m \left(1 - e^{-\frac{t_e}{\tau_m}}\right)\right] e^{-\frac{(t - t_e)}{\tau_m}} + E_m & t > t_e. \end{cases}$$

Note: The $+E_m$ term ensures that as $t \to \infty$, $V(t)$ asymptotically approaches E_m. Without this term, the potential would decay toward zero, which wouldn't match the behavior of a real membrane that stabilizes at the resting potential E_m when no external current is applied.

End of Solutions