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Assignment Sheet Nr. 2

Exercise 1

1.1) R_a

Given the formula for axial resistance:

$$R_a = \frac{\tilde{r}_a \cdot L}{\pi \left(\frac{d}{2}\right)^2}$$

For Axon 1

$L = 10\,\text{mm} = 10 \times 10^{-3}\,\text{m}$

$d = 2\,\mu\text{m} = 2 \times 10^{-6}\,\text{m}$

Therefore,

$$R_a = \frac{1 \cdot (10 \times 10^{-3})}{\pi \left(\frac{2\times10^{-6}}{2}\right)^2} = \frac{10 \times 10^{-3}}{\pi \times (1 \times 10^{-6})^2} = \frac{10 \times 10^{-3}}{\pi \times 10^{-12}}$$

$$R_a \approx 3.183 \times 10^9\,\Omega\,\text{m}$$

For Axon 2

$L = 20\,\text{mm} = 20 \times 10^{-3}\,\text{m}$

$d = 2\,\mu\text{m} = 2 \times 10^{-6}\,\text{m}$

Therefore,

$$R_a = \frac{1 \cdot (20 \times 10^{-3})}{\pi \left(\frac{2\times10^{-6}}{2}\right)^2} = \frac{20 \times 10^{-3}}{\pi \times (1 \times 10^{-6})^2} = \frac{20 \times 10^{-3}}{\pi \times 10^{-12}}$$

$$R_a \approx 6.366 \times 10^9\,\Omega\,\text{m}$$

For Axon 3

$L = 10\,\text{mm} = 10 \times 10^{-3}\,\text{m}$

$d = 4\,\mu\text{m} = 4 \times 10^{-6}\,\text{m}$

$$R_a = \frac{1 \cdot (10 \times 10^{-3})}{\pi \left(\frac{4\times10^{-6}}{2}\right)^2} = \frac{10 \times 10^{-3}}{\pi \times (2 \times 10^{-6})^2} = \frac{10 \times 10^{-3}}{\pi \times 4 \times 10^{-12}}$$

$$R_a \approx 7.96 \times 10^8\,\Omega\,\text{m}$$

1.2) C_m

Given the formula for membrane capacitance:

$$C_m = \tilde{c}_m \cdot \pi d \cdot L$$

For Axon 1

$d = 2\,\mu\text{m} = 2 \times 10^{-6}\,\text{m}$

$L = 10\,\text{mm} = 10 \times 10^{-3}\,\text{m}$:

Therefore,

$$C_m = 10^{-2} \times \pi \times (2 \times 10^{-6}) \times (10 \times 10^{-3}) = 6.2832 \times 10^{-10}\,\text{F}$$

For Axon 2

$d = 2\,\mu\text{m} = 2 \times 10^{-6}\,\text{m}$

$L = 20\,\text{mm} = 20 \times 10^{-3}\,\text{m}$: Therefore,

$$C_m = 10^{-2} \times \pi \times (2 \times 10^{-6}) \times (20 \times 10^{-3}) = 1.25664 \times 10^{-9}\,\text{F}$$

For Axon 3

$d = 4\,\mu\text{m} = 4 \times 10^{-6}\,\text{m}$

$L = 10\,\text{mm} = 10 \times 10^{-3}\,\text{m}$: Therefore,

$$C_m = 10^{-2} \times \pi \times (4 \times 10^{-6}) \times (10 \times 10^{-3}) = 1.25664 \times 10^{-9}\,\text{F}$$

1.3) τ

To calculate τ we first need to calculate R_m, so given the formula for membrane resistance:

$$R_m = \frac{\tilde{r}_m}{\pi d}$$

For Axon 1

$d = 2\,\mu\text{m} = 2 \times 10^{-6}\,\text{m}$

Therefore,

$$R_m = \frac{1}{\pi \times (2 \times 10^{-6})} = 1.5915 \times 10^{5}\,\Omega$$

For Axon 2

$d = 2\,\mu\text{m} = 2 \times 10^{-6}\,\text{m}$

Therefore,

$$R_m = \frac{1}{\pi \times (2 \times 10^{-6})} = 1.5915 \times 10^{5}\,\Omega$$

For Axon 3

$d = 4\,\mu\text{m} = 4 \times 10^{-6}\,\text{m}$

Therefore,

$$R_m = \frac{1}{\pi \times (4 \times 10^{-6})} = 7.9577 \times 10^4\,\Omega$$

Now we can calculate τ given the formula:

$$\tau = C_m \times R_m$$

For Axon 1

$$\tau = 6.2832 \times 10^{-10} \times 1.5915 \times 10^5 = 0.000099997128 \approx 1 \times 10^{-4}\,\text{s}$$

For Axon 2

$$\tau = 1.25664 \times 10^{-9} \times 1.5915 \times 10^5 = 0.000199994256 \approx 2 \times 10^{-4}\,\text{s}$$

For Axon 3

$$\tau = 1.25664 \times 10^{-9} \times 7.9577 \times 10^4 = 0.00009999964128 \approx 1 \times 10^{-4}\,\text{s}$$

1.4) Implications

In the context of axons 1, 2, and 3, the axial resistance R_a, membrane capacitance C_m, and time constant τ are all influenced by the length L and diameter d of the axon. Axon 2, with a longer length, exhibits a higher axial resistance and membrane capacitance, leading to a longer time constant, which results in slower signal propagation. Axon 3, with a larger diameter, has a lower axial resistance, but the increased capacitance balances out, keeping the time constant similar to Axon 1, allowing for more efficient signal transmission. This implies that overall, axons with longer lengths tend to slow down signal propagation because of the increased resistance and capacitance, which result in longer time constants, but larger diameters tends to improve signal propagation by reducing axial resistance, which lowers the time constant.

Exercise 2

We start by looking at the steady-state cable equation which is given by:

$$\frac{d^2V}{dx^2} = \frac{V}{\lambda^2},$$

where:

- $V(x)$ is the voltage at position x,
- λ is the length constant of the cable.

This is a second-order linear differential equation, and cable equation can be rewritten as:

$$\frac{d^2V}{dx^2} - \frac{1}{\lambda^2}V = 0.$$

The characteristic equation for this differential equation is:

$$r^2 - \frac{1}{\lambda^2} = 0,$$

which has roots:

$$r = \pm\frac{1}{\lambda}.$$

Thus, the general solution to the differential equation is:

$$V(x) = Ae^{x/\lambda} + Be^{-x/\lambda},$$

where A and B are constants to be determined from the boundary conditions.

The boundary conditions are:

1. A constant current is injected at $x = 0$: $I(0) = I_0$.
2. The voltage is clamped at $x = l$: $V(l) = V_l$.

Boundary Condition 1: Current at $x = 0$

At $x = 0$, a constant current I_0 is injected. Using Ohm's law, the current at any position x in the cable is related to the voltage gradient:

$$I(x) = -g\lambda^2\frac{dV}{dx},$$

where g is the conductance per unit length of the cable.

At $x = 0$, we have:

$$I(0) = -g\lambda^2\frac{dV}{dx}\Big|_{x=0} = I_0.$$

Substituting the derivative of $V(x)$ from the general solution:

$$\frac{dV}{dx} = \frac{A}{\lambda}e^{x/\lambda} - \frac{B}{\lambda}e^{-x/\lambda}.$$

At $x = 0$:

$$\frac{dV}{dx}\Big|_{x=0} = \frac{A}{\lambda} - \frac{B}{\lambda}.$$

Equating this to $-\frac{I_0}{g\lambda^2}$:

$$\frac{A}{\lambda} - \frac{B}{\lambda} = -\frac{I_0}{g\lambda^2}.$$

Simplifying, we get:

$$\boxed{A - B = -\frac{I_0}{g\lambda}} \tag{1}$$

This is the first equation.

Boundary Condition 2: Voltage at $x = l$

At $x = l$, the voltage is clamped to V_l. From the general solution:

$$V(l) = Ae^{l/\lambda} + Be^{-l/\lambda}.$$

Substituting $V(l) = V_l$, we get:

$$\boxed{Ae^{l/\lambda} + Be^{-l/\lambda} = V_l} \tag{2}$$

This is the second equation.

Solving for A and B

First we substitute $A = B - \frac{I_0}{g\lambda}$ into $Ae^{l/\lambda} + Be^{-l/\lambda} = V_l$:

$$\left(B - \frac{I_0}{g\lambda}\right)e^{l/\lambda} + Be^{-l/\lambda} = V_l.$$

Expand:

$$Be^{l/\lambda} - \frac{I_0}{g\lambda}e^{l/\lambda} + Be^{-l/\lambda} = V_l.$$

Combine terms with B:

$$B\left(e^{l/\lambda} + e^{-l/\lambda}\right) = V_l + \frac{I_0}{g\lambda}e^{l/\lambda}.$$

Simplify using the hyperbolic cosine identity:

$$e^{l/\lambda} + e^{-l/\lambda} = 2\cosh(l/\lambda).$$

Thus:

$$B \cdot 2\cosh(l/\lambda) = V_l + \frac{I_0}{g\lambda}e^{l/\lambda}.$$

Solve for B:

$$B = \frac{V_l + \frac{I_0}{g\lambda}e^{l/\lambda}}{2\cosh(l/\lambda)}.$$

Solving for A

Using $A = B - \frac{I_0}{g\lambda}$:

$$A = \frac{V_l + \frac{I_0}{g\lambda}e^{l/\lambda}}{2\cosh(l/\lambda)} - \frac{I_0}{g\lambda}.$$

Substituting A and B back into the general solution:

$$V(x) = Ae^{x/\lambda} + Be^{-x/\lambda}.$$

we get:

$$V(x) = \left(\frac{V_l + \frac{I_0}{g\lambda} e^{l/\lambda}}{2\cosh(l/\lambda)} - \frac{I_0}{g\lambda} \right) e^{x/\lambda} + \frac{V_l + \frac{I_0}{g\lambda} e^{l/\lambda}}{2\cosh(l/\lambda)} e^{-x/\lambda}.$$

This is the steady-state voltage distribution $V(x)$ along the cable.

End of Solutions