A Groupoidal Natural Model of HoTT in Lean 4

Sina Hazratpour, Joseph Hua

August 23, 2024

0.1 Natural Models

0.1.1 Types

Assume an inaccessible cardinal λ . Write **Set** for the category of all sets. Say that a set A is λ -small if $|A| < \lambda$. Write **Set** $_{\lambda}$ for the full subcategory of **Set** spanned by λ -small sets.

Let $\mathbb C$ be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write $\mathbf{Psh}(\mathbb{C})$ for the category of presheaves over \mathbb{C} ,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $tp \colon Tm \to Ty$ consists of

- contexts as objects $\Gamma, \Delta, ... \in \mathbb{C}$,
- a type in context $y(\Gamma)$ as a map $A: y(\Gamma) \to \mathsf{Ty}$,
- a term of type A in context Γ as a map $a: y(\Gamma) \to Tm$ such that



commutes,

• an operation called "context extension" which given a context Γ and a type $A\colon \mathsf{y}(\Gamma)\to \mathsf{T}\mathsf{y}$ produces a context $\Gamma\cdot A$ which fits into a pullback diagram below.

$$\begin{array}{cccc} \mathbf{y}(\Gamma.A) & & & \mathsf{Tm} \\ & & & & \downarrow \\ & & & \downarrow \\ \mathbf{y}(\Gamma) & & & & \mathsf{Ty} \end{array}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty. We write

$$\begin{matrix} X & \longrightarrow \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{y}(\Gamma) & \xrightarrow{r_X} \mathsf{Ty} \end{matrix}$$

to express this situation, i.e. $X \cong y(\Gamma \cdot \lceil X \rceil)$.

0.1.2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$U : \mathsf{Ty} \qquad \frac{a \colon \mathsf{U}}{\mathsf{El}(a) \colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ , with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{ccc}
\mathsf{U} & \longrightarrow \mathsf{Tm} \\
\downarrow & & \downarrow \\
1 & \longrightarrow \mathsf{Ty}
\end{array}$$

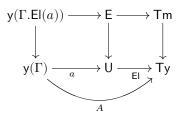
• There is an inclusion of U into Ty

$$\mathsf{EI} \colon \mathsf{U} \rightarrowtail \mathsf{Ty}$$

• $\pi: \mathsf{E} \to \mathsf{U}$ is obtained as pullback of tp ; There is a pullback diagram

$$E \rightarrowtail \mathsf{Tm} \downarrow \downarrow \downarrow \\ \mathsf{U} \rightarrowtail \mathsf{Ty}$$

With the notation above, we get



Both squares above are pullback squares.

0.1.3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

0.2 The Groupoid Model

In this section we construct a natural model in **Psh(grpd)** the presheaf category indexed by the category **grpd** of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on **grpd** - which comes from restricting the "classical Quillen model structure" on **cat** [Joy] to **grpd**.

0.2.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as **set** and the large sets as **Set** (in the previous sections this would have been \mathbf{Set}_{λ} and \mathbf{Set} respectively). We denote the category of small categories as **cat** and the large categories as **Cat**. We denote the category of small groupoids as \mathbf{grpd} .

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$Psh(grpd) = [grpd^{op}, Set]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

Definition 0.2.1 (Pointed). We will take the category of pointed small categories \mathbf{cat}_{\bullet} to have objects as pairs $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$ and morphisms as pairs

$$(F:\mathbb{C}_1 \to \mathbb{C}_0, \phi: Fc_1 \to c_0) \colon (\mathbb{C}_1, c_1) \to (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids \mathbf{grpd}_{\bullet} will be the full subcategory of objects (Γ, c) with Γ a groupoid.

Definition 0.2.2 (The display map classifier). We would like to define a natural transformation in $\mathbf{Psh}(\mathbf{grpd})$

$$\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$$

with representable fibers.

Consider the functor that forgets the point

$$U \colon \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$$
 in **Cat**.

If we apply the Yoneda embedding $y \colon \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U \circ : [-, \mathbf{grpd}_{\bullet}] \to [-, \mathbf{grpd}]$$
 in $\mathbf{Psh}(\mathbf{Cat})$.

Since any small groupoid is also a large category $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict \mathbf{Cat} indexed presheaves to be \mathbf{grpd} indexed presheaves. We define $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$ as the image of $U \circ \mathsf{under}$ this restriction.

$$\begin{array}{ccc} \mathbf{Cat} & \stackrel{\mathsf{y}}{\longrightarrow} \mathbf{Psh}(\mathbf{Cat}) & \stackrel{\mathsf{res}}{\longrightarrow} \mathbf{Psh}(\mathbf{grpd}) \\ \\ \mathbf{grpd} & \longmapsto [-,\mathbf{grpd}] & \longmapsto \mathsf{Ty} \end{array}$$

Note that Tm and Ty are not representable in Psh(grpd).

 $Remark\ 0.2.3.$ By Yoneda we can identify maps with representable domain into the type classifier

$$A: \mathsf{y}\Gamma \to \mathsf{T}\mathsf{y} \qquad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A:\Gamma \to \mathbf{grpd}$$
 in **Cat**

Definition 0.2.4 (Grothendieck construction). From $\mathbb C$ a small category and $F: \mathbb C \to \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in $\mathbb C$ we refer to Fc as the fiber over c. The objects of $\int F$ consist of pairs $(c \in \mathbb C, x \in Fc)$, and morphisms between (c,x) and (d,y) are pairs $(f:c\to d,\phi:Ffx\to y)$. This makes the following pullback in \mathbf{Cat}

$$(c,x)\,\longmapsto\,(Fc,x)$$

$$\begin{array}{ccccc} (c,x) & & \int F & \longrightarrow \mathbf{cat}_{\bullet} & & (C,c) \\ & \downarrow & & \downarrow & & \downarrow \\ c & & \mathbb{C} & \longrightarrow_{F} & \mathbf{cat} & & C \\ \end{array}$$

Definition 0.2.5 (Grothendieck construction for groupoids). Let Γ be a groupoid and $A \colon \Gamma \to \mathbf{grpd}$ a functor, we can compose F with the inclusion $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \operatorname{disp}_A \colon \Gamma \cdot A \to \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids Γ and Ax for $x \in \Gamma$. Furthermore the pullback factors through (pointed) groupoids.

$$egin{array}{cccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{ullet} & \longrightarrow \mathbf{cat}_{ullet} \ \operatorname{disp}_A & & & \downarrow & & \downarrow \ \Gamma & \longrightarrow_A & \mathbf{grpd} & \longrightarrow \mathbf{cat} \end{array}$$

Corollary 0.2.6 (The display map classifier is presentable). For any small groupoid Γ and $A: y\Gamma \to Ty$, the pullback of tp along A can be given by the representable map $ydisp_A$.

$$\begin{array}{ccc} \mathsf{y}\Gamma \cdot A & \longrightarrow & \mathsf{Tm} \\ \mathsf{ydisp}_A & & & \mathsf{tp} \\ & \mathsf{y}\Gamma & \longrightarrow_A & \mathsf{Ty} \end{array}$$

Proof. Consider the pullback in Cat

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow \mathbf{grpd} \end{array}$$

We send this square along res o y in the following

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) \\ & & & \downarrow_{\mathsf{res}} \\ \mathbf{grpd} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{grpd}) \end{array}$$

The Yoneda embedding $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does res since it is a right adjoint (with left Kan extension $\iota_1 \dashv \mathsf{res}$,).

0.2.2 Groupoid fibrations

Definition 0.2.7 (Fibration). Let $p:\mathbb{C}_1\to\mathbb{C}_0$ be a functor. We say p is a split Grothendieck fibration if we have a dependent function lift a f satisfying the following: for any object a in \mathbb{C}_1 and morphism f:p $a\to y$ in the base \mathbb{C}_0 we have lift a $f:a\to b$ in \mathbb{C}_1 such that p(lift af)=f and moreover lift a $g\circ f=\text{lift }b$ $g\circ \text{lift }af$

$$\begin{array}{ccc}
a & \xrightarrow{\text{lift } a f} & b \\
\downarrow & & \uparrow & \downarrow \\
x & \xrightarrow{f} & y
\end{array}$$

In particular, we are intereseted in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid Γ as Fib_{Γ} , which is a full subcategory of the slice grpd/Γ . We will decorate an arrow with \twoheadrightarrow to indicate it is a fibration.

Note that $\operatorname{\sf disp}_A\colon \Gamma\cdot A\to \Gamma$ is a fibration, since for any $(x\in\Gamma, a\in A\,x)$ and $f\colon x\to y$ in Γ we have a morphism $(f,\operatorname{\sf id}_{A\,f\,a})\colon (x,a)\to (y,A\,f\,a)$ lifting f. Furthermore

Proposition 0.2.8. There is an adjoint equivalence

$$[\Gamma,\mathbf{grpd}] \xrightarrow[\text{fiber}]{\text{disp}} \operatorname{Fib}_{\Gamma}$$

where for each fibration $\delta: \Delta \to \Gamma$ and each object $x \in \Gamma$

$$fiber_{\delta} x = full subcategory \{ a \in \Delta \mid \delta a = x \}$$

It follows that all fibrations are pullbacks of the classifier $U: \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$, when viewed as morphisms in \mathbf{Cat} .

Pullback of fibrations along groupoid functors is not strictly coherent, in the sense that for $\tau:\Xi\to\Delta$ and $\sigma:\Delta\to\Gamma$ and a fibration $p\in\mathsf{Fib}_\Gamma$ we only have an isomorphism

$$\tau^*\sigma^*p \cong (\sigma \circ \tau)^*p$$

rather than equality.

In order to interpret reindexing/substitution strictly, it is convenient to work with classifiers $[\Gamma, \mathbf{grpd}]$ instead of fibrations.

Proposition 0.2.9 (Strictly coherent pullback). Let $\sigma: \Delta \to \Gamma$ be a functor between groupoids. Since display maps are pullbacks of the classifier $U: \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$ we have the pasting diagram

This gives us a functor $\circ \sigma : [\Gamma, \mathbf{grpd}] \to [\Delta, \mathbf{grpd}]$ which is our strict version of pullback.

Corollary 0.2.10 (Fibrations are stable under pullback).

$$\begin{split} [\Gamma, \mathbf{grpd}] &\longleftarrow^{\mathsf{fiber}} & \mathsf{Fib}_{\Gamma} \\ & \circ \sigma \!\!\!\! \downarrow \\ [\Delta, \mathbf{grpd}] &\longleftarrow^{\mathsf{disp}} & \mathsf{Fib}_{\Delta} \end{split}$$

We can deduce a corresponding fact about fibrations: since fibrations are closed under isomorphism, and since any pullback in **grpd** of a fibration p is isomorphic to the display map $\operatorname{disp}_{\operatorname{fiberpo}\sigma}$, any pullback of a fibration is a fibration.

A strict interpretation of type theory would require Σ and Π -formers to be stable under pullback (Beck-Chevalley). Thus we again define these as operations on classifiers $[\Gamma, \mathbf{grpd}]$.

Definition 0.2.11 (Σ -former operation). Then given $A:\Gamma\to\operatorname{\mathbf{grpd}}$ and $B:\Gamma\cdot A\to\operatorname{\mathbf{grpd}}$ we define $\Sigma_AB:\Gamma\to\operatorname{\mathbf{grpd}}$ such that Σ_AB acts on objects by forming fiberwise Grothendieck constructions

$$\Sigma_A B(x) := A(x) \cdot B \circ x_A$$

where $x_A:A(x)\to\Gamma\cdot A$ takes $f:a_0\to a_1$ to $(\mathsf{id}_x,f):(x,a_0)\to(x,a_1)$

 $\Sigma_A B$ acts on morphism $f: x \to y$ in Γ and $(a \in A(x), b \in B(x,a))$ by

$$\Sigma_A B f(a,b) := (A f a, B(f, \mathsf{id}_{A f a}) b)$$

and for morphism $(\alpha: a_0 \to a_1 \in A(x), \beta: B(\mathsf{id}_x, \alpha) \, b_0 \to b_1 \in B(x, a_1))$ in $\Sigma_A B x$

$$\Sigma_A B f(\alpha, \beta) := (A f \alpha, B(f, id_{A f a_1}) \beta)$$

Let us also define the natural transformation $fst: \Sigma_A B \to A$ by

$$\mathsf{fst}_x:(a,b)\mapsto a$$

Proposition 0.2.12 (Fibrations are closed under composition). The corresponding fact about fibrations is that the composition of two fibrations is a fibration.

$$\Xi$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \Delta$$
 $\Delta \longrightarrow \Gamma$

We can compare the two fibrations

$$\mathsf{disp}_B \circ \mathsf{disp}_A \qquad \text{and} \qquad \mathsf{disp}_{\Sigma_A(B)}$$

An object in the composition would look like ((x,a),b) for $x \in \Gamma$, $a \in A(x)$ and $b \in B(x,a)$, whereas an object in $\Gamma \cdot \Sigma_A(B)$ would instead be (x,(a,b)).

Proposition 0.2.13 (Strict Beck-Chevalley for Σ). Let $\sigma : \Delta \to \Gamma$, $A : \Gamma \to \operatorname{\mathbf{grpd}}$ and $B : \Gamma \cdot A \to \operatorname{\mathbf{grpd}}$. Then

$$(\Sigma_A B) \circ \sigma = \Sigma_{A \circ \sigma} (B \circ \sigma_A)$$

where σ_A is uniquely determined by the pullback in

Proof. By checking pointwise at $x \in \Delta$, this boils down to showing

$$A(\sigma x) \xrightarrow[]{(\sigma x)_A} \Delta \cdot A\sigma \xrightarrow[]{\sigma_A} \Gamma.A \xrightarrow[]{B} \mathbf{grpd}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

 $(\sigma x)_A = \sigma_A \circ x_{A \circ \sigma} : A(\sigma x) \to \Gamma \cdot A$

which holds because of the universal property of pullback.

Definition 0.2.14 (Π -former operation). Given $A:\Gamma\to\operatorname{\mathbf{grpd}}$ and $B:\Gamma\cdot A\to\operatorname{\mathbf{grpd}}$ we will define $\Pi_AB:\Gamma\to\operatorname{\mathbf{grpd}}$ such that for any $C:\Gamma\to\operatorname{\mathbf{grpd}}$ we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathsf{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

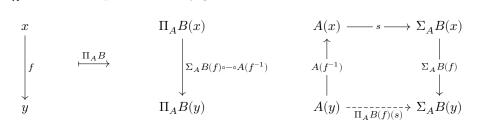
natural in both B and C.

Proof. $\Pi_A B$ acts on objects by taking fiberwise sections

$$\Pi_A B(x) := \{ s \in [A(x), \Sigma_A B(x)] \mid \mathsf{fst}_x \circ s = \mathsf{id}_{A(x)} \}$$

Where we have taken the full subcategory of the functor category $[A(x), \Sigma_A B(x)]$. This is a groupoid since any natural transformation of functors into groupoids are natural isomorphisms.

 $\Pi_A B$ acts on morphisms via conjugation



Note that conjugation is functorial and invertible.

Corollary 0.2.15 (Fibrations are closed under pushforward). Stated in terms of fibrations, we have

$$\begin{array}{ccc}
\Xi & \Gamma_! \sigma_* \tau \\
\downarrow^{\tau} & \downarrow^{\sigma_* \tau} \\
\Delta & \xrightarrow{\sigma} & \Gamma
\end{array}$$

with the universal property of pushforward

$$\mathsf{Fib}_{\Delta}(\sigma^*\rho,\tau) \cong \mathsf{Fib}_{\Gamma}(\rho,\sigma_*\tau)$$

natural in both τ and ρ .

Proposition 0.2.16 (Strict Beck-Chevalley for Π). Let $\sigma : \Delta \to \Gamma$, $A : \Gamma \to \operatorname{\mathbf{grpd}}$ and $B : \Gamma \cdot A \to \operatorname{\mathbf{grpd}}$. Then

$$(\Pi_A B) \circ \sigma = \Pi_{A \circ \sigma} (B \circ \sigma_A)$$

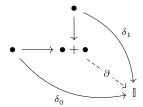
where σ_A is uniquely determined by the pullback in

Proof. By checking pointwise, this boils down to Beck-Chevalley for Σ .

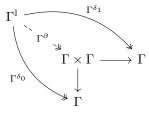
Proposition 0.2.17 (All objects are fibrant). Let \bullet denote the terminal groupoid, namely that with a single object and morphism. Then the unique map $\Gamma \to \bullet$ is a fibration.

Definition 0.2.18 (Interval). Let the interval groupoid \mathbb{I} be the small groupoid with two objects and a single non-identity isomorphism. There are two distinct morphisms $\delta_0, \delta_1 : \bullet \to \mathbb{I}$ and a natural isomorphism $i : \delta_0 \Rightarrow \delta_1$. Note that δ_0 and δ_1 both form adjoint equivalences with the unique map $! : \mathbb{I} \to \bullet$.

Denote by $\bullet + \bullet$ the small groupoid with two objects and only identity morphisms. Then let $\partial : \bullet + \bullet \to \mathbb{I}$ be the unique map factoring δ_0 and δ_1 .



Proposition 0.2.19 (Path object fibration). Let Γ be a small groupoid. Recall that **grpd** is Cartesian closed, so we can take the image of the above diagram under the functor Γ^- .



Then the indicated morphisms are fibrations, and Γ^{δ_0} , Γ^{δ_1} form adjoint equivalences with $\Gamma^!:\Gamma\to\Gamma^{\mathbb{I}}$.

0.2.3 Polynomial endofunctors

Definition 0.2.20 (Polynomial endofunctor on a morphism in an locally Cartesian closed category). Let $\mathbb C$ be a locally Cartesian closed category (we will take presheaves on small groupoids). This means for each morphism $t:B\to A$ we have an adjoint triple

$$\begin{array}{c|c} \mathbb{C}/B \\ t_! \left(\begin{array}{c} \uparrow \\ + \ t^* \end{array} \right) t_* \\ \mathbb{C}/A \end{array}$$

where t^* is pullback, and $t_!$ is composition with t.

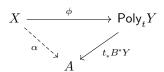
Let $t: B \to A$ be a morphism in \mathbb{C} . Then define $\mathsf{Poly}_t: \mathbb{C} \to \mathbb{C}$ be the composition

$$\mathsf{Poly}_t := A_! \circ t_* \circ B^* \qquad \qquad \mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

Proposition 0.2.21. The data of a map into the polynomial applied to an object in $\mathbb C$

$$X \stackrel{\phi}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} \mathsf{Poly}_t Y$$

corresponds to



Applying the adjunction $A_! \dashv A^*$, this corresponds to

$$\alpha: X \to A$$
 and
$$B_! t^* \alpha \xrightarrow[t^* \alpha]{\tilde{\phi}} B \times Y$$

Applying the adjunction $t^* \dashv t_*$, this corresponds to

$$\alpha: X \to A \qquad and \qquad B_! t^* \alpha \xrightarrow{\beta} Y$$

Proposition 0.2.22 ($\mathsf{Poly}_\mathsf{tp}\mathsf{Ty}$ classifies dependent types). Specialized to $\mathsf{tp}:\mathsf{Tm}\to \mathsf{Ty}$ in $\mathsf{Psh}(\mathsf{grpd})$, the previous proposition says that a map from a representable $\Gamma\to \mathsf{Poly}_\mathsf{tp}\mathsf{Ty}$ corresponds to the data of

$$A:\Gamma\to \mathsf{Ty} \qquad and \qquad B:\Gamma\cdot A\to \mathsf{Ty}$$

which by Yoneda corresponds to the data in Cat of

$$A:\Gamma \to \mathbf{grpd}$$
 and $B:\Gamma \cdot A \to \mathbf{grpd}$

Furthermore, if $\sigma: \Delta \to \Gamma$ were a representable map, then we have a naturality square

0.2.4 Pi and Sigma structure

Lemma 0.2.23. Let \mathbb{C} be a large category, and let $[-,\mathbb{C}] \in \mathbf{Psh}(\mathbf{grpd})$ be the restriction of the Yoneda embedding $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$. Let F be an operation that takes a groupoid Γ , a functor $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbb{C}$ and returns a functor $F_AB : \Gamma \to \mathbb{C}$.

Then
$$\tilde{F}: \mathsf{Poly_{tp}}[-,\mathbb{C}] \to [-,\mathbb{C}]$$

$$\tilde{F}_{\Gamma}(A,B) = F_{\Lambda}B$$

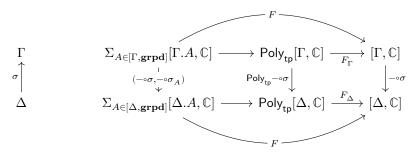
 $\ defines\ a\ natural\ transformation\ if\ and\ only\ if\ F\ satisfies\ the\ strict\ Beck-Chevalley\ condition$

$$(F_AB)\circ\sigma=F_{A\circ\sigma}(B\circ\sigma_A)$$

where σ_A is given by

$$\begin{array}{ccccc} \Delta \cdot A \circ \sigma & \xrightarrow{\sigma_A} & \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} \\ & \downarrow & & \downarrow & & \downarrow \\ & \Delta & \xrightarrow{\sigma} & \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

Proof. Using proposition 0.2.22



Definition 0.2.24 (Interpretation of Π types). We define the natural transformation $\Pi: \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ as that which is induced (lemma 0.2.23) by the Π -former operation (definition 0.2.14).

Then we define the natural transformation $\lambda: \mathsf{Poly}_\mathsf{tp}\mathsf{Ty} \to \mathsf{Ty}$ as the natural transformation induced by the following operation: given $A: \Gamma \to \mathsf{grpd}$ and $\beta: \Gamma \cdot A \to \mathsf{grpd}_\bullet$, $\lambda_A \beta: \Gamma \to \mathsf{grpd}_\bullet$ will be the functor such that on objects $x \in \Gamma$

$$\lambda_A \beta(x) := (\Pi_A B(x), a \mapsto (a, b(x, a)))$$

where $B := U \circ \beta : \Gamma \cdot A \to \mathbf{grpd}$ and b(x, a) is the point in $\beta(x, a)$. On morphisms $f : x \to y$ in Γ we have

$$\lambda_{A}\beta\left(f\right):=\left(\Pi_{A}B\left(f\right),\eta\right)$$

where $\eta:\Pi_AB\,f\,s_x\to s_y$ is a natural isomorphism between functors $A_y\to \Sigma_ABy$ given on objects $a\in A_y$ by

$$\eta_a := (\mathsf{id}_a, \mathsf{id}_{b(y,a)})$$

These combine to give us a pullback square

Proof. We should check that the λ operation satisfied Beck-Chevalley. This follows from the Π satisfying Beck-Chevalley and extensionality results for functors.

The square commutes and is a pullback if and only it pointwise commutes and pointwise gives pullbacks, i.e. for each groupoid Γ

$$\begin{array}{c} \operatorname{\mathsf{Poly}_{\mathsf{tp}}\mathsf{Tm}}\,\Gamma \xrightarrow{\lambda_{\Gamma}} [\Gamma,\operatorname{\mathbf{grpd}_{\bullet}}] \\ \operatorname{\mathsf{Poly}_{\mathsf{tp}}\mathsf{tp}_{\Gamma}} \downarrow & \downarrow_{U^{\circ}-} \\ \operatorname{\mathsf{Poly}_{\mathsf{tp}}}\mathsf{Ty}\,\Gamma \xrightarrow{\Pi_{\Gamma}} [\Gamma,\operatorname{\mathbf{grpd}}] \end{array}$$

by proposition 0.2.22 this holds if and only if

$$\begin{array}{cccc} \Sigma_{A \in [\Gamma, \mathbf{grpd}]}[\Gamma.A, \mathbf{grpd}_{\bullet}] & \xrightarrow{\lambda} & [\Gamma, \mathbf{grpd}_{\bullet}] \\ & & \downarrow U \circ - \\ & \Sigma_{A \in [\Gamma, \mathbf{grpd}]}[\Gamma.A, \mathbf{grpd}] & \xrightarrow{\Pi} & [\Gamma, \mathbf{grpd}] \end{array}$$

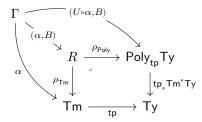
which follows from the definitions of Π and λ .

Lemma 0.2.25. Use R to denote the fiber product

$$\begin{array}{c} R \xrightarrow{\rho_{\mathsf{Poly}}} \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \\ \downarrow^{\mathsf{tp}^*\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty} = \rho_{\mathsf{Tm}}} & \downarrow^{\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}} \\ \mathsf{Tm} \xrightarrow{\mathsf{tp}} \mathsf{Ty} \end{array}$$

By the universal property of pullbacks, The data of a map from a respresentable $\varepsilon:\Gamma\to R$ corresponds to the data of $\alpha:\Gamma\to {\sf Tm}$ and $(U\circ\alpha,B):\Gamma\to {\sf Poly}_{\sf tn}{\sf Ty}.$

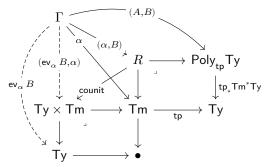
Then by proposition 0.2.22 this corresponds to the data of $\alpha:\Gamma\to \mathsf{Tm}$ and $B:\Gamma\cdot U\circ \alpha\to \mathsf{Ty}.$



Precomposition by a substitution $\sigma: \Delta \to \Gamma$ then act on such a pair by

$$(\alpha, B) \mapsto (\alpha \circ \sigma, B \circ \sigma_{U \circ \alpha})$$

Definition 0.2.26 (Evaluation). Define the operation of evaluation $\operatorname{ev}_{\alpha} B$ to take $\alpha:\Gamma\to\operatorname{\mathbf{grpd}}_{\bullet}$ and $B:\Gamma\cdot U\circ\alpha\to\operatorname{\mathbf{grpd}}$ and return $\operatorname{\mathbf{ev}}_{\alpha} B:\Gamma\to\operatorname{\mathbf{grpd}}$, described below.



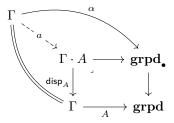
where we write $A := U \circ \alpha$ and treat a map $\Gamma \to \mathbf{grpd}$ as the same as a map $\Gamma \to \mathsf{Ty}$. More concisely, evaluation is a natural transformation $\mathsf{ev} : R \to \mathsf{Ty}$, given by

$$\operatorname{ev} \, = \pi_{\mathsf{Tv}} \circ \operatorname{counit}$$

Lemma 0.2.27. The functor $ev_{\alpha} B : \Gamma \to \mathbf{grpd}$ can be computed as

$$\operatorname{ev}_{\alpha} B = B \circ a$$

where



Proof. Since counit = $(ev, \rho_{Tm}) : R \to Ty$, it suffices to find out how the counit computes. The adjunction $tp^* \dashv tp_*$ suggests that we use the way

$$\widetilde{\mathsf{counit}} = \mathsf{id}_{\mathsf{Poly}_{\mathsf{L}}\mathsf{Ty}}$$

computes. Namely for any $A:\Gamma\to\operatorname{\mathbf{grpd}}$ and $B:\Gamma\cdot A\to\operatorname{\mathbf{grpd}}$

$$\widetilde{\mathsf{counit}} \circ (A,B) = (A,B) : \Gamma \to \mathsf{Poly}_\mathsf{tp} \mathsf{Ty} \tag{0.2.1}$$

Working on both sides of eq. (0.2.1) we get

$$\begin{split} &(\operatorname{ev}_{\operatorname{var}_A} B \circ U^* \operatorname{disp}_A, \operatorname{var}_A) \\ &= (\operatorname{ev}, \rho_{\operatorname{Ty}}) \circ (\operatorname{var}_A, B \circ U^* (\operatorname{disp}_A)) \\ &= (\operatorname{ev}, \rho_{\operatorname{Ty}}) \circ \operatorname{tp}^* (A, B) \\ &= \operatorname{counit} \circ \operatorname{tp}^* (A, B) \\ &= \widetilde{\operatorname{counit}} \circ (A, B) \\ &= \overline{(A, B)} \\ &= (B, \operatorname{var}_A) \end{split}$$

Hence we know that evaluation of B (weakened to the context $\Gamma \cdot A \cdot A$) on a variable of type A is just B.

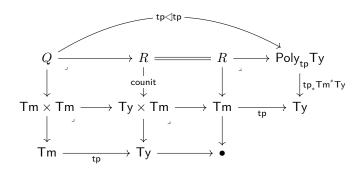
$$\operatorname{ev}_{\operatorname{var}_A} B \circ U^* \operatorname{disp}_A = B$$

Then the naturality square for the natural transformation ${\sf ev}:R\to {\sf Ty}$ on $a:\Gamma\to \Gamma\cdot A$ tells us that

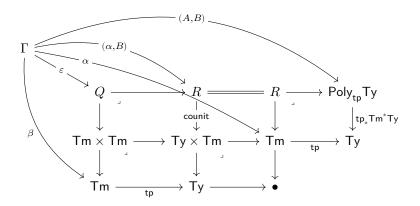
$$\begin{split} \operatorname{ev}_{\alpha} B \\ &= \operatorname{ev}_{\Gamma} \left(\alpha, B \right) \\ &= \operatorname{ev}_{\Gamma} \left(\operatorname{var}_A \circ a, B \circ \operatorname{U}^*(\operatorname{id}_{\Gamma}) \right) \\ &= \operatorname{ev}_{\Gamma} \left(\operatorname{var}_A \circ a, B \circ \operatorname{U}^*(\operatorname{disp}_A \circ a) \right) \\ &= \operatorname{ev}_{\Gamma} \left(\operatorname{var}_A \circ a, B \circ \operatorname{U}^* \operatorname{disp}_A \circ U^* a \right) \\ &= \operatorname{ev}_{\Gamma} \left(\left(\operatorname{var}_A, B \circ \operatorname{U}^* \operatorname{disp}_A \right) \circ a \right) \\ &= \left(\operatorname{ev}_{\Gamma \cdot A} \left(\operatorname{var}_A, B \circ \operatorname{U}^* \operatorname{disp}_A \right) \circ a \right) \\ &= \left(\operatorname{ev}_{\operatorname{var}_A} B \circ \operatorname{U}^* \operatorname{disp}_A \right) \circ a \\ &= \left(\operatorname{ev}_{\operatorname{var}_A} B \circ \operatorname{U}^* \operatorname{disp}_A \right) \circ a \\ &= B \circ a \end{split} \end{split}$$
 by naturality

Definition 0.2.28 (Classifier for dependent pairs). Recall the following definition

of composition of polynomial endofunctors, specialized to our situation



By the universal property of pullbacks, a data of a map with representable domain $\varepsilon:\Gamma\to Q$ corresponds to the data of a triple of maps $\alpha,\beta:\Gamma\to \mathsf{Tm}$ and $(A,B):\Gamma\to \mathsf{Poly}_\mathsf{to}\mathsf{Ty}$ such that $\mathsf{tp}\circ\beta=\pi_\mathsf{Ty}\circ\mathsf{counit}\circ(\alpha,B)$ and $A=\mathsf{tp}\circ\alpha.$



This in turn corresponds to three functors $\alpha, \beta : \Gamma \to \mathbf{grpd}_{\bullet}$ and $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$, such that $U \circ \beta = \mathbf{ev}_{\alpha} B$. So we will write

$$\varepsilon = (\beta, \alpha, B)$$

Type theoretically $\alpha = (A, a : A)$ and $\operatorname{ev}_{\alpha} B = Ba$ and $\beta = (Ba, b : Ba)$. Then composing ε with $\operatorname{tp} \triangleleft \operatorname{tp}$ returns γ , which consists of (A, B). It is in this sense that Q classifies pairs of dependent terms, and $\operatorname{tp} \triangleleft \operatorname{tp}$ extracts the underlying types.

Definition 0.2.29 (Interpretation of Σ). We define the natural transformation

$$\Sigma:\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}\to\mathsf{Ty}$$

as that which is induced (lemma 0.2.23) by the Σ -former operation (definition 0.2.14).

To define $\operatorname{\mathsf{pair}}: Q \to \operatorname{\mathsf{Tm}},$ let Γ be a groupoid and $(\beta, \alpha, B): \Gamma \to Q$ (such that $U \circ \beta = \operatorname{\mathsf{ev}}_\alpha \beta$). We define a functor $\operatorname{\mathsf{pair}}_\Gamma(\beta, \alpha, B): \Gamma \to \operatorname{\mathsf{grpd}}_\bullet$ such that on objects $x \in \Gamma$, the functor returns $(\Sigma_A B \, x, (a_x, b_{a_x}))$, where (using lemma 0.2.27 $U \circ \beta x = \operatorname{\mathsf{ev}}_\alpha B x = B(x, a_x)$)

$$\alpha x = (A x, a_x)$$
 and $\beta x = (B(x, a_x), b_a)$

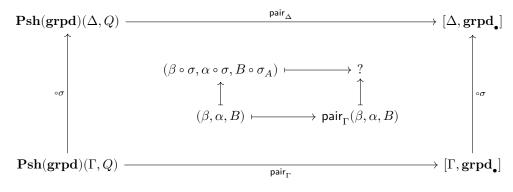
and on morphisms $f:x\to y$, the functor returns $(\Sigma_A B\,f,(\phi_f,\psi_f))$, where (using lemma 0.2.27 $U\circ\beta f=\operatorname{ev}_\alpha Bf=B(f,\phi_f))$

$$\alpha\,f = (A\,f,\phi_f\!\colon A\,f\,a_x \to a_y) \quad \text{ and } \quad \beta\,f = (B(f,\phi_f),\psi_f\!\colon B(f,\phi_f)\,b_{a_x} \to b_{a_y})$$

 Σ and pair combine to give us a pullback square

$$\begin{array}{c} Q \xrightarrow{\quad \text{pair} \quad} \mathsf{Tm} \\ \underset{\mathsf{tp} \lhd \mathsf{tp} \downarrow}{\bigvee} & \qquad \qquad \underset{\Sigma}{\bigvee} \mathsf{tp} \\ \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \xrightarrow{\quad \Sigma \quad} \mathsf{Ty} \end{array}$$

Proof. To show naturality of pair, suppose $\sigma: \Delta \to \Gamma$ is a functor between groupoids.



So we check that for any $x \in \Gamma$,

$$\begin{split} & \operatorname{pair}_{\Delta}(\beta \circ \sigma, \alpha \circ \sigma, B \circ \sigma_{A}) \, x \\ &= (\Sigma_{A \circ \sigma} B \circ \sigma_{A} \, x, (a_{x}, b_{a_{x}})) \\ &= ((\Sigma_{A} B) \circ \sigma \, x, (a_{x}, b_{a_{x}})) \\ &= \operatorname{pair}_{\Gamma}(\beta, \alpha, B) \circ \sigma \, x \end{split}$$

where

$$\alpha \circ \sigma x = (A \circ \sigma x, a_x)$$
 and $\beta \circ \sigma x = (ev_\alpha B \circ \sigma x, b_a)$

and so on.

It follows from the definition of pair that the square commutes. To show that it is pullback, it suffices to show that for each Γ ,

$$\begin{split} \mathbf{Psh}(\mathbf{grpd})(\Gamma, Q) & \xrightarrow{-\mathsf{pair}_{\Gamma}} & [\Gamma, \mathbf{grpd}_{\bullet}] \\ \text{$_{\mathsf{tp}} \in \mathsf{pro}$} & & \downarrow_{U \circ -} \\ \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty}) & \xrightarrow{-\Sigma_{\Gamma}} & [\Gamma, \mathbf{grpd}] \end{split}$$

is a pullback. Since we are in **Set**, it suffices to just show the universal property applied to a point: so for any $A:\Gamma\to\mathbf{grpd}$, any $B:\Gamma\cdot A\to\mathbf{grpd}$, and any $p:\Gamma\to\mathbf{grpd}_{\bullet}$, such that

$$U \circ p = \Sigma_{\Gamma}(A, B)$$

there exists a unique $(\beta, \alpha, B) : \Gamma \to Q$ such that

$$\mathsf{pair}_{\Gamma}(\beta, \alpha, B) = p \quad \text{and} \quad \mathsf{tp} \triangleleft \mathsf{tp} \circ (B, \alpha, B) = (A, B)$$

Indeed if we write

$$px = (\Sigma_A Bx, (a_r \in Ax, b_r \in B(x, a_r)))$$

this uniquely determines α and β as

$$\alpha x = (Ax, a_x)$$
 and $\beta x = (ev_\alpha Bx, b_x)$

Bibliography

- [Awo23] Steve Awodey. On hofmann-streicher universes, 2023.
- [HS98] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In Twenty-five years of constructive type theory (Venice, 1995), volume 36 of Oxford Logic Guides, pages 83–111. Oxford Univ. Press, New York, 1998.
- [Joy] André Joyal. Model structures on cat. https://ncatlab.org/ joyalscatlab/published/Model+structures+on+Cat.