# Universe in the Natural Model of Type Theory

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August 8, 2024

### 1 Types

Assume an inaccessible cardinal  $\lambda$ . Write **Set** for the category of all sets. Say that a set A is  $\lambda$ -small if  $|A| < \lambda$ . Write **Set** $_{\lambda}$  for the full subcategory of **Set** spanned by  $\lambda$ -small sets.

Let  $\mathbb{C}$  be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write  $\mathbf{Psh}(\mathbb{C})$  for the category of presheaves over  $\mathbb{C}$ ,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map  $tp: Tm \to Ty$  consists of

- contexts as objects  $\Gamma, \Delta, \ldots \in \mathbb{C}$ ,
- a type in context  $y(\Gamma)$  as a map  $A: y(\Gamma) \to \mathsf{Ty}$ ,
- a term of type A in context  $\Gamma$  as a map  $a: y(\Gamma) \to Tm$  such that



commutes,

• an operation called "context extension" which given a context  $\Gamma$  and a type  $A \colon \mathsf{y}(\Gamma) \to \mathsf{T}\mathsf{y}$  produces a context  $\Gamma \cdot A$  which fits into a pullback diagram below.

$$\begin{array}{ccc} \mathsf{y}(\Gamma.A) & \longrightarrow \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{y}(\Gamma) & \longrightarrow_A & \mathsf{Ty} \end{array}$$

**Remark.** Sometimes, we first construct a presheaf X over  $\Gamma$  and observe that it can be classified by a map into Ty. We write

to express this situation, i.e.  $X \cong y(\Gamma \cdot \lceil X \rceil)$ .

### 2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U}\colon \mathsf{Ty} \qquad \frac{a\colon \mathsf{U}}{\mathsf{El}(a)\colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal  $\kappa$ , with  $\kappa < \lambda$ .)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{c} \mathsf{U} \longrightarrow \mathsf{Tm} \\ \downarrow \\ \downarrow \\ 1 \xrightarrow{} \mathsf{Ty} \end{array}$$

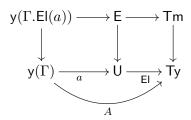
• There is an inclusion of U into Ty

$$\mathsf{EI}\colon\mathsf{U}\rightarrowtail\mathsf{Ty}$$

•  $\pi: E \to U$  is obtained as pullback of tp; There is a pullback diagram

$$\begin{array}{c} E {\longmapsto} \operatorname{Tm} \\ \downarrow \qquad \qquad \downarrow \\ \operatorname{U} {\longmapsto} \operatorname{Ty} \end{array}$$

With the notation above, we get



Both squares above are pullback squares.

- 3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model
- 4 Groupoid Model of HoTT

In this section we construct a natural model in **Psh**(**grpd**) the presheaf category indexed by the category **grpd** of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on **grpd** - which comes from restricting the "classical Quillen model structure" on **cat** [Joy] to **grpd**.

#### 4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as **set** and the large sets as **Set** (in the previous sections this would have been  $\mathbf{Set}_{\lambda}$  and  $\mathbf{Set}$  respectively). We denote the category of small categories as **cat** and the large categories as **Cat**. We denote the category of small groupoids as **grpd**.

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

**Definition 4.1** (Pointed). We will take the category of pointed small categories  $\mathbf{cat}_{\bullet}$  to have objects as pairs ( $\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C}$ ) and morphisms as pairs

$$(F: \mathbb{C}_1 \to \mathbb{C}_0, \phi: Fc_1 \to c_0): (\mathbb{C}_1, c_1) \to (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids  $\mathbf{grpd}_{\bullet}$  will be the full subcategory of objects  $(\Gamma, c)$  with  $\Gamma$  a groupoid.

**Definition 4.2** (The display map classifier). We would like to define a natural transformation in  $\mathbf{Psh}(\mathbf{grpd})$ 

$$tp: Tm \rightarrow Tv$$

with representable fibers.

Consider the functor that forgets the point

$$U : \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$$
 in Cat.

If we apply the Yoneda embedding  $y: \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$  to U we obtain

$$U \circ : [-, \mathbf{grpd}_{\bullet}] \to [-, \mathbf{grpd}]$$
 in  $\mathbf{Psh}(\mathbf{Cat})$ .

Since any small groupoid is also a large category  $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$ , we can restrict  $\mathbf{Cat}$  indexed presheaves to be  $\mathbf{grpd}$  indexed presheaves. We define  $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$  as the image of  $U \circ \mathsf{under}$  this restriction.

$$\mathbf{Cat} \xrightarrow{\quad y \quad} \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\quad res \quad} \mathbf{Psh}(\mathbf{grpd})$$

$$\operatorname{\mathbf{grpd}} \longmapsto [-,\operatorname{\mathbf{grpd}}] \longmapsto \mathsf{Ty}$$

Note that  $\mathsf{Tm}$  and  $\mathsf{Ty}$  are not representable in  $\mathsf{Psh}(\mathsf{grpd})$ .

Remark 4.3. By Yoneda we can identify maps with representable domain into the type classifier

$$A: \mathsf{y}\Gamma \to \mathsf{T}\mathsf{y} \qquad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A:\Gamma\to\mathbf{grpd}$$
 in Cat

**Definition 4.4** (Grothendieck construction). From  $\mathbb{C}$  a small category and  $F:\mathbb{C}\to\mathbf{cat}$  a functor, we construct a small category  $\int F$ . For any c in  $\mathbb{C}$  we refer to Fc as the fiber over c. The objects of  $\int F$  consist of pairs  $(c\in\mathbb{C},x\in Fc)$ , and morphisms between (c,x) and (d,y) are pairs  $(f:c\to d,\phi:Ffx\to y)$ . This makes the following pullback in  $\mathbf{Cat}$ 

$$(c,x) \longmapsto (Fc,x)$$

**Definition 4.5** (Grothendieck construction for groupoids). Let  $\Gamma$  be a groupoid and  $A \colon \Gamma \to \mathbf{grpd}$  a functor, we can compose F with the inclusion  $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$  and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \mathsf{disp}_A \colon \Gamma \cdot A o \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids  $\Gamma$  and Ax for  $x \in \Gamma$ . Furthermore the pullback factors through (pointed) groupoids.

$$\begin{array}{cccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} & \longrightarrow \mathbf{cat}_{\bullet} \\ & \downarrow & & \downarrow & & \downarrow \\ \Gamma & \longrightarrow_A & \mathbf{grpd} & \longrightarrow \mathbf{cat} \end{array}$$

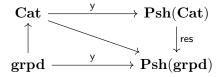
Corollary 4.6 (The display map classifier is presentable). For any small groupoid  $\Gamma$  and  $A: y\Gamma \to Ty$ , the pullback of tp along A can be given by the representable map  $ydisp_A$ .

$$\begin{array}{ccc} \mathsf{y}\Gamma \cdot A & \longrightarrow & \mathsf{Tm} \\ \mathsf{ydisp}_A & & & \mathsf{tp} \\ & \mathsf{y}\Gamma & \longrightarrow & \mathsf{Ty} \end{array}$$

Proof. Consider the pullback in Cat

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow_A & \mathbf{grpd} \end{array}$$

We send this square along  $res \circ y$  in the following



The Yoneda embedding  $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$  preserves pullbacks, as does res since it is a right adjoint (with left Kan extension  $\iota_! \dashv \mathsf{res}_\iota$ ).

#### 4.2 Groupoid fibrations

**Definition 4.7** (Fibration). Let  $p: \mathbb{C}_1 \to \mathbb{C}_0$  be a functor. We say p is a *split Grothendieck fibration* if we have a dependent function lift a f satisfying the following: for any object a in  $\mathbb{C}_1$  and morphism  $f: p \, a \to y$  in the base  $\mathbb{C}_0$  we have lift  $a \, f: a \to b$  in  $\mathbb{C}_1$  such that  $p(\text{lift } a \, f) = f$  and moreover lift  $a \, g \circ f = \text{lift } b \, g \circ \text{lift } a \, f$ 

$$\begin{array}{ccc} a & \xrightarrow{\text{lift } a f} & b \\ \downarrow & & \uparrow & \downarrow \\ \downarrow & & \downarrow & \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

In particular, we are intereseted in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid  $\Gamma$  as  $\mathsf{Fib}_{\Gamma}$ , which is a full subcategory of the slice  $\mathsf{grpd}/\Gamma$ . We will decorate an arrow with  $\twoheadrightarrow$  to indicate it is a fibration.

Note that  $\operatorname{\mathsf{disp}}_A \colon \Gamma \cdot A \to \Gamma$  is a fibration, since for any  $(x \in \Gamma, a \in A \, x)$  and  $f \colon x \to y$  in  $\Gamma$  we have a morphism  $(f, \operatorname{\mathsf{id}}_{A \, f \, a}) \colon (x, a) \to (y, A \, f \, a)$  lifting f. Furthermore **Proposition 4.8.** There is an adjoint equivalence

$$[\Gamma,\mathbf{grpd}] \xrightarrow[\mathsf{fiber}]{\mathsf{disp}} \mathsf{Fib}_{\Gamma}$$

where for each fibration  $\delta: \Delta \to \Gamma$  and each object  $x \in \Gamma$ 

fiber<sub>$$\delta$$</sub>  $x = \text{full subcategory } \{a \in \Delta \mid \delta a = x\}$ 

It follows that all fibrations are pullbacks of the classifier  $U: \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$ , when viewed as morphisms in  $\mathbf{Cat}$ .

Pullback of fibrations along groupoid functors is not strictly coherent, in the sense that for  $\tau: \Xi \to \Delta$  and  $\sigma: \Delta \to \Gamma$  and a fibration  $p \in \mathsf{Fib}_{\Gamma}$  we only have an isomorphism

$$\tau^*\sigma^*p \cong (\sigma \circ \tau)^*p$$

rather than equality.

In order to interpret reindexing/substitution strictly, it is convenient to work with classifiers  $[\Gamma, \mathbf{grpd}]$  instead of fibrations.

**Proposition 4.9** (Strictly coherent pullback). Let  $\sigma : \Delta \to \Gamma$  be a functor between groupoids. Since display maps are pullbacks of the classifier  $U : \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$  we have the pasting diagram

This gives us a functor  $\circ \sigma : [\Gamma, \mathbf{grpd}] \to [\Delta, \mathbf{grpd}]$  which is our strict version of pullback.

Corollary 4.10 (Fibrations are stable under pullback).

$$\begin{array}{ccc} [\Gamma, \operatorname{\mathbf{grpd}}] & \longleftarrow & \operatorname{\mathsf{Fib}}_{\Gamma} \\ \circ \sigma & & & & \downarrow \sigma^* \\ [\Delta, \operatorname{\mathbf{grpd}}] & \longleftarrow & \operatorname{\mathsf{Fib}}_{\Delta} \end{array}$$

We can deduce a corresponding fact about fibrations: since fibrations are closed under isomorphism, and since any pullback in  $\operatorname{\mathbf{grpd}}$  of a fibration p is isomorphic to the display map  $\operatorname{\mathsf{disp}}_{\operatorname{fiberpoo}}$ , any pullback of a fibration is a fibration.

A strict interpretation of type theory would require  $\Sigma$  and  $\Pi$ -formers to be stable under pullback (Beck-Chevalley). Thus we again define these as operations on classifiers  $[\Gamma, \mathbf{grpd}]$ .

**Definition 4.11** ( $\Sigma$ -former operation). Then given  $A:\Gamma\to\operatorname{\mathbf{grpd}}$  and  $B:\Gamma\cdot A\to\operatorname{\mathbf{grpd}}$  we define  $\Sigma_AB:\Gamma\to\operatorname{\mathbf{grpd}}$  such that  $\Sigma_AB$  acts on objects by forming fiberwise Grothendieck constructions

$$\Sigma_A B(x) := A(x) \cdot B \circ x_A$$

where  $x_A: A(x) \to \Gamma \cdot A$  takes  $f: a_0 \to a_1$  to  $(id_x, f): (x, a_0) \to (x, a_1)$ 

 $\Sigma_A B$  acts on morphism  $f: x \to y$  in  $\Gamma$  and  $(a \in A(x), b \in B(x, a))$  by

$$\Sigma_A B f(a,b) := (A f a, B (f, \mathsf{id}_{A f a}) b)$$

and for morphism  $(\alpha: a_0 \to a_1 \in A(x), \beta: B(\mathsf{id}_x, \alpha) b_0 \to b_1 \in B(x, a_1))$  in  $\Sigma_A B x$ 

$$\Sigma_A B f(\alpha, \beta) := (A f \alpha, B (f, id_{A f a_1}) \beta)$$

Let us also define the natural transformation  $fst: \Sigma_A B \to A$  by

$$\mathsf{fst}_r:(a,b)\mapsto a$$

**Proposition 4.12** (Fibrations are closed under composition). The corresponding fact about fibrations is that the composition of two fibrations is a fibration.



We can compare the two fibrations

$$\operatorname{\mathsf{disp}}_B \circ \operatorname{\mathsf{disp}}_A \qquad \text{and} \qquad \operatorname{\mathsf{disp}}_{\Sigma_A(B)}$$

An object in the composition would look like ((x, a), b) for  $x \in \Gamma$ ,  $a \in A(x)$  and  $b \in B(x, a)$ , whereas an object in  $\Gamma \cdot \Sigma_A(B)$  would instead be (x, (a, b)).

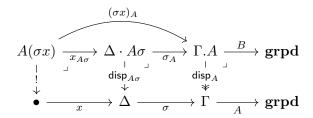
**Proposition 4.13** (Strict Beck-Chevalley for  $\Sigma$ ). Let  $\sigma : \Delta \to \Gamma$ ,  $A : \Gamma \to \mathbf{grpd}$  and  $B : \Gamma \cdot A \to \mathbf{grpd}$ . Then

$$(\Sigma_A B) \circ \sigma = \Sigma_{A \circ \sigma} (B \circ \sigma_A)$$

where  $\sigma_A$  is uniquely determined by the pullback in

*Proof.* By checking pointwise at  $x \in \Delta$ , this boils down to showing

$$(\sigma x)_A = \sigma_A \circ x_{A \circ \sigma} : A(\sigma x) \to \Gamma \cdot A$$



which holds because of the universal property of pullback.

**Definition 4.14** ( $\Pi$ -former operation). Given  $A: \Gamma \to \mathbf{grpd}$  and  $B: \Gamma \cdot A \to \mathbf{grpd}$  we will define  $\Pi_A B: \Gamma \to \mathbf{grpd}$  such that for any  $C: \Gamma \to \mathbf{grpd}$  we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathsf{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

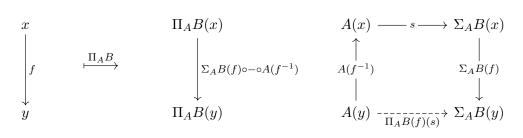
natural in both B and C.

*Proof.*  $\Pi_A B$  acts on objects by taking fiberwise sections

$$\Pi_A B(x) := \{ s \in [A(x), \Sigma_A B(x)] \mid \mathsf{fst}_x \circ s = \mathsf{id}_{A(x)} \}$$

Where we have taken the full subcategory of the functor category  $[A(x), \Sigma_A B(x)]$ . This is a groupoid since any natural transformation of functors into groupoids are natural isomorphisms.

 $\Pi_A B$  acts on morphisms via conjugation



Note that conjugation is functorial and invertible.

Corollary 4.15 (Fibrations are closed under pushforward). Stated in terms of fibrations, we have

$$\begin{array}{ccc}
\Xi & & \Gamma_! \sigma_* \tau \\
\tau \downarrow & & \downarrow \sigma_* \tau \\
\Delta & & & \Gamma
\end{array}$$

with the universal property of pushforward

$$\operatorname{Fib}_{\Delta}(\sigma^*\rho,\tau) \cong \operatorname{Fib}_{\Gamma}(\rho,\sigma_*\tau)$$

natural in both  $\tau$  and  $\rho$ .

**Proposition 4.16** (Strict Beck-Chevalley for  $\Pi$ ). Let  $\sigma : \Delta \to \Gamma$ ,  $A : \Gamma \to \mathbf{grpd}$  and  $B : \Gamma \cdot A \to \mathbf{grpd}$ . Then

$$(\Pi_A B) \circ \sigma = \Pi_{A \circ \sigma} (B \circ \sigma_A)$$

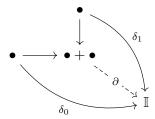
where  $\sigma_A$  is uniquely determined by the pullback in

*Proof.* By checking pointwise, this boils down to Beck-Chevalley for  $\Sigma$ .

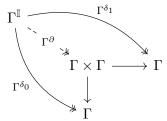
**Proposition 4.17** (All objects are fibrant). Let  $\bullet$  denote the terminal groupoid, namely that with a single object and morphism. Then the unique map  $\Gamma \to \bullet$  is a fibration.

**Definition 4.18** (Interval). Let the interval groupoid  $\mathbb{I}$  be the small groupoid with two objects and a single non-identity isomorphism. There are two distinct morphisms  $\delta_0, \delta_1 : \bullet \to \mathbb{I}$  and a natural isomorphism  $i : \delta_0 \Rightarrow \delta_1$ . Note that  $\delta_0$  and  $\delta_1$  both form adjoint equivalences with the unique map  $! : \mathbb{I} \to \bullet$ .

Denote by  $\bullet + \bullet$  the small groupoid with two objects and only identity morphisms. Then let  $\partial : \bullet + \bullet \to \mathbb{I}$  be the unique map factoring  $\delta_0$  and  $\delta_1$ .



**Proposition 4.19** (Path object fibration). Let  $\Gamma$  be a small groupoid. Recall that **grpd** is Cartesian closed, so we can take the image of the above diagram under the functor  $\Gamma^-$ .



Then the indicated morphisms are fibrations, and  $\Gamma^{\delta_0}$ ,  $\Gamma^{\delta_1}$  form adjoint equivalences with  $\Gamma^!:\Gamma\to\Gamma^{\mathbb{I}}$ .

#### 4.3 Polynomial endofunctors

**Definition 4.20** (Polynomial endofunctor on a morphism in an locally Cartesian closed category). Let  $\mathbb{C}$  be a locally Cartesian closed category (we will take presheaves on small groupoids). This means for each morphism  $t: B \to A$  we have an adjoint triple

$$\begin{array}{c|c}
\mathbb{C}/B \\
t_! \left( \begin{array}{c} \uparrow \\ + t^* \end{array} \right) \\
\mathbb{C}/A
\end{array}$$

where  $t^*$  is pullback, and  $t_!$  is composition with t.

Let  $t: B \to A$  be a morphism in  $\mathbb{C}$ . Then define  $\mathsf{Poly}_t: \mathbb{C} \to \mathbb{C}$  be the composition

$$\mathsf{Poly}_t := A_! \circ t_* \circ B^* \qquad \qquad \mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

**Proposition 4.21.** The data of a map into the polynomial applied to an object in  $\mathbb{C}$ 

$$X \stackrel{\phi}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} \mathsf{Poly}_t Y$$

corresponds to

$$X \xrightarrow{\phi} \operatorname{Poly}_t Y$$

$$A \xrightarrow{t_*B^*Y}$$

Applying the adjunction  $A_! \dashv A^*$ , this corresponds to

$$\alpha: X \to A$$
 and 
$$B!t^*\alpha \xrightarrow{\tilde{\phi}} B \times Y$$
 
$$B*Y$$

Applying the adjunction  $t^* \dashv t_*$ , this corresponds to

$$\alpha: X \to A$$
 and  $B_! t^* \alpha \xrightarrow{\beta} Y$ 

**Proposition 4.22** (Poly<sub>tp</sub>Ty classifies dependent types). Specialized to tp: Tm  $\rightarrow$  Ty in  $\mathbf{Psh}(\mathbf{grpd})$ , the previous proposition says that a map from a representable  $\Gamma \rightarrow \mathsf{Poly}_\mathsf{tp}\mathsf{Ty}$  corresponds to the data of

$$A:\Gamma \to \mathsf{Ty}$$
 and  $B:\Gamma \cdot A \to \mathsf{Ty}$ 

which by Yoneda corresponds to the data in Cat of

$$A: \Gamma \to \mathbf{grpd}$$
 and  $B: \Gamma \cdot A \to \mathbf{grpd}$ 

Furthermore, if  $\sigma: \Delta \to \Gamma$  were a representable map, then we have a naturality square

#### 4.4 $\Pi$ and $\Sigma$ structure

**Lemma 4.23.** Let  $\mathbb{C}$  be a large category, and let  $[-,\mathbb{C}] \in \mathbf{Psh}(\mathbf{grpd})$  be the restriction of the Yoneda embedding  $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ . Let F be an operation that takes a groupoid  $\Gamma$ , a functor  $A : \Gamma \to \mathbf{grpd}$  and  $B : \Gamma \cdot A \to \mathbb{C}$  and returns a functor  $F_AB : \Gamma \to \mathbb{C}$ .

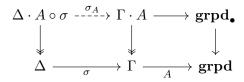
Then 
$$\tilde{F}: \mathsf{Poly_{tp}}[-,\mathbb{C}] \to [-,\mathbb{C}]$$

$$\tilde{F}_{\Gamma}(A,B) = F_A B$$

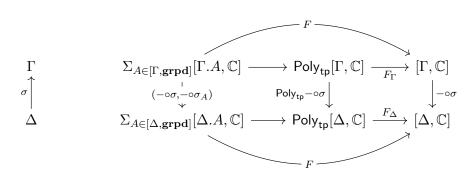
 $defines\ a\ natural\ transformation\ if\ and\ only\ if\ F\ satisfies\ the\ strict\ Beck-Chevalley\ condition$ 

$$(F_A B) \circ \sigma = F_{A \circ \sigma}(B \circ \sigma_A)$$

where  $\sigma_A$  is given by



*Proof.* Using proposition 4.22



**Definition 4.24** (Interpretation of  $\Pi$  types). We define the natural transformation  $\Pi: \mathsf{Poly_{tp}Ty} \to \mathsf{Ty}$  as that which is induced (lemma 4.23) by the  $\Pi$ -former operation (definition 4.14).

Then we define the natural transformation  $\lambda : \mathsf{Poly}_\mathsf{tp} \mathsf{Ty} \to \mathsf{Ty}$  as the natural transformation induced by the following operation: given  $A : \Gamma \to \mathsf{grpd}$  and  $\beta : \Gamma \cdot A \to \mathsf{grpd}_{\bullet}$ ,  $\lambda_A \beta : \Gamma \to \mathsf{grpd}_{\bullet}$  will be the functor such that on objects  $x \in \Gamma$ 

$$\lambda_A \beta(x) := (\Pi_A B(x), a \mapsto (a, b(x, a)))$$

where  $B := U \circ \beta : \Gamma \cdot A \to \mathbf{grpd}$  and b(x,a) is the point in  $\beta(x,a)$ . On morphisms  $f : x \to y$  in  $\Gamma$  we have

$$\lambda_A \beta(f) := (\Pi_A B(f), \eta)$$

where  $\eta: \Pi_A B f s_x \to s_y$  is a natural isomorphism between functors  $A_y \to \Sigma_A B y$  given on objects  $a \in A_y$  by

$$\eta_a := (\mathsf{id}_a, \mathsf{id}_{b(y,a)})$$

These combine to give us a pullback square

*Proof.* We should check that the  $\lambda$  operation satisfied Beck-Chevalley. This follows from the  $\Pi$  satisfying Beck-Chevalley and extensionality results for functors.

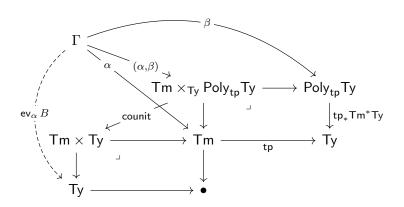
The square commutes and is a pullback if and only it pointwise commutes and pointwise gives pullbacks, i.e. for each groupoid  $\Gamma$ 

by proposition 4.22 this holds if and only if

$$\begin{array}{ccc} \Sigma_{A \in [\Gamma, \mathbf{grpd}]}[\Gamma.A, \mathbf{grpd}_{\bullet}] & \xrightarrow{\lambda} & [\Gamma, \mathbf{grpd}_{\bullet}] \\ & \xrightarrow{(-\circ \sigma, -\circ \sigma_A)} & & & \downarrow U \circ - \\ & \Sigma_{A \in [\Gamma, \mathbf{grpd}]}[\Gamma.A, \mathbf{grpd}] & \xrightarrow{\Pi} & [\Gamma, \mathbf{grpd}] \end{array}$$

which follows from the definitions of  $\Pi$  and  $\lambda$ .

**Definition 4.25** (Evaluation). Define the operation of evaluation ev to take  $\alpha : \Gamma \to \mathbf{grpd}_{\bullet}$  and  $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$  and return  $\mathbf{ev}_{\alpha} B : \Gamma \to \mathbf{grpd}$ , described below.



The input data corresponds to  $\alpha:\Gamma\to\mathsf{Tm}$  and and some  $\beta:\Gamma\to\mathsf{Poly}_\mathsf{tp}\mathsf{Ty}$  such that

$$\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}\circ\beta=\mathsf{tp}\circ\alpha$$

This in turn corresponds to a map into the fiber product

$$(\alpha, \beta) : \Gamma \to \mathsf{tp} \times_{\mathsf{Tv}} \mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty}$$

We then compose this the counit of the adjunction  $tp^* \dashv tp_*$  to get

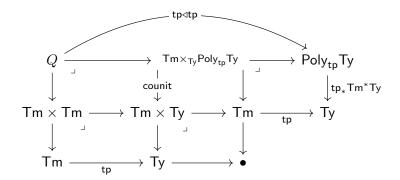
$$\mathsf{counit} \circ (\alpha, \beta) : \Gamma \to \mathsf{tp} \times_{\mathsf{Tv}} \mathsf{tp}_* \mathsf{Ty}$$

then extract the type by composing with the projection to Ty

$$\pi_{\mathsf{Tv}} \circ \mathsf{counit} \circ (\alpha, \beta) : \Gamma \to \mathsf{Ty}$$

Finally, this corresponds to a unique map  $ev_{\alpha}B:\Gamma\to \mathbf{grpd}$ .

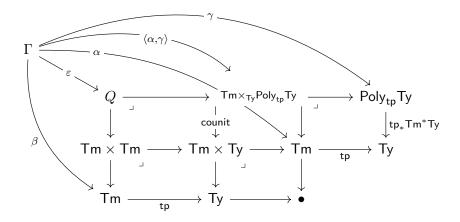
**Definition 4.26** (Classifier for dependent pairs). Recall the following definition of composition of polynomial endofunctors, specialized to our situation



**Proposition 4.27.** Q classifies the data of a pair of dependent terms (a,b), and  $tp \triangleleft tp$  extracts the underlying type and dependent type (A,B).

$$Q\Gamma \cong \{(\beta: \gamma \to \mathbf{grpd}_{\bullet}, \alpha: \Gamma \to \mathbf{grpd}_{\bullet}, B: \Gamma \cdot U \circ \alpha \to \mathbf{grpd}) \mid U \circ \beta = \mathsf{ev}_{\alpha}\beta\}$$

*Proof.* Extending the diagram from definition 4.26



By the universal property of pullbacks, a data of a map with representable domain  $\varepsilon:\Gamma\to Q$  corresponds to the data of a triple of maps  $\alpha,\beta:\Gamma\to \mathsf{Tm}$  and  $\gamma:\Gamma\to \mathsf{Poly}_\mathsf{tp}\mathsf{Ty}$  such that  $\mathsf{tp}\circ\beta=\pi_\mathsf{Ty}\circ\mathsf{counit}\circ\langle\alpha,\gamma\rangle$  and  $\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}\circ\gamma=\mathsf{tp}\circ\alpha$ .

This in turn corresponds to three functors  $\alpha, \beta : \Gamma \to \mathbf{grpd}_{\bullet}$  and  $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$ , such that  $U \circ \beta = \mathsf{ev}_{\alpha} B$ . Type theoretically  $\alpha = (A, a : A)$  and  $\mathsf{ev}_{\alpha} B = Ba$  and  $\beta = (Ba, b : Ba)$ . Then composing  $\varepsilon$  with  $\mathsf{tp} \triangleleft \mathsf{tp}$  returns  $\gamma$ , which consists of (A, B). It is in this sense that Q classifies pairs of dependent terms, and  $\mathsf{tp} \triangleleft \mathsf{tp}$  extracts the underlying types.

**Definition 4.28** (Interpretation of  $\Sigma$ ). We define the natural transformation

$$\Sigma:\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}\to\mathsf{Ty}$$

as that which is induced (lemma 4.23) by the  $\Sigma$ -former operation (definition 4.14).

Define  $\mathsf{pair}:Q\to\mathsf{Tm}$  such that

$$\begin{split} &Q\,\Gamma\\ \cong &\{(\beta:\gamma\to\mathbf{grpd}_\bullet,\alpha:\Gamma\to\mathbf{grpd}_\bullet,B:\Gamma\cdot U\circ\alpha\to\mathbf{grpd})\,|\,U\circ\beta=\mathsf{ev}_\alpha\,\beta\}\\ \to &[\Gamma,\mathbf{grpd}_\bullet]\\ &(\beta,\alpha,B) \end{split}$$

 $\Sigma$  and pair combine to give us a pullback square

Proof. TODO: naturality.

TODO: prove pullback.

## References

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