Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ. Write **Set** for the category of all sets. Say that a set A is λ-small if $|A| < \lambda$. Write $\mathbf{Set}_\lambda$ for the full subcategory of **Set** spanned by λ-small sets.

Let $\mathbb{C}$ be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write $\mathbf{Psh}(\mathbb{C})$ for the category of presheaves over $\mathbb{C}$,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, \ldots \in \mathbb{C}$,
- a type in context $\mathsf{y}(\Gamma)$ as a map $A \colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$,
- a term of type A in context Γ as a map $a \colon \mathsf{y}(\Gamma) \to \mathsf{Tm}$ such that

$$\begin{tikzcd} & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

  commutes,
- an operation called "context extension" which given a context Γ and a type $A \colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{tikzcd} \mathsf{y}(\Gamma.A) \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{y}(\Gamma) \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty. We write

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{y}(\Gamma) \arrow[r, "\ulcorner X \urcorner"'] & \mathsf{Ty} \end{tikzcd}$$

to express this situation, i.e. $X \cong \mathsf{y}(\Gamma \cdot \ulcorner X \urcorner)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} \colon \mathsf{Ty} \qquad \frac{a \colon \mathsf{U}}{\mathsf{El}(a) \colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ, with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{tikzcd} \mathsf{U} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ 1 \arrow[r, "\ulcorner \mathsf{U} \urcorner"'] & \mathsf{Ty} \end{tikzcd}$$

- There is an inclusion of U into Ty

$$\mathsf{El} \colon \mathsf{U} \rightarrowtail \mathsf{Ty}$$

- $\pi : \mathsf{E} \to \mathsf{U}$ is obtained as pullback of tp; There is a pullback diagram

$$\begin{tikzcd} E \arrow[r, tail] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{U} \arrow[r, tail, "\mathsf{El}"'] & \mathsf{Ty} \end{tikzcd}$$

With the notation above, we get

$$\begin{tikzcd}
\mathsf{y}(\Gamma.\mathsf{El}(a)) \arrow[r] \arrow[d] & \mathsf{E} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\
\mathsf{y}(\Gamma) \arrow[r, "a"'] \arrow[rr, bend right, "A"'] & \mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd}$$

Both squares above are pullback squares.

3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

4 Groupoid Model of HoTT

In this section we construct a natural model in $\mathbf{Psh}(\mathbf{grpd})$ the presheaf category indexed by the category $\mathbf{grpd}$ of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on $\mathbf{grpd}$ - which comes from restricting the "classical Quillen model structure" on $\mathbf{cat}$ [Joy] to $\mathbf{grpd}$.

4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as $\mathbf{set}$ and the large sets as $\mathbf{Set}$ (in the previous sections this would have been $\mathbf{Set}_\lambda$ and $\mathbf{Set}$ respectively). We denote the category of small categories as $\mathbf{cat}$ and the large categories as $\mathbf{Cat}$. We denote the category of small groupoids as $\mathbf{grpd}$.

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

Definition 4.1 (Pointed)**.** We will take the category of pointed small categories $\mathbf{cat}_\bullet$ to have objects as pairs $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$ and morphisms as pairs

$$(F : \mathbb{C}_1 \to \mathbb{C}_0, \phi : Fc_1 \to c_0) \colon (\mathbb{C}_1, c_1) \to (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids $\mathbf{grpd}_\bullet$ will be the full subcategory of objects (Γ, c) with Γ a groupoid.

Definition 4.2 (The disply map classifier)**.** We would like to define a natural transformation in $\mathbf{Psh}(\mathbf{grpd})$

$$\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$$

with representable fibers.

Consider the functor that forgets the point

$$U \colon \mathbf{grpd}_\bullet \to \mathbf{grpd} \qquad \text{in} \quad \mathbf{Cat}.$$

If we apply the Yoneda embedding $\mathsf{y} \colon \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U\circ \colon [-, \mathbf{grpd}_\bullet] \to [-, \mathbf{grpd}] \qquad \text{in} \quad \mathbf{Psh}(\mathbf{Cat}).$$

Since any small groupoid is also a large category $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict $\mathbf{Cat}$ indexed presheaves to be $\mathbf{grpd}$ indexed presheaves. We define $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$ as the image of $U\circ$ under this restriction.

$$\begin{array}{ccccc} \mathbf{Cat} & \xrightarrow{\mathsf{y}} & \mathbf{Psh}(\mathbf{Cat}) & \xrightarrow{\mathsf{res}} & \mathbf{Psh}(\mathbf{grpd}) \\ \mathbf{grpd} & \longmapsto & [-, \mathbf{grpd}] & \longmapsto & \mathsf{Ty} \end{array}$$

Note that Tm and Ty are not representable in $\mathbf{Psh}(\mathbf{grpd})$.

Remark 4.3. By Yoneda we can identify maps with representable domain into the type classifier

$$A : \mathsf{y}\Gamma \to \mathsf{Ty} \qquad \text{in} \qquad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A : \Gamma \to \mathbf{grpd} \qquad \text{in} \qquad \mathbf{Cat}$$

Definition 4.4 (Grothendieck construction)**.** From $\mathbb{C}$ a small category and $F : \mathbb{C} \to \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in $\mathbb{C}$ we refer to Fc as the fiber over c. The objects of $\int F$ consist of pairs $(c \in \mathbb{C}, x \in Fc)$, and morphisms between (c, x) and (d, y) are pairs $(f : c \to d, \phi : F\, f\, x \to y)$. This makes the following pullback in $\mathbf{Cat}$

$$(c, x) \longmapsto (Fc, x)$$

$$\begin{tikzcd}
(c,x) \arrow[d, mapsto] & \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{cat}_\bullet \arrow[d] & (C,c) \arrow[d, mapsto] \\
c & \mathbb{C} \arrow[r, "F"'] & \mathbf{cat} & C
\end{tikzcd}$$

Definition 4.5 (Grothendieck construction for groupoids)**.** Let Γ be a groupoid and $A \colon \Gamma \to \mathbf{grpd}$ a functor, we can compose F with the inclusion $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \mathsf{disp}_A \colon \Gamma \cdot A \to \Gamma$$

This is also a small groupoid since the underlying morphisms are from the groupoid Γ. Furthermore the pullback factors through (pointed) groupoids.

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{cat}_\bullet \arrow[d] \\
\Gamma \arrow[r, "A"'] & \mathbf{grpd} \arrow[r] & \mathbf{cat}
\end{tikzcd}$$

Corollary 4.6 (The display map classifier is presentable)**.** *For any small groupoid* Γ *and* $A : \mathsf{y}\Gamma \to \mathsf{Ty}$*, the pullback of* tp *along* A *can be given by the representable map* ydisp_A*.*

$$\begin{tikzcd}
\mathsf{y}\Gamma \cdot A \arrow[r] \arrow[d, "\mathsf{y}h\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{y}\Gamma \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}$$

Proof. Consider the pullback in **Cat**

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d] \\
\Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

We send this square along $\mathsf{res} \circ \mathsf{y}$ in the following

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "\mathsf{y}"] \arrow[dr] & \mathbf{Psh}(\mathbf{Cat}) \arrow[d, "\mathsf{res}"] \\
\mathbf{grpd} \arrow[u] \arrow[r, "\mathsf{y}"] & \mathbf{Psh}(\mathbf{grpd})
\end{tikzcd}$$

The Yoneda embedding $\mathsf{y} : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does res since it is a right adjoint (with left Kan extension $\iota_! \dashv \mathsf{res}_\iota$). ☐

4.2 Groupoid fibrations

Definition 4.7 (Fibration)**.** Let $p : \mathbb{C}_1 \to \mathbb{C}_0$ be a functor. We say p is a *split Grothendieck fibration* if we have a dependent function $\mathsf{lift}\, a\, f$ satisfying the following: for any object a in $\mathbb{C}_1$ and morphism $f : p\, a \to y$ in the base $\mathbb{C}_0$ we have $\mathsf{lift}\, a\, f : a \to b$ in $\mathbb{C}_1$ such that $p(\mathsf{lift}\, a\, f) = f$ and moreover $\mathsf{lift}\, a\, g \circ f = \mathsf{lift}\, b\, g \circ \mathsf{lift}\, a\, f$

$$\begin{tikzcd}
a \arrow[r, dashed, "\mathsf{lift}\, a\, f"] \arrow[d, mapsto] \arrow[dr, phantom, "\Downarrow"] & b \arrow[d, dashed, mapsto] \\
x \arrow[r, "f"'] & y
\end{tikzcd}$$

In particular, we are intereseted in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid Γ as Fib_Γ, which is a full subcategory of the slice $\mathbf{grpd}/\Gamma$.

Note that $\mathsf{disp}_A \colon \Gamma \cdot A \to \Gamma$ is a fibration, since for any $(x \in \Gamma, a \in A\,x)$ and $f \colon x \to y$ in Γ we have a morphism $(f, \mathsf{id}_{A\,f\,a}) : (x, a) \to (y, A\,f\,a)$ lifting f. Furthermore

Proposition 4.8. *There is an adjoint equivalence*

$$\begin{tikzcd} {[\Gamma, \mathbf{grpd}]} \arrow[r, shift left, "\mathsf{disp}"] \arrow[r, phantom, "\simeq"] & \mathsf{Fib}_\Gamma \arrow[l, shift left, "\mathsf{fiber}"] \end{tikzcd}$$

where for each fibration $\delta : \Delta \to \Gamma$ and each object $x \in \Gamma$

$$\mathsf{fiber}_\delta\, x = \text{full subcategory}\,\{a \in \Delta \mid \delta\, a = x\}$$

It follows that all fibrations are pullbacks of the classifier U : $\mathbf{grpd}_\bullet \to \mathbf{grpd}$, when viewed as morphisms in **Cat**. *From now on, we will use disp_A to represent any groupoid fibration, which we can adjust up to isomorphism using this equivalence.*

Proposition 4.9 (Pullback of fibrations)**.** *Let σ : $\Delta \to Ga$ be a functor between groupoids. Since display maps are pullbacks of the classifier U : $\mathbf{grpd}_\bullet \to \mathbf{grpd}$ we have the pasting diagram*

$$\begin{tikzcd} \Delta.A\sigma \arrow[r, dashed] \arrow[d, two heads, "\mathsf{disp}_{A\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left] & \Gamma.A \arrow[r] \arrow[d, two heads, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd} \end{tikzcd}$$

Note that this avoids coherence issues, since we take the pullback to be the map $\mathsf{disp}_{A\sigma}$ specifically. It follows that fibrations are stable under pullback along all groupoid functors.

$$\begin{tikzcd} {[\Gamma, \mathbf{grpd}]} \arrow[d, "\circ\sigma"'] & \mathsf{Fib}_\Gamma \arrow[l, "fiber"'] \arrow[d, dashed, "\sigma^*"] \\ {[\Delta, \mathbf{grpd}]} \arrow[r, "\mathsf{disp}"'] & \mathsf{Fib}_\Delta \end{tikzcd}$$

Definition 4.10 (Composition of fibrations)**.** The composition of two fibrations is a fibration.

$$\begin{tikzcd} \Gamma \cdot A \cdot B \arrow[d, two heads] \arrow[dr, dashed, two heads] & \\ \Gamma \cdot A \arrow[r, two heads] & \Gamma \end{tikzcd}$$

Then given $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$ we define

$$\Sigma_A B := \mathsf{fiber}(\mathsf{disp}_B \circ \mathsf{disp}_A) : \Gamma \to \mathbf{grpd}$$

Proof. Easy by unfolding the "lift" definition of fibrations. □

Definition 4.11 (Pushforward of fibrations)**.** Given $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma{\cdot}A \to \mathbf{grpd}$ we will define $\Pi_A B : \Gamma \to \mathbf{grpd}$ such that for any $C : \Gamma \to \mathbf{grpd}$ we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathsf{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

natural in both B and C. Stated in terms of fibrations we have

$$\begin{tikzcd} \Gamma \cdot A \cdot B \arrow[d, two heads, "\mathsf{disp}_B"'] & \Gamma \cdot \Pi_A B \arrow[d, two heads, dashed] \\ \Gamma \cdot A \arrow[r, two heads, "\mathsf{disp}_A"'] & \Gamma \end{tikzcd}$$

with the universal property of pushforward

$$\mathsf{Fib}_{\Gamma \cdot A}(\mathsf{disp}_A^* \mathsf{disp}_C, \mathsf{disp}_B) \cong \mathsf{Fib}_\Gamma(\mathsf{disp}_C, \mathsf{disp}_{\Pi_A B})$$

Proof. $\Pi_A B$ takes on objects by taking fiberwise sections and act on morphisms via conjugation

$$\begin{tikzcd} x \arrow[d, "f"] \\ y \end{tikzcd} \quad \overset{\Pi_A B}{\longmapsto} \quad \begin{tikzcd} {[A(x), \Sigma_A B(x)]} \arrow[d, "\Sigma_A B(f) \circ - \circ A(f^{-1})"] \\ {[A(y), \Sigma_A B(y)]} \end{tikzcd} \qquad \begin{tikzcd} A(x) \arrow[r, "s"] & \Sigma_A B(x) \arrow[d, "\Sigma_A B(f)"] \\ A(y) \arrow[u, "A(f^{-1})"] \arrow[r, dashed, "\Pi_A B(f)(s)"'] & \Sigma_A B(y) \end{tikzcd}$$

The functor categories are groupoids since any natural transformation of functors into groupoids are natural isomorphisms. Note that conjugation is functorial and invertible. □

Proposition 4.12 (All objects are fibrant)**.** *Let $\bullet$ denote the terminal groupoid, namely that with a single object and morphism. Then the unique map $\Gamma \to \bullet$ is a fibration.*

Proposition 4.13. *TODO (*Id*) Path object fibration*

4.3 Polynomial endofunctors

Definition 4.14 (Polynomial endofunctor in an LCCC)**.** TODO

Proposition 4.15 (Universal property of polynomial endofunctors)**.** *TODO*

4.4 Π and Σ structure

Definition 4.16 (Interpretation of Π and λ)**.** Sketch: we define the natural transformation $\Pi : \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ by first taking some small groupoid Γ and defining

$$\Pi_\Gamma : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}) \to \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathsf{Ty})$$

Unfolding the universal property of $\mathsf{Poly}_{\mathsf{tp}}$ this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

$$\begin{tikzcd} \Gamma \cdot A \cdot B \arrow[d, two heads, "\mathsf{disp}_B"'] & & \mapsto & \Gamma \cdot \Pi_A B \arrow[d, two heads, "(\mathsf{disp}_A)_* \mathsf{disp}_B"] \\ \Gamma \cdot A \arrow[r, two heads, "\mathsf{disp}_A"'] & \Gamma & & \Gamma \end{tikzcd}$$

As indicated in the diagram, we take this to be the pushforward of the dependent display map disp_B along the display map it depends on disp_A. Note that this pushforward is in **grpd**, and this pushforward is only defined on fibrations.

TODO: define λ.

Proof. TODO: naturality.

TODO: prove pullback. $\square$

Definition 4.17 (Interpretation of Σ)**.** Sketch: we define the natural transformation $\Sigma : \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ by first taking some small groupoid Γ and defining

$$\Sigma_\Gamma : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}) \to \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathsf{Ty})$$

Again, this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

$$\begin{tikzcd} \Gamma \cdot A \cdot B \arrow[d, two heads, "\mathsf{disp}_B"'] & & \mapsto & \Gamma \cdot \Sigma_A B \arrow[d, two heads, "(\mathsf{disp}_A)_! \mathsf{disp}_B"] \\ \Gamma \cdot A \arrow[r, two heads, "\mathsf{disp}_A"'] & \Gamma & & \Gamma \end{tikzcd}$$

As indicated in the diagram, we take this to be the composition of disp_B and disp_A, recalling that fibrations are closed under composition.

TODO: define pair.

Proof. TODO: naturality.

TODO: prove pullback. $\square$

References

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