# Universe in the Natural Model of Type Theory

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### 1 Types

Assume an inaccessible cardinal  $\lambda$ . Write **Set** for the category of all sets. Say that a set A is  $\lambda$ -small if  $|A| < \lambda$ . Write **Set** $_{\lambda}$  for the full subcategory of **Set** spanned by  $\lambda$ -small sets.

Let  $\mathbb C$  be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write  $\mathbf{Psh}(\mathbb{C})$  for the category of presheaves over  $\mathbb{C}$ ,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map  $tp: Tm \to Ty$  consists of

- contexts as objects  $\Gamma, \Delta, \ldots \in \mathbb{C}$ ,
- a type in context  $y(\Gamma)$  as a map  $A: y(\Gamma) \to \mathsf{Ty}$ ,
- a term of type A in context  $\Gamma$  as a map  $a: y(\Gamma) \to Tm$  such that



commutes,

• an operation called "context extension" which given a context  $\Gamma$  and a type  $A \colon \mathsf{y}(\Gamma) \to \mathsf{T}\mathsf{y}$  produces a context  $\Gamma \cdot A$  which fits into a pullback diagram below.

$$\begin{array}{ccc} \mathsf{y}(\Gamma.A) & \longrightarrow \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{y}(\Gamma) & \longrightarrow_A & \mathsf{Ty} \end{array}$$

**Remark.** Sometimes, we first construct a presheaf X over  $\Gamma$  and observe that it can be classified by a map into Ty. We write

$$\begin{array}{c} X \longrightarrow \mathsf{Tm} \\ \downarrow \\ \mathsf{y}(\Gamma) \xrightarrow{\Gamma X \to \mathsf{Ty}} \mathsf{Ty} \end{array}$$

to express this situation, i.e.  $X \cong y(\Gamma \cdot \lceil X \rceil)$ .

## 2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U}\colon \mathsf{Ty} \qquad \frac{a\colon \mathsf{U}}{\mathsf{El}(a)\colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal  $\kappa$ , with  $\kappa < \lambda$ .)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{c} \mathsf{U} \longrightarrow \mathsf{Tm} \\ \downarrow \\ \downarrow \\ 1 \xrightarrow{} \mathsf{Ty} \end{array}$$

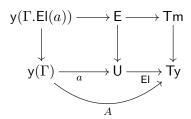
• There is an inclusion of U into Ty

$$\mathsf{EI}\colon\mathsf{U}\rightarrowtail\mathsf{Ty}$$

•  $\pi: E \to U$  is obtained as pullback of tp; There is a pullback diagram

$$\begin{array}{c} E {\longmapsto} \operatorname{Tm} \\ \downarrow \qquad \qquad \downarrow \\ \operatorname{U} {\longmapsto} \operatorname{Ty} \end{array}$$

With the notation above, we get



Both squares above are pullback squares.

- 3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model
- 4 Groupoid Model of HoTT

In this section we construct a natural model in **Psh**(**grpd**) the presheaf category indexed by the category **grpd** of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on **grpd** - which comes from restricting the "classical Quillen model structure" on **cat** [Joy] to **grpd**.

#### 4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as **set** and the large sets as **Set** (in the previous sections this would have been  $\mathbf{Set}_{\lambda}$  and  $\mathbf{Set}$  respectively). We denote the category of small categories as **cat** and the large categories as **Cat**. We denote the category of small groupoids as **grpd**. The category of small pointed groupoids will be  $\mathbf{grpd}_{\bullet}$ .

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$Psh(grpd) = [grpd^{op}, Set]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

**Definition 4.1** (The disply map classifier). We would like to define a natural transformation in **Psh**(**grpd**)

$$\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$$

with representable fibers.

Consider the functor that forgets the point

$$U \colon \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$$
 in **Cat**.

If we apply the Yoneda embedding y:  $Cat \rightarrow Psh(Cat)$  to U we obtain

$$U \circ : [-, \mathbf{grpd}_{\bullet}] \to [-, \mathbf{grpd}]$$
 in  $\mathbf{Psh}(\mathbf{Cat})$ .

Since any small groupoid is also a large category  $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$ , we can restrict  $\mathbf{Cat}$  indexed presheaves to be  $\mathbf{grpd}$  indexed presheaves. We define  $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$  as the image of  $U \circ \mathsf{under}$  this restriction.

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) & \xrightarrow{\mathsf{res}} & \mathbf{Psh}(\mathbf{grpd}) \\ \\ \mathbf{grpd} & \longmapsto & [-,\mathbf{grpd}] & \longmapsto & \mathsf{Ty} \end{array}$$

Note that Tm and Ty are not representable in Psh(grpd).

Remark 4.2. By Yoneda we can identify maps with representable domain into the type classifier

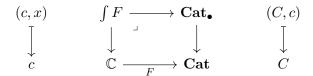
$$A: \mathsf{y}\Gamma \to \mathsf{T}\mathsf{y} \qquad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A:\Gamma\to\mathbf{grpd}$$
 in Cat

**Definition 4.3** (Grothendieck construction). From  $\mathbb{C}$  a small category and  $F: \mathbb{C} \to \mathbf{Cat}$  a functor, we construct a small category  $\int F$ . For any c in  $\mathbb{C}$  we refer to Fc as the fiber over c. The objects of  $\int F$  consist of pairs  $(c \in \mathbb{C}, x \in Fc)$ , and morphisms are those in  $\mathbb{C}$  preserving the points in the fiber. This makes the following pullback

$$(c,x) \longmapsto (Fc,x)$$



where Cat<sub>•</sub> is the category of large pointed categories.

**Definition 4.4** (Grothendieck construction for groupoids). Let  $\Gamma$  be a groupoid and  $A \colon \Gamma \to \mathbf{grpd}$  a functor, we can compose F with the inclusion  $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$  and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \mathsf{disp}_A \colon \Gamma \cdot A \to \Gamma$$

This is also a small groupoid since the underlying morphisms are from the groupoid  $\Gamma$ . Furthermore the pullback factors through (pointed) groupoids.

$$\begin{array}{cccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} & \longrightarrow \mathbf{Cat}_{\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \longrightarrow_A & \mathbf{grpd} & \longrightarrow \mathbf{Cat} \end{array}$$

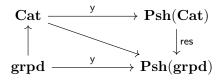
Corollary 4.5 (The display map classifier is presentable). For any small groupoid  $\Gamma$  and  $A: y\Gamma \to Ty$ , the pullback of tp along A can be given by the representable map  $ydisp_A$ .

$$\begin{array}{ccc} \mathbf{y}\Gamma \cdot A & \longrightarrow & \mathsf{Tm} \\ \mathsf{disp}_A & & & \mathsf{tp} \\ & \mathbf{y}\Gamma & \longrightarrow & \mathsf{Ty} \end{array}$$

*Proof.* Consider the pullback in Cat

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow \mathbf{grpd} \end{array}$$

We send this square along  $res \circ y$  in the following



The Yoneda embedding  $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$  preserves pullbacks, as does res since it is a right adjoint (with left Kan extension  $\iota_! \dashv \mathsf{res}_\iota$ ).

### 4.2 Groupoid fibrations

**Definition 4.6** (Isofibration). TODO

Unless specified otherwise, by a (Groupoid) *fibration* we will mean an isofibration. **Proposition 4.7** (Display map properties of fibrations). *TODO* 

- 1. (subst) Stable under pullback
- 2.  $(\Sigma)$  Closed under composition
- 3.  $(\Pi)$  Closed under pushforward
- 4. (Fibrant objects) Map to terminal is a fibration
- 5. (Id) Path object fibration

### 4.3 Polynomial endofunctors

**Definition 4.8** (Polynomial endofunctor in an LCCC). TODO

**Proposition 4.9** (Universal property of polynomial endofunctors). *TODO* 

#### 4.4 $\Pi$ and $\Sigma$ structure

**Definition 4.10** (Interpretation of  $\Pi$  and  $\lambda$ ). Sketch: we define the natural transformation  $\Pi : \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \to \mathsf{Ty}$  by first taking some small groupoid  $\Gamma$  and defining

$$\Pi_{\Gamma}:\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty})\to\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Ty})$$

Unfolding the universal property of  $\mathsf{Poly}_\mathsf{tp}$  this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

As indicated in the diagram, we take this to be the pushforward of the dependent display map  $\operatorname{disp}_B$  along the display map it depends on  $\operatorname{disp}_A$ . Note that this pushforward is in  $\operatorname{\mathbf{grpd}}$ , and this pushforward is only defined on fibrations.

TODO: define  $\lambda$ .

Proof. TODO: naturality.

TODO: prove pullback.  $\Box$ 

**Definition 4.11** (Interpretation of  $\Sigma$ ). Sketch: we define the natural transformation  $\Sigma$ : Poly<sub>tp</sub>Ty  $\to$  Ty by first taking some small groupoid  $\Gamma$  and defining

$$\Sigma_{\Gamma}:\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty})\to\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Ty})$$

Again, this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

As indicated in the diagram, we take this to be the composition of  $\mathsf{disp}_B$  and  $\mathsf{disp}_A$ , recalling that fibrations are closed under composition.

TODO: define pair.

*Proof.* TODO: naturality.

TODO: prove pullback.  $\Box$ 

# References

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- [HS98] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.
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