

Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ . Write **Set** for the category of all sets. Say that a set A is λ -small if $|A| < \lambda$. Write **Set** $_{\lambda}$ for the full subcategory of **Set** spanned by λ -small sets.

Let \mathbb{C} be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write **Psh**(\mathbb{C}) for the category of presheaves over \mathbb{C} ,

$$\mathbf{Psh}(\mathbb{C}) =_{\text{def}} [\mathbb{C}^{\text{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $\text{tp}: \mathsf{Tm} \rightarrow \mathsf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, \dots \in \mathbb{C}$,
- a type in context $y(\Gamma)$ as a map $A: y(\Gamma) \rightarrow \mathsf{Ty}$,
- a term of type A in context Γ as a map $a: y(\Gamma) \rightarrow \mathsf{Tm}$ such that

$$\begin{array}{ccc} & \mathsf{Tm} & \\ & \uparrow a & \downarrow \text{tp} \\ \Gamma & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

commutes,

- an operation called “context extension” which given a context Γ and a type $A: y(\Gamma) \rightarrow \mathsf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{array}{ccc} y(\Gamma \cdot A) & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty . We write

$$\begin{array}{ccc} X & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{\ulcorner X \urcorner} & \mathsf{Ty} \end{array}$$

to express this situation, i.e. $X \cong y(\Gamma \cdot \ulcorner X \urcorner)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U , not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} : \mathsf{Ty} \quad \frac{a : \mathsf{U}}{\mathsf{El}(a) : \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ , with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi : \mathsf{E} \rightarrow \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\ulcorner \mathsf{U} \urcorner} & \mathsf{Ty} \end{array}$$

- There is an inclusion of U into Ty

$$\mathsf{El} : \mathsf{U} \rightarrowtail \mathsf{Ty}$$

- $\pi : \mathsf{E} \rightarrow \mathsf{U}$ is obtained as pullback of tp ; There is a pullback diagram

$$\begin{array}{ccc} \mathsf{E} & \twoheadrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{U} & \twoheadrightarrow_{\mathsf{El}} & \mathsf{Ty} \end{array}$$

With the notation above, we get

$$\begin{array}{ccccc}
 y(\Gamma, \text{El}(a)) & \longrightarrow & E & \longrightarrow & Tm \\
 \downarrow & & \downarrow & & \downarrow \\
 y(\Gamma) & \xrightarrow{a} & U & \xrightarrow{\text{El}} & Ty \\
 & \searrow \scriptstyle A & \nearrow & &
 \end{array}$$

Both squares above are pullback squares.

3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

4 Groupoid Model of HoTT

In this section we construct a natural model in $\mathbf{Psh}(\mathbf{grpd})$ the presheaf category indexed by the category \mathbf{grpd} of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on \mathbf{grpd} - which comes from restricting the “classical Quillen model structure” on \mathbf{cat} [Joy] to \mathbf{grpd} .

4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as \mathbf{set} and the large sets as \mathbf{Set} (in the previous sections this would have been \mathbf{Set}_λ and \mathbf{Set} respectively). We denote the category of small categories as \mathbf{cat} and the large categories as \mathbf{Cat} . We denote the category of small groupoids as \mathbf{grpd} .

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section, \mathbf{Tm} and \mathbf{Ty} and so on will refer to the natural model semantics in this specific model.

Definition 4.1 (Pointed). We will take the category of pointed small categories \mathbf{cat}_\bullet to have objects as pairs $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$ and morphisms as pairs

$$(F : \mathbb{C}_1 \rightarrow \mathbb{C}_0, \phi : Fc_1 \rightarrow c_0) : (\mathbb{C}_1, c_1) \rightarrow (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids \mathbf{grpd}_\bullet will be the full subcategory of objects (Γ, c) with Γ a groupoid.

Definition 4.2 (The display map classifier). We would like to define a natural transformation in $\mathbf{Psh}(\mathbf{grpd})$

$$\mathbf{tp} : \mathbf{Tm} \rightarrow \mathbf{Ty}$$

with representable fibers.

Consider the functor that forgets the point

$$U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}.$$

If we apply the Yoneda embedding $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U \circ [-, \mathbf{grpd}_\bullet] \rightarrow [-, \mathbf{grpd}] \quad \text{in} \quad \mathbf{Psh}(\mathbf{Cat}).$$

Since any small groupoid is also a large category $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict \mathbf{Cat} indexed presheaves to be \mathbf{grpd} indexed presheaves. We define $\mathbf{tp} : \mathbf{Tm} \rightarrow \mathbf{Ty}$ as the image of $U \circ$ under this restriction.

$$\begin{aligned} \mathbf{Cat} &\xrightarrow{y} \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\text{res}} \mathbf{Psh}(\mathbf{grpd}) \\ \mathbf{grpd} &\longmapsto [-, \mathbf{grpd}] \longmapsto \mathbf{Ty} \end{aligned}$$

Note that \mathbf{Tm} and $\mathbf{T}\mathbf{y}$ are not representable in $\mathbf{Psh}(\mathbf{grpd})$.

Remark 4.3. By Yoneda we can identify maps with representable domain into the type classifier

$$A : \mathbf{y}\Gamma \rightarrow \mathbf{T}\mathbf{y} \quad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A : \Gamma \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}$$

Definition 4.4 (Grothendieck construction). From \mathbb{C} a small category and $F : \mathbb{C} \rightarrow \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in \mathbb{C} we refer to Fc as the fiber over c . The objects of $\int F$ consist of pairs $(c \in \mathbb{C}, x \in Fc)$, and morphisms between (c, x) and (d, y) are pairs $(f : c \rightarrow d, \phi : Ff x \rightarrow y)$. This makes the following pullback in \mathbf{Cat}

$$\begin{array}{ccccc} (c, x) & & \int F & \longrightarrow & \mathbf{cat}_\bullet & & (C, c) \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ c & & \mathbb{C} & \xrightarrow{F} & \mathbf{cat} & & C \end{array}$$

Definition 4.5 (Grothendieck construction for groupoids). Let Γ be a groupoid and $A : \Gamma \rightarrow \mathbf{grpd}$ a functor, we can compose F with the inclusion $i : \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \quad \text{disp}_A : \Gamma \cdot A \rightarrow \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids Γ and Ax for $x \in \Gamma$. Furthermore the pullback factors through (pointed) groupoids.

$$\begin{array}{ccccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet & \longrightarrow & \mathbf{cat}_\bullet \\ \text{disp}_A \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} & \longrightarrow & \mathbf{cat} \end{array}$$

Corollary 4.6 (The display map classifier is presentable). *For any small groupoid Γ and $A : y\Gamma \rightarrow \mathbf{Ty}$, the pullback of \mathbf{tp} along A can be given by the representable map $y\mathbf{disp}_A$.*

$$\begin{array}{ccc} y\Gamma \cdot A & \longrightarrow & \mathbf{Tm} \\ y\mathbf{disp}_A \downarrow & \lrcorner & \downarrow \mathbf{tp} \\ y\Gamma & \xrightarrow{A} & \mathbf{Ty} \end{array}$$

Proof. Consider the pullback in **Cat**

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

We send this square along $\mathbf{res} \circ y$ in the following

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) \\ \uparrow & \searrow & \downarrow \mathbf{res} \\ \mathbf{grpd} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{grpd}) \end{array}$$

The Yoneda embedding $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does \mathbf{res} since it is a right adjoint (with left Kan extension $u_! \dashv \mathbf{res}_!$). \square

4.2 Groupoid fibrations

Definition 4.7 (Fibration). Let $p : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ be a functor. We say p is a *split Grothendieck fibration* if we have a dependent function $\mathbf{lift} \, a \, f$ satisfying the following: for any object a in \mathbb{C}_1 and morphism $f : pa \rightarrow y$ in the base \mathbb{C}_0 we have $\mathbf{lift} \, a \, f : a \rightarrow b$ in \mathbb{C}_1 such that $p(\mathbf{lift} \, a \, f) = f$ and moreover $\mathbf{lift} \, a \, g \circ f = \mathbf{lift} \, b \, g \circ \mathbf{lift} \, a \, f$

$$\begin{array}{ccc} a & \xrightarrow{\mathbf{lift} \, a \, f} & b \\ \downarrow & \Downarrow & \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

In particular, we are interested in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid Γ as \mathbf{Fib}_Γ , which is a full subcategory of the slice \mathbf{grpd}/Γ . We will decorate an arrow with \rightarrow to indicate it is a fibration.

Note that $\mathrm{disp}_A : \Gamma \cdot A \rightarrow \Gamma$ is a fibration, since for any $(x \in \Gamma, a \in Ax)$ and $f : x \rightarrow y$ in Γ we have a morphism $(f, \mathrm{id}_{Afa}) : (x, a) \rightarrow (y, Afa)$ lifting f . Furthermore

Proposition 4.8. *There is an adjoint equivalence*

$$[\Gamma, \mathbf{grpd}] \begin{array}{c} \xrightarrow{\mathrm{disp}} \\ \xleftarrow[\mathrm{fiber}]{\simeq} \end{array} \mathbf{Fib}_\Gamma$$

where for each fibration $\delta : \Delta \rightarrow \Gamma$ and each object $x \in \Gamma$

$$\mathrm{fiber}_\delta x = \text{full subcategory } \{a \in \Delta \mid \delta a = x\}$$

It follows that all fibrations are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd}$, when viewed as morphisms in \mathbf{Cat} .

Pullback of fibrations along groupoid functors is not strictly coherent, in the sense that for $\tau : \Xi \rightarrow \Delta$ and $\sigma : \Delta \rightarrow \Gamma$ and a fibration $p \in \mathbf{Fib}_\Gamma$ we only have an isomorphism

$$\tau^* \sigma^* p \cong (\sigma \circ \tau)^* p$$

rather than equality.

In order to interpret reindexing/substitution strictly, it is convenient to work with classifiers $[\Gamma, \mathbf{grpd}]$ instead of fibrations.

Proposition 4.9 (Strictly coherent pullback). *Let $\sigma : \Delta \rightarrow \Gamma$ be a functor between groupoids. Since display maps are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd}$ we have the pasting diagram*

$$\begin{array}{ccccc} & & \Delta.A\sigma & \overset{\sigma_A}{\dashrightarrow} & \Gamma.A & \longrightarrow & \mathbf{grpd}_\bullet \\ & & \downarrow \mathrm{disp}_{A\sigma} & \lrcorner & \downarrow \mathrm{disp}_A & \lrcorner & \downarrow \\ \Delta & \xrightarrow{\sigma} & \Gamma & \xrightarrow{A} & \mathbf{grpd} & & \end{array}$$

This gives us a functor $\circ\sigma : [\Gamma, \mathbf{grpd}] \rightarrow [\Delta, \mathbf{grpd}]$ which is our strict version of pullback.

Corollary 4.10 (Fibrations are stable under pullback).

$$\begin{array}{ccc} [\Gamma, \mathbf{grpd}] & \xleftarrow{\mathrm{fiber}} & \mathbf{Fib}_\Gamma \\ \circ\sigma \downarrow & & \downarrow \sigma^* \\ [\Delta, \mathbf{grpd}] & \xrightarrow{\mathrm{disp}} & \mathbf{Fib}_\Delta \end{array}$$

We can deduce a corresponding fact about fibrations: since fibrations are closed under isomorphism, and since any pullback in **grpd** of a fibration p is isomorphic to the display map $\text{disp}_{\text{fiber}_{p \circ \sigma}}$, any pullback of a fibration is a fibration.

A strict interpretation of type theory would require Σ and Π -formers to be stable under pullback (Beck-Chevalley). Thus we again define these as operations on classifiers $[\Gamma, \mathbf{grpd}]$.

Definition 4.11 (Σ -former operation). Then given $A : \Gamma \rightarrow \mathbf{grpd}$ and $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$ we define $\Sigma_A B : \Gamma \rightarrow \mathbf{grpd}$ such that $\Sigma_A B$ acts on objects by forming fiberwise Grothendieck constructions

$$\Sigma_A B(x) := A(x) \cdot B \circ x_A$$

where $x_A : A(x) \rightarrow \Gamma \cdot A$ takes $f : a_0 \rightarrow a_1$ to $(\text{id}_x, f) : (x, a_0) \rightarrow (x, a_1)$

$$\begin{array}{ccccc} A(x) \cdot B \circ x_A & \dashrightarrow & \Gamma \cdot A \cdot B & \longrightarrow & \bullet \\ \downarrow \text{disp}_{B \circ x_A} & & \downarrow \text{disp}_B & & \\ A(x) & \xrightarrow{x_A} & \Gamma \cdot A & \xrightarrow{B} & \mathbf{grpd} \\ \downarrow ! & \lrcorner & \downarrow \text{disp}_A & \lrcorner & \\ \bullet & \xrightarrow{x} & \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

$\Sigma_A B$ acts on morphism $f : x \rightarrow y$ in Γ and $(a \in A(x), b \in B(x, a))$ by

$$\Sigma_A B f(a, b) := (A f a, B(f, \text{id}_{A f a}) b)$$

and for morphism $(\alpha : a_0 \rightarrow a_1 \in A(x), \beta : B(\text{id}_x, \alpha) b_0 \rightarrow b_1 \in B(x, a_1))$ in $\Sigma_A B x$

$$\Sigma_A B f(\alpha, \beta) := (A f \alpha, B(f, \text{id}_{A f a_1}) \beta)$$

Let us also define the natural transformation $\text{fst} : \Sigma_A B \rightarrow A$ by

$$\text{fst}_x : (a, b) \mapsto a$$

Proposition 4.12 (Fibrations are closed under composition). *The corresponding fact about fibrations is that the composition of two fibrations is a fibration.*

$$\begin{array}{ccc} \Xi & & \\ \downarrow & \searrow & \\ \Delta & \longrightarrow & \Gamma \end{array}$$

We can compare the two fibrations

$$\text{disp}_B \circ \text{disp}_A \quad \text{and} \quad \text{disp}_{\Sigma_A(B)}$$

An object in the composition would look like $((x, a), b)$ for $x \in \Gamma$, $a \in A(x)$ and $b \in B(x, a)$, whereas an object in $\Gamma \cdot \Sigma_A(B)$ would instead be $(x, (a, b))$.

Proposition 4.13 (Strict Beck-Chevalley for Σ). *Let $\sigma : \Delta \rightarrow \Gamma$, $A : \Gamma \rightarrow \mathbf{grpd}$ and $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$. Then*

$$(\Sigma_A B) \circ \sigma = \Sigma_{A \circ \sigma} (B \circ \sigma_A)$$

where σ_A is uniquely determined by the pullback in

$$\begin{array}{ccccc} \Delta \cdot A\sigma \cdot B \circ \sigma_A & \xrightarrow{\sigma_{A \cdot B}} & \Gamma \cdot A \cdot B & & \\ \downarrow \text{disp}_{B \circ \sigma_A} & & \downarrow \text{disp}_B & & \\ \Delta \cdot A\sigma & \xrightarrow{\sigma_A} & \Gamma \cdot A & \xrightarrow{B} & \mathbf{grpd} \\ \downarrow \text{disp}_{A\sigma} & \lrcorner & \downarrow \text{disp}_A & \lrcorner & \\ \mathbf{grpd} \xleftarrow[\Sigma_{A \circ \sigma}(B \circ \sigma_A)]{(\Sigma_A B) \circ \sigma} \Delta & \xrightarrow{\sigma} & \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

Proof. By checking pointwise at $x \in \Delta$, this boils down to showing

$$(\sigma x)_A = \sigma_A \circ x_{A \circ \sigma} : A(\sigma x) \rightarrow \Gamma \cdot A$$

$$\begin{array}{ccccccc} & & (\sigma x)_A & & & & \\ & \searrow & & \nearrow & & & \\ A(\sigma x) & \xrightarrow{x_{A\sigma}} & \Delta \cdot A\sigma & \xrightarrow{\sigma_A} & \Gamma \cdot A & \xrightarrow{B} & \mathbf{grpd} \\ \downarrow \text{!} & \lrcorner & \downarrow \text{disp}_{A\sigma} & \lrcorner & \downarrow \text{disp}_A & \lrcorner & \\ \bullet & \xrightarrow{x} & \Delta & \xrightarrow{\sigma} & \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

which holds because of the universal property of pullback. \square

Definition 4.14 (Π -former operation). Given $A : \Gamma \rightarrow \mathbf{grpd}$ and $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$ we will define $\Pi_A B : \Gamma \rightarrow \mathbf{grpd}$ such that for any $C : \Gamma \rightarrow \mathbf{grpd}$ we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\text{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

natural in both B and C .

Proof. $\Pi_A B$ acts on objects by taking fiberwise sections

$$\Pi_A B(x) := \{s \in [A(x), \Sigma_A B(x)] \mid \text{fst}_x \circ s = \text{id}_{A(x)}\}$$

Where we have taken the full subcategory of the functor category $[A(x), \Sigma_A B(x)]$. This is a groupoid since any natural transformation of functors into groupoids are natural isomorphisms.

$\Pi_A B$ acts on morphisms via conjugation

$$\begin{array}{ccccc}
 x & & \Pi_A B(x) & & A(x) \xrightarrow{s} \Sigma_A B(x) \\
 \downarrow f & \xrightarrow{\Pi_A B} & \downarrow \Sigma_A B(f) \circ - \circ A(f^{-1}) & & \uparrow A(f^{-1}) \quad \downarrow \Sigma_A B(f) \\
 y & & \Pi_A B(y) & & A(y) \xrightarrow{\Pi_A B(f)(s)} \Sigma_A B(y)
 \end{array}$$

Note that conjugation is functorial and invertible. \square

Corollary 4.15 (Fibrations are closed under pushforward). *Stated in terms of fibrations, we have*

$$\begin{array}{ccc}
 \Xi & & \Gamma \downarrow \sigma_* \tau \\
 \tau \downarrow & & \downarrow \sigma_* \tau \\
 \Delta & \xrightarrow{\sigma} & \Gamma
 \end{array}$$

with the universal property of pushforward

$$\text{Fib}_\Delta(\sigma^* \rho, \tau) \cong \text{Fib}_\Gamma(\rho, \sigma_* \tau)$$

natural in both τ and ρ .

Proposition 4.16 (Strict Beck-Chevalley for Π). *Let $\sigma : \Delta \rightarrow \Gamma$, $A : \Gamma \rightarrow \mathbf{grpd}$ and $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$. Then*

$$(\Pi_A B) \circ \sigma = \Pi_{A \circ \sigma}(B \circ \sigma_A)$$

where σ_A is uniquely determined by the pullback in

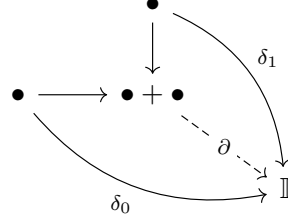
$$\begin{array}{ccccc}
 \Delta \cdot A \sigma \cdot B \circ \sigma_A & \xrightarrow{\sigma_{A \cdot B}} & \Gamma \cdot A \cdot B & & \\
 \downarrow \text{disp}_{B \circ \sigma_A} & & \downarrow \text{disp}_B & & \\
 \Delta \cdot A \sigma & \xrightarrow{\sigma_A} & \Gamma \cdot A & \xrightarrow{B} & \mathbf{grpd} \\
 \downarrow \text{disp}_{A \sigma} & \lrcorner & \downarrow \text{disp}_A & \lrcorner & \\
 \mathbf{grpd} & \xleftarrow[\Pi_{A \circ \sigma}(B \circ \sigma_A)]{\Pi_A B \circ \sigma} \Delta & \xrightarrow{\sigma} & \Gamma & \xrightarrow{A} \mathbf{grpd}
 \end{array}$$

Proof. By checking pointwise, this boils down to Beck-Chevalley for Σ . \square

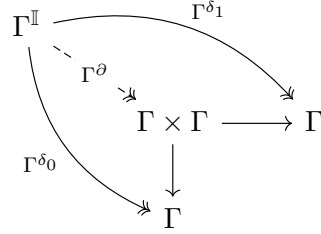
Proposition 4.17 (All objects are fibrant). *Let \bullet denote the terminal groupoid, namely that with a single object and morphism. Then the unique map $\Gamma \rightarrow \bullet$ is a fibration.*

Definition 4.18 (Interval). Let the interval groupoid \mathbb{I} be the small groupoid with two objects and a single non-identity isomorphism. There are two distinct morphisms $\delta_0, \delta_1 : \bullet \rightarrow \mathbb{I}$ and a natural isomorphism $i : \delta_0 \Rightarrow \delta_1$. Note that δ_0 and δ_1 both form adjoint equivalences with the unique map $! : \mathbb{I} \rightarrow \bullet$.

Denote by $\bullet + \bullet$ the small groupoid with two objects and only identity morphisms. Then let $\partial : \bullet + \bullet \rightarrow \mathbb{I}$ be the unique map factoring δ_0 and δ_1 .



Proposition 4.19 (Path object fibration). *Let Γ be a small groupoid. Recall that **grpd** is Cartesian closed, so we can take the image of the above diagram under the functor Γ^- .*



Then the indicated morphisms are fibrations, and $\Gamma^{\delta_0}, \Gamma^{\delta_1}$ form adjoint equivalences with $\Gamma^! : \Gamma \rightarrow \Gamma^{\mathbb{I}}$.

4.3 Polynomial endofunctors

Definition 4.20 (Polynomial endofunctor on a morphism in an locally Cartesian closed category). Let \mathbb{C} be a locally Cartesian closed category (we will take presheaves on small groupoids). This means for each morphism $t : B \rightarrow A$ we have an adjoint triple

$$\begin{array}{c} \mathbb{C}/B \\ \left(\begin{array}{c} \uparrow \\ \dashv \quad t^* \quad \dashv \\ \downarrow \end{array} \right)_{t_!} \\ \mathbb{C}/A \end{array}$$

where t^* is pullback, and $t_!$ is composition with t .

Let $t : B \rightarrow A$ be a morphism in \mathbb{C} . Then define $\text{Poly}_t : \mathbb{C} \rightarrow \mathbb{C}$ be the composition

$$\text{Poly}_t := A_! \circ t_* \circ B^* \quad \mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

Proposition 4.21. *The data of a map into the polynomial applied to an object in \mathbb{C}*

$$X \xrightarrow{\phi} \text{Poly}_t Y$$

corresponds to

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \text{Poly}_t Y \\ & \searrow \alpha & \swarrow t_* B^* Y \\ & A & \end{array}$$

Applying the adjunction $A_! \dashv A^$, this corresponds to*

$$\alpha : X \rightarrow A \quad \text{and} \quad \begin{array}{ccc} B_! t^* \alpha & \xrightarrow{\tilde{\phi}} & B \times Y \\ & \searrow t^* \alpha & \swarrow B^* Y \\ & B & \end{array}$$

Applying the adjunction $t^ \dashv t_*$, this corresponds to*

$$\alpha : X \rightarrow A \quad \text{and} \quad B_! t^* \alpha \xrightarrow{\beta} Y$$

Proposition 4.22 ($\text{Poly}_{\text{tp}} \text{Ty}$ classifies dependent types). *Specialized to $\text{tp} : \text{Tm} \rightarrow \text{Ty}$ in $\mathbf{Psh}(\mathbf{grpd})$, the previous proposition says that a map from a representable $\Gamma \rightarrow \text{Poly}_{\text{tp}} \text{Ty}$ corresponds to the data of*

$$A : \Gamma \rightarrow \text{Ty} \quad \text{and} \quad B : \Gamma \cdot A \rightarrow \text{Ty}$$

which by Yoneda corresponds to the data in \mathbf{Cat} of

$$A : \Gamma \rightarrow \mathbf{grpd} \quad \text{and} \quad B : \Gamma \cdot A \rightarrow \mathbf{grpd}$$

Furthermore, if $\sigma : \Delta \rightarrow \Gamma$ were a representable map, then we have a naturality square

$$\begin{array}{ccc} \Gamma & \Sigma_{A \in [\Gamma, \mathbf{grpd}]}[\Gamma.A, \mathbb{C}] & \xrightarrow{\cong} \text{Poly}_{\text{tp}}[\Gamma, \mathbb{C}] \\ \sigma \uparrow & \downarrow (- \circ \sigma, - \circ \sigma_A) & \downarrow \text{Poly}_{\text{tp}}(- \circ \sigma) \\ \Delta & \Sigma_{A \in [\Delta, \mathbf{grpd}]}[\Delta.A, \mathbb{C}] & \xrightarrow{\cong} \text{Poly}_{\text{tp}}[\Delta, \mathbb{C}] \end{array}$$

4.4 Π and Σ structure

Lemma 4.23. *Let \mathbb{C} be a large category, and let $[-, \mathbb{C}] \in \mathbf{Psh}(\mathbf{grpd})$ be the restriction of the Yoneda embedding $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$. Let F be an operation that takes a groupoid Γ , a functor $A : \Gamma \rightarrow \mathbf{grpd}$ and $B : \Gamma \cdot A \rightarrow \mathbb{C}$ and returns a functor $F_A B : \Gamma \rightarrow \mathbb{C}$.*

Then $\tilde{F} : \text{Poly}_{\text{tp}}[-, \mathbb{C}] \rightarrow [-, \mathbb{C}]$

$$\tilde{F}_{\Gamma}(A, B) = F_A B$$

defines a natural transformation if and only if F satisfies the strict Beck-Chevalley condition

$$(F_A B) \circ \sigma = F_{A \circ \sigma}(B \circ \sigma_A)$$

where σ_A is given by

$$\begin{array}{ccccc} \Delta \cdot A \circ \sigma & \xrightarrow{\sigma_A} & \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_{\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \xrightarrow{\sigma} & \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

Proof. Using proposition 4.22

$$\begin{array}{ccccc} & & F & & \\ & \curvearrowright & & \curvearrowleft & \\ \Gamma & \Sigma_{A \in [\Gamma, \mathbf{grpd}]} [\Gamma.A, \mathbb{C}] & \longrightarrow & \text{Poly}_{\text{tp}}[\Gamma, \mathbb{C}] & \xrightarrow{F_{\Gamma}} [\Gamma, \mathbb{C}] \\ \sigma \uparrow & \downarrow (-\circ \sigma, -\circ \sigma_A) & & \downarrow \text{Poly}_{\text{tp}} - \circ \sigma & \downarrow -\circ \sigma \\ \Delta & \Sigma_{A \in [\Delta, \mathbf{grpd}]} [\Delta.A, \mathbb{C}] & \longrightarrow & \text{Poly}_{\text{tp}}[\Delta, \mathbb{C}] & \xrightarrow{F_{\Delta}} [\Delta, \mathbb{C}] \\ & & F & & \end{array}$$

□

Definition 4.24 (Interpretation of Π types). We define the natural transformation $\Pi : \text{Poly}_{\text{tp}} \text{Ty} \rightarrow \text{Ty}$ as that which is induced (lemma 4.23) by the Π -former operation (definition 4.14).

Then we define the natural transformation $\lambda : \text{Poly}_{\text{tp}} \text{Ty} \rightarrow \text{Ty}$ as the natural transformation induced by the following operation: given $A : \Gamma \rightarrow \mathbf{grpd}$ and $\beta : \Gamma \cdot A \rightarrow \mathbf{grpd}_{\bullet}$, $\lambda_A \beta : \Gamma \rightarrow \mathbf{grpd}_{\bullet}$ will be the functor such that on objects $x \in \Gamma$

$$\lambda_A \beta(x) := (\Pi_A B(x), a \mapsto (a, b(x, a)))$$

where $B := U \circ \beta : \Gamma \cdot A \rightarrow \mathbf{grpd}$ and $b(x, a)$ is the point in $\beta(x, a)$. On morphisms $f : x \rightarrow y$ in Γ we have

$$\lambda_A \beta(f) := (\Pi_A B(f), \eta)$$

where $\eta : \Pi_A B f s_x \rightarrow s_y$ is a natural isomorphism between functors $A_y \rightarrow \Sigma_A B y$ given on objects $a \in A_y$ by

$$\eta_a := (\text{id}_a, \text{id}_{b(y, a)})$$

These combine to give us a pullback square

$$\begin{array}{ccc} \text{Poly}_{\text{tp}} \text{Tm} & \xrightarrow{\lambda} & \text{Tm} \\ \text{Poly}_{\text{tp}} \text{tp} \downarrow & \lrcorner & \downarrow \text{tp} \\ \text{Poly}_{\text{tp}} \text{Ty} & \xrightarrow{\Pi} & \text{Ty} \end{array}$$

Proof. We should check that the λ operation satisfied Beck-Chevalley. This follows from the Π satisfying Beck-Chevalley and extensionality results for functors.

The square commutes and is a pullback if and only if it pointwise commutes and pointwise gives pullbacks, i.e. for each groupoid Γ

$$\begin{array}{ccc} \text{Poly}_{\text{tp}} \text{ Tm } \Gamma & \xrightarrow{\lambda_{\Gamma}} & [\Gamma, \mathbf{grpd.}] \\ \text{Poly}_{\text{tp}} \text{tp}_{\Gamma} \downarrow & \lrcorner & \downarrow U_{\circ-} \\ \text{Poly}_{\text{tp}} \text{ Ty } \Gamma & \xrightarrow{\Pi_{\Gamma}} & [\Gamma, \mathbf{grpd}] \end{array}$$

by proposition 4.22 this holds if and only if

$$\begin{array}{ccc} \Sigma_{A \in [\Gamma, \mathbf{grpd}]} [\Gamma.A, \mathbf{grpd.}] & \xrightarrow{\lambda} & [\Gamma, \mathbf{grpd.}] \\ (-\circ\sigma, -\circ\sigma_A) \downarrow & \lrcorner & \downarrow U\circ- \\ \Sigma_{A \in [\Gamma, \mathbf{grpd}]} [\Gamma.A, \mathbf{grpd}] & \xrightarrow{\Pi} & [\Gamma, \mathbf{grpd}] \end{array}$$

which follows from the definitions of Π and λ .

Definition 4.25 (Evaluation). Define the operation of evaluation ev to take $\alpha : \Gamma \rightarrow \mathbf{grpd}_\bullet$ and $B : \Gamma \cdot U \circ \alpha \rightarrow \mathbf{grpd}$ and return $\text{ev}_\alpha B : \Gamma \rightarrow \mathbf{grpd}$, described below.

Commutative diagram illustrating the relationship between various types and terms:

- Top node: Γ
- Second row nodes: $Tm \times_{Ty}$, $Poly_{tp} Ty$, $Poly_{tp} Ty$
- Third row nodes: $Tm \times Ty$, Tm , Ty
- Bottom row nodes: Ty , \bullet

Arrows and their labels:

- $\Gamma \rightarrow Tm \times_{Ty}$ (labeled α)
- $\Gamma \rightarrow Poly_{tp} Ty$ (labeled β)
- $\Gamma \rightarrow Poly_{tp} Ty$ (labeled (α, β))
- $Tm \times_{Ty} \rightarrow Tm$ (labeled $counit$)
- $Tm \times_{Ty} \rightarrow Ty$ (labeled $ev_{\alpha} B$)
- $Tm \rightarrow Ty$ (labeled tp)
- $Tm \rightarrow Ty$ (labeled $tp_* Tm * Ty$)
- $Ty \rightarrow Ty$ (labeled tp)
- $Tm \rightarrow \bullet$ (labeled \lrcorner)
- $Ty \rightarrow \bullet$ (labeled \lrcorner)

The input data corresponds to $\alpha : \Gamma \rightarrow \mathsf{Tm}$ and some $\beta : \Gamma \rightarrow \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty}$ such that

$$\mathrm{tp}_* \mathrm{Tm}^* \mathrm{T}y \circ \beta = \mathrm{tp} \circ \alpha$$

This in turn corresponds to a map into the fiber product

$$(\alpha, \beta) : \Gamma \rightarrow \mathbf{tp} \times_{\mathbf{Ty}} \mathbf{tp}_* \mathbf{Tm}^* \mathbf{Ty}$$

We then compose this the counit of the adjunction $\mathbf{tp}^* \dashv \mathbf{tp}_*$ to get

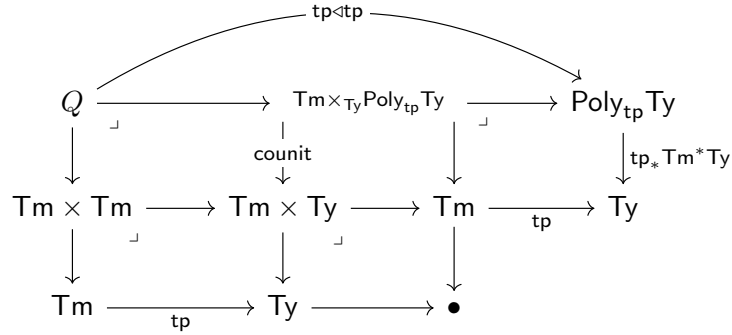
$$\mathbf{counit} \circ (\alpha, \beta) : \Gamma \rightarrow \mathbf{tp} \times_{\mathbf{Ty}} \mathbf{tp}_* \mathbf{Ty}$$

then extract the type by composing with the projection to \mathbf{Ty}

$$\pi_{\mathbf{Ty}} \circ \mathbf{counit} \circ (\alpha, \beta) : \Gamma \rightarrow \mathbf{Ty}$$

Finally, this corresponds to a unique map $\mathbf{ev}_\alpha B : \Gamma \rightarrow \mathbf{grpd}$.

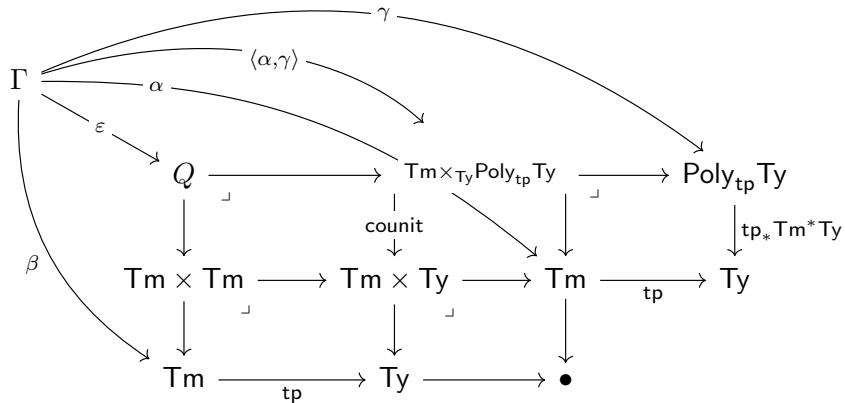
Definition 4.26 (Classifier for dependent pairs). Recall the following definition of composition of polynomial endofunctors, specialized to our situation



Proposition 4.27. Q classifies the data of a pair of dependent terms (a, b) , and $\mathbf{tp} \triangleleft \mathbf{tp}$ extracts the underlying type and dependent type (A, B) .

$$Q \Gamma \cong \{(\beta : \gamma \rightarrow \mathbf{grpd}_\bullet, \alpha : \Gamma \rightarrow \mathbf{grpd}_\bullet, B : \Gamma \cdot U \circ \alpha \rightarrow \mathbf{grpd}) \mid U \circ \beta = \mathbf{ev}_\alpha \beta\}$$

Proof. Extending the diagram from definition 4.26



By the universal property of pullbacks, a data of a map with representable domain $\varepsilon : \Gamma \rightarrow Q$ corresponds to the data of to a triple of maps $\alpha, \beta : \Gamma \rightarrow \mathsf{Tm}$ and $\gamma : \Gamma \rightarrow \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty}$ such that $\mathsf{tp} \circ \beta = \pi_{\mathsf{Ty}} \circ \mathsf{counit} \circ \langle \alpha, \gamma \rangle$ and $\mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty} \circ \gamma = \mathsf{tp} \circ \alpha$.

This in turn corresponds to three functors $\alpha, \beta : \Gamma \rightarrow \mathbf{grpd}_\bullet$ and $B : \Gamma \cdot A \rightarrow \mathbf{grpd}_\bullet$, such that $U \circ \beta = \mathsf{ev}_\alpha B$. Type theoretically $\alpha = (A, a : A)$ and $\mathsf{ev}_\alpha B = Ba$ and $\beta = (Ba, b : Ba)$. Then composing ε with $\mathsf{tp} \triangleleft \mathsf{tp}$ returns γ , which consists of (A, B) . It is in this sense that Q classifies pairs of dependent terms, and $\mathsf{tp} \triangleleft \mathsf{tp}$ extracts the underlying types. \square

Definition 4.28 (Interpretation of Σ). We define the natural transformation $\Sigma : \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \rightarrow \mathsf{Ty}$ as that which is induced (lemma 4.23) by the Σ -former operation (definition 4.14).

Define $\mathsf{pair} : Q \rightarrow \mathsf{Tm}$ such that TODO.

Σ and pair combine to give us a pullback square

$$\begin{array}{ccc} Q & \xrightarrow{\mathsf{pair}} & \mathsf{Tm} \\ \mathsf{tp} \triangleleft \mathsf{tp} \downarrow & \lrcorner & \downarrow \mathsf{tp} \\ \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} & \xrightarrow{\Sigma} & \mathsf{Ty} \end{array}$$

Proof. TODO: naturality.

TODO: prove pullback. \square

References

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