Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ. Write **Set** for the category of all sets. Say that a set A is λ-small if $|A| < \lambda$. Write $\mathbf{Set}_\lambda$ for the full subcategory of **Set** spanned by λ-small sets.

Let $\mathbb{C}$ be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write $\mathrm{Psh}(\mathbb{C})$ for the category of presheaves over $\mathbb{C}$,

$$\mathrm{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $\mathtt{tp}\colon \mathsf{Tm} \to \mathsf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, \ldots \in \mathbb{C}$,
- a type in context $\mathsf{y}(\Gamma)$ as a map $A\colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$,
- a term of type A in context Γ as a map $a\colon \mathsf{y}(\Gamma) \to \mathsf{Tm}$ such that

$$\begin{tikzcd} & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

  commutes,

- an operation called "context extension" which given a context Γ and a type $A\colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{tikzcd} \mathsf{y}(\Gamma.A) \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{y}(\Gamma) \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty. We write

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{y}(\Gamma) \arrow[r, "\ulcorner X \urcorner"'] & \mathsf{Ty} \end{tikzcd}$$

to express this situation, i.e. $X \cong \mathsf{y}(\Gamma \cdot \ulcorner X \urcorner)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} \colon \mathsf{Ty} \qquad \frac{a \colon \mathsf{U}}{\mathsf{El}(a) \colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ, with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model,

- There is a pullback diagram of the form

$$\begin{tikzcd} \mathsf{U} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ 1 \arrow[r, "\ulcorner \mathsf{U} \urcorner"'] & \mathsf{Ty} \end{tikzcd}$$

- There is a map

$$\mathsf{El} \colon \mathsf{U} \to \mathsf{Ty}$$

- $\pi : \mathsf{E} \to \mathsf{U}$ is obtained as pullback of tp along El; There is a pullback diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\pi"'] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty} \end{tikzcd} \tag{2.1}$$

With the notation above, we get

$$\begin{tikzcd}
\mathsf{y}(\Gamma.\mathsf{El}(a)) \arrow[r] \arrow[d] & \mathsf{E} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\
\mathsf{y}(\Gamma) \arrow[r, "a"'] \arrow[rr, bend right, "A"'] & \mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd}$$

Both squares above are pullback squares.

3 Defining Π and Σ type formers for the universe $\pi\colon \mathsf{E} \to \mathsf{U}$

Take the pullback diagram eq. (2.1). That is a morphism in the category of polynomials. We have a cartesian natural transformation $P_\pi \to P_{\mathsf{tp}}$ induced by the pullback eq. (2.1). This cartesian natural transformation induces the diagrams of the left cube in below; all of the squares in the left cube are pullback squares.

Now, consider the right cube only. The right-side face of the cube is a pullback square due to the definition of π. The left-side is also because polynomials are left exact functors and therefore they preserve pullbacks. Observe also that the front face is a pullback square by the definition of Π for Ty.

$$\begin{tikzcd}
& P_\pi E \arrow[rr] \arrow[dl] \arrow[dd] & & P_{\mathsf{tp}} E \arrow[rr] \arrow[dl] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\
P_\pi \mathsf{Tm} \arrow[rr] \arrow[dd] & & P_{\mathsf{tp}} \mathsf{Tm} \arrow[rr, "\lambda"] \arrow[dd] & & \mathsf{Tm} \arrow[dd] & \\
& P_\pi \mathsf{U} \arrow[rr] \arrow[dl] & & P_{\mathsf{tp}} U \arrow[rr, dashed] \arrow[dl] & & U \arrow[dl] \\
P_\pi \mathsf{Ty} \arrow[rr] & & P_{\mathsf{tp}} \mathsf{Ty} \arrow[rr, "\Pi"'] & & \mathsf{Ty} &
\end{tikzcd}$$

We define $\Pi_U\colon P_\pi U \to U$ as a dashed arrow which makes the bottom square in the right cube commute, that is the following square commutes.

$$\begin{tikzcd}
P_{\mathsf{tp}} \mathsf{U} \arrow[r] \arrow[d, dashed, "\Pi_{\mathsf{U}}"'] & P_{\mathsf{tp}} \mathsf{Ty} \arrow[d, "\Pi_{\mathsf{Ty}}"] \\
\mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd} \tag{3.1}$$

By the universal property of the pullback which defines π we get a unique arrow $P_{\mathsf{tp}}E \to E$ which we shall name λ_{U}. By the pullback pasting lemma it follows that the square involving Π_U and λ_U is a pullback square.

$$\begin{tikzcd}
P_{\mathsf{tp}}\mathsf{E} \arrow[r, "\lambda_{\mathsf{U}}"] \arrow[d, "P_{\mathsf{tp}}\pi"'] & E \arrow[d, "P_{\mathsf{tp}}\pi"] \\
P_{\mathsf{tp}}\mathsf{U} \arrow[r, "\Pi_{\mathsf{U}}"'] & \mathsf{U}
\end{tikzcd} \tag{3.2}$$

This concludes the construction of Π type former for the universe U. The only data we needed to supply for the definition of Π_U was a lift of $\Pi\colon P_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ to U.

4 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model