

Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ . Write **Set** for the category of all sets. Say that a set A is λ -small if $|A| < \lambda$. Write **Set** $_{\lambda}$ for the full subcategory of **Set** spanned by λ -small sets.

Let \mathbb{C} be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write **Psh**(\mathbb{C}) for the category of presheaves over \mathbb{C} ,

$$\mathbf{Psh}(\mathbb{C}) =_{\text{def}} [\mathbb{C}^{\text{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $\text{tp}: \mathsf{Tm} \rightarrow \mathsf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, \dots \in \mathbb{C}$,
- a type in context $y(\Gamma)$ as a map $A: y(\Gamma) \rightarrow \mathsf{Ty}$,
- a term of type A in context Γ as a map $a: y(\Gamma) \rightarrow \mathsf{Tm}$ such that

$$\begin{array}{ccc} & \mathsf{Tm} & \\ & \uparrow a & \downarrow \text{tp} \\ \Gamma & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

commutes,

- an operation called “context extension” which given a context Γ and a type $A: y(\Gamma) \rightarrow \mathsf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{array}{ccc} y(\Gamma \cdot A) & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty . We write

$$\begin{array}{ccc} X & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{\ulcorner X \urcorner} & \mathsf{Ty} \end{array}$$

to express this situation, i.e. $X \cong y(\Gamma \cdot \ulcorner X \urcorner)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U , not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} : \mathsf{Ty} \quad \frac{a : \mathsf{U}}{\mathsf{El}(a) : \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ , with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi : \mathsf{E} \rightarrow \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\ulcorner \mathsf{U} \urcorner} & \mathsf{Ty} \end{array}$$

- There is an inclusion of U into Ty

$$\mathsf{El} : \mathsf{U} \rightarrowtail \mathsf{Ty}$$

- $\pi : \mathsf{E} \rightarrow \mathsf{U}$ is obtained as pullback of tp ; There is a pullback diagram

$$\begin{array}{ccc} \mathsf{E} & \twoheadrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{U} & \xrightarrow{\mathsf{El}} & \mathsf{Ty} \end{array}$$

With the notation above, we get

$$\begin{array}{ccccc}
 y(\Gamma, \text{El}(a)) & \longrightarrow & E & \longrightarrow & Tm \\
 \downarrow & & \downarrow & & \downarrow \\
 y(\Gamma) & \xrightarrow{a} & U & \xrightarrow{\text{El}} & Ty \\
 & \searrow & & \nearrow & \\
 & A & & &
 \end{array}$$

Both squares above are pullback squares.

3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

4 Groupoid Model of HoTT

In this section we construct a natural model in $\mathbf{Psh}(\mathbf{grpd})$ the presheaf category indexed by the category \mathbf{grpd} of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on \mathbf{grpd} - which comes from restricting the “classical Quillen model structure” on \mathbf{cat} [Joy] to \mathbf{grpd} .

4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as \mathbf{set} and the large sets as \mathbf{Set} (in the previous sections this would have been \mathbf{Set}_λ and \mathbf{Set} respectively). We denote the category of small categories as \mathbf{cat} and the large categories as \mathbf{Cat} . We denote the category of small groupoids as \mathbf{grpd} . The category of small pointed groupoids will be \mathbf{grpd}_\bullet .

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\text{op}}, \mathbf{Set}]$$

In this section, \mathbf{Tm} and \mathbf{T}_y and so on will refer to the natural model semantics in this specific model.

Definition 4.1 (The display map classifier). We would like to define a natural transformation in $\mathbf{Psh}(\mathbf{grpd})$

$$\mathbf{tp}: \mathbf{Tm} \rightarrow \mathbf{T}_y$$

with representable fibers.

Consider the functor that forgets the point

$$U: \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}.$$

If we apply the Yoneda embedding $y: \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U \circ [-, \mathbf{grpd}_\bullet] \rightarrow [-, \mathbf{grpd}] \quad \text{in} \quad \mathbf{Psh}(\mathbf{Cat}).$$

Since any small groupoid is also a large category $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict \mathbf{Cat} indexed presheaves to be \mathbf{grpd} indexed presheaves. We define $\mathbf{tp}: \mathbf{Tm} \rightarrow \mathbf{T}_y$ as the image of $U \circ [-, \mathbf{grpd}_\bullet]$ under this restriction.

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\text{res}} \mathbf{Psh}(\mathbf{grpd}) \\ \mathbf{grpd} & \longmapsto & [-, \mathbf{grpd}] \longmapsto \mathbf{T}_y \end{array}$$

Note that \mathbf{Tm} and \mathbf{T}_y are not representable in $\mathbf{Psh}(\mathbf{grpd})$.

Remark 4.2. By Yoneda we can identify maps with representable domain into the type classifier

$$A : y\Gamma \rightarrow \mathbf{Ty} \quad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A : \Gamma \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}$$

Definition 4.3 (Grothendieck construction). From \mathbb{C} a small category and $F : \mathbb{C} \rightarrow \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in \mathbb{C} we refer to Fc as the fiber over c . The objects of $\int F$ consist of pairs $(c \in \mathbb{C}, x \in Fc)$, and morphisms are those in \mathbb{C} preserving the points in the fiber. This makes the following pullback

$$(c, x) \longmapsto (Fc, x)$$

$$\begin{array}{ccccc} (c, x) & & \int F & \longrightarrow & \mathbf{cat}_\bullet & & (C, c) \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ c & & \mathbb{C} & \xrightarrow{F} & \mathbf{cat} & & C \end{array}$$

where \mathbf{cat}_\bullet is the category of large pointed categories.

Definition 4.4 (Grothendieck construction for groupoids). Let Γ be a groupoid and $A : \Gamma \rightarrow \mathbf{grpd}$ a functor, we can compose F with the inclusion $i : \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \quad \text{disp}_A : \Gamma \cdot A \rightarrow \Gamma$$

This is also a small groupoid since the underlying morphisms are from the groupoid Γ . Furthermore the pullback factors through (pointed) groupoids.

$$\begin{array}{ccccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet & \longrightarrow & \mathbf{cat}_\bullet \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} & \longrightarrow & \mathbf{cat} \end{array}$$

Corollary 4.5 (The display map classifier is presentable). *For any small groupoid Γ and $A : y\Gamma \rightarrow \mathbf{Ty}$, the pullback of \mathbf{tp} along A can be given by the representable map $y\text{disp}_A$.*

$$\begin{array}{ccc} y\Gamma \cdot A & \longrightarrow & \mathbf{Tm} \\ \text{disp}_A \downarrow & \lrcorner & \downarrow \mathbf{tp} \\ y\Gamma & \xrightarrow{A} & \mathbf{Ty} \end{array}$$

Proof. Consider the pullback in **Cat**

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

We send this square along $\text{res} \circ y$ in the following

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) \\ \uparrow & \searrow & \downarrow \text{res} \\ \mathbf{grpd} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{grpd}) \end{array}$$

The Yoneda embedding $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does res since it is a right adjoint (with left Kan extension $\iota_! \dashv \text{res}_!$). \square

4.2 Groupoid fibrations

Definition 4.6 (Isofibration). *TODO*

Unless specified otherwise, by a (Groupoid) *fibration* we will mean an isofibration.

Proposition 4.7 (Display map properties of fibrations). *TODO*

1. (*subst*) *Stable under pullback*
2. (Σ) *Closed under composition*
3. (Π) *Closed under pushforward*
4. (*Fibrant objects*) *Map to terminal is a fibration*
5. (*Id*) *Path object fibration*

4.3 Polynomial endofunctors

Definition 4.8 (Polynomial endofunctor in an LCCC). *TODO*

Proposition 4.9 (Universal property of polynomial endofunctors). *TODO*

4.4 Π and Σ structure

Definition 4.10 (Interpretation of Π and λ). Sketch: we define the natural transformation $\Pi : \text{Poly}_{\text{tp}} \text{Ty} \rightarrow \text{Ty}$ by first taking some small groupoid Γ and defining

$$\Pi_{\Gamma} : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Poly}_{\text{tp}} \text{Ty}) \rightarrow \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Ty})$$

Unfolding the universal property of Poly_{tp} this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & \mapsto & \Gamma \cdot \Pi_A B \\ \text{disp}_B \downarrow & & \downarrow (\text{disp}_A)_* \text{disp}_B \\ \Gamma \cdot A & \xrightarrow{\text{disp}_A} & \Gamma \end{array}$$

As indicated in the diagram, we take this to be the pushforward of the dependent display map disp_B along the display map it depends on disp_A . Note that this pushforward is in \mathbf{grpd} , and this pushforward is only defined on fibrations.

TODO: define λ .

Proof. TODO: naturality.

TODO: prove pullback. □

Definition 4.11 (Interpretation of Σ). Sketch: we define the natural transformation $\Sigma : \text{Poly}_{\text{tp}} \text{Ty} \rightarrow \text{Ty}$ by first taking some small groupoid Γ and defining

$$\Sigma_{\Gamma} : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Poly}_{\text{tp}} \text{Ty}) \rightarrow \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Ty})$$

Again, this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & \mapsto & \Gamma \cdot \Sigma_A B \\ \text{disp}_B \downarrow & & \downarrow (\text{disp}_A)! \text{disp}_B \\ \Gamma \cdot A & \xrightarrow{\text{disp}_A} & \Gamma \end{array}$$

As indicated in the diagram, we take this to be the composition of disp_B and disp_A , recalling that fibrations are closed under composition.

TODO: define pair .

Proof. TODO: naturality.

TODO: prove pullback. □

References

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- [Joy] André Joyal. Model structures on cat. <https://ncatlab.org/joyalscatlab/published/Model+structures+on+Cat>.