

Universe in the Natural Model of Type Theory

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July 22, 2024

1 Types

Assume an inaccessible cardinal λ . Write **Set** for the category of all sets. Say that a set A is λ -small if $|A| < \lambda$. Write **Set** $_{\lambda}$ for the full subcategory of **Set** spanned by λ -small sets.

Let \mathbb{C} be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write **Psh**(\mathbb{C}) for the category of presheaves over \mathbb{C} ,

$$\mathbf{Psh}(\mathbb{C}) =_{\text{def}} [\mathbb{C}^{\text{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $\text{tp}: \mathsf{Tm} \rightarrow \mathsf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, \dots \in \mathbb{C}$,
- a type in context $y(\Gamma)$ as a map $A: y(\Gamma) \rightarrow \mathsf{Ty}$,
- a term of type A in context Γ as a map $a: y(\Gamma) \rightarrow \mathsf{Tm}$ such that

$$\begin{array}{ccc} & \mathsf{Tm} & \\ & \uparrow a & \downarrow \text{tp} \\ \Gamma & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

commutes,

- an operation called “context extension” which given a context Γ and a type $A: y(\Gamma) \rightarrow \mathsf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{array}{ccc} y(\Gamma \cdot A) & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty . We write

$$\begin{array}{ccc} X & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{\ulcorner X \urcorner} & \mathsf{Ty} \end{array}$$

to express this situation, i.e. $X \cong y(\Gamma \cdot \ulcorner X \urcorner)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U , not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} : \mathsf{Ty} \quad \frac{a : \mathsf{U}}{\mathsf{El}(a) : \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ , with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi : \mathsf{E} \rightarrow \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\ulcorner \mathsf{U} \urcorner} & \mathsf{Ty} \end{array}$$

- There is an inclusion of U into Ty

$$\mathsf{El} : \mathsf{U} \rightarrow \mathsf{Ty}$$

- $\pi : \mathsf{E} \rightarrow \mathsf{U}$ is obtained as pullback of tp ; There is a pullback diagram

$$\begin{array}{ccc} \mathsf{E} & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{U} & \xrightarrow{\mathsf{El}} & \mathsf{Ty} \end{array}$$

With the notation above, we get

$$\begin{array}{ccccc}
 y(\Gamma, \text{El}(a)) & \longrightarrow & E & \longrightarrow & Tm \\
 \downarrow & & \downarrow & & \downarrow \\
 y(\Gamma) & \xrightarrow{a} & U & \xrightarrow{\text{El}} & Ty \\
 & \searrow \scriptstyle A & \nearrow & &
 \end{array}$$

Both squares above are pullback squares.

3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

4 Groupoid Model of HoTT

In this section we construct a natural model in $\mathbf{Psh}(\mathbf{grpd})$ the presheaf category indexed by the category \mathbf{grpd} of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on \mathbf{grpd} - which comes from restricting the “classical Quillen model structure” on \mathbf{cat} [Joy] to \mathbf{grpd} .

4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as \mathbf{set} and the large sets as \mathbf{Set} (in the previous sections this would have been \mathbf{Set}_λ and \mathbf{Set} respectively). We denote the category of small categories as \mathbf{cat} and the large categories as \mathbf{Cat} . We denote the category of small groupoids as \mathbf{grpd} .

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section, \mathbf{Tm} and \mathbf{T}_y and so on will refer to the natural model semantics in this specific model.

Definition 4.1 (Pointed). We will take the category of pointed small categories \mathbf{cat}_\bullet to have objects as pairs $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$ and morphisms as pairs

$$(F : \mathbb{C}_1 \rightarrow \mathbb{C}_0, \phi : Fc_1 \rightarrow c_0) : (\mathbb{C}_1, c_1) \rightarrow (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids \mathbf{grpd}_\bullet will be the full subcategory of objects (Γ, c) with Γ a groupoid.

Definition 4.2 (The display map classifier). We would like to define a natural transformation in $\mathbf{Psh}(\mathbf{grpd})$

$$\mathbf{tp} : \mathbf{Tm} \rightarrow \mathbf{T}_y$$

with representable fibers.

Consider the functor that forgets the point

$$U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}.$$

If we apply the Yoneda embedding $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U \circ [-, \mathbf{grpd}_\bullet] \rightarrow [-, \mathbf{grpd}] \quad \text{in} \quad \mathbf{Psh}(\mathbf{Cat}).$$

Since any small groupoid is also a large category $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict \mathbf{Cat} indexed presheaves to be \mathbf{grpd} indexed presheaves. We define $\mathbf{tp} : \mathbf{Tm} \rightarrow \mathbf{T}_y$ as the image of $U \circ$ under this restriction.

$$\begin{aligned} \mathbf{Cat} &\xrightarrow{y} \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\text{res}} \mathbf{Psh}(\mathbf{grpd}) \\ \mathbf{grpd} &\longmapsto [-, \mathbf{grpd}] \longmapsto \mathbf{T}_y \end{aligned}$$

Note that \mathbf{Tm} and $\mathbf{T}\mathbf{y}$ are not representable in $\mathbf{Psh}(\mathbf{grpd})$.

Remark 4.3. By Yoneda we can identify maps with representable domain into the type classifier

$$A : \mathbf{y}\Gamma \rightarrow \mathbf{T}\mathbf{y} \quad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A : \Gamma \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}$$

Definition 4.4 (Grothendieck construction). From \mathbb{C} a small category and $F : \mathbb{C} \rightarrow \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in \mathbb{C} we refer to Fc as the fiber over c . The objects of $\int F$ consist of pairs $(c \in \mathbb{C}, x \in Fc)$, and morphisms between (c, x) and (d, y) are pairs $(f : c \rightarrow d, \phi : Ff x \rightarrow y)$. This makes the following pullback in \mathbf{Cat}

$$\begin{array}{ccccc} (c, x) & & \int F & \longrightarrow & \mathbf{cat}_\bullet & & (C, c) \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ c & & \mathbb{C} & \xrightarrow{F} & \mathbf{cat} & & C \end{array}$$

Definition 4.5 (Grothendieck construction for groupoids). Let Γ be a groupoid and $A : \Gamma \rightarrow \mathbf{grpd}$ a functor, we can compose F with the inclusion $i : \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \quad \text{disp}_A : \Gamma \cdot A \rightarrow \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids Γ and Ax for $x \in \Gamma$. Furthermore the pullback factors through (pointed) groupoids.

$$\begin{array}{ccccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet & \longrightarrow & \mathbf{cat}_\bullet \\ \text{disp}_A \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} & \longrightarrow & \mathbf{cat} \end{array}$$

Corollary 4.6 (The display map classifier is presentable). *For any small groupoid Γ and $A : y\Gamma \rightarrow \mathbf{Ty}$, the pullback of \mathbf{tp} along A can be given by the representable map $y\mathbf{disp}_A$.*

$$\begin{array}{ccc} y\Gamma \cdot A & \longrightarrow & \mathbf{Tm} \\ y\mathbf{disp}_A \downarrow & \lrcorner & \downarrow \mathbf{tp} \\ y\Gamma & \xrightarrow{A} & \mathbf{Ty} \end{array}$$

Proof. Consider the pullback in **Cat**

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

We send this square along $\mathbf{res} \circ y$ in the following

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) \\ \uparrow & \searrow & \downarrow \mathbf{res} \\ \mathbf{grpd} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{grpd}) \end{array}$$

The Yoneda embedding $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does \mathbf{res} since it is a right adjoint (with left Kan extension $u_! \dashv \mathbf{res}_*$). \square

4.2 Groupoid fibrations

Definition 4.7 (Fibration). Let $p : \mathbb{C}_1 \rightarrow \mathbb{C}_0$ be a functor. We say p is a *split Grothendieck fibration* if we have a dependent function $\mathbf{lift} a f$ satisfying the following: for any object a in \mathbb{C}_1 and morphism $f : pa \rightarrow y$ in the base \mathbb{C}_0 we have $\mathbf{lift} a f : a \rightarrow b$ in \mathbb{C}_1 such that $p(\mathbf{lift} a f) = f$ and moreover $\mathbf{lift} a g \circ f = \mathbf{lift} b g \circ \mathbf{lift} a f$

$$\begin{array}{ccc} a & \xrightarrow{\mathbf{lift} a f} & b \\ \downarrow & \Downarrow & \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

In particular, we are interested in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid Γ as \mathbf{Fib}_Γ , which is a full subcategory of the slice \mathbf{grpd}/Γ .

Note that $\mathbf{disp}_A : \Gamma \cdot A \rightarrow \Gamma$ is a fibration, since for any $(x \in \Gamma, a \in Ax)$ and $f : x \rightarrow y$ in Γ we have a morphism $(f, \mathrm{id}_{Afa}) : (x, a) \rightarrow (y, Afa)$ lifting f . Furthermore

Proposition 4.8. *There is an adjoint equivalence*

$$[\Gamma, \mathbf{grpd}] \begin{array}{c} \xrightarrow{\mathbf{disp}} \\ \xleftarrow[\mathbf{fiber}]{\simeq} \end{array} \mathbf{Fib}_\Gamma$$

where for each fibration $\delta : \Delta \rightarrow \Gamma$ and each object $x \in \Gamma$

$$\mathbf{fiber}_\delta x = \text{full subcategory } \{a \in \Delta \mid \delta a = x\}$$

It follows that all fibrations are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd}$, when viewed as morphisms in \mathbf{Cat} . From now on, we will use \mathbf{disp}_A to represent any groupoid fibration, which we can adjust up to isomorphism using this equivalence.

Proposition 4.9 (Pullback of fibrations). *Let $\sigma : \Delta \rightarrow \Gamma$ be a functor between groupoids. Since display maps are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd}$ we have the pasting diagram*

$$\begin{array}{ccccc} & & \Delta \cdot A\sigma & \xrightarrow{\quad} & \Gamma \cdot A & \xrightarrow{\quad} & \mathbf{grpd}_\bullet \\ & \searrow & \downarrow \mathbf{disp}_{A\sigma} & \lrcorner & \downarrow \mathbf{disp}_A & \lrcorner & \downarrow \\ \Delta & \xrightarrow{\quad \sigma \quad} & \Gamma & \xrightarrow{\quad A \quad} & \mathbf{grpd} \end{array}$$

Note that this avoids coherence issues, since we take the pullback to be the map $\mathbf{disp}_{A\sigma}$ specifically. It follows that fibrations are stable under pullback along all groupoid functors.

$$\begin{array}{ccc} [\Gamma, \mathbf{grpd}] & \xleftarrow{\mathbf{fiber}} & \mathbf{Fib}_\Gamma \\ \circ \sigma \downarrow & & \downarrow \sigma^* \\ [\Delta, \mathbf{grpd}] & \xrightarrow{\mathbf{disp}} & \mathbf{Fib}_\Delta \end{array}$$

Definition 4.10 (Composition of fibrations). The composition of two fibrations is a fibration.

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & & \\ \downarrow & \searrow & \\ \Gamma \cdot A & \xrightarrow{\quad} & \Gamma \end{array}$$

Then given $A : \Gamma \rightarrow \mathbf{grpd}$ and $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$ we define

$$\Sigma_A B := \mathbf{fiber}(\mathbf{disp}_B \circ \mathbf{disp}_A) : \Gamma \rightarrow \mathbf{grpd}$$

Proof. Easy by unfolding the “lift” definition of fibrations. \square

Definition 4.11 (Pushforward of fibrations). Given $A : \Gamma \rightarrow \mathbf{grpd}$ and $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$ we will define $\Pi_A B : \Gamma \rightarrow \mathbf{grpd}$ such that for any $C : \Gamma \rightarrow \mathbf{grpd}$ we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathrm{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

natural in both B and C . Stated in terms of fibrations we have

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & & \Gamma \cdot \Pi_A B \\ \mathrm{disp}_B \downarrow & & \downarrow \\ \Gamma \cdot A & \xrightarrow{\mathrm{disp}_A} & \Gamma \end{array}$$

with the universal property of pushforward

$$\mathrm{Fib}_{\Gamma \cdot A}(\mathrm{disp}_A^* \mathrm{disp}_C, \mathrm{disp}_B) \cong \mathrm{Fib}_{\Gamma}(\mathrm{disp}_C, \mathrm{disp}_{\Pi_A B})$$

Proof. $\Pi_A B$ takes on objects by taking fiberwise sections and act on morphisms via conjugation

$$\begin{array}{ccccc} x & & [A(x), \Sigma_A B(x)] & & A(x) \xrightarrow{s} \Sigma_A B(x) \\ \downarrow f & \xrightarrow{\Pi_A B} & \downarrow \Sigma_A B(f) \circ - \circ A(f^{-1}) & & \uparrow A(f^{-1}) \\ y & & [A(y), \Sigma_A B(y)] & & A(y) \xrightarrow{\Pi_A B(f)(s)} \Sigma_A B(y) \end{array}$$

The functor categories are groupoids since any natural transformation of functors into groupoids are natural isomorphisms. Note that conjugation is functorial and invertible. \square

Proposition 4.12 (All objects are fibrant). *Let \bullet denote the terminal groupoid, namely that with a single object and morphism. Then the unique map $\Gamma \rightarrow \bullet$ is a fibration.*

Proposition 4.13. *TODO (Id) Path object fibration*

4.3 Polynomial endofunctors

Definition 4.14 (Polynomial endofunctor in an LCCC). *TODO*

Proposition 4.15 (Universal property of polynomial endofunctors). *TODO*

4.4 Π and Σ structure

Definition 4.16 (Interpretation of Π and λ). Sketch: we define the natural transformation $\Pi : \text{Poly}_{\text{tp}} \text{Ty} \rightarrow \text{Ty}$ by first taking some small groupoid Γ and defining

$$\Pi_{\Gamma} : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Poly}_{\text{tp}} \text{Ty}) \rightarrow \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Ty})$$

Unfolding the universal property of Poly_{tp} this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & \mapsto & \Gamma \cdot \Pi_A B \\ \text{disp}_B \downarrow & & \downarrow (\text{disp}_A)_* \text{disp}_B \\ \Gamma \cdot A & \xrightarrow{\text{disp}_A} & \Gamma \end{array}$$

As indicated in the diagram, we take this to be the pushforward of the dependent display map disp_B along the display map it depends on disp_A . Note that this pushforward is in \mathbf{grpd} , and this pushforward is only defined on fibrations.

TODO: define λ .

Proof. TODO: naturality.

TODO: prove pullback. □

Definition 4.17 (Interpretation of Σ). Sketch: we define the natural transformation $\Sigma : \text{Poly}_{\text{tp}} \text{Ty} \rightarrow \text{Ty}$ by first taking some small groupoid Γ and defining

$$\Sigma_{\Gamma} : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Poly}_{\text{tp}} \text{Ty}) \rightarrow \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Ty})$$

Again, this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & \mapsto & \Gamma \cdot \Sigma_A B \\ \text{disp}_B \downarrow & & \downarrow (\text{disp}_A)! \text{disp}_B \\ \Gamma \cdot A & \xrightarrow{\text{disp}_A} & \Gamma \end{array}$$

As indicated in the diagram, we take this to be the composition of disp_B and disp_A , recalling that fibrations are closed under composition.

TODO: define pair .

Proof. TODO: naturality.

TODO: prove pullback. □

References

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