Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ . Write **Set** for the category of all sets. Say that a set A is λ -small if $|A| < \lambda$. Write **Set** $_{\lambda}$ for the full subcategory of **Set** spanned by λ -small sets.

Let \mathbb{C} be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write $Psh(\mathbb{C})$ for the category of presheaves over \mathbb{C} ,

$$\operatorname{Psh}(\mathbb{C}) =_{\operatorname{def}} [\mathbb{C}^{\operatorname{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $tp: Tm \to Ty$ consists of

- contexts as objects $\Gamma, \Delta, \ldots \in \mathbb{C}$,
- a type in context $y(\Gamma)$ as a map $A: y(\Gamma) \to \mathsf{Ty}$,
- a term of type A in context Γ as a map $a: y(\Gamma) \to Tm$ such that



commutes,

• an operation called "context extension" which given a context Γ and a type $A \colon \mathsf{y}(\Gamma) \to \mathsf{T}\mathsf{y}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{array}{ccc} \mathsf{y}(\Gamma.A) & \longrightarrow \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{y}(\Gamma) & \longrightarrow_A & \mathsf{Ty} \end{array}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty. We write

$$\begin{array}{c} X \longrightarrow \mathsf{Tm} \\ \downarrow \\ \mathsf{y}(\Gamma) \xrightarrow{\Gamma} \mathsf{Ty} \end{array}$$

to express this situation, i.e. $X \cong y(\Gamma \cdot \lceil X \rceil)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U}\colon \mathsf{Ty} \qquad \frac{a\colon \mathsf{U}}{\mathsf{El}(a)\colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ , with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{c} \mathsf{U} \longrightarrow \mathsf{Tm} \\ \downarrow \\ \downarrow \\ 1 \xrightarrow{} \mathsf{Ty} \end{array}$$

• There is an inclusion of U into Ty

$$\mathsf{EI}\colon\mathsf{U}\rightarrowtail\mathsf{Ty}$$

• $\pi: \mathsf{E} \to \mathsf{U}$ is obtained as pullback of tp; There is a pullback diagram

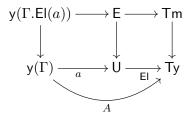
$$E \longmapsto \mathsf{Tm}$$

$$\pi \downarrow \qquad \qquad \downarrow \mathsf{tp}$$

$$\mathsf{U} \longmapsto \mathsf{Ty}$$

$$(2.1)$$

With the notation above, we get

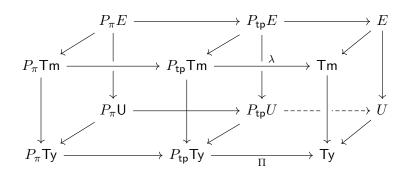


Both squares above are pullback squares.

3 Defining Π and Σ type formers for the universe $\pi \colon \mathsf{E} \to \mathsf{U}$

Take the pullback diagram eq. (2.1). That is a morphism in the category of polynomials. We have a cartesian natural transformation $P_{\pi} \to P_{\mathsf{tp}}$ induced by the pullback eq. (2.1). All the squares in the left cube are pullback squares.

Now, consider the right cube only. The right-side face of the cube is a pullback square due to the definition of π . The left-side is also because polynomials are left exact functors and therefore they preserve pullbacks. Observe also that the front face is a pullback square by the definition of Π for Ty.



We define $\Pi_U : P_{\pi}U \to U$ to be a dashed arrow which makes the bottom square commute, that is the following square commutes

By the universal property of the pullback which defines π we get a unique arrow $\lambda_U \colon P_{\mathsf{tp}}E \to E$ which we shall name λ_U . By the pullback pasting lemma it follows

that the square involving Π_U and λ_U is a pullback square. This concludes the construction of Π type former for the universe U . The only data we needed to supply for the definition of Π_U was a lift of $\Pi\colon P_\mathsf{tp}\mathsf{Ty}\to \mathsf{Ty}$ to U.

4 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model