# Universe in the Natural Model of Type Theory

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## 1 Types

Assume an inaccessible cardinal  $\lambda$ . Write **Set** for the category of all sets. Say that a set A is  $\lambda$ -small if  $|A| < \lambda$ . Write **Set** $_{\lambda}$  for the full subcategory of **Set** spanned by  $\lambda$ -small sets.

Let  $\mathbb C$  be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write  $\mathbf{Psh}(\mathbb{C})$  for the category of presheaves over  $\mathbb{C}$ ,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map  $tp: Tm \to Ty$  consists of

- contexts as objects  $\Gamma, \Delta, \ldots \in \mathbb{C}$ ,
- a type in context  $y(\Gamma)$  as a map  $A: y(\Gamma) \to \mathsf{Ty}$ ,
- a term of type A in context  $\Gamma$  as a map  $a: y(\Gamma) \to Tm$  such that



commutes,

• an operation called "context extension" which given a context  $\Gamma$  and a type  $A \colon \mathsf{y}(\Gamma) \to \mathsf{T}\mathsf{y}$  produces a context  $\Gamma \cdot A$  which fits into a pullback diagram below.

$$\begin{array}{ccc} \mathsf{y}(\Gamma.A) & \longrightarrow \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{y}(\Gamma) & \longrightarrow_A & \mathsf{Ty} \end{array}$$

**Remark.** Sometimes, we first construct a presheaf X over  $\Gamma$  and observe that it can be classified by a map into Ty. We write

to express this situation, i.e.  $X \cong y(\Gamma \cdot \lceil X \rceil)$ .

# 2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U}\colon \mathsf{Ty} \qquad \frac{a\colon \mathsf{U}}{\mathsf{El}(a)\colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal  $\kappa$ , with  $\kappa < \lambda$ .)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{c} \mathsf{U} \longrightarrow \mathsf{Tm} \\ \downarrow \\ \downarrow \\ 1 \xrightarrow{\vdash_{\mathsf{\Gamma}\mathsf{U}^{\neg}}} \mathsf{Ty} \end{array}$$

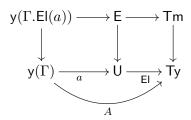
• There is an inclusion of U into Ty

$$\mathsf{EI}\colon\mathsf{U}\rightarrowtail\mathsf{Ty}$$

•  $\pi: E \to U$  is obtained as pullback of tp; There is a pullback diagram

$$\begin{array}{c} E {\longmapsto} \operatorname{Tm} \\ \downarrow \qquad \qquad \downarrow \\ \operatorname{U} {\longmapsto} \operatorname{Ty} \end{array}$$

With the notation above, we get



Both squares above are pullback squares.

- 3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model
- 4 Groupoid Model of HoTT

In this section we construct a natural model in **Psh(grpd)** the presheaf category indexed by the category **grpd** of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on **grpd** - which comes from restricting the "classical Quillen model structure" on **cat** [Joy] to **grpd**.

### 4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as **set** and the large sets as **Set** (in the previous sections this would have been  $\mathbf{Set}_{\lambda}$  and  $\mathbf{Set}$  respectively). We denote the category of small categories as **cat** and the large categories as **Cat**. We denote the category of small groupoids as **grpd**.

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

**Definition 4.1** (Pointed). We will take the category of pointed small categories  $\mathbf{cat}_{\bullet}$  to have objects as pairs ( $\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C}$ ) and morphisms as pairs

$$(F: \mathbb{C}_1 \to \mathbb{C}_0, \phi: Fc_1 \to c_0): (\mathbb{C}_1, c_1) \to (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids  $\mathbf{grpd}_{\bullet}$  will be the full subcategory of objects  $(\Gamma, c)$  with  $\Gamma$  a groupoid.

**Definition 4.2** (The disply map classifier). We would like to define a natural transformation in **Psh**(**grpd**)

$$tp\colon Tm\to Ty$$

with representable fibers.

Consider the functor that forgets the point

$$U : \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$$
 in Cat.

If we apply the Yoneda embedding  $y: \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$  to U we obtain

$$U \circ : [-, \mathbf{grpd}_{\bullet}] \to [-, \mathbf{grpd}]$$
 in  $\mathbf{Psh}(\mathbf{Cat})$ .

Since any small groupoid is also a large category  $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$ , we can restrict  $\mathbf{Cat}$  indexed presheaves to be  $\mathbf{grpd}$  indexed presheaves. We define  $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$  as the image of  $U \circ \mathsf{under}$  this restriction.

$$\mathbf{Cat} \xrightarrow{\quad y \quad} \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\quad res \quad} \mathbf{Psh}(\mathbf{grpd})$$

$$\operatorname{\mathbf{grpd}} \longmapsto [-,\operatorname{\mathbf{grpd}}] \longmapsto \mathsf{Ty}$$

Note that Tm and Ty are not representable in Psh(grpd).

Remark 4.3. By Yoneda we can identify maps with representable domain into the type classifier

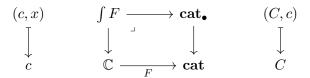
$$A: \mathsf{y}\Gamma \to \mathsf{T}\mathsf{y} \qquad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A:\Gamma\to\mathbf{grpd}$$
 in Cat

**Definition 4.4** (Grothendieck construction). From  $\mathbb{C}$  a small category and  $F:\mathbb{C}\to\mathbf{cat}$  a functor, we construct a small category  $\int F$ . For any c in  $\mathbb{C}$  we refer to Fc as the fiber over c. The objects of  $\int F$  consist of pairs  $(c\in\mathbb{C},x\in Fc)$ , and morphisms between (c,x) and (d,y) are pairs  $(f:c\to d,\phi:Ffx\to y)$ . This makes the following pullback in  $\mathbf{Cat}$ 

$$(c,x) \longmapsto (Fc,x)$$



**Definition 4.5** (Grothendieck construction for groupoids). Let  $\Gamma$  be a groupoid and  $A \colon \Gamma \to \mathbf{grpd}$  a functor, we can compose F with the inclusion  $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$  and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \mathsf{disp}_A \colon \Gamma \cdot A o \Gamma$$

This is also a small groupoid since the underlying morphisms are from the groupoid  $\Gamma$ . Furthermore the pullback factors through (pointed) groupoids.

$$\begin{array}{cccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} & \longrightarrow \mathbf{cat}_{\bullet} \\ & \downarrow & & \downarrow & & \downarrow \\ \Gamma & \longrightarrow_A & \mathbf{grpd} & \longrightarrow \mathbf{cat} \end{array}$$

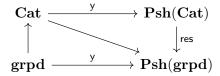
Corollary 4.6 (The display map classifier is presentable). For any small groupoid  $\Gamma$  and  $A: y\Gamma \to Ty$ , the pullback of tp along A can be given by the representable map  $ydisp_A$ .

$$\begin{array}{ccc} \mathsf{y}\Gamma \cdot A & \longrightarrow & \mathsf{Tm} \\ \mathsf{y}h\mathsf{disp}_A & & & \mathsf{tp} \\ & \mathsf{y}\Gamma & \longrightarrow & \mathsf{Ty} \end{array}$$

Proof. Consider the pullback in Cat

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow_A & \mathbf{grpd} \end{array}$$

We send this square along  $res \circ y$  in the following



The Yoneda embedding  $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$  preserves pullbacks, as does res since it is a right adjoint (with left Kan extension  $\iota_! \dashv \mathsf{res}_\iota$ ).

### 4.2 Groupoid fibrations

**Definition 4.7** (Fibration). Let  $p: \mathbb{C}_1 \to \mathbb{C}_0$  be a functor. We say p is a *split Grothendieck fibration* if we have a dependent function lift af satisfying the following: for any object a in  $\mathbb{C}_1$  and morphism  $f: pa \to y$  in the base  $\mathbb{C}_0$  we have lift  $af: a \to b$  in  $\mathbb{C}_1$  such that p(lift af) = f and moreover lift  $ag \circ f = \text{lift } bg \circ \text{lift } af$ 

$$\begin{array}{ccc} a & \xrightarrow{\text{lift } a f} & b \\ \downarrow & & \uparrow & \downarrow \\ \downarrow & & \downarrow & \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

In particular, we are intereseted in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid  $\Gamma$  as  $\mathsf{Fib}_{\Gamma}$ , which is a full subcategory of the slice  $\mathsf{grpd}/\Gamma$ .

Note that  $\operatorname{\mathsf{disp}}_A \colon \Gamma \cdot A \to \Gamma$  is a fibration, since for any  $(x \in \Gamma, a \in Ax)$  and  $f \colon x \to y$  in  $\Gamma$  we have a morphism  $(f, \operatorname{\mathsf{id}}_{Afa}) \colon (x, a) \to (y, Afa)$  lifting f. Furthermore **Proposition 4.8.** There is an adjoint equivalence

$$[\Gamma, \operatorname{\mathbf{grpd}}] \xrightarrow{\operatorname{\mathsf{disp}}} \operatorname{\mathsf{Fib}}_{\Gamma}$$

where for each fibration  $\delta: \Delta \to \Gamma$  and each object  $x \in \Gamma$ 

$$fiber_{\delta} x = full subcategory \{ a \in \Delta \mid \delta a = x \}$$

It follows that all fibrations are pullbacks of the classifier  $U: \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$ , when viewed as morphisms in  $\mathbf{Cat}$ . From now on, we will use  $\mathsf{disp}_A$  to represent any groupoid fibration, which we can adjust up to isomorphism using this equivalence.

**Proposition 4.9** (Pullback of fibrations). Let  $\sigma: \Delta \to Ga$  be a functor between groupoids. Since display maps are pullbacks of the classifier  $U: \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$  we have the pasting diagram

Note that this avoids coherence issues, since we take the pullback to be the map  $\operatorname{\mathsf{disp}}_{A\sigma}$  specifically. It follows that fibrations are stable under pullback along all groupoid functors.

$$\begin{array}{ccc} [\Gamma,\mathbf{grpd}] & \longleftarrow & \mathsf{fiber} & & \mathsf{Fib}_{\Gamma} \\ & \circ \sigma & & & & \downarrow \sigma^* \\ [\Delta,\mathbf{grpd}] & \longleftarrow & \mathsf{fib}_{\Delta} \end{array}$$

**Definition 4.10** (Composition of fibrations). The composition of two fibrations is a fibration.

$$\begin{array}{c} \Gamma \cdot A \cdot B \\ \downarrow \\ \Gamma \cdot A & \stackrel{\text{\tiny 4}}{\longrightarrow} \Gamma \end{array}$$

Then given  $A:\Gamma\to\operatorname{\mathbf{grpd}}$  and  $B:\Gamma\cdot A\to\operatorname{\mathbf{grpd}}$  we define

$$\Sigma_A B := \mathsf{fiber}(\mathsf{disp}_B \circ \mathsf{disp}_A) : \Gamma \to \mathbf{grpd}$$

*Proof.* Easy by unfolding the "lift" definition of fibrations.

**Definition 4.11** (Pushforward of fibrations). Given  $A : \Gamma \to \mathbf{grpd}$  and  $B : \Gamma \cdot A \to \mathbf{grpd}$  we will define  $\Pi_A B : \Gamma \to \mathbf{grpd}$  such that for any  $C : \Gamma \to \mathbf{grpd}$  we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathsf{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

natural in both B and C. Stated in terms of fibrations we have

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & \Gamma \cdot \Pi_A B \\ \operatorname{disp}_B \downarrow & \downarrow \\ \Gamma \cdot A & \xrightarrow{\operatorname{disp}_A} & \Gamma \end{array}$$

with the universal property of pushforward

$$\mathsf{Fib}_{\Gamma \cdot A}(\mathsf{disp}_A^*\mathsf{disp}_C,\mathsf{disp}_B) \cong \mathsf{Fib}_{\Gamma}(\mathsf{disp}_C,\mathsf{disp}_{\Pi_AB})$$

*Proof.*  $\Pi_A B$  takes on objects by taking fiberwise sections and act on morphisms via conjugation

The functor categories are groupoids since any natural transformation of functors into groupoids are natural isomorphisms. Note that conjugation is functorial and invertible.

**Proposition 4.12** (All objects are fibrant). Let  $\bullet$  denote the terminal groupoid, namely that with a single object and morphism. Then the unique map  $\Gamma \to \bullet$  is a fibration.

Proposition 4.13. TODO (Id) Path object fibration

#### 4.3 Polynomial endofunctors

**Definition 4.14** (Polynomial endofunctor in an LCCC). TODO

**Proposition 4.15** (Universal property of polynomial endofunctors). *TODO* 

### 4.4 $\Pi$ and $\Sigma$ structure

**Definition 4.16** (Interpretation of  $\Pi$  and  $\lambda$ ). Sketch: we define the natural transformation  $\Pi$ : Poly<sub>tp</sub>Ty  $\rightarrow$  Ty by first taking some small groupoid  $\Gamma$  and defining

$$\Pi_{\Gamma}:\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_\mathsf{tp}\mathsf{Ty})\to\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Ty})$$

Unfolding the universal property of  $\mathsf{Poly}_\mathsf{tp}$  this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

As indicated in the diagram, we take this to be the pushforward of the dependent display map  $\operatorname{disp}_B$  along the display map it depends on  $\operatorname{disp}_A$ . Note that this pushforward is in  $\operatorname{\mathbf{grpd}}$ , and this pushforward is only defined on fibrations.

TODO: define  $\lambda$ .

Proof. TODO: naturality.

TODO: prove pullback.  $\Box$ 

**Definition 4.17** (Interpretation of  $\Sigma$ ). Sketch: we define the natural transformation  $\Sigma : \mathsf{Poly}_\mathsf{tp}\mathsf{Ty} \to \mathsf{Ty}$  by first taking some small groupoid  $\Gamma$  and defining

$$\Sigma_{\Gamma}:\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty})\to\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Ty})$$

Again, this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

As indicated in the diagram, we take this to be the composition of  $\mathsf{disp}_B$  and  $\mathsf{disp}_A$ , recalling that fibrations are closed under composition.

TODO: define pair.

*Proof.* TODO: naturality.

TODO: prove pullback.  $\Box$ 

# References

- [Awo23] Steve Awodey. On hofmann-streicher universes, 2023.
- [HS98] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.
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