A Groupoidal Natural Model of HoTT in Lean 4

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0.1 Natural Models

In this section we describe the categorical semantics of HoTT via Natural Models. This will not be a detailed account of the syntax of HoTT, but will be a detailed account of what is needed to interpret such syntax. It will follow [Awo17], but with a more compact description of identity types using the technology of polynomial endofunctors, and a universe of small types.

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as **set** and the large sets as **Set**. For example, we could take the small sets **set** to be those in **Set** bounded in cardinality by some inaccessible cardinal.

0.1.1 Types

Let $\mathbb{C}$ be a locally small category, i.e. a category whose class of objects is a **Set** and whose hom-classes are from **set**. We write **Psh**($\mathbb{C}$) for the category of (large) presheaves over $\mathbb{C}$,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

Definition 0.1.1. Following Awodey [Awo17], we say that a map $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ is presentable when any fiber of a representable is representable. In other words, given any $\Gamma \in \mathbb{C}$ and a map $A : \mathsf{y}(\Gamma) \to \mathsf{Ty}$, there is some representable $\Gamma \cdot A \in \mathbb{C}$ and maps $\mathsf{disp}_A : \Gamma \cdot A \to \Gamma$ and $\mathsf{var}_A : \mathsf{y}(\Gamma \cdot A) \to \mathsf{Tm}$ forming a pullback

$$\begin{tikzcd}
\mathsf{y}(\Gamma \cdot A) \arrow[r, "\mathsf{var}_A"] \arrow[d, "\mathsf{y}(\mathsf{disp}_A)"'] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{y}(\Gamma) \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}$$

The Natural Model associated to a presentable map $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, ... \in \mathbb{C}$,
- a type in context $\mathsf{y}(\Gamma)$ as a map $A \colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$,
- a term of type A in context Γ as a map $a \colon \mathsf{y}(\Gamma) \to \mathsf{Tm}$ such that

$$\begin{tikzcd}
 & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}$$

  commutes,
- an operation called "context extension" which given a context Γ and a type $A \colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{tikzcd}
\mathsf{y}(\Gamma.A) \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\
\mathsf{y}(\Gamma) \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty. We write

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{y}(\Gamma) \arrow[r, "\ulcorner X \urcorner"'] & \mathsf{Ty} \end{tikzcd}$$

to express this situation, i.e. $X \cong \mathsf{y}(\Gamma \cdot \ulcorner X \urcorner)$.

0.1.2 Pi types

We will use P_{tp} to denote the polynomial endofunctor (definition 0.3.1) associated with our presentable map tp. Then an interpretation of Π types consists of a pullback square

$$\begin{tikzcd} P_{\mathsf{tp}}\mathsf{Tm} \arrow[r, "\lambda"] \arrow[d, "P_{\mathsf{tp}}\mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ P_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Pi"'] & \mathsf{Ty} \end{tikzcd} \tag{0.1.1}$$

0.1.3 Sigma types

An interpretation of Σ types consists of a pullback square

$$\begin{tikzcd} Q \arrow[r, "\mathsf{pair}"] \arrow[d, "\mathsf{tp} \lhd \mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ P_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Sigma"'] & \mathsf{Ty} \end{tikzcd} \tag{0.1.2}$$

where the composition of polynomials $\mathsf{tp} \lhd \mathsf{tp} : Q \to P_{\mathsf{tp}}\mathsf{Ty}$ is given by

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[rrr, bend left, "\mathsf{tp} \lhd \mathsf{tp}"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[r, equal] \arrow[d, "\mathsf{counit}"] & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}"] \\ \mathsf{Tm} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Ty} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] \arrow[d] & \mathsf{Ty} \\ \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty} \arrow[r] & \bullet & \end{tikzcd}$$

Here, $\mathsf{counit} : \mathsf{tp}^*\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty} \to \mathsf{Tm}^*\mathsf{Ty}$ is the counit of the adjunction $\mathsf{tp}^* \dashv \mathsf{tp}_*$ at $\mathsf{Tm}^*\mathsf{Ty} \in \mathbf{Psh}(\mathbf{grpd})/\mathsf{Tm}$.

0.1.4 Identity types

To interpret the formation and introduction rules for identity types we require a commutative square (this need not be pullback)

$$\begin{tikzcd} \mathsf{Tm} \arrow[r, "\mathsf{refl}"] \arrow[d, "\delta"'] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[r, "\mathsf{Id}"'] & \mathsf{Ty} \end{tikzcd} \tag{0.1.3}$$

where δ is the diagonal:

$$
\begin{tikzcd}
\mathsf{Tm} \arrow[dr, "\delta"] \arrow[drr, equals, bend left] \arrow[ddr, equals, bend right] & & \\
& \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
& \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty}
\end{tikzcd}
$$

Then let I be the pullback. We get a comparison map ρ

$$
\begin{tikzcd}
\mathsf{Tm} \arrow[dr, dashed, "\rho"] \arrow[drr, bend left, "\mathsf{refl}"] \arrow[ddr, bend right, "\delta"'] & & \\
& I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
& \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[r, "\mathsf{Id}"'] & \mathsf{Ty}
\end{tikzcd}
$$

Then view $\rho : \mathsf{tp} \to q$ as a map in the slice over Ty.

$$
\begin{tikzcd}
\mathsf{Tm} \arrow[dr, dashed, "\rho"] \arrow[ddr, "\delta"] \arrow[dddr, equals, bend right] \arrow[ddddr, bend right, "\mathsf{tp}"'] & \\
& I \arrow[d] \arrow[ddd, bend left, "q"] \\
& \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[d, "\mathsf{fst}"'] \\
& \mathsf{Tm} \arrow[d] \\
& \mathsf{Ty}
\end{tikzcd}
$$

Now (by definition 0.3.6) applying $P_{\_} : (\mathbf{Psh}(\mathbb{C})/\mathsf{Ty})^{\mathrm{op}} \to [\mathbf{Psh}(\mathbb{C}), \mathbf{Psh}(\mathbb{C})]$ to $\rho : \mathsf{tp} \to q$ gives us a naturality square (this also need not be pullback).

$$
\begin{tikzcd}
P_q\mathsf{Tm} \arrow[r, "\rho^\star_{\mathsf{Tm}}"] \arrow[d, "P_q\mathsf{tp}"'] & P_{\mathsf{tp}}\mathsf{Tm} \arrow[d, "P_{\mathsf{tp}}\mathsf{tp}"] \\
P_q\mathsf{Ty} \arrow[r, "\rho^\star_{\mathsf{Ty}}"'] & P_{\mathsf{tp}}\mathsf{Tm}
\end{tikzcd}
\tag{0.1.4}
$$

Taking the pullback T and the comparison map ε we have

$$
\begin{tikzcd}
P_q\mathsf{Tm} \arrow[dr, dashed, "\varepsilon"] \arrow[drr, bend left, "\rho^\star_{\mathsf{Tm}}"] \arrow[ddr, bend right, "P_q\mathsf{tp}"'] & & \\
& T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Tm} \arrow[d, "P_{\mathsf{tp}}\mathsf{tp}"] \\
& P_q\mathsf{Ty} \arrow[r, "\rho^\star_{\mathsf{Ty}}"'] & P_{\mathsf{tp}}\mathsf{Ty}
\end{tikzcd}
\tag{0.1.5}
$$

Finally, we require a section $\mathsf{J} : T \to P_q\mathsf{Tm}$ of ε, to interpret the identity elimination rule.

0.1.5 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} \colon \mathsf{Ty} \qquad \frac{a \colon \mathsf{U}}{\mathsf{El}(a) \colon \mathsf{Ty}}$$

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{tikzcd} \mathsf{U} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ 1 \arrow[r, "\ulcorner \mathsf{U} \urcorner"'] & \mathsf{Ty} \end{tikzcd} \tag{0.1.6}$$

- There is an inclusion of U into Ty

$$\mathsf{El} \colon \mathsf{U} \rightarrowtail \mathsf{Ty}$$

- $\pi : \mathsf{E} \to \mathsf{U}$ is obtained as pullback of tp; There is a pullback diagram

$$\begin{tikzcd} E \arrow[r, tail] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{U} \arrow[r, tail, "\mathsf{El}"'] & \mathsf{Ty} \end{tikzcd} \tag{0.1.7}$$

With the notation above, we get

$$\begin{tikzcd} \mathsf{y}(\Gamma.\mathsf{El}(a)) \arrow[r] \arrow[d] & \mathsf{E} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{y}(\Gamma) \arrow[r, "a"'] \arrow[rr, bend right, "A"'] & \mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty} \end{tikzcd}$$

Both squares above are pullback squares.

0.1.6 Stablity of the universe under type formers

Take the pullback diagram eq. (0.1.7). That is a morphism in the category of polynomials. By definition 0.3.7 we have a cartesian natural transformation $P_\pi \to P_{\mathsf{tp}}$ induced by the pullback eq. (0.1.7). This cartesian natural transformation induces the cube diagram below; all of the squares in the cube are pullback squares.

$$\begin{tikzcd}
& P_\pi E \arrow[rr] \arrow[dl] \arrow[dd] & & P_{\mathsf{tp}} E \arrow[dl] \arrow[dd] \\
P_\pi \mathsf{Tm} \arrow[rr] \arrow[dd] & & P_{\mathsf{tp}} \mathsf{Tm} \arrow[dd] & \\
& P_\pi \mathsf{U} \arrow[rr] \arrow[dl] & & P_{\mathsf{tp}} U \arrow[dl] \\
P_\pi \mathsf{Ty} \arrow[rr] & & P_{\mathsf{tp}} \mathsf{Ty} &
\end{tikzcd} \tag{0.1.8}$$

We will use the compositions $P_\pi U \to P_{\mathsf{tp}} \mathsf{Ty}$ and $P_\pi E \to P_{\mathsf{tp}} \mathsf{Tm}$ below.

Definition 0.1.2. We will say that universe U is closed under formation of Π-types when we have some map $\Pi_U : P_\pi \mathsf{U} \to \mathsf{U}$ making the following square commute

$$\begin{tikzcd}
P_\pi \mathsf{U} \arrow[r] \arrow[d, dashed, "\Pi_{\mathsf{U}}"'] & P_{\mathsf{tp}} \mathsf{Ty} \arrow[d, "\Pi"] \\
\mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd} \tag{0.1.9}$$

Note that this is merely propositional when El is a monomorphism.

From the universal property of pullbacks we can define $\lambda_U P_\pi \mathsf{E} \to \mathsf{E}$.

$$\begin{tikzcd}
& P_{\mathsf{tp}} E \arrow[rr, dashed, "\lambda_U"] \arrow[dl] \arrow[dd] & & E \arrow[dl] \arrow[dd] \\
P_{\mathsf{tp}} \mathsf{Tm} \arrow[rr, "\lambda"] \arrow[dd] & & \mathsf{Tm} \arrow[dd] & \\
& P_\pi U \arrow[rr, dashed, "\Pi_U"] \arrow[dl] & & U \arrow[dl, "\mathsf{El}"] \\
P_{\mathsf{tp}} \mathsf{Ty} \arrow[rr, "\Pi"'] & & \mathsf{Ty} &
\end{tikzcd}$$

The top and bottom squares in the cube above are not pullbacks, but we know three of the vertical faces are pullback squares. By the pullback pasting lemma it follows that the back square involving Π_U and λ_U is also a pullback square. This concludes the construction of the Π type former for the universe U. The only data we needed to supply was the lift Π_U of $\Pi \colon P_{\mathsf{tp}} \mathsf{Ty} \to \mathsf{Ty}$ to the universe U.

Definition 0.1.3. We will say that universe U is closed under formation of Σ-types when we have some map $\Sigma_U : P_\pi \mathsf{U} \to \mathsf{U}$ making the following square commute

$$\begin{tikzcd}
P_\pi \mathsf{U} \arrow[r] \arrow[d, dashed, "\Sigma_{\mathsf{U}}"'] & P_{\mathsf{tp}} \mathsf{Ty} \arrow[d, "\Sigma"] \\
\mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd} \tag{0.1.10}$$

Again, this is merely propositional when El is a monomorphism.

Now consider the polynomial composition $\pi \lhd \pi$.

$$
\begin{tikzcd}
Q_U \arrow[r] \arrow[d] \arrow[rrr, bend left, "\pi\lhd\pi"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dddr, dashed] & R_U \arrow[r, equal] \arrow[d] \arrow[dddr, dashed] & R_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dddr, dashed] & P_\pi U \arrow[d] \arrow[dddr] & \\
E \times E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dddr, dashed] & E \times U \arrow[r] \arrow[d] \arrow[dddr, dashed] & E \arrow[r, "\pi"] \arrow[dddr] & U \arrow[dddr] & \\
E \arrow[r] \arrow[dddr] & U \arrow[dddr] & & & \\
& Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[r, equal] \arrow[d] & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty}"] \\
& \mathsf{Tm} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \times \mathsf{Ty} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"] & \mathsf{Ty} \\
& \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty} & &
\end{tikzcd}
$$

Using pullback pasting, we see that the horizontal faces of the left cuboid and the right cube are all pullbacks. Hence we have a pullback

$$
\begin{tikzcd}
Q_U \arrow[r] \arrow[d] \arrow[rr, bend left, "\pi\lhd\pi"] \arrow[dr, phantom, "\lrcorner", very near start] & R_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_\pi U \arrow[d] \\
Q \arrow[r] \arrow[rr, bend right, "\mathsf{tp}\lhd\mathsf{tp}"'] & R \arrow[r] & P_{\mathsf{tp}}\mathsf{Ty}
\end{tikzcd}
$$

which is the left side of the following cube

$$
\begin{tikzcd}
& Q_U \arrow[rr, dashed, "\mathsf{pair}_U"] \arrow[dl] \arrow[dd, "\pi\lhd\pi"'] & & E \arrow[dl] \arrow[dd] \\
Q \arrow[rr, "\mathsf{pair}"] \arrow[dd, "\mathsf{tp}\lhd\mathsf{tp}"'] & & \mathsf{Tm} \arrow[dd] & \\
& P_\pi U \arrow[rr, "\Sigma_U"] \arrow[dl] & & U \arrow[dl, "\mathsf{El}"] \\
P_{\mathsf{tp}}\mathsf{Ty} \arrow[rr, "\Sigma"'] & & \mathsf{Ty} &
\end{tikzcd}
$$

As was the case for Π, pullback pasting along the vertical faces shows that the back face involving Σ_U and pair_U is also a pullback square. This concludes the construction of the Σ type former for the universe U. Again, the only data we needed to supply was the lift Σ_U of $\Sigma\colon P_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ to the universe U.

$$
\begin{tikzcd}
& \pi \times_{\mathsf{U}} \pi \arrow[rr] \arrow[dl, dashed] \arrow[dd] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathsf{E} \arrow[dl] \arrow[dd] \\
\mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[rr, crossing over] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{Tm} & \\
& \mathsf{E} \arrow[rr] \arrow[dl] & & \mathsf{U} \arrow[dl] \\
\mathsf{Tm} \arrow[rr] & & \mathsf{Ty} \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

Definition 0.1.4. We will say that universe U is closed under formation of Id-types when we have some map $\mathsf{Id}_U : \pi \times_{\mathsf{U}} \pi \to \mathsf{U}$ making the following square commute

$$
\begin{tikzcd}
\pi \times_{\mathsf{U}} \pi \arrow[r] \arrow[d, dashed, "\mathsf{Id}_U"'] & \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[d, "\mathsf{Id}"] \\
\mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd}
$$

Again, this is merely propositional when El is a monomorphism.

From this we can obtain refl_U since E is a pulback.

$$
\begin{tikzcd}
& E \arrow[rr, dashed, "\mathsf{refl}_U"] \arrow[dl] \arrow[dd, "\delta_U"'] & & E \arrow[dl] \arrow[dd] \\
\mathsf{Tm} \arrow[rr, crossing over, "\mathsf{refl}"] \arrow[dd, "\delta"'] & & \mathsf{Tm} & \\
& \pi \times_{\mathsf{U}} \pi \arrow[rr, "\mathsf{Id}_U"] \arrow[dl] & & U \arrow[dl, "\mathsf{El}"] \\
\mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[rr, "\mathsf{Id}"'] & & \mathsf{Ty} \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

We must ensure that the left face commutes - which we can prove using the universal property of $\mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp}$. It remains to construct J_U, a section of ε_U, given below

$$
\begin{tikzcd}
\mathsf{E} \arrow[r, dashed, "\rho_U"] \arrow[rr, bend left, "\mathsf{refl}_U"] \arrow[dr, "\delta"'] & I_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{E} \arrow[d, "\pi"] \\
& \pi \times_{\mathsf{U}} \pi \arrow[r, "\mathsf{Id}_U"'] & \mathsf{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathsf{E} \arrow[r] & \mathsf{Tm} \arrow[d] \\
I_U \arrow[ur] \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dddd, bend right, "q_U"'] & I \arrow[ur] \arrow[d] & \mathsf{Ty} \\
\pi \times_{\mathsf{U}} \pi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[ur] \arrow[d] & \\
\mathsf{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d] & \\
\mathsf{U} \arrow[r] & \mathsf{Ty} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
P_{q_U}\mathsf{E} \arrow[r, dashed, "\varepsilon_U"] \arrow[rr, bend left, "(\rho_U{}^\star)_{\mathsf{E}}"] \arrow[dr, "P_{q_U}\mathsf{tp}"'] & T_U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_\pi \mathsf{E} \arrow[d, "P_\pi \pi"] \\
& P_{q_U}\mathsf{U} \arrow[r, "(\rho_U{}^\star)_{\mathsf{U}}"'] & P_\pi \mathsf{U}
\end{tikzcd}
$$

(0.1.11)

Now the above give us the following pullback factorization of eq. (0.1.7)

$$\pi \overset{}{\underset{\kappa}{\Longrightarrow}} \mathsf{tp}$$

$$\begin{tikzcd}
\mathsf{E} \arrow[r, dashed] \arrow[d, "\rho_U"] \arrow[dd, bend right=60, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\rho"'] \arrow[dd, bend left=60, "\mathsf{tp}"] \\
I_U \arrow[r] \arrow[d, dashed, "q_U"] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, dashed, "q"] \\
\mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd}$$

If we apply definition 0.3.7 to the lower pullback square, i.e. κ_q then we will get a cartesian natural tranformation $P_{\kappa_U} : P_{q_U} \to P_q$. Like in eq. (0.1.8), evaluating this at the pullback square eq. (0.1.7) induces the cube below with pullbacks on every face.

$$\begin{tikzcd}
& P_{q_U}\mathsf{E} \arrow[rr] \arrow[dl] \arrow[dd] & & P_q\mathsf{E} \arrow[dl] \arrow[dd] \\
P_{q_U}\mathsf{Tm} \arrow[rr, crossing over] \arrow[dd] & & P_q\mathsf{Tm} & \\
& P_{q_U}\mathsf{U} \arrow[rr] \arrow[dl] & & P_q\mathsf{U} \arrow[dl] \\
P_{q_U}\mathsf{Ty} \arrow[rr] & & P_q\mathsf{Ty} \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{0.1.12}$$

Now we consider the diagrams for both ε and ε_U

$$\begin{tikzcd}
& P_{q_U}\mathsf{E} \arrow[r, "\varepsilon_U"] \arrow[d] \arrow[ddl] & T_U \arrow[r] \arrow[dl] \arrow[ddl, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & P_\pi\mathsf{E} \arrow[d] \arrow[ddl] \\
& P_{q_U}\mathsf{U} \arrow[rr] \arrow[ddl] & & P_\pi\mathsf{U} \arrow[ddl] \\
P_q\mathsf{Tm} \arrow[r, "\varepsilon"] \arrow[d] & T \arrow[r] \arrow[dl] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Tm} \arrow[d] & \\
P_q\mathsf{Ty} \arrow[rr] & & P_{\mathsf{tp}}\mathsf{Ty} &
\end{tikzcd}$$

The right face is the composed diagonal pullback square from eq. (0.1.8). The left face of the outer cube is the composed diagonal pullback square from eq. (0.1.12). The front face of the outer cube is the naturality square for $\rho^\star$ from eq. (0.1.5) and similarly the back face is the naturality square for ${\rho_U}^\star$ from eq. (0.1.11).

Since the left face is a pullback square, to make $J_U : T_U \to P_{q_U}\mathsf{E}$ it suffices to consider

$$\begin{tikzcd}
T_U \arrow[rr, dashed, "d"] \arrow[dr, dashed, "J_U"] \arrow[ddr] & & T \arrow[d, dashed, "J"] \\
& P_{q_U}\mathsf{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_q\mathsf{Tm} \arrow[d] \\
& P_{q_U}\mathsf{U} \arrow[r] & P_q\mathsf{Ty}
\end{tikzcd}$$

It follows from uniqueness of maps into pullbakcs that J_U so defined is a section of ε_U. This concludes the construction of the Id type former for the universe U. Again, the only data we needed to supply was the lift Id_U of Id to the universe U.

0.1.7 Binary products and Exponentials

It is convenient to specialize Σ and Π types to their non-dependent counterparts. In the natural model we can construct these by considering first the map

$$(\mathsf{fst}, \mathsf{snd}) : \mathsf{Ty} \times \mathsf{Ty} \to P_{\mathsf{tp}}\mathsf{Ty}$$

which we can visualize in

$$\begin{tikzcd}
\mathsf{Ty} & \mathsf{Tm} \times \mathsf{Ty} \arrow[l, dashed, "\mathsf{snd}"'] \arrow[r] \arrow[d, "\mathsf{fst}^*\mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
& \mathsf{Ty} \times \mathsf{Ty} \arrow[r, "\mathsf{fst}"'] & \mathsf{Ty}
\end{tikzcd}$$

Then, respectively, the pullback of the diagrams eq. (0.1.1) and eq. (0.1.2) for interpreting Π and Σ rules along this map give us pullback diagrams for interpreting function types and product types.

$$\begin{tikzcd}
F \arrow[r, "(\mathsf{dom},\mathsf{fun})"] \arrow[d, "(\mathsf{dom},\mathsf{cod})"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\lambda"] & P_{\mathsf{tp}}\mathsf{Tm} \arrow[r, "\lambda"] \arrow[d, "P_{\mathsf{tp}}\mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{Ty} \times \mathsf{Ty} \arrow[r, "(\mathsf{fst},\mathsf{snd})"'] \arrow[rr, bend right, "\mathsf{Exp}"'] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Pi"'] & \mathsf{Ty}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Tm} \times \mathsf{Tm} \arrow[r, "(\mathsf{snd},\mathsf{fst},\mathsf{tp}\circ\mathsf{snd})"] \arrow[d, "\mathsf{tp}\times\mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\mathsf{pair}"] & Q \arrow[r, "\mathsf{pair}"] \arrow[d, "\mathsf{tp}\lhd\mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{Ty} \times \mathsf{Ty} \arrow[r, "(\mathsf{fst},\mathsf{snd})"'] \arrow[rr, bend right, "\times"'] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Sigma"'] & \mathsf{Ty}
\end{tikzcd}$$

By the universal property of pullbacks and proposition 0.3.2 We can write a map $\Gamma \to F$ as a triple (A, B, f) such that $A, B : \Gamma \to \mathsf{Ty}$ and

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r, "f"] \arrow[d, "\mathsf{disp}_A"'] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\Gamma \arrow[r, "B"'] & \mathsf{Ty}
\end{tikzcd}$$

This gives us four equivalent ways we can view a function. Namely, as $f : \Gamma{\cdot}A \to \mathsf{Tm}$ in the above diagram, $\lambda \circ f : \Gamma \to \mathsf{Tm}$, as $(A, B, f) : \Gamma \to F$, or as a map between the displays $\mathsf{disp}_A \to \mathsf{disp}_B$

$$\begin{tikzcd}
\Gamma \cdot A \arrow[drr, bend left, "f"] \arrow[dr, dashed, "(\mathsf{disp}_A, f)"] \arrow[ddr, bend right, "\mathsf{disp}_A"'] & & \\
& \Gamma \cdot B \arrow[r] \arrow[d, "\mathsf{disp}_B"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d] \\
& \Gamma \arrow[r, "B"'] & \mathsf{Ty}
\end{tikzcd}$$

For the formalization, we need not prove that the pullback of $\mathsf{tp} \lhd \mathsf{tp}$ is $\mathsf{tp} \times \mathsf{tp}$. Rather, we can also use the universal property of pullbacks and proposition 0.3.2 to classify a map into the pullback (whatever it may be) as a pair (α, β), where $\alpha, \beta : \Gamma \to \mathsf{Tm}$. This could then be adapted to a proof that the pullback is what the diagram claims it to be.

Naturally, there are the same constructions bounded by the universe, which will exist when Π_U and Σ_U exist.

$$\begin{tikzcd}
\mathsf{U} \times \mathsf{U} \arrow[r, "\mathsf{Exp}_U"] \arrow[d, "\mathsf{El} \times \mathsf{El}"'] & \mathsf{U} \arrow[d, "\mathsf{El}"] \\
\mathsf{Ty} \times \mathsf{Ty} \arrow[r, "\mathsf{Exp}"'] & \mathsf{Ty}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{U} \times \mathsf{U} \arrow[r, "\times_U"] \arrow[d, "\mathsf{El} \times \mathsf{El}"'] & \mathsf{U} \arrow[d, "\mathsf{El}"] \\
\mathsf{Ty} \times \mathsf{Ty} \arrow[r, "\times"'] & \mathsf{Ty}
\end{tikzcd}$$

The identity function $\mathsf{id}_A : \Gamma \to \mathsf{Tm}$ of type $\mathsf{Exp} \circ (A, A) : \Gamma \to \mathsf{Ty}$ can be defined by the following

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r, "\mathsf{var}_A"'] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\Gamma \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \arrow[drr, dashed, bend left, "\mathsf{id}_A"] \arrow[dr, "(A, A, \mathsf{var}_A)"] \arrow[ddr, bend right, "(A, A)"'] & & \\
& F \arrow[r, "\lambda"] \arrow[d, "(\mathsf{dom}, \mathsf{cod})"] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
& \mathsf{Ty} \times \mathsf{Ty} \arrow[r, "\mathsf{Exp}"'] & \mathsf{Ty}
\end{tikzcd}$$

Viewed as a map between the display maps, this is simply the identity $\Gamma \cdot A \to \Gamma \cdot A$.

$$\begin{tikzcd}
\Gamma \cdot A \arrow[drr, bend left, "\mathsf{var}_A"] \arrow[dr, equal] \arrow[ddr, bend right, "\mathsf{disp}_A"'] & & \\
& \Gamma \cdot A \arrow[r, "\mathsf{var}_A"'] \arrow[d, "\mathsf{disp}_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
& \Gamma \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}$$

Composition is also simplest when viewed as an operation on maps between fibers. Given $f : \mathsf{disp}_A \to \mathsf{disp}_B$ and $g : \mathsf{disp}_B \to \mathsf{disp}_C$, the composition is $g \circ f : \mathsf{disp}_A \to \mathsf{disp}_C$.

0.1.8 Univalence

For two types $AB : \Gamma \to \mathsf{Ty}$ and two functions $f, g : A \to B$ we can define internally a *homotopy* from f to g as

$$f \sim g := \Pi_{a:A} \mathsf{Id}(f\, a, g\, a)$$

We define the types of left and right inverses of $f : A \to B$ as

$$\mathsf{BigLinv} f := \Sigma_{g : B \to A}\, g \circ f \sim \mathsf{id}_A$$

$$\mathsf{BigRinv} f := \Sigma_{g : B \to A}\, f \circ g \sim \mathsf{id}_B$$

and the property of being an equivalence

$$\mathsf{IsBigEquiv} f := \mathsf{BigLinv} f \times \mathsf{BigRinv} f$$

We could do the same for two small types $A, B : \Gamma \to \mathsf{U}$

$$\mathsf{IsEquiv} f := \mathsf{Linv} f \times \mathsf{Rinv} f$$

$$\mathsf{Equiv} A\, B := \Sigma_{f : A \to B} \mathsf{IsEquiv} f$$

Again, internally we can define a function

$$\mathsf{IdToEquiv} A\, B : \mathsf{Id}(A, B) \to \mathsf{Equiv} A\, B$$

which uses J to transport along the proof of equality to produce an equivalence. Univalence for universe U states that this itself is an equivalence

$$\mathsf{ua} : \mathsf{IsBigEquiv}(\mathsf{IdToEquiv} A\, B)$$

Note that this statement is large, i.e. not a type in the universe U.

$$\begin{tikzcd}
\mathsf{U} \cdot \mathsf{U} \cdot \mathsf{Id} \arrow[rr, "\mathsf{IdToEquiv}"] \arrow[dr] & & \mathsf{U} \cdot \mathsf{U} \cdot \mathsf{Equiv} \arrow[dl] \\
& \mathsf{U} \cdot \mathsf{U} &
\end{tikzcd}$$

0.1.9 Extensional identity types and UIP

In this section we outline variations on the identity type in the natural model. We will describe these as additional structure on Id, as opposed to introducing different identity types. The first option is fully extensional identity types, i.e. those satisfying equality reflection and uniqueness of identity proofs (UIP). Equality reflection says that if one can construct a term satisfying $\mathsf{Id}(a, b)$ then we have that definitionally $a \equiv b$, i.e. they are equal morphisms in the natural model. This amounts to requiring that eq. (0.1.3) is a pullback, i.e. ρ is an isomorphism

$$\begin{tikzcd}
\mathsf{Tm} \arrow[r, "\mathsf{refl}"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[r, "\mathsf{Id}"'] & \mathsf{Ty}
\end{tikzcd}$$

Note that this means $\rho^\star$ is an isomorphism, from which it follows that eq. (0.1.4) is also a pullback, i.e. ε is an isomorphism.

$$\begin{tikzcd}
P_q \mathsf{Tm} \arrow[r, "\rho^\star_{\mathsf{Tm}}"] \arrow[d, "P_q \mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}} \mathsf{Tm} \arrow[d, "P_{\mathsf{tp}} \mathsf{tp}"] \\
P_q \mathsf{Ty} \arrow[r, "\rho^\star_{\mathsf{Ty}}"'] & P_{\mathsf{tp}} \mathsf{Tm}
\end{tikzcd}$$

If we were to only require UIP then this is asking that $I \to \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp}$ is a strict proposition, meaning for any $(a,b) : \Gamma \to \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp}$ there it at most one lift

$$\begin{tikzcd} & I \arrow[d] \\ \Gamma \arrow[ur, dashed, "!"] \arrow[r, "(a,b)"'] & \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \end{tikzcd}$$

One might wonder what other variations we could come up with by tweaking the pullback conditions. In fact, only requiring that ρ has a section is equivalent to requiring that ρ is an isomorphism. So this just the extensional case again.

If we require instead that ε is an isomorphism then this is giving an η-rule for J, from which we can prove equality reflection and UIP [?]. So this just the extensional case again.

0.2 The Groupoid Model

In this section we construct a natural model in **Psh**(**grpd**) the presheaf category indexed by the category **grpd** of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on **grpd** - which comes from restricting the "classical Quillen model structure" on **cat** [Joy] to **grpd**.

0.2.1 Classifying display maps

Notation. We denote the category of small categories as **cat** and the large categories as **Cat**. We denote the category of small groupoids as **grpd**.

We are primarily working in the category of large presheaves indexed by the (large, locally small) category of small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

Definition 0.2.1 (Pointed)**.** We will take the category of pointed small categories $\mathbf{cat}_\bullet$ to have objects as pairs $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$ and morphisms as pairs

$$(F : \mathbb{C}_1 \to \mathbb{C}_0, \phi : Fc_1 \to c_0) \colon (\mathbb{C}_1, c_1) \to (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids $\mathbf{grpd}_\bullet$ will be the full subcategory of objects (Γ, c) with Γ a groupoid.

Definition 0.2.2 (The display map classifier)**.** We would like to define a natural transformation in **Psh**(**grpd**)

$$\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$$

with representable fibers.

Consider the functor that forgets the point

$$U \colon \mathbf{grpd}_\bullet \to \mathbf{grpd} \qquad \text{in} \qquad \mathbf{Cat}.$$

If we apply the Yoneda embedding $\mathsf{y}\colon \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U\circ\colon [-, \mathbf{grpd}_\bullet] \to [-, \mathbf{grpd}] \qquad \text{in} \qquad \mathbf{Psh}(\mathbf{Cat}).$$

Since any small groupoid is also a large category $i : \mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict $\mathbf{Cat}$ indexed presheaves to be $\mathbf{grpd}$ indexed presheaves (this the nerve in $i_! \dashv \mathsf{res}$). We define $\mathsf{tp}\colon \mathsf{Tm} \to \mathsf{Ty}$ as the image of $U\circ$ under this restriction.

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "\mathsf{y}"] & \mathbf{Psh}(\mathbf{Cat}) \arrow[r, "\mathsf{res}"] & \mathbf{Psh}(\mathbf{grpd}) \\
\mathbf{grpd} \arrow[r, mapsto] & {[-, \mathbf{grpd}]} \arrow[r, mapsto] & \mathsf{Ty}
\end{tikzcd}$$

Note that Tm and Ty are not representable in $\mathbf{Psh}(\mathbf{grpd})$.

Remark 0.2.3. By Yoneda we can identify maps with representable domain into the type classifier

$$A : \mathsf{y}\Gamma \to \mathsf{Ty} \qquad \text{in} \qquad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A : \Gamma \to \mathbf{grpd} \qquad \text{in} \qquad \mathbf{Cat}$$

Definition 0.2.4 (Grothendieck construction)**.** From $\mathbb{C}$ a small category and $F : \mathbb{C} \to \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in $\mathbb{C}$ we refer to Fc as the fiber over c. The objects of $\int F$ consist of pairs $(c \in \mathbb{C}, x \in Fc)$, and morphisms between (c, x) and (d, y) are pairs $(f : c \to d, \phi : F\,f\,x \to y)$. This makes the following pullback in $\mathbf{Cat}$

$$(c, x) \longmapsto (Fc, x)$$

$$\begin{tikzcd}
(c,x) \arrow[d, mapsto] & \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{cat}_\bullet \arrow[d] & (C,c) \arrow[d, mapsto] \\
c & \mathbb{C} \arrow[r, "F"'] & \mathbf{cat} & C
\end{tikzcd}$$

Definition 0.2.5 (Grothendieck construction for groupoids)**.** Let Γ be a groupoid and $A\colon \Gamma \to \mathbf{grpd}$ a functor, we can compose F with the inclusion $i\colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \mathsf{disp}_A\colon \Gamma \cdot A \to \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids Γ and Ax for $x \in \Gamma$. Furthermore the pullback factors through (pointed) groupoids.

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{cat}_\bullet \arrow[d] \\
\Gamma \arrow[r, "A"'] & \mathbf{grpd} \arrow[r] & \mathbf{cat}
\end{tikzcd}$$

Corollary 0.2.6 (The display map classifier is presentable)**.** *For any small groupoid Γ and $A : \mathsf{y}\Gamma \to \mathsf{Ty}$, the pullback of tp along A can be given by the representable map ydisp_A.*

$$\begin{tikzcd}
\mathsf{y}\Gamma \cdot A \arrow[r] \arrow[d, "\mathsf{ydisp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{y}\Gamma \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}$$

Proof. Consider the pullback in **Cat**

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d] \\
\Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

We send this square along $\mathsf{res} \circ \mathsf{y}$ in the following

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "\mathsf{y}"] \arrow[dr] & \mathbf{Psh}(\mathbf{Cat}) \arrow[d, "\mathsf{res}"] \\
\mathbf{grpd} \arrow[u] \arrow[r, "\mathsf{y}"] & \mathbf{Psh}(\mathbf{grpd})
\end{tikzcd}$$

The Yoneda embedding $\mathsf{y} : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does res since it is a right adjoint (with left Kan extension $\iota_! \dashv \mathsf{res}_\iota$). $\square$

0.2.2 Groupoid fibrations

Definition 0.2.7 (Fibration)**.** Let $p : \mathbb{C}_1 \to \mathbb{C}_0$ be a functor. We say p is a *split Grothendieck fibration* if we have a dependent function $\mathsf{lift}\, a\, f$ satisfying the following: for any object a in $\mathbb{C}_1$ and morphism $f : p\, a \to y$ in the base $\mathbb{C}_0$ we have $\mathsf{lift}\, a\, f : a \to b$ in $\mathbb{C}_1$ such that $p(\mathsf{lift}\, a\, f) = f$ and moreover $\mathsf{lift}\, a\, g \circ f = \mathsf{lift}\, b\, g \circ \mathsf{lift}\, a\, f$

$$\begin{tikzcd}
a \arrow[r, dashed, "\mathsf{lift}\, a\, f"] \arrow[d, mapsto] \arrow[dr, phantom, "\Downarrow"] & b \arrow[d, dashed, mapsto] \\
x \arrow[r, "f"'] & y
\end{tikzcd}$$

In particular, we are interested in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid Γ as Fib_Γ, which is a full subcategory of the slice $\mathbf{grpd}/\Gamma$. We will decorate an arrow with $\twoheadrightarrow$ to indicate it is a fibration.

Note that $\mathsf{disp}_A \colon \Gamma \cdot A \to \Gamma$ is a fibration, since for any $(x \in \Gamma, a \in A\,x)$ and $f \colon x \to y$ in Γ we have a morphism $(f, \mathsf{id}_{A\,f\,a}) : (x, a) \to (y, A\,f\,a)$ lifting f. Furthermore

Proposition 0.2.8. *There is an adjoint equivalence*

$$\begin{tikzcd}[\Gamma, \mathbf{grpd}] \arrow[r, shift left, "\mathsf{disp}"] \arrow[r, phantom, "\simeq"] & \mathsf{Fib}_\Gamma \arrow[l, shift left, "\mathsf{fiber}"]\end{tikzcd}$$

where for each fibration $\delta : \Delta \to \Gamma$ and each object $x \in \Gamma$

$$\mathsf{fiber}_\delta\, x = \text{full subcategory}\, \{a \in \Delta \mid \delta\, a = x\}$$

It follows that all fibrations are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \to \mathbf{grpd}$, when viewed as morphisms in **Cat**.

Pullback of fibrations along groupoid functors is not strictly coherent, in the sense that for $\tau : \Xi \to \Delta$ and $\sigma : \Delta \to \Gamma$ and a fibration $p \in \mathsf{Fib}_\Gamma$ we only have an isomorphism

$$\tau^* \sigma^* p \cong (\sigma \circ \tau)^* p$$

rather than equality.

In order to interpret reindexing/substitution strictly, it is convenient to work with classifiers $[\Gamma, \mathbf{grpd}]$ instead of fibrations.

Proposition 0.2.9 (Strictly coherent pullback)**.** *Let $\sigma : \Delta \to \Gamma$ be a functor between groupoids. Since display maps are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \to \mathbf{grpd}$ we have the pasting diagram*

$$\begin{tikzcd}
\Delta.A\sigma \arrow[r, dashed, "\sigma_A"'] \arrow[d, two heads, "\mathsf{disp}_{A\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left] & \Gamma.A \arrow[r] \arrow[d, two heads, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d] \\
\Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

This gives us a functor $\circ\sigma : [\Gamma, \mathbf{grpd}] \to [\Delta, \mathbf{grpd}]$ which is our strict version of pullback.

Corollary 0.2.10 (Fibrations are stable under pullback)**.**

$$\begin{tikzcd}
{[\Gamma, \mathbf{grpd}]} \arrow[d, "\circ\sigma"'] & \mathsf{Fib}_\Gamma \arrow[l, "\mathsf{fiber}"'] \arrow[d, dashed, "\sigma^*"] \\
{[\Delta, \mathbf{grpd}]} \arrow[r, "\mathsf{disp}"'] & \mathsf{Fib}_\Delta
\end{tikzcd}$$

We can deduce a corresponding fact about fibrations: since fibrations are closed under isomorphism, and since any pullback in **grpd** *of a fibration p is isomorphic to the display map $\mathsf{disp}_{\mathsf{fiber}_p \circ \sigma}$, any pullback of a fibration is a fibration.*

A strict interpretation of type theory would require Σ and Π-formers to be stable under pullback (Beck-Chevalley). Thus we again define these as operations on classifiers $[\Gamma, \mathbf{grpd}]$.

Definition 0.2.11 (Σ-former operation)**.** Then given $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$ we define $\Sigma_A B : \Gamma \to \mathbf{grpd}$ such that $\Sigma_A B$ acts on objects by forming fiberwise Grothendieck constructions

$$\Sigma_A B(x) := A(x) \cdot B \circ x_A$$

where $x_A : A(x) \to \Gamma \cdot A$ takes $f : a_0 \to a_1$ to $(\mathsf{id}_x, f) : (x, a_0) \to (x, a_1)$

$$\begin{tikzcd}
A(x) \cdot B \circ x_A \arrow[r, dashed] \arrow[d, "\mathsf{disp}_{B \circ x_A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.B \arrow[r] \arrow[d, "\mathsf{disp}_B"] & \bullet \\
A(x) \arrow[r, "x_A"'] \arrow[d, "!"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] & \mathbf{grpd} \\
\bullet \arrow[r, "x"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

$\Sigma_A B$ acts on morphism $f : x \to y$ in Γ and $(a \in A(x), b \in B(x, a))$ by

$$\Sigma_A B\, f\, (a, b) := (A\, f\, a, B\, (f, \mathsf{id}_{A\, f\, a})\, b)$$

and for morphism $(\alpha : a_0 \to a_1 \in A(x), \beta : B\, (\mathsf{id}_x, \alpha)\, b_0 \to b_1 \in B(x, a_1))$ in $\Sigma_A B\, x$

$$\Sigma_A B\, f\, (\alpha, \beta) := (A\, f\, \alpha, B\, (f, \mathsf{id}_{A\, f\, a_1})\, \beta)$$

Let us also define the natural transformation $\mathsf{fst} : \Sigma_A B \to A$ by

$$\mathsf{fst}_x : (a, b) \mapsto a$$

Proposition 0.2.12 (Fibrations are closed under composition)**.** *The corresponding fact about fibrations is that the composition of two fibrations is a fibration.*

$$\begin{tikzcd}
\Xi \arrow[d, two heads] \arrow[dr, dashed, two heads] & \\
\Delta \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

We can compare the two fibrations

$$\mathsf{disp}_B \circ \mathsf{disp}_A \qquad \text{and} \qquad \mathsf{disp}_{\Sigma_A(B)}$$

An object in the composition would look like $((x, a), b)$ for $x \in \Gamma$, $a \in A(x)$ and $b \in B(x, a)$, whereas an object in $\Gamma \cdot \Sigma_A(B)$ would instead be $(x, (a, b))$.

Proposition 0.2.13 (Strict Beck-Chevalley for Σ)**.** *Let $\sigma : \Delta \to \Gamma$, $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$. Then*

$$(\Sigma_A B) \circ \sigma = \Sigma_{A \circ \sigma}(B \circ \sigma_A)$$

where σ_A is uniquely determined by the pullback in

$$\begin{tikzcd}
 & \Delta \cdot A\sigma \cdot B \circ \sigma_A \arrow[r, "\sigma_{A \cdot B}"'] \arrow[d, "\mathsf{disp}_{B \circ \sigma_A}"'] & \Gamma.A.B \arrow[d, two heads, "\mathsf{disp}_B"] & \\
 & \Delta \cdot A\sigma \arrow[r, "\sigma_A"'] \arrow[d, "\mathsf{disp}_{A\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd} \\
\mathbf{grpd} & \Delta \arrow[l, shift right, "(\Sigma_A B) \circ \sigma"'] \arrow[l, shift left, "\Sigma_{A \circ \sigma}(B \circ \sigma_A)"] \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

Proof. By checking pointwise at $x \in \Delta$, this boils down to showing

$$(\sigma x)_A = \sigma_A \circ x_{A \circ \sigma} : A(\sigma x) \to \Gamma \cdot A$$

$$\begin{tikzcd}
A(\sigma x) \arrow[r, "x_{A\sigma}"'] \arrow[d, dashed, "!"] \arrow[rr, bend left, "(\sigma x)_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \cdot A\sigma \arrow[r, "\sigma_A"'] \arrow[d, "\mathsf{disp}_{A\sigma}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd} \\
\bullet \arrow[r, "x"'] & \Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

which holds because of the universal property of pullback. □

Definition 0.2.14 (Π-former operation)**.** Given $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$ we will define $\Pi_A B : \Gamma \to \mathbf{grpd}$ such that for any $C : \Gamma \to \mathbf{grpd}$ we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathsf{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

natural in both B and C.

Proof. $\Pi_A B$ acts on objects by taking fiberwise sections

$$\Pi_A B(x) := \{s \in [A(x), \Sigma_A B(x)] \,|\, \mathsf{fst}_x \circ s = \mathsf{id}_{A(x)}\}$$

Where we have taken the full subcategory of the functor category $[A(x), \Sigma_A B(x)]$. This is a groupoid since any natural transformation of functors into groupoids are natural isomorphisms.

$\Pi_A B$ acts on morphisms via conjugation

$$\begin{tikzcd}
x \arrow[d, "f"] & & \Pi_A B(x) \arrow[d, "\Sigma_A B(f) \circ - \circ A(f^{-1})"] & A(x) \arrow[r, "s"] & \Sigma_A B(x) \arrow[d, "\Sigma_A B(f)"] \\
y & \arrow[r, mapsto, "\Pi_A B"] & \Pi_A B(y) & A(y) \arrow[u, "A(f^{-1})"] \arrow[r, dashed, "\Pi_A B(f)(s)"'] & \Sigma_A B(y)
\end{tikzcd}$$

Note that conjugation is functorial and invertible. □

Corollary 0.2.15 (Fibrations are closed under pushforward)**.** *Stated in terms of fibrations, we have*

$$\begin{tikzcd}
\Xi \arrow[d, two heads, "\tau"'] & \Gamma_! \sigma_* \tau \arrow[d, two heads, "\sigma_* \tau"] \\
\Delta \arrow[r, two heads, "\sigma"'] & \Gamma
\end{tikzcd}$$

with the universal property of pushforward

$$\mathsf{Fib}_\Delta(\sigma^* \rho, \tau) \cong \mathsf{Fib}_\Gamma(\rho, \sigma_* \tau)$$

natural in both τ and ρ.

Proposition 0.2.16 (Strict Beck-Chevalley for Π)**.** *Let $\sigma : \Delta \to \Gamma$, $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$. Then*

$$(\Pi_A B) \circ \sigma = \Pi_{A \circ \sigma}(B \circ \sigma_A)$$

where σ_A is uniquely determined by the pullback in

$$\begin{tikzcd}
 & \Delta \cdot A\sigma \cdot B \circ \sigma_A \arrow[r, "\sigma_{A\cdot B}"'] \arrow[d, "\mathsf{disp}_{B\circ\sigma_A}"'] & \Gamma.A.B \arrow[d, two heads, "\mathsf{disp}_B"] & \\
 & \Delta \cdot A\sigma \arrow[r, "\sigma_A"'] \arrow[d, "\mathsf{disp}_{A\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd} \\
\mathbf{grpd} & \Delta \arrow[l, shift right, "\Pi_A B\circ\sigma"'] \arrow[l, shift left, "\Pi_{A\circ\sigma}(B\circ\sigma_A)"] \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

Proof. By checking pointwise, this boils down to Beck-Chevalley for Σ. □

Proposition 0.2.17 (All objects are fibrant)**.** *Let $\bullet$ denote the terminal groupoid, namely that with a single object and morphism. Then the unique map $\Gamma \to \bullet$ is a fibration.*

Definition 0.2.18 (Interval)**.** Let the interval groupoid $\mathbb{I}$ be the small groupoid with two objects and a single non-identity isomorphism. There are two distinct morphisms $\delta_0, \delta_1 : \bullet \to \mathbb{I}$ and a natural isomorphism $i : \delta_0 \Rightarrow \delta_1$. Note that δ_0 and δ_1 both form adjoint equivalences with the unique map $! : \mathbb{I} \to \bullet$.

Denote by $\bullet + \bullet$ the small groupoid with two objects and only identity morphisms. Then let $\partial : \bullet + \bullet \to \mathbb{I}$ be the unique map factoring δ_0 and δ_1.

$$\begin{tikzcd}
 & \bullet \arrow[d] \arrow[dr, bend left, "\delta_1"] & \\
\bullet \arrow[r] \arrow[rr, bend right, "\delta_0"'] & \bullet + \bullet \arrow[r, dashed, "\partial"] & \mathbb{I}
\end{tikzcd}$$

Proposition 0.2.19 (Path object fibration)**.** *Let A be a small groupoid. Recall that* **grpd** *is Cartesian closed, so we can take the image of the above diagram under the functor A^-.*

$$\begin{tikzcd}
A^{\mathbb{I}} \arrow[dr, dashed, two heads, "A^{\partial}"] \arrow[drr, bend left, two heads, "A^{\delta_1}"] \arrow[ddr, bend right, two heads, "A^{\delta_0}"'] & & \\
 & A \times A \arrow[r] \arrow[d] & A \\
 & A &
\end{tikzcd}$$

Then the indicated morphisms are fibrations, and $A^{\delta_0}, A^{\delta_1}$ form adjoint equivalences with $A^{!} : A \to A^{\mathbb{I}}$.

We can use this to justify the interpretation of the identity type later, where we will have the strictified versions (as in strictly stable under substitution) of the above

$$\begin{tikzcd}
A \arrow[r, "\cong" description] \arrow[d] & \bullet \cdot A \arrow[r, "\mathsf{var}_A"] \arrow[d, dashed, "A^*\rho'"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{grpd}_\bullet \arrow[d, dashed, "\rho'"] & \mathbf{grpd}_\bullet \arrow[d, "U"] \\
A^{\mathbb{I}} \arrow[r, "\cong" description] \arrow[d, dashed, two heads, "A^\partial"] & \bullet \cdot A \cdot A \cdot \mathsf{Id} \arrow[r] \arrow[d, dashed, "\mathsf{disp}_{\mathsf{Id}' \circ U^*\mathsf{var}_A}"] \arrow[dr, phantom, very near start, "\lrcorner"] & I' \arrow[ur] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{grpd} \\
A \times A \arrow[r, "\cong" description] \arrow[d, "\mathsf{fst}"] & \bullet \cdot A \cdot A \arrow[r] \arrow[d, dashed, "\mathsf{disp}_{U \circ \mathsf{var}_A}"] \arrow[dr, phantom, very near start, "\lrcorner"] & U \times_{\mathbf{grpd}} U \arrow[r, "\mathsf{snd}"'] \arrow[ur, "\mathsf{Id}'"] \arrow[d, dashed, "\mathsf{fst}"] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{grpd}_\bullet \arrow[d, "U"'] \\
A \arrow[r, "\cong" description] & \bullet \cdot A \arrow[r, "\mathsf{var}_A" description] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{grpd}_\bullet \arrow[r, "U"'] \arrow[d, "U"] & \mathbf{grpd} \\
& \bullet \arrow[r, "A"'] & \mathbf{grpd} &
\end{tikzcd}$$

In general, we will want to build a pathspace for a type in any context, which requires us to pull back the interval along the context, and rebuild the required fibration by exponentiation in the slice.

0.2.3 Classifying type dependency

Proposition 0.2.20 (P_{tp} classifies type dependency)**.** *Specialized to* $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ *in* $\mathbf{Psh}(\mathbf{grpd})$*, the characterizing property of polynomial endofunctors proposition 0.3.2 says that a map from a representable* $\Gamma \to P_{\mathsf{tp}}X$ *corresponds to the data of*

$$A : \Gamma \to \mathsf{Ty} \qquad \textit{and} \qquad B : \Gamma \cdot A \to X$$

The special case of when X *is also* Ty *gives us a classifier for dependent types; by Yoneda the above corresponds to the data in* $\mathbf{Cat}$ *of*

$$A : \Gamma \to \mathbf{grpd} \qquad \textit{and} \qquad B : \Gamma \cdot A \to \mathbf{grpd}$$

Furthermore, precomposition by a substitution $\sigma : \Delta \to \Gamma$ *acts on such a pair by*

$$\begin{tikzcd}
\Delta \arrow[d, "\sigma"'] \arrow[dr, "(A \circ \sigma, B \circ \mathsf{tp}^*\sigma)"] & \\
\Gamma \arrow[r, "(A,B)"'] & P_{\mathsf{tp}}X
\end{tikzcd}$$

where $\mathsf{tp}^*\sigma$ *is given by*

$$\begin{tikzcd}
\Delta \cdot A \circ \sigma \arrow[r, dashed, "\mathsf{tp}^*\sigma"] \arrow[d, two heads] & \Gamma \cdot A \arrow[r] \arrow[d, two heads] & \mathbf{grpd}_\bullet \arrow[d] \\
\Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

0.2.4 Pi and Sigma structure

Lemma 0.2.21. $X \in \mathbf{Psh}(\mathbf{grpd})$ *be a presheaf. Let* F *be an operation that takes a groupoid* Γ*, a functor* $A : \Gamma \to \mathbf{grpd}$ *and* $B : \Gamma \cdot A \to X$ *and returns a natural transformation* $F_A B : \Gamma \to X$*.*

Then using Yoneda to define $\tilde{F} : P_{\mathsf{tp}}X \to X$ pointwise as

$$\begin{aligned}\tilde{F}_\Gamma : \mathbf{Psh}(\mathbf{grpd})(\Gamma, P_{\mathsf{tp}}X) &\to \mathbf{Psh}(\mathbf{grpd})(\Gamma, X)\\ (A, B) &\mapsto F_A B\end{aligned}$$

gives us a natural transformation if and only if F satisfies the strict Beck-Chevalley condition

$$(F_A B) \circ \sigma = F_{A\circ\sigma}(B \circ \mathsf{tp}^*\sigma)$$

for every $\sigma : \Delta \to \Gamma$ in $\mathbf{grpd}$.

Proof. Using proposition 0.2.20

$$\begin{tikzcd}
(A, B) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & F_A B \arrow[ddd, mapsto] \\
& \mathbf{Psh}(\mathbf{grpd})(\Gamma, P_{\mathsf{tp}}X) \arrow[r, "\tilde{F}_\Gamma"] \arrow[d, "- \circ \sigma"'] & \mathbf{Psh}(\mathbf{grpd})(\Gamma, X) \arrow[d, "- \circ \sigma"] & \\
& \mathbf{Psh}(\mathbf{grpd})(\Delta, P_{\mathsf{tp}}X) \arrow[r, "\tilde{F}_\Delta"'] & \mathbf{Psh}(\mathbf{grpd})(\Delta, X) & \\
(A \circ \sigma, B \circ \mathsf{tp}^*\sigma) \arrow[rr, mapsto] & & F_{A\circ\sigma} B \circ \mathsf{tp}^*\sigma \arrow[r, equal, dashed] & (F_A B) \circ \sigma
\end{tikzcd}$$

□

Definition 0.2.22 (Interpretation of Π types)**.** We define the natural transformation $\Pi : P_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ as that which is induced (lemma 0.2.21) by the Π-former operation (definition 0.2.14).

Then we define the natural transformation $\lambda : P_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ as the natural transformation induced by the following operation: given $A : \Gamma \to \mathbf{grpd}$ and $\beta : \Gamma \cdot A \to \mathbf{grpd}_\bullet$, $\lambda_A\beta : \Gamma \to \mathbf{grpd}_\bullet$ will be the functor such that on objects $x \in \Gamma$

$$\lambda_A\beta\,(x) := (\Pi_A B\,(x), a \mapsto (a, b(x, a)))$$

where $B := U \circ \beta : \Gamma \cdot A \to \mathbf{grpd}$ and $b(x, a)$ is the point in $\beta(x, a)$. On morphisms $f : x \to y$ in Γ we have

$$\lambda_A\beta\,(f) := (\Pi_A B\,(f), \eta)$$

where $\eta : \Pi_A B\, f\, s_x \to s_y$ is a natural isomorphism between functors $A_y \to \Sigma_A By$ given on objects $a \in A_y$ by

$$\eta_a := (\mathsf{id}_a, \mathsf{id}_{b(y,a)})$$

These combine to give us a pullback square

$$\begin{tikzcd}
P_{\mathsf{tp}}\mathsf{Tm} \arrow[r, "\lambda"] \arrow[d, "P_{\mathsf{tp}}\mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
P_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Pi"'] & \mathsf{Ty}
\end{tikzcd}$$

Proof. We should check that the λ operation satisfied Beck-Chevalley. This follows from the Π satisfying Beck-Chevalley and extensionality results for functors.

The square commutes and is a pullback if and only it pointwise commutes and pointwise gives pullbacks, i.e. for each groupoid Γ

$$
\begin{tikzcd}
(A,\beta) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \lambda_A \beta \arrow[ddd, mapsto] \\
& \mathbf{Psh}(\mathbf{grpd})(\Gamma, P_{\mathsf{tp}}\mathsf{Tm}) \arrow[r, "\lambda_\Gamma"] \arrow[d, "\mathbf{Psh}(\mathbf{grpd})(\Gamma, P_{\mathsf{tp}}\mathsf{tp})"'] \arrow[dr, phantom, "\lrcorner" very near start] & {[\Gamma, \mathbf{grpd}_\bullet]} \arrow[d, "U\circ -"] & \\
& \mathbf{Psh}(\mathbf{grpd})(\Gamma, P_{\mathsf{tp}}\mathsf{Ty}) \arrow[r, "\Pi_\Gamma"'] & {[\Gamma, \mathbf{grpd}]} & \\
(A, U\circ\beta) \arrow[rr, mapsto] & & \Pi_\Gamma U\circ\beta \arrow[r, equal] & U\circ\lambda_A\beta
\end{tikzcd}
$$

where we have used proposition 0.2.20. That this commutes follows from the definitions of Π and λ.

To show it is pullback it suffices to note that for any $f : \Gamma \to \mathbf{grpd}_\bullet$ and $(A,B) : \Gamma \to P_{\mathsf{tp}}\mathsf{Ty}$ such that $U\circ f = \Pi_A B$, there exists a unique $(A,\beta) : \Gamma \to P_{\mathsf{tp}}\mathsf{Tm}$ such that $U\circ\beta = B$ and $\lambda_A\beta = f$. Indeed β is fully determined by the above conditions to be

$$
\begin{aligned}
\beta : \Gamma \cdot A &\to \mathbf{grpd}_\bullet \\
(x,a) &\mapsto (B(x,a), f\,x\,a)
\end{aligned}
$$

$\square$

Lemma 0.2.23. *This is a specialization of lemma 0.3.3. Use R to denote the fiber product*

$$
\begin{tikzcd}
R \arrow[r, "\rho_P"] \arrow[d, "\mathsf{tp}^*\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty} = \rho_{\mathsf{Tm}}"'] \arrow[dr, phantom, "\lrcorner" very near start] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}"] \\
\mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty}
\end{tikzcd}
$$

By the universal property of pullbacks, The data of a map from a representable $\varepsilon : \Gamma \to R$ corresponds to the data of $\alpha : \Gamma \to \mathsf{Tm}$ and $(U\circ\alpha, B) : \Gamma \to P_{\mathsf{tp}}\mathsf{Ty}$. Then by proposition 0.2.20 this corresponds to the data of $\alpha : \Gamma \to \mathsf{Tm}$ and $B : \Gamma \cdot U\circ\alpha \to \mathsf{Ty}$.

$$
\begin{tikzcd}
\Gamma \arrow[drr, bend left, "(U\circ\alpha, B)"] \arrow[dr, "(\alpha,B)"] \arrow[ddr, bend right, "\alpha"'] & & \\
& R \arrow[r, "\rho_P"] \arrow[d, "\rho_{\mathsf{Tm}}"'] \arrow[dr, phantom, "\lrcorner" very near start] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}"] \\
& \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty}
\end{tikzcd}
$$

Precomposition by a substitution $\sigma : \Delta \to \Gamma$ then acts on such a pair by

$$
\begin{tikzcd}
\Delta \arrow[d, "\sigma"'] \arrow[dr, "(\alpha\circ\sigma, B\circ\mathsf{tp}^*\sigma)"] & \\
\Gamma \arrow[r, "(\alpha,B)"'] & R
\end{tikzcd}
$$

Definition 0.2.24 (Evaluation)**.** Define the operation of evaluation $\mathsf{ev}_\alpha\, B$ to take $\alpha : \Gamma \to \mathbf{grpd}_\bullet$ and $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$ and return $\mathsf{ev}_\alpha\, B : \Gamma \to \mathbf{grpd}$, described below.

$$
\begin{tikzcd}
\Gamma \arrow[drr, bend left, "{(A,B)}"] \arrow[dr, "{(\alpha,B)}"] \arrow[ddr, "\alpha"'] \arrow[dd, dashed, "{(\mathsf{ev}_\alpha\, B,\alpha)}"'] \arrow[ddd, dashed, bend right=60, "\mathsf{ev}_\alpha\, B"'] & & \\
& R \arrow[r] \arrow[d] \arrow[dl, "\mathsf{counit}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}"] \\
\mathsf{Ty} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] \arrow[d] & \mathsf{Ty} \\
\mathsf{Ty} \arrow[r] & \bullet &
\end{tikzcd}
$$

where we write $A := U \circ \alpha$ and treat a map $\Gamma \to \mathbf{grpd}$ as the same as a map $\Gamma \to \mathsf{Ty}$.

More concisely, evaluation is a natural transformation $\mathsf{ev}\ : R \to \mathsf{Ty}$, given by

$$\mathsf{ev}\ = \pi_{\mathsf{Ty}} \circ \mathsf{counit}$$

Lemma 0.2.25. *The functor* $\mathsf{ev}_\alpha\, B : \Gamma \to \mathbf{grpd}$ *can be computed as*

$$\mathsf{ev}_\alpha\, B = B \circ a$$

where

$$
\begin{tikzcd}
\Gamma \arrow[drr, bend left, "\alpha"] \arrow[dr, dashed, "a"] \arrow[ddr, equals, bend right] & & \\
& \Gamma \cdot A \arrow[r] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d] \\
& \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}
$$

Proof. This is a specialization of lemma 0.3.4 with liberal applications of Yoneda. ☐

Definition 0.2.26 (Classifier for dependent pairs)**.** Recall the following definition of composition of polynomial endofunctors, specialized to our situation

$$
\begin{tikzcd}
Q \arrow[rrr, bend left, "\mathsf{tp} \triangleleft \mathsf{tp}"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[r, equals] \arrow[d, "\mathsf{counit}"] & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}"] \\
\mathsf{Tm} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Ty} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] \arrow[d] & \mathsf{Ty} \\
\mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty} \arrow[r] & \bullet &
\end{tikzcd}
$$

By the universal property of pullbacks, the data of a map with representable domain $\varepsilon : \Gamma \to Q$ corresponds to the data of a triple of maps $\alpha, \beta : \Gamma \to \mathsf{Tm}$ and (A, B) :

$\Gamma \to P_{\mathsf{tp}}\mathsf{Ty}$ such that $\mathsf{tp} \circ \beta = \pi_{\mathsf{Ty}} \circ \mathsf{counit} \circ (\alpha, B)$ and $A = \mathsf{tp} \circ \alpha$.

$$\begin{tikzcd}
\Gamma \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "(\alpha,B)"] \arrow[ddrrr, "\alpha"] \arrow[drrrr, bend left, "(A,B)"] \arrow[dddr, bend right, "\beta"'] & & & & \\
& Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "\mathsf{counit}"] \arrow[r, equal] & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty}"] \\
& \mathsf{Tm} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Ty} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] \arrow[d] & \mathsf{Ty} \\
& \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty} \arrow[r] & \bullet &
\end{tikzcd}$$

This in turn corresponds to three functors $\alpha, \beta : \Gamma \to \mathbf{grpd}_\bullet$ and $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$, such that $U \circ \beta = \mathsf{ev}_\alpha B$. So we will write

$$\varepsilon = (\beta, \alpha, B)$$

Type theoretically $\alpha = (A, a : A)$ and $\mathsf{ev}_\alpha B = Ba$ and $\beta = (Ba, b : Ba)$. Then composing ε with $\mathsf{tp} \lhd \mathsf{tp}$ returns γ, which consists of (A, B). It is in this sense that Q classifies pairs of dependent terms, and $\mathsf{tp} \lhd \mathsf{tp}$ extracts the underlying types.

Precomposition with a substitution $\sigma : \Delta \to \Gamma$ acts on this triple by

$$\begin{tikzcd}
\Delta \arrow[d, "\sigma"'] \arrow[dr, "(\beta \circ \sigma, \alpha \circ \sigma, B \circ \mathsf{tp}^* \sigma)"] & \\
\Gamma \arrow[r, "(\beta, \alpha, B)"'] & Q
\end{tikzcd}$$

Definition 0.2.27 (Interpretation of Σ)**.** We define the natural transformation

$$\Sigma : P_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$$

as that which is induced (lemma 0.2.21) by the Σ-former operation (definition 0.2.14).

To define $\mathsf{pair} : Q \to \mathsf{Tm}$, let Γ be a groupoid and $(\beta, \alpha, B) : \Gamma \to Q$ (such that $U \circ \beta = \mathsf{ev}_\alpha \beta$). We define a functor $\mathsf{pair}_\Gamma(\beta, \alpha, B) : \Gamma \to \mathbf{grpd}_\bullet$ such that on objects $x \in \Gamma$, the functor returns $(\Sigma_A B\, x, (a_x, b_{a_x}))$, where (using lemma 0.2.25 $U \circ \beta x = \mathsf{ev}_\alpha B x = B(x, a_x)$)

$$\alpha\, x = (A\, x, a_x) \quad \text{and} \quad \beta\, x = (B(x, a_x), b_{a_x})$$

and on morphisms $f : x \to y$, the functor returns $(\Sigma_A B\, f, (\phi_f, \psi_f))$, where (using lemma 0.2.25 $U \circ \beta f = \mathsf{ev}_\alpha B f = B(f, \phi_f)$)

$$\alpha\, f = (A\, f, \phi_f \colon A\, f\, a_x \to a_y) \quad \text{and} \quad \beta\, f = (B(f, \phi_f), \psi_f \colon B(f, \phi_f)\, b_{a_x} \to b_{a_y})$$

Σ and pair combine to give us a pullback square

$$\begin{tikzcd}
Q \arrow[r, "\mathsf{pair}"] \arrow[d, "\mathsf{tp} \lhd \mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
P_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Sigma"'] & \mathsf{Ty}
\end{tikzcd}$$

Proof. To show naturality of pair, suppose $\sigma : \Delta \to \Gamma$ is a functor between groupoids.

$$\begin{tikzcd}
\mathbf{Psh}(\mathbf{grpd})(\Delta, Q) \arrow[rrr, "\mathsf{pair}_\Delta"] & & & {[\Delta, \mathbf{grpd}_\bullet]} \\
& (\beta \circ \sigma, \alpha \circ \sigma, B \circ \mathsf{tp}^* \sigma) \arrow[r, mapsto] & ? & \\
& (\beta, \alpha, B) \arrow[u, mapsto] \arrow[r, mapsto] & \mathsf{pair}_\Gamma(\beta, \alpha, B) \arrow[u, mapsto] & \\
\mathbf{Psh}(\mathbf{grpd})(\Gamma, Q) \arrow[uuu, "\circ\sigma"] \arrow[rrr, "\mathsf{pair}_\Gamma"'] & & & {[\Gamma, \mathbf{grpd}_\bullet]} \arrow[uuu, "\circ\sigma"']
\end{tikzcd}$$

So we check that for any $x \in \Gamma$,

$$\begin{aligned}
& \mathsf{pair}_\Delta(\beta \circ \sigma, \alpha \circ \sigma, B \circ \sigma_A)\, x \\
&= (\Sigma_{A \circ \sigma} B \circ \sigma_A\, x, (a_x, b_{a_x})) \\
&= ((\Sigma_A B) \circ \sigma\, x, (a_x, b_{a_x})) \\
&= \mathsf{pair}_\Gamma(\beta, \alpha, B) \circ \sigma\, x
\end{aligned}$$

where

$$\alpha \circ \sigma\, x = (A \circ \sigma x, a_x) \quad \text{and} \quad \beta \circ \sigma\, x = (\mathsf{ev}_\alpha\, B \circ \sigma\, x, b_{a_x})$$

and so on.

It follows from the definition of pair that the square commutes. To show that it is pullback, it suffices to show that for each Γ,

$$\begin{tikzcd}
\mathbf{Psh}(\mathbf{grpd})(\Gamma, Q) \arrow[r, "\mathsf{pair}_\Gamma"] \arrow[d, "\mathsf{tp} \lhd \mathsf{tp} \circ -"'] & {[\Gamma, \mathbf{grpd}_\bullet]} \arrow[d, "U \circ -"] \\
\mathbf{Psh}(\mathbf{grpd})(\Gamma, P_{\mathsf{tp}} \mathsf{Ty}) \arrow[r, "\Sigma_\Gamma"'] & {[\Gamma, \mathbf{grpd}]}
\end{tikzcd}$$

is a pullback. Since we are in **Set**, it suffices to just show the universal property applied to a point: so for any $A : \Gamma \to \mathbf{grpd}$, any $B : \Gamma \cdot A \to \mathbf{grpd}$, and any $p : \Gamma \to \mathbf{grpd}_\bullet$, such that

$$U \circ p = \Sigma_\Gamma(A, B)$$

there exists a unique $(\beta, \alpha, B) : \Gamma \to Q$ such that

$$\mathsf{pair}_\Gamma(\beta, \alpha, B) = p \quad \text{and} \quad \mathsf{tp} \lhd \mathsf{tp} \circ (B, \alpha, B) = (A, B)$$

Indeed if we write

$$p\, x = (\Sigma_A B\, x, (a_x \in Ax, b_x \in B(x, a_x)))$$

this uniquely determines α and β as

$$\alpha\, x = (Ax, a_x) \quad \text{and} \quad \beta\, x = (\mathsf{ev}_\alpha\, B\, x, b_x)$$

$\square$

0.2.5 Identity types

Definition 0.2.28 (Identity formation and introduction)**.** To define the commutative square in $\mathbf{Psh}(\mathbf{grpd})$

$$\begin{tikzcd} \mathsf{Tm} \arrow[r, "\mathsf{refl}"] \arrow[d, "\delta"'] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} \arrow[r, "\mathsf{Id}"'] & \mathsf{Ty} \end{tikzcd}$$

We first note that both δ and tp in the are in the essential image of the composition from definition 0.2.2

$$\mathbf{Cat} \xrightarrow{\mathsf{y}} \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\mathsf{res}} \mathbf{Psh}(\mathbf{grpd})$$

since the composition preserves pullbacks. So we first define in $\mathbf{Cat}$

$$\begin{tikzcd} \mathbf{grpd}_\bullet \arrow[r, "\mathsf{refl}'"] \arrow[d, "\delta"'] & \mathbf{grpd}_\bullet \arrow[d, "U"] \\ U \times_{\mathbf{grpd}} U \arrow[r, "\mathsf{Id}'"'] & \mathbf{grpd} \end{tikzcd} \tag{0.2.1}$$

Then obtain Id and refl in $\mathbf{Psh}(\mathbf{grpd})$ by applying $\mathsf{res} \circ \mathsf{y}$ to this diagram.

To this end, let $\mathsf{Id}' : U \times_{\mathbf{grpd}} U \to \mathbf{grpd}$ act on objects by taking the *set* - the discrete groupoid - of isomorphisms

$$(A, a_0, a_1) \mapsto A(a_0, a_1)$$

and on morphisms $(f, \phi_0, \phi_1) : (A, a_0, a_1) \to (B, b_0, b_1)$ by

$$(f : A \to B, \phi_0 : fa_0 \to b_0, \phi_1 : fa_1 \to b_1) \mapsto \phi_1 \circ f(-) \circ \phi_0^{-1}$$

Let $\mathsf{refl}' : \mathbf{grpd}_\bullet \to \mathbf{grpd}_\bullet$ act on objects by

$$(A, a) \mapsto (A(a, a), \mathsf{id}_a)$$

and on morphisms $(f, \phi) : (A, a) \to (B, b)$ by

$$(f : A \to B, \phi : (A, a) \to (B, b)) \mapsto (\phi \circ f(-) \circ \phi^{-1}, \phi \circ f(\mathsf{id}_a) \circ \phi^{-1} = \mathsf{id}_b)$$

where the second component has to be the identity on the object id_d, since $B(b, b)$ is a discrete groupoid. So we need a merely propositional proof that the two maps are equal, which in this case is clear.

Proof. Since $\delta(A, a) = (A, a, a)$, it follows that the square in eq. (0.2.1) commutes. ☐

Lemma 0.2.29. *We can then construct the pullback I'*

$$\begin{tikzcd} \mathbf{grpd}_\bullet \arrow[dr, dashed, "\rho'"] \arrow[drr, bend left, "\mathsf{refl}'"] \arrow[ddr, bend right, "\delta"'] & & \\ & I' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d, "U"] \\ & U \times_{\mathbf{grpd}} U \arrow[r, "\mathsf{Id}'"'] & \mathbf{grpd} \end{tikzcd}$$

as the groupoid with objects (A, a_0, a_1, h) where A is a groupoid with $a_0, a_1 \in A$ and $h : a_0 \to a_1$, and morphisms

$$(f, \phi_0, \phi_1, Ah = k) : (A, a_0, a_1, h : a_0 \to a_1) \to (B, b_0, b_1, k : b_0 \to b_1)$$

where $f : A \to B$, $\phi_i : fa_i \to b_i$ and $Ah = k$ represents a merely propositional proof of equality. Then we can also compute

$$\rho'(A, a) = (A, a, a, \mathsf{id}_a)$$

Lemma 0.2.30. *Specialized to $q : I \to \mathsf{Ty}$ in $\mathbf{Psh}(\mathbf{grpd})$, the characterizing property of polynomial endofunctors proposition 0.3.2 says that a map from a representable $\varepsilon : \Gamma \to P_q X$ corresponds to the data of*

$$A : \Gamma \to \mathsf{Ty} \qquad \textit{and} \qquad C : \Gamma \cdot A \cdot A \cdot \mathsf{Id} \to X$$

where $A = q \circ \varepsilon$ and

$$\begin{tikzcd}
X & \Gamma \cdot A \cdot A \cdot \mathsf{Id} \arrow[l, "C"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & I' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d, "U"] \\
& \Gamma \cdot A \cdot A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \times_{\mathbf{grpd}} U \arrow[r, "\mathsf{Id}'"'] \arrow[d, "\mathsf{fst}"'] & \mathbf{grpd} \\
& \Gamma \cdot A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d, "U"] & \\
& \Gamma \arrow[r, "A"'] & \mathbf{grpd} &
\end{tikzcd}$$

Lemma 0.2.31. *The data of a map $(A, C, \gamma_{\mathsf{refl}}) : \Gamma \to T$ corresponds to the data of*

$$\begin{aligned}
A &: \Gamma \to \mathbf{grpd} \\
C &: \Gamma \cdot A \cdot A \cdot \mathsf{Id} \to \mathbf{grpd} \\
\gamma_{\mathsf{refl}} &: \Gamma \cdot A \to \mathbf{grpd}_\bullet
\end{aligned}$$

such that $C \circ A^*\rho' = U \circ \gamma_{\mathsf{refl}}$

$$\begin{tikzcd}
\mathbf{grpd}_\bullet \arrow[d, "U"'] & \Gamma \cdot A \arrow[l, "\gamma_{\mathsf{refl}}"'] \arrow[r, "\mathsf{var}_A"] \arrow[d, dashed, "A^*\rho'"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d, dashed, "\rho'"] & \mathbf{grpd}_\bullet \arrow[d, "U"] \\
\mathbf{grpd} & \Gamma \cdot A \cdot A \cdot \mathsf{Id} \arrow[l, "C"'] \arrow[r] \arrow[d, "\mathsf{disp}_{\mathsf{Id}' \circ U^* \mathsf{var}_A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & I' \arrow[ur] \arrow[d] & \mathbf{grpd} \\
& \Gamma \cdot A \cdot A \arrow[r] \arrow[d, "\mathsf{disp}_{U \circ \mathsf{var}_A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times_{\mathbf{grpd}} U \arrow[ur, "\mathsf{Id}'"] \arrow[r, "\mathsf{snd}"'] \arrow[d, dashed, "\mathsf{fst}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d, "U"'] \\
& \Gamma \cdot A \arrow[r, "\mathsf{var}_A"] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[r, "U"'] \arrow[d, "U"] & \mathbf{grpd} \\
& \Gamma \arrow[r, "A"'] & \mathbf{grpd} &
\end{tikzcd}$$

Then precomposition with $\sigma : \Delta \to \Gamma$ acts on such a triple via

$$\begin{tikzcd}
\Delta \arrow[d, "\sigma"'] \arrow[dr, "{(A \circ \sigma, C \circ q^* \sigma, \gamma_{\mathsf{refl}} \circ \mathsf{tp}^* \sigma)}"] & \\
\Gamma \arrow[r, "{(A, C, \gamma_{\mathsf{refl}})}"'] & T
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
\Gamma \arrow[dr, dashed] \arrow[drr, dashed, bend left, "{(A,\gamma_{\mathsf{refl}})}"] \arrow[ddr, dashed, bend right, "{(A,C)}"'] & & \\
& T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P_{\mathsf{tp}}\mathsf{Tm} \arrow[d, "P_{\mathsf{tp}}\mathsf{tp}"] \\
& P_q\mathsf{Ty} \arrow[r, "\rho^\star_{\mathsf{Ty}}"'] & P_{\mathsf{tp}}\mathsf{Tm}
\end{tikzcd}$$

By the universal property of pullbacks, The data of a map from a respresentable $\Gamma \to T$ corresponds to the data of $(A, C) : \Gamma \to P_q\mathsf{Ty}$ and $(A', \gamma_{\mathsf{refl}}) : \Gamma \to P_{\mathsf{tp}}\mathsf{Tm}$ such that

$$\rho^\star_{\mathsf{Ty}} \circ (A, C) = P_{\mathsf{tp}}\mathsf{tp} \circ (A', \gamma_{\mathsf{refl}})$$

By definition 0.3.6 and proposition 0.3.2 this says

$$(A, C \circ A^*\rho) = (A', \mathsf{tp} \circ \gamma_{\mathsf{refl}})$$

so the above is equivalent to having $A = A', C, \gamma_{\mathsf{refl}}$ such that

$$C \circ A^*\rho = \mathsf{tp} \circ \gamma_{\mathsf{refl}} \text{ in } \mathbf{Psh}(\mathbf{grpd})$$

By Yoneda this is equivalent to requiring

$$C \circ A^*\rho' = U \circ \gamma_{\mathsf{refl}} \text{ in } \mathbf{Cat}$$

$\square$

Proposition 0.2.32. *We can compute $\varepsilon : P_q\mathsf{Tm} \to T$ via*

$$\begin{aligned}
\varepsilon_\Gamma : \mathbf{Psh}(\mathbf{grpd})(\Gamma, P_q\mathsf{Tm}) &\to \mathbf{Psh}(\mathbf{grpd})(\Gamma, T) \\
(A, \gamma) &\mapsto (A, U \circ \gamma, \gamma \circ A^*\rho')
\end{aligned}$$

Proof. This follows from the computation for T lemma 0.2.31, the polynomial action on slice morphisms definition 0.3.6, and proposition 0.3.2. $\square$

Definition 0.2.33 (Identity elimination)**.** We want to define $J : T \to P_q\mathsf{Tm}$

$$\begin{aligned}
J_\Gamma : \mathbf{Psh}(\mathbf{grpd})(\Gamma, T) &\to \mathbf{Psh}(\mathbf{grpd})(\Gamma, P_q\mathsf{Tm}) \\
(A, C, \gamma_{\mathsf{refl}}) &\mapsto (A, \gamma)
\end{aligned}$$

for some $\gamma : \Gamma \cdot A \cdot A \cdot \mathsf{Id} \to \mathbf{grpd}_\bullet$ which we will define below. We first use T lemma 0.2.31 to describe the given data:

$$\begin{tikzcd}
\mathbf{grpd}_\bullet \arrow[d, "U"'] & \Gamma \cdot A \arrow[l, "\gamma_{\mathsf{refl}}"'] \arrow[r, "\mathsf{var}_A"] \arrow[d, dashed, "A^*\rho'"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d, dashed, "\rho'"] & \mathbf{grpd}_\bullet \arrow[d, "U"] \\
\mathbf{grpd} & \Gamma \cdot A \cdot A \cdot \mathsf{Id} \arrow[l, "C"] \arrow[ul, dashed, "\gamma"'] \arrow[r] \arrow[d, "\mathsf{disp}_{\mathsf{Id}' \circ U^*\mathsf{var}_A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & I' \arrow[ur] \arrow[d] & \mathbf{grpd} \\
& \Gamma \cdot A \cdot A \arrow[r] \arrow[d, "\mathsf{disp}_{U \circ \mathsf{var}_A}"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \times_{\mathbf{grpd}} U \arrow[ur, "\mathsf{Id}'"] \arrow[r, "\mathsf{snd}"'] \arrow[d, "\mathsf{fst}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d, "U"'] \\
& \Gamma \cdot A \arrow[r, "\mathsf{var}_A"] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[r, "U"'] \arrow[d, "U"] & \mathbf{grpd} \\
& \Gamma \arrow[r, "A"'] & \mathbf{grpd} &
\end{tikzcd}$$

Let us name the fibers over the diagonal

$$C_{\mathsf{refl}} := U \circ \gamma_{\mathsf{refl}} = C \circ A^*\rho' : \Gamma \cdot A \to \mathbf{grpd}$$

and its given points

$$\gamma_{\mathsf{refl}} = (C_{\mathsf{refl}}, c_{\mathsf{refl}})$$

(Note that c_{refl} is not a functor, but will give us an object per object (x, a), and morphism $c_{\mathsf{refl}}(f, \phi) : C_{\mathsf{refl}}(f, \phi)c_{\mathsf{refl}}(x, a) \to c_{\mathsf{refl}}(y, b)$ per morphism (f, ϕ).) Then γ will be defined by using C to lift the path

$$(\mathsf{id}_x, \mathsf{id}_{a_0}, h, \_) : (x, a_0, a_0, \mathsf{id}_a) \to (x, a_0, a_1, h) \in \Gamma \cdot A \cdot A \cdot \mathsf{Id}$$

that starts on the diagonal, to give us a point in any fiber, using c_{refl}. Note that we unfolded $\Gamma \cdot A \cdot A \cdot \mathsf{Id}$ as the domain of the nested display maps so that $x \in \Gamma$, $a_0 \in Ax$,

$$a_1 \in U \circ \mathsf{var}_A(x, a_0) = U(Ax, a_0) = Ax$$

and

$$h \in Id' \circ U^*\mathsf{var}_A(x, a_0, a_1) = \mathsf{Id}'(Ax, a_0, a_1) = Ax(a_0, a_1)$$

We also check $(\mathsf{id}_x, \mathsf{id}_{a_0}, h, \_)$ is a path in $\Gamma \cdot A \cdot A \cdot \mathsf{Id}$ by proving "$\_$", the ommited equality

$$(\mathsf{Id}' \circ U^*\mathsf{var}_A(\mathsf{id}_x, \mathsf{id}_{a_0}, h))\mathsf{id}_{a_0} = (\mathsf{Id}'(A\mathsf{id}_x, \mathsf{id}_{a_0}, h))\mathsf{id}_{a_0} = h \circ A\mathsf{id}_x\mathsf{id}_{a_0} \circ \mathsf{id}_{a_0}^{-1} = h$$

So we define $\gamma : \Gamma \cdot A \cdot A \cdot \mathsf{Id} \to \mathbf{grpd}_\bullet$ on objects by

$$(x, a_0, a_1, h) \mapsto (C(x, a_0, a_1, h), C(\mathsf{id}_x, \mathsf{id}_{a_0}, h, \_)\, c_{\mathsf{refl}}(x, a_0))$$

noting that from the computation of ρ' given in lemma 0.2.29 it follows that

$$c_{\mathsf{refl}}(x, a_0) \in C \circ A^*\rho'(x, a_0) = C(x, a_0, a_1, h)$$

Define γ on morphism $(f, \phi_0, \phi_1, \phi_1 \circ Afh \circ \phi_0^{-1} = k) : (x, a_0, a_1, h) \to (y, b_0, b_1, k)$ by

$$(f, \phi_0, \phi_1, \_) \mapsto (C(f, \phi_0, \phi_1, \_), C(\mathsf{id}_y, \mathsf{id}_{b_0}, k, \_)\, c_{\mathsf{refl}}(f, \phi_0))$$

We type check $C(\mathsf{id}_y, \mathsf{id}_{b_0}, k, \_)\, c_{\mathsf{refl}}(f, \phi_0)$

$$\begin{aligned}
C(\mathsf{id}_y, \mathsf{id}_{b_0}, k, _)c_{\mathsf{refl}}(f, \phi_0) \quad : \quad & C(f, \phi_0, \phi_1, _) \circ C(\mathsf{id}_x, \mathsf{id}_{a_0}, h, _)\, c_{\mathsf{refl}}(x, a_0) \\
&= C(f, \phi_0, \phi_1 \circ Afh, _)\, c_{\mathsf{refl}}(x, a_0) \\
&= C(f, \phi_0, k \circ \phi_0, _)\, c_{\mathsf{refl}}(x, a_0) \\
&= C(\mathsf{id}_y, \mathsf{id}_{b_0}, k, _) \circ C(f, \phi_0, \phi_0, _)\, c_{\mathsf{refl}}(x, a_0) \\
&= C(\mathsf{id}_y, \mathsf{id}_{b_0}, k, _) \circ C_{\mathsf{refl}}(f, \phi_0)\, c_{\mathsf{refl}}(x, a_0) \\
&\to C(\mathsf{id}_y, \mathsf{id}_{b_0}, k, _)\, c_{\mathsf{refl}}(y, b_0)
\end{aligned}$$

Proof. Functoriality of γ is routine. We show naturality of J. Suppose $\sigma : \Delta \to \Gamma$

is representable

$$\begin{tikzcd}
(A \circ \sigma, C \circ q^*\sigma, \gamma_{\mathsf{refl}} \circ \mathsf{tp}^*\sigma) \arrow[r, mapsto] & (A \circ \sigma, \gamma_\Delta) \arrow[d, dashed, equal] \\
& (A \circ \sigma, \gamma_\Gamma \circ q^*\sigma)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Psh}(\mathbf{grpd})(\Delta, T) \arrow[r, "J_\Delta"] & \mathbf{Psh}(\mathbf{grpd})(\Delta, P_q\mathsf{Tm}) \\
\mathbf{Psh}(\mathbf{grpd})(\Gamma, T) \arrow[r, "J_\Gamma"'] \arrow[u, "-\circ\sigma"] & \mathbf{Psh}(\mathbf{grpd})(\Gamma, P_q\mathsf{Tm}) \arrow[u, "-\circ\sigma"']
\end{tikzcd}$$

$$(A, C, \gamma_{\mathsf{refl}}) \longmapsto (A, \gamma_\Gamma)$$

So we want to show that on objects $(x, a_0, a_1, h) \in \Delta \cdot A \circ \sigma \cdot A \circ \sigma \cdot \mathsf{Id}$

$$\gamma_\Delta\,(x, a_0, a_1, h) = \gamma_\Gamma \circ q^*\sigma\,(x, a_0, a_1, h)$$

Let us denote $q^*\sigma(x, a_0, a_1, h) = (\sigma x, a_0', a_1', h')$. Then

$$\begin{aligned}
&\gamma_\Delta\,(x, a_0, a_1, h) \\
&= (C \circ q^*\sigma\,(x, a_0, a_1, h), (C \circ q^*\sigma\,(\mathsf{id}_x, \mathsf{id}_{a_0}, h, _))(c_{\mathsf{refl}}(\mathsf{tp}^*\sigma(x, a_0)))) \\
&= (C\,(\sigma x, a_0', a_1', h'), (C\,(\mathsf{id}_{\sigma x}, \mathsf{id}_{a_0'}, h', _))(c_{\mathsf{refl}}\,(\sigma x, a_0'))) \\
&= \gamma_\Gamma(\sigma x, a_0', a_1', h') \\
&= \gamma_\Gamma \circ q^*\sigma\,(x, a_0, a_1, h)
\end{aligned}$$

and similarly for morphisms. □

Proposition 0.2.34. *$J : T \to P_q\mathsf{Tm}$, as defined above is a section of ε.*

Proof. Let $(A, C, \gamma_{\mathsf{refl}}) : \Gamma \to T$ be a map from a representable. Then using the definition of J and the computation of ε proposition 0.2.32

$$\varepsilon_\Gamma \circ J_\Gamma\,(A, C, \gamma_{\mathsf{refl}}) = \varepsilon_\Gamma(A, \gamma) = (A, U \circ \gamma, \gamma \circ A^*\rho')$$

By definition of γ from J we can see that $U \circ \gamma = C$, so it suffices to show that $\gamma \circ A^*\rho' = \gamma_{\mathsf{refl}}$. On an object (x, a_0)

$$\begin{aligned}
&\gamma \circ A^*\rho'(x, a_0) \\
&\qquad = \gamma(x, a_0, a_0, \mathsf{id}_{a_0}) \\
&\qquad = (C(x, a_0, a_0, \mathsf{id}_{a_0}), C(\mathsf{id}_x, \mathsf{id}_{a_0}, \mathsf{id}_{a_0})\, c_{\mathsf{refl}}) \\
&\qquad = (C_{\mathsf{refl}}(x, a_0), c_{\mathsf{refl}}(x, a_0))
\end{aligned}$$

□

0.2.6 Universe of Discrete Groupoids

In this section we assume *three* different universe sizes, which we will distinguish by all lowercase (small), capitalized first letter (large), and all-caps (extra large), respectively. For example, the three categories of sets will be nested as follows

$$\mathbf{set} \hookrightarrow \mathbf{Set} \hookrightarrow \mathbf{SET}$$

We shift all of our previous work up by one universe level, so that we are working in the category $\mathbf{PSH}(\mathbf{Grpd})$ of extra large presheaves, indexed by the (extra large, locally large) category of large groupoids. We would then have $\mathsf{Ty} = [-, \mathbf{Grpd}]$ and $\mathsf{Tm} = [-, \mathbf{Grpd}_{\bullet}]$.

Definition 0.2.35 (Universe of discrete groupoids)**.** Let U be the (large) groupoid of small sets, i.e. let U have **set** as its objects and morphisms between two small sets as all the bijections between them. This gives us $\ulcorner \mathsf{U} \urcorner : \bullet \to \mathsf{Ty}$.

Then we define $\mathsf{El} : \mathsf{yU} \to \mathsf{Ty}$ by defining $\mathsf{El} : \mathsf{U} \to \mathbf{Grpd}$ as the inclusion - any small set can be regarded as a large discrete groupoid.

$$\begin{tikzcd} \mathsf{U} \arrow[r, hook] \arrow[dr, hook, "\mathsf{El}"'] & \mathbf{grpd} \arrow[d, hook] \\ & \mathbf{Grpd} \end{tikzcd}$$

Then we take $\pi := \mathsf{disp}_{\mathsf{El}}$, giving us

$$\begin{tikzcd} \mathsf{E} \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty} \end{tikzcd}$$

We can compute the groupoid E as that with objects that are pairs (X, x) where $x \in X \in \mathbf{set}$, and morphisms

$$E((X, x), (Y, y)) = \{f : X \to Y \mid f\, x = y\}$$

Then $\pi : E \to U$ is the forgetful functor $(X, x) \mapsto X$.

Showing that this universe is closed under Π, Σ, Id formation depends on how we formalize $\mathbf{set} \hookrightarrow \mathbf{Set}$. In both cases we need to check that discreteness is preserved by the type formers, which is straightforward. If we are working with sets and cardinality, i.e. taking $\mathbf{set} = \mathbf{Set}_{<\lambda} \subset \mathbf{Set}_{<\kappa} = \mathbf{Set}$ for some inaccessible cardinals $\lambda < \kappa$, then it is straightforward to check that the type formers do not make "larger" types. If we are working with type theoretic universes with a lift operation $\mathsf{ULift} : \mathbf{set} \to \mathbf{Set}$ then it may *not* be true that ULift commutes with our type formers.

0.3 Polynomial Endofunctors

In this section we develop some of the definitions and lemmas related to polynomial endofunctors that we will use in the rest of the notes.

Definition 0.3.1 (Polynomial endofunctor)**.** Let $\mathbb{C}$ be a locally Cartesian closed category (in our case, presheaves on the category of contexts). This means for each morphism $t : B \to A$ we have an adjoint triple

$$\begin{tikzcd} \mathbb{C}/B \arrow[d, bend right=60, "t_!"'] \arrow[d, bend left=60, "t_*"] \arrow[d, phantom, "\dashv\ t^*\ \dashv"] \\ \mathbb{C}/A \arrow[u] \end{tikzcd}$$

where t^* is pullback, and $t_!$ is composition with t.

Let $t : B \to A$ be a morphism in $\mathbb{C}$. Then define $P_t : \mathbb{C} \to \mathbb{C}$ be the composition

$$P_t := A_! \circ t_* \circ B^* \qquad \mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

Proposition 0.3.2 (Characterising property of Polynomial Endofunctors)**.** *The data of a map into the polynomial applied to an object in* $\mathbb{C}$

$$\Gamma \longrightarrow P_t Y$$

corresponds to

$$\begin{tikzcd} \Gamma \arrow[rr, "\phi"] \arrow[dr, dashed, "\alpha"'] & & P_t Y \arrow[dl, "t_* B^* Y"] \\ & A & \end{tikzcd}$$

Applying the adjunction $A_! \dashv A^*$*, this corresponds to*

$$\alpha : \Gamma \to A \qquad \text{and} \qquad \begin{tikzcd} B_! t^* \alpha \arrow[rr, dashed, "\tilde{\phi}"] \arrow[dr, "t^* \alpha"'] & & B \times Y \arrow[dl, "B^* Y"] \\ & B & \end{tikzcd}$$

Applying the adjunction $t^* \dashv t_*$*, this corresponds to*

$$\alpha : \Gamma \to A \qquad \text{and} \qquad \Gamma \cdot \alpha := B_! t^* \alpha \overset{\beta}{\dashrightarrow} Y$$

Henceforth we will write

$$(\alpha, \beta) : \Gamma \to P_t Y$$

for this map, since it is uniquely determined by this data. Furthermore, precomposition by $\sigma : \Delta \to \Gamma$*, acts on such a pair by*

$$\begin{tikzcd} \Delta \arrow[d, "\sigma"'] \arrow[dr, "(\alpha \circ \sigma, \beta \circ t^* \sigma)"] & \\ \Gamma \arrow[r, "(\alpha,\beta)"'] & P_t Y \end{tikzcd}$$

and given a morphism $f : X \to Y$*, the morphism* $P_t f$ *acts on such a pair by*

$$\begin{tikzcd} \Gamma \arrow[r, "(\alpha,\beta)"] \arrow[dr, "(\alpha, f \circ \beta)"'] & P_t X \arrow[d, "P_t f"] \\ & P_t Y \end{tikzcd}$$

Lemma 0.3.3. *Use* R *to denote the fiber product*

$$\begin{tikzcd} R \arrow[r, "\rho_P"] \arrow[d, "t^* t_* B^* Y = \rho_{\mathsf{Tm}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_t Y \arrow[d, "t_* B^* Y"] \\ B \arrow[r, "t"'] & A \end{tikzcd}$$

By the universal property of pullbacks and proposition 0.3.2, The data of a map $\Gamma \to R$ *corresponds to the data of* $\beta : \Gamma \to B$ *and* $(t \circ \beta, y) : \Gamma \to P_t Y$*, or just* $\beta : \Gamma \to B$ *and* $y : \Gamma \cdot t \circ \beta \to Y$

$$
\begin{tikzcd}
\Gamma \arrow[dr, dashed, "(\beta,y)"] \arrow[drr, bend left, "(t\circ\beta,y)"] \arrow[ddr, bend right, "\beta"'] & & \\
& R \arrow[r, "\rho_P"] \arrow[d, "\rho_{\mathsf{Tm}}"] \arrow[dr, phantom, "\lrcorner", very near start] & P_t Y \arrow[d, "t_* B^* Y"] \\
& B \arrow[r, "t"'] & A
\end{tikzcd}
$$

By uniqueness in the universal property of pullbacks and proposition 0.3.2, Precomposition by a map $\sigma : \Delta \to \Gamma$ *acts on such a pair by*

$$
\begin{tikzcd}
\Delta \arrow[d, "\sigma"'] \arrow[dr, "(\beta\circ\sigma, y\circ t^*\sigma)"] & \\
\Gamma \arrow[r, "(\beta,y)"'] & R
\end{tikzcd}
$$

Lemma 0.3.4 (Evaluation)**.** *Suppose* $(\beta, y) : \Gamma \to R$*, as in lemma 0.3.3*

$$\beta : \Gamma \to B \quad \text{and} \quad y : \Gamma \cdot t \circ \beta \to Y$$

Then the evaluation of y *at* β *can be described in the following two ways*

$$y \circ b = \pi_Y \circ \mathsf{counit} \circ (\beta, y)$$

where

$$
\begin{tikzcd}
\Gamma \arrow[dr, dashed, "b"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, equal] & & \\
& \Gamma \cdot t \circ \beta \arrow[r, "v"'] \arrow[d, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
& \Gamma \arrow[r, "t\circ\beta"'] & A
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& \Gamma \arrow[ddl, dashed, bend right, "y\circ b"'] \arrow[dd, dashed, "(y\circ b,\beta)"] \arrow[dr, "(\beta,y)"] \arrow[drr, bend left, "(t\circ\beta,y)"] & & \\
& & R \arrow[r] \arrow[d] \arrow[dl, "\mathsf{counit}"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_t Y \arrow[d, "t_* B^* Y"] \\
Y & Y \times B \arrow[l, "\pi_Y"] \arrow[r, "\pi_B"'] & B \arrow[r, "t"'] & A
\end{tikzcd}
$$

Proof. It suffices to show $(\mathsf{counit} \circ (\beta, y)) = (y \circ b, \beta)$ instead.

$$
\begin{aligned}
&\mathsf{counit} \circ (\beta, y) && \\
&= \mathsf{counit} \circ (v \circ b, y \circ t^*d \circ t^*b) && \textit{fig. } 1 \\
&= \mathsf{counit} \circ (v, y \circ t^*d) \circ b && \textit{lemma } 0.3.3\textit{and fig. } 2 \\
&= \mathsf{counit} \circ t^*(t \circ \beta, y) \circ b && \textit{fig. } 3 \\
&= \overline{(t \circ \beta, y)} \circ b && \textit{fig. } 4 \\
&= (y, v) \circ b && \textit{fig. } 5 \\
&= (y \circ b, v \circ b) && \\
&= (y \circ b, \beta) &&
\end{aligned}
$$

$$
\begin{tikzcd}
\Gamma \cdot t \circ \beta \arrow[r, dashed, "t^*b"] \arrow[rr, bend left, equal] \arrow[d, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \cdot t \circ \beta \cdot t \circ \beta \arrow[r, dashed, "t^*d"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \cdot t \circ \beta \arrow[r, "v"] \arrow[d, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
\Gamma \arrow[r, "b"'] \arrow[rr, bend right, equal] & \Gamma \cdot t \circ \beta \arrow[r, "d"'] & \Gamma \arrow[r, "t \circ \beta"'] & A
\end{tikzcd}
$$

Figure 1: $t^*d \circ t^*b = \mathsf{id}_{\Gamma \cdot t \circ \beta}$

$$
\begin{tikzcd}
\Gamma \arrow[d, "b"'] \arrow[dr, "{(v \circ b, y \circ t^*d \circ t^*b)}"] & \\
\Gamma \cdot t \circ \beta \arrow[r, "{(v, y \circ t^*d)}"'] & R
\end{tikzcd}
$$

Figure 2: $(v, y \circ t^*d) \circ b = (v \circ b, y \circ t^*d \circ t^*b)$

$$
\begin{tikzcd}
\Gamma \cdot t \circ \beta \arrow[r, dashed, "{(v, y \circ t^*d)}"] \arrow[rr, bend left, "v"] \arrow[d, dashed, "d"] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
\Gamma \arrow[r, "{(t \circ \beta, y)}"'] \arrow[rr, bend right, "t \circ \beta"'] & P_tY \arrow[r, "t_*B^*Y"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \cdot t \circ \beta \arrow[d, "d"'] \arrow[dr, "{(t \circ \beta \circ d, y \circ t^*d)}"] & \\
\Gamma \arrow[r, "{(t \circ \beta, y)}"'] & P_tY
\end{tikzcd}
$$

using proposition 0.3.2 and lemma 0.3.3

Figure 3: $t^*(t \circ \beta, y) = (v, y \circ t^*d)$

$\square$

Definition 0.3.5. Suppose

$$
\begin{tikzcd}
C \arrow[rr, "\rho"] \arrow[dr, "s"'] & & B \arrow[dl, "t"] \\
& A &
\end{tikzcd}
$$

$$
\begin{tikzcd}
t^*(t\circ\beta) \arrow[d, "t^*(t\circ\beta,y)"'] \arrow[dr, dashed, "\overline{(t\circ\beta,y)}"] & \\
t^*t_*B^*Y \arrow[r, "\mathsf{counit}"'] & B^*Y
\end{tikzcd}
\qquad
\begin{array}{c} \Big\| \\ t^* \dashv t_* \end{array}
\qquad
\begin{tikzcd}
t\circ\beta \arrow[d, "(t\circ\beta,y)"'] \arrow[dr, "(t\circ\beta,y)"] & \\
t_*B^*Y \arrow[r, equal, "\widetilde{\mathsf{counit}}"'] & t_*B^*Y
\end{tikzcd}
$$

Figure 4: $\mathsf{counit} \circ t^*(t\circ\beta, y) = \overline{(t\circ\beta, y)}$

$$
\begin{tikzcd}
\Gamma\cdot t\circ\beta \arrow[r, dashed, "(y,v)"] \arrow[d, "v=t^*(t\circ\beta)"'] & Y\times B \arrow[dl, "B^*Y"] \\
B &
\end{tikzcd}
\qquad
\begin{array}{c} \Big\| \\ t^* \dashv t_* \end{array}
\qquad
\begin{tikzcd}
\Gamma \arrow[r, "(t\circ\beta,y)"] \arrow[d, "t\circ\beta"'] & P_tY \arrow[dl, "t_*B^*Y"] \\
A &
\end{tikzcd}
$$

Figure 5: $\overline{(t\circ\beta, y)} = (y, v)$

Then we have a mate $\mu_! : \rho_! \circ s^* \Rightarrow t^*$. This is given by the universal property of pullbacks: given $f : x \to y$ in the slice $\mathbb{C}/A$ we have

$$
\begin{tikzcd}
\bullet \arrow[r, "\mu_{!x}"] \arrow[d, "s^*f"'] \arrow[dr, phantom, "= \mu_! \Rightarrow"] & \bullet \arrow[r] \arrow[d, "t^*f"] & X \arrow[d, "f"] \arrow[dd, bend left=60, "x"] \\
\bullet \arrow[r, "\mu_{!y}"'] \arrow[d, "s^*y"'] & \bullet \arrow[r] \arrow[d, "t^*y"'] & Y \arrow[d, "y"] \\
C \arrow[r, "\rho"'] & B \arrow[r, "t"'] & A
\end{tikzcd}
$$

By the calculus of mates we also have a reversed mate between the right adjoints $\mu^* : t_* \to s_* \circ \rho^*$. Explicitly μ^* is the composition

$$
t_* \xrightarrow{\mathsf{unit}\ t_*} s_*\rho^*\rho_!s^*t_* \xrightarrow{s_*\rho^*\mu_!t_*} s_*\rho^*t^*t_* \xrightarrow{s_*\rho^*\ \mathsf{counit}} s_*\rho^*
$$

Definition 0.3.6 (Contravariant of $P\_$ on a slice)**.** Let $P\_ : (\mathbb{C}/A)^{\mathrm{op}} \to [\mathbb{C}, \mathbb{C}]$ be defined by taking $s \mapsto P_s$ on objects and act on a morphism by

$$
\begin{tikzcd}
& B \arrow[dl, "t"'] \\
A & \\
& C \arrow[ul, "s"] \arrow[uu, "\rho"']
\end{tikzcd}
\qquad \longmapsto \qquad
\begin{tikzcd}
P_t \arrow[dd, "\rho^\star"] \\
\\
P_s
\end{tikzcd}
$$

where

$$
\rho^\star := A_!(s_*\eta \circ \mu B^*) : P_t \to P_s
$$

$$
\begin{tikzcd}
\mathbb{C} \arrow[d, "C^*"'] \arrow[dr, "B^*"] \arrow[dd, phantom, ""] \arrow[ddd, bend right=60, "P_s"'] \arrow[ddd, bend left=90, "P_t"] & \\
\mathbb{C}/C \arrow[d, "s_*"'] \arrow[u, phantom, shift left=6ex, "\Leftarrow\eta\cong"] & \mathbb{C}/B \arrow[l, "\rho^*"] \arrow[dl, "t_*"] \arrow[dl, phantom, shift right=3ex, "\Leftarrow\mu\cong"] \\
\mathbb{C}/A \arrow[d, "A_!"] & \\
\mathbb{C} &
\end{tikzcd}
$$

where $\mu = \mu^*$ is the mate from definition 0.3.5, and η is the natural isomorphism given by pullback pasting.

Pointwise, this natural transformation acts on a pair $(\alpha, \beta) : \Gamma \to P_t X$ by

$$
\begin{tikzcd}
\Gamma \arrow[r, "{(\alpha,\beta)}"] \arrow[dr, "{(\alpha,\beta\circ\alpha^*\rho)}"'] & P_t X \arrow[d, "\rho^\star_X"] \\
& P_s X
\end{tikzcd}
$$

where $\alpha^*\rho$ is defined as

$$
\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[r, "s^*\alpha"] \arrow[d, dashed, "\alpha^*\rho"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\rho"] \\
\Gamma \cdot_t \alpha \arrow[r, "t^*\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
\Gamma \arrow[r, "\alpha"'] & A
\end{tikzcd}
$$

We prove this now.

Proof. Firstly $\rho^\star_X = A_!(s_*\eta_X \circ \mu_{B^*X})$, so the first component $\alpha : \Gamma \to A$ is preserved by $\rho^\star_X$ and it suffices to show, in $\mathbb{C}/A$

$$
\begin{tikzcd}
\alpha \arrow[r, "{(\alpha,\beta)}"] \arrow[dr, "{(\alpha,\beta\circ\alpha^*\rho)}"'] & t_* B^* X \arrow[d, "s_*\eta_X\circ\mu_{B^*X}"] \\
& s_* C^* X
\end{tikzcd}
$$

By the adjunction $s^* \dashv s_*$, it suffices to show, in $\mathbb{C}/C$

$$
\begin{tikzcd}
s^*\alpha \arrow[r, "{s^*(\alpha,\beta)}"] \arrow[dr, "\overline{(\alpha,\beta\circ\alpha^*\rho)}"'] & s^* t_* B^* X \arrow[d, "\overline{s_*\eta_X\circ\mu_{B^*X}}"] \\
& C^* X
\end{tikzcd}
$$

Now we calculate $\overline{s_*\eta_X \circ \mu_{B^*X}} = \eta_X \circ \overline{\mu_{B^*X}}$. So that our goal is to show

$$
\begin{tikzcd}
s^*\alpha \arrow[r, "{s^*(\alpha,\beta)}"] \arrow[dr, "\overline{(\alpha,\beta\circ\alpha^*\rho)}"'] & s^* t_* B^* X \arrow[r, "\overline{\mu_{B^*X}}"] & \rho^* B^* X \arrow[dl, "\sim"', "\eta_X"] \\
& C^* X &
\end{tikzcd}
$$

Since η_X is an isomorphism between two limits of the same diagram, namely $X \times C \cong C_! C^* X \cong C_! \rho^* B^* X$, it suffices to show that both $\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)$ and $\overline{(\alpha, \beta \circ \alpha^* \rho)}$ are uniquely determined by the same two maps into X and C.

By the characterising property of polynomial endofunctors (proposition 0.3.2) we calculate

$$\overline{(\alpha, \beta \circ \alpha^* \rho)} = (\beta \circ \alpha^* \rho, s^* \alpha)$$

$$\alpha \xrightarrow{(\alpha,\beta\circ\alpha^*\rho)} s_* C^* X \quad \Bigg\| \quad s^*\alpha \xrightarrow[(\beta\circ\alpha^*\rho, s^*\alpha)]{\overline{(\alpha,\beta\circ\alpha^*\rho)}} C^* X \quad \Bigg\| \quad C_! s^* \alpha \xrightarrow[\beta\circ\alpha^*\rho]{} X$$

More formally, this means $\beta \circ \alpha^* \rho : C_! s^* \alpha \to X$ and $s^* \alpha : C_! s^* \alpha \to C$ are the two maps that uniquely determine the map $C_! \overline{\alpha, \beta \circ \alpha^* \rho} : C_! s^* \alpha \to X \times C$.

On the other hand,

$$\begin{tikzcd}
\alpha \arrow[r, "(\alpha,\beta)"] & t_* B^* X \arrow[r, "\mathsf{unit}_{t_* B^* X}"] \arrow[rrr, bend right, "\mu_{B^*X}"'] & s_* \rho^* \rho_! s^* t_* B^* X \arrow[r, "s_*\rho^*\mu_! t_* B^* X"] & s_* \rho^* t^* t_* B^* X \arrow[r, "s_*\rho^* \mathsf{counit}_{B^*X}"] & s_* \rho * B^* X
\end{tikzcd}$$

$$\overline{} \; s^* \dashv s_*$$

$$\begin{tikzcd}
s^*\alpha \arrow[r, "s^*(\alpha,\beta)"] & s^* t_* B^* X \arrow[r, "\overline{\mathsf{unit}_{t_* B^* X}}"] \arrow[rrr, bend right, "\overline{\mu_{B^*X}}"'] & \rho^* \rho_! s^* t_* B^* X \arrow[r, "\rho^*\mu_! t_* B^* X"] & \rho^* t^* t_* B^* X \arrow[r, "\rho^* \mathsf{counit}_{B^*X}"] & \rho^* B^* X
\end{tikzcd}$$

$$\overline{} \; \rho_! \dashv \rho^*$$

$$\begin{tikzcd}
\rho_! s^*\alpha \arrow[r, "\rho_! s^*(\alpha,\beta)"] \arrow[d, equal] & \rho_! s^* t_* B^* X \arrow[r, equal, "\overline{\overline{\mathsf{unit}_{t_* B^* X}}}"] \arrow[d, equal] & \rho_! s^* t_* B^* X \arrow[r, "\mu_! t_* B^* X"] & t^* t_* B^* X \arrow[r, "\mathsf{counit}_{B^*X}"] \arrow[d, equal] & B^* X \arrow[d, equal] \\
\Gamma \cdot_s \alpha & S & & R & X \times B
\end{tikzcd}$$

The mate $\mu_!$ is calculated via the universal map into the pullback R (dotted below).

$$\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[r] \arrow[d, "s^*(\alpha,\beta)"'] \arrow[dd, bend right=60, "s^*\alpha"'] & \Gamma \cdot_t \alpha \arrow[r] \arrow[d] & \Gamma \arrow[d, "(\alpha,\beta)"] \arrow[dd, bend left=60, "\alpha"] \\
S \arrow[r, dashed, "\mu_! t_* B^* X"'] \arrow[d, "s^* t_* B^* X"'] & R \arrow[r] \arrow[d, "t^* t_* B^* X"] \arrow[dr, phantom, "\lrcorner", very near start] & P_t X \arrow[d, "t_* B^* X"] \\
C \arrow[r, "\rho"'] & B \arrow[r, "t"'] & A
\end{tikzcd}$$

Using the characterization of maps into R from lemma 0.3.3 we can calculate

$$\mu_! t_* B^* X \circ s^*(\alpha, \beta) = (\rho \circ s^* \alpha, \beta \circ t^* \alpha^* s)$$

since the first component is simply the map $\Gamma \cdot_s \alpha \to B$ and the second component is the second component of the map

$$(\alpha \circ \alpha^* s, \beta \circ t^* \alpha^* s) = (\alpha, \beta) \circ \alpha^* s : \Gamma \cdot_s \alpha \to P_t X$$

Then using lemma 0.3.4

$$\begin{aligned}
&\overline{\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)} && (0.3.1)\\
&= \mathsf{counit}_{B^*X} \circ \mu_! t_* B^* X \circ s^*(\alpha, \beta) && (0.3.2)\\
&= \mathsf{counit}_{B^*X} \circ (\rho \circ s^* \alpha, \beta \circ t^* \alpha^* s) && (0.3.3)\\
&= (\beta \circ t^* \alpha^* s \circ r, \rho \circ s^* \alpha) && (0.3.4)\\
&= (\beta \circ \alpha^* \rho, \rho \circ s^* \alpha) && (0.3.5)\\
&: \Gamma \cdot_s \alpha \to X \times B && (0.3.6)
\end{aligned}$$

where

$$\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[dr, dashed, "r"'] \arrow[drr, bend left, "\rho\circ s^*\alpha"] \arrow[ddr, equal, bend right] & & \\
& \Gamma \cdot_s \alpha \cdot_t \alpha \circ \alpha^* s \arrow[r] \arrow[d] & B \arrow[d, "t"] \\
& \Gamma \cdot_s \alpha \arrow[r, "\alpha\circ\alpha^* s"'] & A
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[r, equal] \arrow[d, dashed, "r"'] \arrow[dd, equal, bend right=60] & \Gamma \cdot_s \alpha \arrow[r, "s^*\alpha"] \arrow[d, "\alpha^*\rho"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\rho"] \arrow[dd, bend left=60, "s"] \\
\Gamma \cdot_s \alpha \cdot_t \alpha \circ \alpha^* s \arrow[r, dashed, "t^*\alpha^* s"'] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \cdot_t \alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "t"] \\
\Gamma \cdot_s \alpha \arrow[r, "\alpha * s"'] & \Gamma \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

Moving back along the adjunction $\rho_! \dashv \rho^*$ eq. (0.3.1) tells us that

$$\begin{tikzcd}
\Gamma \cdot_s \alpha \arrow[dr, "\overline{\mu_{B^*X}}\circ s^*(\alpha,\beta)"] \arrow[drr, bend left, "s^*\alpha"] \arrow[ddr, bend right, "\overline{\overline{\mu_{B^*X}}\circ s^*(\alpha,\beta)}"] \arrow[dddr, bend right=60, "\beta\circ\alpha^*\rho"'] & & \\
& X \times C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\rho"] \\
& X \times B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& X \arrow[r] & 1
\end{tikzcd}$$

So that, as required, $\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)$ and $\overline{(\alpha, \beta \circ \alpha^* \rho)}$ are uniquely determined by the same two maps into X and C. $\square$

Definition 0.3.7 (Covariant of $P\_$ on a cartesian square)**.** We can also view taking polynomial endofunctors as a covariant functor on the category of arrows with cartesian squares as morphisms

$$P\_ : \mathsf{CartArr}(\mathbb{C}) \to [\mathbb{C}, \mathbb{C}]$$

where the action on a cartesian square is

$$
\begin{tikzcd}
C \arrow[d, "\theta"'] \arrow[dr, phantom, very near start, "\lrcorner"] & D \arrow[l, "s"'] \arrow[d, "\rho"] & & P_s \arrow[d, "P_\kappa"] \\
A & B \arrow[l, "t"] \arrow[ul, phantom, "\overset{\kappa}{\Downarrow}"] & \longmapsto & P_t
\end{tikzcd}
$$

given by the whiskered natural transformations

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, equal] \arrow[d, "C^*"'] \arrow[dr, phantom, "\eta^{-1}"] \arrow[ddd, bend right=60, "P_s"'] & \mathbb{C} \arrow[d, "B^*"] \arrow[ddd, bend left=60, "P_t"] \\
\mathbb{C}/D \arrow[d, "s_*"'] \arrow[dr, phantom, "\mu^{*-1}"] & \mathbb{C}/B \arrow[l, dashed, "\rho^*"'] \arrow[d, "t_*"] \\
\mathbb{C}/C \arrow[d, "C_!"'] \arrow[dr, Rightarrow, "\mu_!"'] & \mathbb{C}/A \arrow[l, "\theta^*"] \arrow[d, "A_!"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}
$$

Furthermore, the natural transformation P_κ is cartesian. meaning each naturality square is a pullback square.

$$
\begin{tikzcd}
P_sX \arrow[r, "P_{\kappa_Y}"] \arrow[d, "P_sf"'] \arrow[dr, phantom, very near start, "\lrcorner"] & P_tX \arrow[d, "P_sf"] \\
P_sY \arrow[r, "P_{\kappa_Y}"'] & P_tY
\end{tikzcd}
$$

The natural tranformation P_κ computes in the following way

$$
\begin{tikzcd}
\Gamma \cdot_t \theta \circ \alpha \arrow[dr, dashed, "i"] \arrow[drr, bend left=10] \arrow[ddr] & & & & & \\
& \Gamma \cdot_s \alpha \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & D \arrow[r, dashed, "\rho"] \arrow[d, "s"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "t"] & \Gamma \arrow[d, "(\alpha,\beta)"'] \arrow[dr, "(\theta\circ\alpha,\beta\circ i)"] & \\
& \Gamma \arrow[r, "\alpha"'] & C \arrow[r, "\theta"'] & A & P_sX \arrow[r, "P_{\kappa_X}"'] & P_tX
\end{tikzcd}
$$

using the fact that $\Gamma \cdot_s \alpha$ and $\Gamma \cdot_t \theta \circ \alpha$ are limits of the same diagram.

Proof. We can use the computation of P_{κ_X} and P_sf to show that the natural transformation P_κ is cartesian. Essentially, the first component of a map $\Gamma \to P_sX$ is determined by its composition with P_sf and its second component is determined by its composition with P_{κ_X}. $\square$

Corollary 0.3.8. *If we have*

$$
\begin{tikzcd}
D' \arrow[r] \arrow[d, "\rho_1"] \arrow[dd, bend right=40, "q_1'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B' \arrow[d, "\rho_2"'] \arrow[dd, bend left=40, "q_2'"] \\
D \arrow[r] \arrow[d, "q_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "q_2"'] \\
C \arrow[r, "\theta"'] & A
\end{tikzcd}
$$

then the two possible ways of obtaining composing the covariant and contravariant actions of $P\_$ form a (strictly commuting) pullback square in $[\mathbb{C}, \mathbb{C}]$.

$$\begin{tikzcd}
P_{q_1} \arrow[r, "P_\kappa"] \arrow[d, "{\rho_1}^\star"'] \arrow[dr, phantom, "\lrcorner", very near start] & P_{q_2} \arrow[d, "{\rho_2}^\star"] \\
P_{q_1'} \arrow[r, "P_{\kappa'}"'] & P_{q_2'}
\end{tikzcd}$$

Proof. To check that it commutes and is a pullback, it suffices to do this pointwise, for some $X \in \mathbb{C}$. Then we simply unfold the computation for each of P_κ and $\rho^\star$. $\square$

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