

# Universe in the Natural Model of Type Theory

Sina Hazratpour

July 25, 2024

## 1 Types

Assume an inaccessible cardinal  $\lambda$ . Write **Set** for the category of all sets. Say that a set  $A$  is  $\lambda$ -small if  $|A| < \lambda$ . Write **Set** $_{\lambda}$  for the full subcategory of **Set** spanned by  $\lambda$ -small sets.

Let  $\mathbb{C}$  be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write **Psh**( $\mathbb{C}$ ) for the category of presheaves over  $\mathbb{C}$ ,

$$\mathbf{Psh}(\mathbb{C}) =_{\text{def}} [\mathbb{C}^{\text{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map  $\text{tp}: \mathsf{Tm} \rightarrow \mathsf{Ty}$  consists of

- contexts as objects  $\Gamma, \Delta, \dots \in \mathbb{C}$ ,
- a type in context  $y(\Gamma)$  as a map  $A: y(\Gamma) \rightarrow \mathsf{Ty}$ ,
- a term of type  $A$  in context  $\Gamma$  as a map  $a: y(\Gamma) \rightarrow \mathsf{Tm}$  such that

$$\begin{array}{ccc} & \mathsf{Tm} & \\ & \uparrow a & \downarrow \text{tp} \\ \Gamma & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

commutes,

- an operation called “context extension” which given a context  $\Gamma$  and a type  $A: y(\Gamma) \rightarrow \mathsf{Ty}$  produces a context  $\Gamma \cdot A$  which fits into a pullback diagram below.

$$\begin{array}{ccc} y(\Gamma \cdot A) & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

**Remark.** Sometimes, we first construct a presheaf  $X$  over  $\Gamma$  and observe that it can be classified by a map into  $\mathsf{Ty}$ . We write

$$\begin{array}{ccc} X & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{\ulcorner X \urcorner} & \mathsf{Ty} \end{array}$$

to express this situation, i.e.  $X \cong y(\Gamma \cdot \ulcorner X \urcorner)$ .

## 2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written  $\mathsf{U}$ , not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} : \mathsf{Ty} \quad \frac{a : \mathsf{U}}{\mathsf{El}(a) : \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal  $\kappa$ , with  $\kappa < \lambda$ .)

In the Natural Model, a universe  $\mathsf{U}$  is postulated by a map

$$\pi : \mathsf{E} \rightarrow \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\ulcorner \mathsf{U} \urcorner} & \mathsf{Ty} \end{array}$$

- There is an inclusion of  $\mathsf{U}$  into  $\mathsf{Ty}$

$$\mathsf{El} : \mathsf{U} \rightarrow \mathsf{Ty}$$

- $\pi : \mathsf{E} \rightarrow \mathsf{U}$  is obtained as pullback of  $\mathsf{tp}$ ; There is a pullback diagram

$$\begin{array}{ccc} \mathsf{E} & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{U} & \xrightarrow{\mathsf{El}} & \mathsf{Ty} \end{array}$$

With the notation above, we get

$$\begin{array}{ccccc}
 y(\Gamma, \text{El}(a)) & \longrightarrow & E & \longrightarrow & Tm \\
 \downarrow & & \downarrow & & \downarrow \\
 y(\Gamma) & \xrightarrow{a} & U & \xrightarrow{\text{El}} & Ty \\
 & \searrow & & \nearrow & \\
 & A & & & 
 \end{array}$$

Both squares above are pullback squares.

### 3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

### 4 Groupoid Model of HoTT

In this section we construct a natural model in  $\mathbf{Psh}(\mathbf{grpd})$  the presheaf category indexed by the category  $\mathbf{grpd}$  of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on  $\mathbf{grpd}$  - which comes from restricting the “classical Quillen model structure” on  $\mathbf{cat}$  [Joy] to  $\mathbf{grpd}$ .

## 4.1 Classifying display maps

*Notation.* We will have two universe sizes - one small and one large. We denote the category of small sets as  $\mathbf{set}$  and the large sets as  $\mathbf{Set}$  (in the previous sections this would have been  $\mathbf{Set}_\lambda$  and  $\mathbf{Set}$  respectively). We denote the category of small categories as  $\mathbf{cat}$  and the large categories as  $\mathbf{Cat}$ . We denote the category of small groupoids as  $\mathbf{grpd}$ .

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section,  $\mathbf{Tm}$  and  $\mathbf{T}_y$  and so on will refer to the natural model semantics in this specific model.

**Definition 4.1** (Pointed). We will take the category of pointed small categories  $\mathbf{cat}_\bullet$  to have objects as pairs  $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$  and morphisms as pairs

$$(F : \mathbb{C}_1 \rightarrow \mathbb{C}_0, \phi : Fc_1 \rightarrow c_0) : (\mathbb{C}_1, c_1) \rightarrow (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids  $\mathbf{grpd}_\bullet$  will be the full subcategory of objects  $(\Gamma, c)$  with  $\Gamma$  a groupoid.

**Definition 4.2** (The display map classifier). We would like to define a natural transformation in  $\mathbf{Psh}(\mathbf{grpd})$

$$\mathbf{tp} : \mathbf{Tm} \rightarrow \mathbf{T}_y$$

with representable fibers.

Consider the functor that forgets the point

$$U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}.$$

If we apply the Yoneda embedding  $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$  to  $U$  we obtain

$$U \circ [-, \mathbf{grpd}_\bullet] \rightarrow [-, \mathbf{grpd}] \quad \text{in} \quad \mathbf{Psh}(\mathbf{Cat}).$$

Since any small groupoid is also a large category  $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$ , we can restrict  $\mathbf{Cat}$  indexed presheaves to be  $\mathbf{grpd}$  indexed presheaves. We define  $\mathbf{tp} : \mathbf{Tm} \rightarrow \mathbf{T}_y$  as the image of  $U \circ$  under this restriction.

$$\begin{aligned} \mathbf{Cat} &\xrightarrow{y} \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\text{res}} \mathbf{Psh}(\mathbf{grpd}) \\ \mathbf{grpd} &\longmapsto [-, \mathbf{grpd}] \longmapsto \mathbf{T}_y \end{aligned}$$

Note that  $\mathbf{Tm}$  and  $\mathbf{T}\mathbf{y}$  are not representable in  $\mathbf{Psh}(\mathbf{grpd})$ .

*Remark 4.3.* By Yoneda we can identify maps with representable domain into the type classifier

$$A : \mathbf{y}\Gamma \rightarrow \mathbf{T}\mathbf{y} \quad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A : \Gamma \rightarrow \mathbf{grpd} \quad \text{in} \quad \mathbf{Cat}$$

**Definition 4.4** (Grothendieck construction). From  $\mathbb{C}$  a small category and  $F : \mathbb{C} \rightarrow \mathbf{cat}$  a functor, we construct a small category  $\int F$ . For any  $c$  in  $\mathbb{C}$  we refer to  $Fc$  as the fiber over  $c$ . The objects of  $\int F$  consist of pairs  $(c \in \mathbb{C}, x \in Fc)$ , and morphisms between  $(c, x)$  and  $(d, y)$  are pairs  $(f : c \rightarrow d, \phi : Ff x \rightarrow y)$ . This makes the following pullback in  $\mathbf{Cat}$

$$\begin{array}{ccccc} (c, x) & & \int F & \longrightarrow & \mathbf{cat}_\bullet & & (C, c) \\ \downarrow & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ c & & \mathbb{C} & \xrightarrow{F} & \mathbf{cat} & & C \end{array}$$

**Definition 4.5** (Grothendieck construction for groupoids). Let  $\Gamma$  be a groupoid and  $A : \Gamma \rightarrow \mathbf{grpd}$  a functor, we can compose  $F$  with the inclusion  $i : \mathbf{grpd} \hookrightarrow \mathbf{Cat}$  and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \quad \text{disp}_A : \Gamma \cdot A \rightarrow \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids  $\Gamma$  and  $Ax$  for  $x \in \Gamma$ . Furthermore the pullback factors through (pointed) groupoids.

$$\begin{array}{ccccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet & \longrightarrow & \mathbf{cat}_\bullet \\ \text{disp}_A \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} & \longrightarrow & \mathbf{cat} \end{array}$$

**Corollary 4.6** (The display map classifier is presentable). *For any small groupoid  $\Gamma$  and  $A : y\Gamma \rightarrow \mathbf{Ty}$ , the pullback of  $\mathbf{tp}$  along  $A$  can be given by the representable map  $y\mathbf{disp}_A$ .*

$$\begin{array}{ccc} y\Gamma \cdot A & \longrightarrow & \mathbf{Tm} \\ y\mathbf{disp}_A \downarrow & \lrcorner & \downarrow \mathbf{tp} \\ y\Gamma & \xrightarrow{A} & \mathbf{Ty} \end{array}$$

*Proof.* Consider the pullback in **Cat**

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow & \mathbf{grpd}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \Gamma & \xrightarrow{A} & \mathbf{grpd} \end{array}$$

We send this square along  $\mathbf{res} \circ y$  in the following

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) \\ \uparrow & \searrow & \downarrow \mathbf{res} \\ \mathbf{grpd} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{grpd}) \end{array}$$

The Yoneda embedding  $y : \mathbf{Cat} \rightarrow \mathbf{Psh}(\mathbf{Cat})$  preserves pullbacks, as does  $\mathbf{res}$  since it is a right adjoint (with left Kan extension  $u_! \dashv \mathbf{res}_*$ ).  $\square$

## 4.2 Groupoid fibrations

**Definition 4.7** (Fibration). Let  $p : \mathbb{C}_1 \rightarrow \mathbb{C}_0$  be a functor. We say  $p$  is a *split Grothendieck fibration* if we have a dependent function  $\mathbf{lift} \, a \, f$  satisfying the following: for any object  $a$  in  $\mathbb{C}_1$  and morphism  $f : pa \rightarrow y$  in the base  $\mathbb{C}_0$  we have  $\mathbf{lift} \, a \, f : a \rightarrow b$  in  $\mathbb{C}_1$  such that  $p(\mathbf{lift} \, a \, f) = f$  and moreover  $\mathbf{lift} \, a \, g \circ f = \mathbf{lift} \, b \, g \circ \mathbf{lift} \, a \, f$

$$\begin{array}{ccc} a & \xrightarrow{\mathbf{lift} \, a \, f} & b \\ \downarrow & \Downarrow & \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

In particular, we are interested in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid  $\Gamma$  as  $\mathbf{Fib}_\Gamma$ , which is a full subcategory of the slice  $\mathbf{grpd}/\Gamma$ . We will decorate an arrow with  $\rightarrow$  to indicate it is a fibration.

Note that  $\mathrm{disp}_A : \Gamma \cdot A \rightarrow \Gamma$  is a fibration, since for any  $(x \in \Gamma, a \in Ax)$  and  $f : x \rightarrow y$  in  $\Gamma$  we have a morphism  $(f, \mathrm{id}_{Afa}) : (x, a) \rightarrow (y, Afa)$  lifting  $f$ . Furthermore

**Proposition 4.8.** *There is an adjoint equivalence*

$$[\Gamma, \mathbf{grpd}] \begin{array}{c} \xrightarrow{\mathrm{disp}} \\ \xleftarrow[\mathrm{fiber}]{\simeq} \end{array} \mathbf{Fib}_\Gamma$$

where for each fibration  $\delta : \Delta \rightarrow \Gamma$  and each object  $x \in \Gamma$

$$\mathrm{fiber}_\delta x = \text{full subcategory } \{a \in \Delta \mid \delta a = x\}$$

It follows that all fibrations are pullbacks of the classifier  $U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd}$ , when viewed as morphisms in  $\mathbf{Cat}$ . From now on, we will use  $\mathrm{disp}_A$  to represent any groupoid fibration, which we can adjust up to isomorphism using this equivalence.

**Proposition 4.9** (Pullback of fibrations). *Let  $\sigma : \Delta \rightarrow \Gamma$  be a functor between groupoids. Since display maps are pullbacks of the classifier  $U : \mathbf{grpd}_\bullet \rightarrow \mathbf{grpd}$  we have the pasting diagram*

$$\begin{array}{ccccc} & & \Delta & \xrightarrow{\quad} & \Gamma & \xrightarrow{\quad} & \mathbf{grpd}_\bullet \\ & \searrow & \downarrow \mathrm{disp}_{A\sigma} & \lrcorner & \downarrow \mathrm{disp}_A & \lrcorner & \downarrow \\ \Delta & \xrightarrow{\quad \sigma \quad} & \Gamma & \xrightarrow{\quad A \quad} & \mathbf{grpd} \end{array}$$

Note that this avoids coherence issues, since we take the pullback to be the map  $\mathrm{disp}_{A\sigma}$  specifically. It follows that fibrations are stable under pullback along all groupoid functors.

$$\begin{array}{ccc} [\Gamma, \mathbf{grpd}] & \xleftarrow{\mathrm{fiber}} & \mathbf{Fib}_\Gamma \\ \circ \sigma \downarrow & & \downarrow \sigma^* \\ [\Delta, \mathbf{grpd}] & \xrightarrow{\mathrm{disp}} & \mathbf{Fib}_\Delta \end{array}$$

**Definition 4.10** (Composition of fibrations). The composition of two fibrations is a fibration.

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & & \\ \downarrow & \searrow & \\ \Gamma \cdot A & \xrightarrow{\quad} & \Gamma \end{array}$$

Then given  $A : \Gamma \rightarrow \mathbf{grpd}$  and  $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$  we define

$$\Sigma_A B := \text{fiber}(\text{disp}_B \circ \text{disp}_A) : \Gamma \rightarrow \mathbf{grpd}$$

Let us also define the natural transformation  $\text{fst} : \Sigma_A B \rightarrow A$  by

$$\begin{array}{ccc} \text{disp}_A \circ \text{disp}_B & & \Sigma_A B \\ \downarrow \text{disp}_B & \xrightarrow{\text{fiber}} & \downarrow \text{fiber}(\text{disp}_B) \\ \text{disp}_A & & \text{fiber}(\text{disp}_A) \end{array} \quad \begin{array}{ccc} & \text{fst} & \\ & \dashrightarrow & \\ & \cong & \\ & \nearrow & \end{array} \quad \begin{array}{c} A \end{array}$$

**Definition 4.11** (Pushforward of fibrations). Given  $A : \Gamma \rightarrow \mathbf{grpd}$  and  $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$  we will define  $\Pi_A B : \Gamma \rightarrow \mathbf{grpd}$  such that for any  $C : \Gamma \rightarrow \mathbf{grpd}$  we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\text{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

natural in both  $B$  and  $C$ . Stated in terms of fibrations we have

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & & \Gamma \cdot \Pi_A B \\ \text{disp}_B \downarrow & & \downarrow \\ \Gamma \cdot A & \xrightarrow{\text{disp}_A} & \Gamma \end{array}$$

with the universal property of pushforward

$$\text{Fib}_{\Gamma \cdot A}(\text{disp}_A^* \text{disp}_C, \text{disp}_B) \cong \text{Fib}_{\Gamma}(\text{disp}_C, \text{disp}_{\Pi_A B})$$

*Proof.*  $\Pi_A B$  acts on objects by taking fiberwise sections

$$\Pi_A B(x) := \{s \in [A(x), \Sigma_A B(x)] \mid \text{fst}_x \circ s = \text{id}_{A(x)}\}$$

Where we have taken the full subcategory of the functor category  $[A(x), \Sigma_A B(x)]$ . This is a groupoid since any natural transformation of functors into groupoids are natural isomorphisms.

$\Pi_A B$  acts on morphisms via conjugation

$$\begin{array}{ccc} x & & A(x) \\ \downarrow f & \xrightarrow{\Pi_A B} & \uparrow A(f^{-1}) \\ y & & A(y) \end{array} \quad \begin{array}{ccc} \Pi_A B(x) & & \Sigma_A B(x) \\ \downarrow \Sigma_A B(f) \circ \circ A(f^{-1}) & & \downarrow \Sigma_A B(f) \\ \Pi_A B(y) & & \Sigma_A B(y) \end{array} \quad \begin{array}{ccc} A(x) & \xrightarrow{s} & \Sigma_A B(x) \\ & & \downarrow \Sigma_A B(f) \\ A(y) & \xrightarrow{\Pi_A B(f)(s)} & \Sigma_A B(y) \end{array}$$

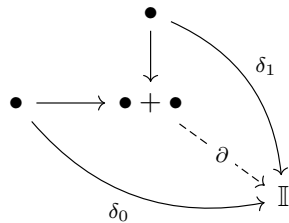
Note that conjugation is functorial and invertible. □



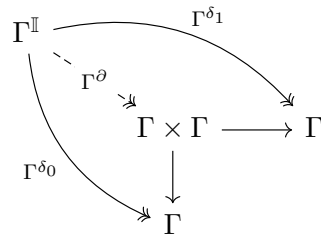
**Proposition 4.12** (All objects are fibrant). *Let  $\bullet$  denote the terminal groupoid, namely that with a single object and morphism. Then the unique map  $\Gamma \rightarrow \bullet$  is a fibration.*

**Definition 4.13** (Interval). Let the interval groupoid  $\mathbb{I}$  be the small groupoid with two objects and a single non-identity isomorphism. There are two distinct morphisms  $\delta_0, \delta_1 : \bullet \rightarrow \mathbb{I}$  and a natural isomorphism  $i : \delta_0 \Rightarrow \delta_1$ . Note that  $\delta_0$  and  $\delta_1$  both form adjoint equivalences with the unique map  $! : \mathbb{I} \rightarrow \bullet$ .

Denote by  $\bullet + \bullet$  the small groupoid with two objects and only identity morphisms. Then let  $\partial : \bullet + \bullet \rightarrow \mathbb{I}$  be the unique map factoring  $\delta_0$  and  $\delta_1$ .



**Proposition 4.14** (Path object fibration). *Let  $\Gamma$  be a small groupoid. Recall that **grpd** is Cartesian closed, so we can take the image of the above diagram under the functor  $\Gamma^-$ .*



*Then the indicated morphisms are fibrations, and  $\Gamma^{\delta_0}, \Gamma^{\delta_1}$  form adjoint equivalences with  $\Gamma^! : \Gamma \rightarrow \Gamma^{\mathbb{I}}$ .*

### 4.3 Polynomial endofunctors

**Definition 4.15** (Polynomial endofunctor on a morphism in an locally Cartesian closed category). Let  $\mathbb{C}$  be a locally Cartesian closed category (we will take presheaves on small groupoids). This means for each morphism  $t : B \rightarrow A$  we have an adjoint triple

$$\begin{array}{c} \mathbb{C}/B \\ \left( \begin{array}{c} \uparrow \\ \dashv \quad t^* \quad \dashv \\ \downarrow \end{array} \right) \\ \mathbb{C}/A \end{array} \quad \begin{array}{c} t_! \\ \downarrow \\ t_* \end{array}$$

where  $t^*$  is pullback, and  $t_!$  is composition with  $t$ .

Let  $t : B \rightarrow A$  be a morphism in  $\mathbb{C}$ . Then define  $\text{Poly}_t : \mathbb{C} \rightarrow \mathbb{C}$  be the composition

$$\text{Poly}_t := A_! \circ t_* \circ B^* \quad \mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

**Proposition 4.16.** *The data of a map into the polynomial applied to an object in  $\mathbb{C}$*

$$X \xrightarrow{\phi} \text{Poly}_t Y$$

corresponds to

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \text{Poly}_t Y \\ & \searrow \alpha & \swarrow t_* B^* Y \\ & A & \end{array}$$

Applying the adjunction  $A_! \dashv A^*$ , this corresponds to

$$\alpha : X \rightarrow A \quad \text{and} \quad \begin{array}{ccc} B_! t^* \alpha & \xrightarrow{\tilde{\phi}} & B \times Y \\ & \searrow t^* \alpha & \swarrow B^* Y \\ & B & \end{array}$$

Applying the adjunction  $t^* \dashv t_*$ , this corresponds to

$$\alpha : X \rightarrow A \quad \text{and} \quad B_! t^* \alpha \xrightarrow{\beta} Y$$

**Proposition 4.17** ( $\text{Poly}_{\text{tp}} \text{Ty}$  classifies dependent types). *Specialized to  $\text{tp} : \text{Tp} \rightarrow \text{Ty}$  in  $\mathbf{Psh}(\mathbf{grpd})$ , the previous proposition says that a map from a representable  $\Gamma \rightarrow \text{Poly}_{\text{tp}} \text{Ty}$  corresponds to the data of*

$$A : \Gamma \rightarrow \text{Ty} \quad \text{and} \quad B : \Gamma \cdot A \rightarrow \text{Ty}$$

which by Yoneda corresponds to the data in  $\mathbf{Cat}$  of

$$A : \Gamma \rightarrow \mathbf{grpd} \quad \text{and} \quad B : \Gamma \cdot A \rightarrow \mathbf{grpd}$$

#### 4.4 $\Pi$ and $\Sigma$ structure

**Definition 4.18** (Interpretation of  $\Pi$  and  $\lambda$ ). We define the natural transformation  $\Pi : \text{Poly}_{\text{tp}} \text{Ty} \rightarrow \text{Ty}$  by first taking some small groupoid  $\Gamma$  and defining

$$\Pi_\Gamma : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Poly}_{\text{tp}} \text{Ty}) \rightarrow \mathbf{Psh}(\mathbf{grpd})(\Gamma, \text{Ty})$$

Let  $(A, B) : \Gamma \rightarrow \text{Poly}_{\text{tp}} \text{Ty}$ , corresponding to  $A : \Gamma \rightarrow \mathbf{grpd}$  and  $B : \Gamma \cdot A \rightarrow \mathbf{grpd}$ . Taking the pushforward of fibrations in  $\mathbf{grpd}$  (formally defined as operations on the

classifying maps), we obtain  $\Pi_A B : \Gamma \rightarrow \mathbf{grpd}$  corresponding by Yoneda to an element of  $\mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathbf{Ty})$ .

As indicated in the diagram, we take this to be the pushforward of the dependent display map  $\mathbf{disp}_B$  along the display map it depends on  $\mathbf{disp}_A$ . Note that this pushforward is in  $\mathbf{grpd}$ , and this pushforward is only defined on fibrations.

TODO: define  $\lambda$ .

*Proof.* TODO: naturality.

TODO: prove pullback. □

**Definition 4.19** (Interpretation of  $\Sigma$ ). Sketch: we define the natural transformation  $\Sigma : \mathbf{Poly}_{\mathbf{tp}} \mathbf{Ty} \rightarrow \mathbf{Ty}$  by first taking some small groupoid  $\Gamma$  and defining

$$\Sigma_\Gamma : \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathbf{Poly}_{\mathbf{tp}} \mathbf{Ty}) \rightarrow \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathbf{Ty})$$

Again, this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

$$\begin{array}{ccc} \Gamma \cdot A \cdot B & \mapsto & \Gamma \cdot \Sigma_A B \\ \mathbf{disp}_B \downarrow & & \downarrow (\mathbf{disp}_A)_! \mathbf{disp}_B \\ \Gamma \cdot A & \xrightarrow{\mathbf{disp}_A} & \Gamma \end{array}$$

As indicated in the diagram, we take this to be the composition of  $\mathbf{disp}_B$  and  $\mathbf{disp}_A$ , recalling that fibrations are closed under composition.

TODO: define  $\mathbf{pair}$ .

*Proof.* TODO: naturality.

TODO: prove pullback. □

## References

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