

Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ . Write **Set** for the category of all sets. Say that a set A is λ -small if $|A| < \lambda$. Write **Set** $_\lambda$ for the full subcategory of **Set** spanned by λ -small sets.

Let \mathbb{C} be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write $\mathbf{Psh}(\mathbb{C})$ for the category of presheaves over \mathbb{C} ,

$$\mathbf{Psh}(\mathbb{C}) =_{\text{def}} [\mathbb{C}^{\text{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $\text{tp}: \mathbf{Tm} \rightarrow \mathbf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, \dots \in \mathbb{C}$,
- a type in context $y(\Gamma)$ as a map $A: y(\Gamma) \rightarrow \mathbf{Ty}$,
- a term of type A in context Γ as a map $a: y(\Gamma) \rightarrow \mathbf{Tm}$ such that

$$\begin{array}{ccc} & \mathbf{Tm} & \\ & \uparrow a & \downarrow \text{tp} \\ \Gamma & \xrightarrow{A} & \mathbf{Ty} \end{array}$$

commutes,

- an operation called “context extension” which given a context Γ and a type $A: y(\Gamma) \rightarrow \mathbf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{array}{ccc} y(\Gamma \cdot A) & \longrightarrow & \mathbf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{A} & \mathbf{Ty} \end{array}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty . We write

$$\begin{array}{ccc} X & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ y(\Gamma) & \xrightarrow{\ulcorner X \urcorner} & \mathsf{Ty} \end{array}$$

to express this situation, i.e. $X \cong y(\Gamma \cdot \ulcorner X \urcorner)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U , not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} : \mathsf{Ty} \quad \frac{a : \mathsf{U}}{\mathsf{El}(a) : \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ , with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi : \mathsf{E} \rightarrow \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{array}{ccc} \mathsf{U} & \longrightarrow & \mathsf{Tm} \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\ulcorner \mathsf{U} \urcorner} & \mathsf{Ty} \end{array}$$

- There is an inclusion of U into Ty

$$\mathsf{El} : \mathsf{U} \rightarrow \mathsf{Ty}$$

- $\pi : \mathsf{E} \rightarrow \mathsf{U}$ is obtained as pullback of tp ; There is a pullback diagram

$$\begin{array}{ccc} \mathsf{E} & \longrightarrow & \mathsf{Tm} \\ \pi \downarrow & & \downarrow \mathsf{tp} \\ \mathsf{U} & \xrightarrow{\mathsf{El}} & \mathsf{Ty} \end{array} \tag{2.1}$$

With the notation above, we get

$$\begin{array}{ccccc}
y(\Gamma.\text{El}(a)) & \longrightarrow & E & \longrightarrow & \text{Tm} \\
\downarrow & & \downarrow & & \downarrow \\
y(\Gamma) & \xrightarrow{a} & U & \xrightarrow{\text{El}} & \text{Ty} \\
& \searrow \scriptstyle A & & \nearrow & \\
& & & &
\end{array}$$

Both squares above are pullback squares.

3 Defining Π and Σ type formers for the universe $\pi: E \rightarrow U$

Take the pullback diagram eq. (2.1). That is a morphism in the category of polynomials. We have a cartesian natural transformation $P_\pi \rightarrow P_{\text{tp}}$ induced by the pullback eq. (2.1). All the squares in the left cube are pullback squares.

Now, consider the right cube only. The right-side face of the cube is a pullback square due to the definition of π . The left-side is also because polynomials are left exact functors and therefore they preserve pullbacks. Observe also that the front face is a pullback square by the definition of Π for Ty .

$$\begin{array}{ccccccc}
& & P_\pi E & \longrightarrow & P_{\text{tp}} E & \longrightarrow & E \\
& \swarrow & \downarrow & & \downarrow & \swarrow & \downarrow \\
P_\pi \text{Tm} & \longrightarrow & P_{\text{tp}} \text{Tm} & \xrightarrow{\lambda} & \text{Tm} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & P_\pi U & \longrightarrow & P_{\text{tp}} U & \xrightarrow{\quad \quad} & U \\
& \swarrow & \downarrow & & \downarrow & \swarrow & \downarrow \\
P_\pi \text{Ty} & \longrightarrow & P_{\text{tp}} \text{Ty} & \xrightarrow{\Pi} & \text{Ty} & &
\end{array}$$

We define $\Pi_U: P_\pi U \rightarrow U$ to be a dashed arrow which makes the bottom square commute, that is the following square commutes

$$\begin{array}{ccc}
P_{\text{tp}} U & \longrightarrow & P_{\text{tp}} \text{Ty} \\
\Pi_U \downarrow & & \downarrow \Pi_{\text{Ty}} \\
U & \xrightarrow{\text{El}} & \text{Ty}
\end{array} \tag{3.1}$$

By the universal property of the pullback which defines π we get a unique arrow $P_{\text{tp}}: P_{\text{tp}} E \rightarrow E$ which we shall name λ_U . By the pullback pasting lemma it follows

that the square involving Π_U and λ_U is a pullback square. This concludes the construction of Π type former for the universe \mathcal{U} . The only data we needed to supply for the definition of Π_U was a lift of $\Pi: P_{\mathbf{tp}}\mathbf{Ty} \rightarrow \mathbf{Ty}$ to U .

4 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model