Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ. Write **Set** for the category of all sets. Say that a set A is λ-small if $|A| < \lambda$. Write $\mathbf{Set}_\lambda$ for the full subcategory of **Set** spanned by λ-small sets.

Let $\mathbb{C}$ be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write $\mathbf{Psh}(\mathbb{C})$ for the category of presheaves over $\mathbb{C}$,

$$\mathbf{Psh}(\mathbb{C}) =_{\text{def}} [\mathbb{C}^{\text{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$ consists of

- contexts as objects $\Gamma, \Delta, \ldots \in \mathbb{C}$,
- a type in context $\mathsf{y}(\Gamma)$ as a map $A \colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$,
- a term of type A in context Γ as a map $a \colon \mathsf{y}(\Gamma) \to \mathsf{Tm}$ such that

$$\begin{tikzcd} & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

  commutes,

- an operation called "context extension" which given a context Γ and a type $A \colon \mathsf{y}(\Gamma) \to \mathsf{Ty}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{tikzcd} \mathsf{y}(\Gamma.A) \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\ \mathsf{y}(\Gamma) \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty. We write

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\
\mathsf{y}(\Gamma) \arrow[r, "\ulcorner X \urcorner"'] & \mathsf{Ty}
\end{tikzcd}$$

to express this situation, i.e. $X \cong \mathsf{y}(\Gamma \cdot \ulcorner X \urcorner)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U} \colon \mathsf{Ty} \qquad \frac{a \colon \mathsf{U}}{\mathsf{El}(a) \colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ, with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

- There is a pullback diagram of the form

$$\begin{tikzcd}
\mathsf{U} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\
1 \arrow[r, "\ulcorner \mathsf{U} \urcorner"'] & \mathsf{Ty}
\end{tikzcd}$$

- There is an inclusion of U into Ty

$$\mathsf{El} \colon \mathsf{U} \rightarrowtail \mathsf{Ty}$$

- $\pi : \mathsf{E} \to \mathsf{U}$ is obtained as pullback of $\mathtt{tp}$; There is a pullback diagram

$$\begin{tikzcd}
E \arrow[r, tail] \arrow[d] & \mathsf{Tm} \arrow[d] \\
\mathsf{U} \arrow[r, tail, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd}$$

With the notation above, we get

$$\begin{tikzcd}
\mathsf{y}(\Gamma.\mathsf{El}(a)) \arrow[r] \arrow[d] & \mathsf{E} \arrow[r] \arrow[d] & \mathsf{Tm} \arrow[d] \\
\mathsf{y}(\Gamma) \arrow[r, "a"'] \arrow[rr, bend right, "A"'] & \mathsf{U} \arrow[r, "\mathsf{El}"'] & \mathsf{Ty}
\end{tikzcd}$$

Both squares above are pullback squares.

3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

4 Groupoid Model of HoTT

In this section we construct a natural model in $\mathbf{Psh}(\mathbf{grpd})$ the presheaf category indexed by the category $\mathbf{grpd}$ of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on $\mathbf{grpd}$ - which comes from restricting the "classical Quillen model structure" on $\mathbf{cat}$ [Joy] to $\mathbf{grpd}$.

4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as $\mathbf{set}$ and the large sets as $\mathbf{Set}$ (in the previous sections this would have been $\mathbf{Set}_\lambda$ and $\mathbf{Set}$ respectively). We denote the category of small categories as $\mathbf{cat}$ and the large categories as $\mathbf{Cat}$. We denote the category of small groupoids as $\mathbf{grpd}$.

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

Definition 4.1 (Pointed)**.** We will take the category of pointed small categories $\mathbf{cat}_\bullet$ to have objects as pairs $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$ and morphisms as pairs

$$(F : \mathbb{C}_1 \to \mathbb{C}_0, \phi : Fc_1 \to c_0) \colon (\mathbb{C}_1, c_1) \to (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids $\mathbf{grpd}_\bullet$ will be the full subcategory of objects (Γ, c) with Γ a groupoid.

Definition 4.2 (The display map classifier)**.** We would like to define a natural transformation in $\mathbf{Psh}(\mathbf{grpd})$

$$\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$$

with representable fibers.

Consider the functor that forgets the point

$$U \colon \mathbf{grpd}_\bullet \to \mathbf{grpd} \qquad \text{in} \qquad \mathbf{Cat}.$$

If we apply the Yoneda embedding $\mathsf{y} \colon \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U\circ \colon [-, \mathbf{grpd}_\bullet] \to [-, \mathbf{grpd}] \qquad \text{in} \qquad \mathbf{Psh}(\mathbf{Cat}).$$

Since any small groupoid is also a large category $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict $\mathbf{Cat}$ indexed presheaves to be $\mathbf{grpd}$ indexed presheaves. We define $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$ as the image of $U\circ$ under this restriction.

$$\begin{array}{ccccc} \mathbf{Cat} & \xrightarrow{\mathsf{y}} & \mathbf{Psh}(\mathbf{Cat}) & \xrightarrow{\mathsf{res}} & \mathbf{Psh}(\mathbf{grpd}) \\ \mathbf{grpd} & \longmapsto & [-, \mathbf{grpd}] & \longmapsto & \mathsf{Ty} \end{array}$$

Note that Tm and Ty are not representable in $\mathbf{Psh}(\mathbf{grpd})$.

Remark 4.3. By Yoneda we can identify maps with representable domain into the type classifier

$$A : \mathsf{y}\Gamma \to \mathsf{Ty} \qquad \text{in} \qquad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A : \Gamma \to \mathbf{grpd} \qquad \text{in} \qquad \mathbf{Cat}$$

Definition 4.4 (Grothendieck construction)**.** From $\mathbb{C}$ a small category and $F : \mathbb{C} \to \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in $\mathbb{C}$ we refer to Fc as the fiber over c. The objects of $\int F$ consist of pairs $(c \in \mathbb{C}, x \in Fc)$, and morphisms between (c, x) and (d, y) are pairs $(f : c \to d, \phi : F\,f\,x \to y)$. This makes the following pullback in $\mathbf{Cat}$

$$(c, x) \longmapsto (Fc, x)$$

$$\begin{tikzcd}
(c,x) \arrow[d, mapsto] & \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{cat}_\bullet \arrow[d] & (C,c) \arrow[d, mapsto] \\
c & \mathbb{C} \arrow[r, "F"'] & \mathbf{cat} & C
\end{tikzcd}$$

Definition 4.5 (Grothendieck construction for groupoids)**.** Let Γ be a groupoid and $A \colon \Gamma \to \mathbf{grpd}$ a functor, we can compose F with the inclusion $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \mathsf{disp}_A \colon \Gamma \cdot A \to \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids Γ and Ax for $x \in \Gamma$. Furthermore the pullback factors through (pointed) groupoids.

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r] \arrow[d, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{cat}_\bullet \arrow[d] \\
\Gamma \arrow[r, "A"'] & \mathbf{grpd} \arrow[r] & \mathbf{cat}
\end{tikzcd}$$

Corollary 4.6 (The display map classifier is presentable)**.** *For any small groupoid* Γ *and* $A : \mathsf{y}\Gamma \to \mathsf{Ty}$*, the pullback of* tp *along* A *can be given by the representable map* ydisp_A*.*

$$\begin{tikzcd}
\mathsf{y}\Gamma \cdot A \arrow[r] \arrow[d, "\mathsf{ydisp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{y}\Gamma \arrow[r, "A"'] & \mathsf{Ty}
\end{tikzcd}$$

Proof. Consider the pullback in **Cat**

$$\begin{tikzcd}
\Gamma \cdot A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d] \\
\Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

We send this square along $\mathsf{res} \circ \mathsf{y}$ in the following

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "\mathsf{y}"] \arrow[dr] & \mathbf{Psh}(\mathbf{Cat}) \arrow[d, "\mathsf{res}"] \\
\mathbf{grpd} \arrow[u] \arrow[r, "\mathsf{y}"] & \mathbf{Psh}(\mathbf{grpd})
\end{tikzcd}$$

The Yoneda embedding $\mathsf{y} : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does res since it is a right adjoint (with left Kan extension $\iota_! \dashv \mathsf{res}_\iota$). □

4.2 Groupoid fibrations

Definition 4.7 (Fibration)**.** Let $p : \mathbb{C}_1 \to \mathbb{C}_0$ be a functor. We say p is a *split Grothendieck fibration* if we have a dependent function $\mathsf{lift}\, a\, f$ satisfying the following: for any object a in $\mathbb{C}_1$ and morphism $f : p\, a \to y$ in the base $\mathbb{C}_0$ we have $\mathsf{lift}\, a\, f : a \to b$ in $\mathbb{C}_1$ such that $p(\mathsf{lift}\, a\, f) = f$ and moreover $\mathsf{lift}\, a\, g \circ f = \mathsf{lift}\, b\, g \circ \mathsf{lift}\, a\, f$

$$\begin{tikzcd}
a \arrow[r, dashed, "\mathsf{lift}\, a\, f"] \arrow[d, mapsto] \arrow[dr, phantom, "\Downarrow"] & b \arrow[d, dashed, mapsto] \\
x \arrow[r, "f"'] & y
\end{tikzcd}$$

In particular, we are intereseted in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid Γ as Fib_Γ, which is a full subcategory of the slice $\mathbf{grpd}/\Gamma$. We will decorate an arrow with $\twoheadrightarrow$ to indicate it is a fibration.

Note that $\mathsf{disp}_A \colon \Gamma \cdot A \to \Gamma$ is a fibration, since for any $(x \in \Gamma, a \in A\,x)$ and $f \colon x \to y$ in Γ we have a morphism $(f, \mathsf{id}_{A\,f\,a}) : (x, a) \to (y, A\,f\,a)$ lifting f. Furthermore

Proposition 4.8. *There is an adjoint equivalence*

$$[\Gamma, \mathbf{grpd}] \underset{\mathsf{fiber}}{\overset{\mathsf{disp}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} \mathsf{Fib}_\Gamma$$

where for each fibration $\delta : \Delta \to \Gamma$ and each object $x \in \Gamma$

$$\mathsf{fiber}_\delta\, x = \text{full subcategory } \{a \in \Delta \mid \delta\, a = x\}$$

It follows that all fibrations are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \to \mathbf{grpd}$, when viewed as morphisms in $\mathbf{Cat}$.

Pullback of fibrations along groupoid functors is not strictly coherent, in the sense that for $\tau : \Xi \to \Delta$ and $\sigma : \Delta \to \Gamma$ and a fibration $p \in \mathsf{Fib}_\Gamma$ we only have an isomorphism

$$\tau^* \sigma^* p \cong (\sigma \circ \tau)^* p$$

rather than equality.

In order to interpret reindexing/substitution strictly, it is convenient to work with classifiers $[\Gamma, \mathbf{grpd}]$ instead of fibrations.

Proposition 4.9 (Strictly coherent pullback). *Let $\sigma : \Delta \to \Gamma$ be a functor between groupoids. Since display maps are pullbacks of the classifier $U : \mathbf{grpd}_\bullet \to \mathbf{grpd}$ we have the pasting diagram*

$$\begin{tikzcd}
\Delta.A\sigma \arrow[r, dashed, "\sigma_A"'] \arrow[rr, bend left] \arrow[d, two heads, "\mathsf{disp}_{A\sigma}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r] \arrow[d, two heads, "\mathsf{disp}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd}_\bullet \arrow[d] \\
\Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

This gives us a functor $\circ\sigma : [\Gamma, \mathbf{grpd}] \to [\Delta, \mathbf{grpd}]$ which is our strict version of pullback.

Corollary 4.10 (Fibrations are stable under pullback)**.**

$$\begin{tikzcd}
{[\Gamma, \mathbf{grpd}]} \arrow[d, "\circ\sigma"'] & \mathsf{Fib}_\Gamma \arrow[l, "\mathsf{fiber}"'] \arrow[d, dashed, "\sigma^*"] \\
{[\Delta, \mathbf{grpd}]} \arrow[r, "\mathsf{disp}"'] & \mathsf{Fib}_\Delta
\end{tikzcd}$$

We can deduce a corresponding fact about fibrations: since fibrations are closed under isomorphism, and since any pullback in **grpd** *of a fibration p is isomorphic to the display map $\mathsf{disp}_{\mathsf{fiber} p \circ \sigma}$, any pullback of a fibration is a fibration.*

A strict interpretation of type theory would require Σ and Π-formers to be stable under pullback (Beck-Chevalley). Thus we again define these as operations on classifiers $[\Gamma, \mathbf{grpd}]$.

Definition 4.11 (Σ-former operation)**.** Then given $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$ we define $\Sigma_A B : \Gamma \to \mathbf{grpd}$ such that $\Sigma_A B$ acts on objects by forming fiberwise Grothendieck constructions

$$\Sigma_A B(x) := A(x) \cdot B \circ x_A$$

where $x_A : A(x) \to \Gamma \cdot A$ takes $f : a_0 \to a_1$ to $(\mathsf{id}_x, f) : (x, a_0) \to (x, a_1)$

$$\begin{tikzcd}
A(x) \cdot B \circ x_A \arrow[r, dashed] \arrow[d, dashed, "\mathsf{disp}_{B \circ x_A}"] & \Gamma.A.B \arrow[r] \arrow[d, "\mathsf{disp}_B"] & \bullet \\
A(x) \arrow[r, "x_A"'] \arrow[d, "!"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd} \\
\bullet \arrow[r, "x"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

$\Sigma_A B$ acts on morphism $f : x \to y$ in Γ and $(a \in A(x), b \in B(x, a))$ by

$$\Sigma_A B\, f\, (a, b) := (A\, f\, a, B\, (f, \mathsf{id}_{A\, f\, a})\, b)$$

and for morphism $(\alpha : a_0 \to a_1 \in A(x), \beta : B\, (\mathsf{id}_x, \alpha)\, b_0 \to b_1 \in B(x, a_1))$ in $\Sigma_A B\, x$

$$\Sigma_A B\, f\, (\alpha, \beta) := (A\, f\, \alpha, B\, (f, \mathsf{id}_{A\, f\, a_1})\, \beta)$$

Let us also define the natural transformation $\mathsf{fst} : \Sigma_A B \to A$ by

$$\mathsf{fst}_x : (a, b) \mapsto a$$

Proposition 4.12 (Fibrations are closed under composition)**.** *The corresponding fact about fibrations is that the composition of two fibrations is a fibration.*

$$\begin{tikzcd}
\Xi \arrow[d, two heads] \arrow[dr, dashed, two heads] & \\
\Delta \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

We can compare the two fibrations

$$\mathsf{disp}_B \circ \mathsf{disp}_A \qquad \text{and} \qquad \mathsf{disp}_{\Sigma_A(B)}$$

An object in the composition would look like $((x, a), b)$ for $x \in \Gamma$, $a \in A(x)$ and $b \in B(x, a)$, whereas an object in $\Gamma \cdot \Sigma_A(B)$ would instead be $(x, (a, b))$.

Proposition 4.13 (Strict Beck-Chevalley for Σ). *Let $\sigma : \Delta \to \Gamma$, $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$. Then*

$$(\Sigma_A B) \circ \sigma = \Sigma_{A \circ \sigma}(B \circ \sigma_A)$$

where σ_A is uniquely determined by the pullback in

$$\begin{tikzcd}
 & \Delta \cdot A\sigma \cdot B \circ \sigma_A \arrow[r, "\sigma_{A \cdot B}"'] \arrow[d, "\mathsf{disp}_{B \circ \sigma_A}"] & \Gamma.A.B \arrow[d, two heads, "\mathsf{disp}_B"] & \\
 & \Delta \cdot A\sigma \arrow[r, "\sigma_A"'] \arrow[d, "\mathsf{disp}_{A\sigma}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd} \\
\mathbf{grpd} & \Delta \arrow[l, shift right, "(\Sigma_A B)\circ\sigma"'] \arrow[l, shift left, "\Sigma_{A\circ\sigma}(B\circ\sigma_A)"] \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

Proof. By checking pointwise at $x \in \Delta$, this boils down to showing

$$(\sigma x)_A = \sigma_A \circ x_{A \circ \sigma} : A(\sigma x) \to \Gamma \cdot A$$

$$\begin{tikzcd}
A(\sigma x) \arrow[r, "x_{A\sigma}"'] \arrow[rr, bend left, "(\sigma x)_A"] \arrow[d, "!"] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta \cdot A\sigma \arrow[r, "\sigma_A"'] \arrow[d, "\mathsf{disp}_{A\sigma}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{grpd} \\
\bullet \arrow[r, "x"'] & \Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

which holds because of the universal property of pullback. $\square$

Definition 4.14 (Π-former operation). Given $A : \Gamma \to \mathbf{grpd}$ and $B : \Gamma \cdot A \to \mathbf{grpd}$ we will define $\Pi_A B : \Gamma \to \mathbf{grpd}$ such that for any $C : \Gamma \to \mathbf{grpd}$ we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathsf{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

natural in both B and C.

Proof. $\Pi_A B$ acts on objects by taking fiberwise sections

$$\Pi_A B(x) := \{s \in [A(x), \Sigma_A B(x)] \,|\, \mathsf{fst}_x \circ s = \mathsf{id}_{A(x)}\}$$

Where we have taken the full subcategory of the functor category $[A(x), \Sigma_A B(x)]$. This is a groupoid since any natural transformation of functors into groupoids are natural isomorphisms.

$\Pi_A B$ acts on morphisms via conjugation

$$
\begin{tikzcd}
x \arrow[d, "f"] & & \Pi_A B(x) \arrow[d, "\Sigma_A B(f)\circ - \circ A(f^{-1})"] & A(x) \arrow[r, "s"] & \Sigma_A B(x) \arrow[d, "\Sigma_A B(f)"] \\
y \arrow[r, phantom, "\overset{\Pi_A B}{\longmapsto}"] & & \Pi_A B(y) & A(y) \arrow[u, "A(f^{-1})"] \arrow[r, dashed, "\Pi_A B(f)(s)"'] & \Sigma_A B(y)
\end{tikzcd}
$$

Note that conjugation is functorial and invertible. □

Corollary 4.15 (Fibrations are closed under pushforward)**.** *Stated in terms of fibrations, we have*

$$
\begin{tikzcd}
\Xi \arrow[d, two heads, "\tau"'] & \Gamma_! \sigma_* \tau \arrow[d, two heads, "\sigma_* \tau"] \\
\Delta \arrow[r, two heads, "\sigma"'] & \Gamma
\end{tikzcd}
$$

with the universal property of pushforward

$$\mathsf{Fib}_\Delta(\sigma^* \rho, \tau) \cong \mathsf{Fib}_\Gamma(\rho, \sigma_* \tau)$$

natural in both τ and ρ.

Proposition 4.16 (Strict Beck-Chevalley for Π)**.** *Let $\sigma : \Delta \to \Gamma$, $A : \Gamma \to$* **grpd** *and $B : \Gamma \cdot A \to$* **grpd**. *Then*

$$(\Pi_A B) \circ \sigma = \Pi_{A \circ \sigma}(B \circ \sigma_A)$$

where σ_A is uniquely determined by the pullback in

$$
\begin{tikzcd}
& \Delta \cdot A\sigma \cdot B \circ \sigma_A \arrow[r, "\sigma_{A \cdot B}"'] \arrow[d, two heads, "\mathsf{disp}_{B \circ \sigma_A}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A.B \arrow[d, two heads, "\mathsf{disp}_B"] & \\
& \Delta \cdot A\sigma \arrow[r, "\sigma_A"'] \arrow[d, two heads, "\mathsf{disp}_{A\sigma}"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[r, "B"] \arrow[d, two heads, "\mathsf{disp}_A"] & \mathbf{grpd} \\
\mathbf{grpd} & \Delta \arrow[l, shift right, "\Pi_A B \circ \sigma"'] \arrow[l, shift left, "\Pi_{A \circ \sigma}(B \circ \sigma_A)"] \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}
$$

Proof. By checking pointwise, this boils down to Beck-Chevalley for Σ. □

Proposition 4.17 (All objects are fibrant)**.** *Let $\bullet$ denote the terminal groupoid, namely that with a single object and morphism. Then the unique map $\Gamma \to \bullet$ is a fibration.*

Definition 4.18 (Interval)**.** Let the interval groupoid $\mathbb{I}$ be the small groupoid with two objects and a single non-identity isomorphism. There are two distinct morphisms $\delta_0, \delta_1 : \bullet \to \mathbb{I}$ and a natural isomorphism $i : \delta_0 \Rightarrow \delta_1$. Note that δ_0 and δ_1 both form adjoint equivalences with the unique map $! : \mathbb{I} \to \bullet$.

Denote by $\bullet + \bullet$ the small groupoid with two objects and only identity morphisms. Then let $\partial : \bullet + \bullet \to \mathbb{I}$ be the unique map factoring δ_0 and δ_1.

$$\begin{tikzcd}
 & \bullet \arrow[d] \arrow[ddr, bend left, "\delta_1"] & \\
\bullet \arrow[r] \arrow[drr, bend right, "\delta_0"'] & \bullet + \bullet \arrow[dr, dashed, "\partial"] & \\
 & & \mathbb{I}
\end{tikzcd}$$

Proposition 4.19 (Path object fibration)**.** *Let Γ be a small groupoid. Recall that* **grpd** *is Cartesian closed, so we can take the image of the above diagram under the functor Γ^{-}.*

$$\begin{tikzcd}
\Gamma^{\mathbb{I}} \arrow[dr, dashed, two heads, "\Gamma^{\partial}"] \arrow[drr, bend left, two heads, "\Gamma^{\delta_1}"] \arrow[ddr, bend right, two heads, "\Gamma^{\delta_0}"'] & & \\
 & \Gamma \times \Gamma \arrow[r] \arrow[d] & \Gamma \\
 & \Gamma &
\end{tikzcd}$$

Then the indicated morphisms are fibrations, and $\Gamma^{\delta_0}, \Gamma^{\delta_1}$ form adjoint equivalences with $\Gamma^{!} : \Gamma \to \Gamma^{\mathbb{I}}$.

4.3 Polynomial endofunctors

Definition 4.20 (Polynomial endofunctor on a morphism in an locally Cartesian closed category)**.** Let $\mathbb{C}$ be a locally Cartesian closed category (we will take presheaves on small groupoids). This means for each morphism $t : B \to A$ we have an adjoint triple

$$\begin{tikzcd}
\mathbb{C}/B \arrow[d, bend right=60, "t_!"'] \arrow[d, bend left=60, "t_*"] \\
\mathbb{C}/A \arrow[u, "t^*" description]
\end{tikzcd}
\qquad t_! \dashv t^* \dashv t_*$$

where t^* is pullback, and $t_!$ is composition with t.

Let $t : B \to A$ be a morphism in $\mathbb{C}$. Then define $\mathsf{Poly}_t : \mathbb{C} \to \mathbb{C}$ be the composition

$$\mathsf{Poly}_t := A_! \circ t_* \circ B^* \qquad \mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

Proposition 4.21. *The data of a map into the polynomial applied to an object in* $\mathbb{C}$

$$X \xrightarrow{\phi} \mathsf{Poly}_t Y$$

corresponds to

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, dashed, "\alpha"'] & & \mathsf{Poly}_t Y \arrow[dl, "t_* B^* Y"] \\ & A & \end{tikzcd}$$

Applying the adjunction $A_! \dashv A^*$*, this corresponds to*

$$\alpha : X \to A \qquad \textit{and} \qquad \begin{tikzcd} B_! t^* \alpha \arrow[rr, dashed, "\tilde{\phi}"] \arrow[dr, "t^*\alpha"'] & & B \times Y \arrow[dl, "B^* Y"] \\ & B & \end{tikzcd}$$

Applying the adjunction $t^* \dashv t_*$*, this corresponds to*

$$\alpha : X \to A \qquad \textit{and} \qquad B_! t^* \alpha \overset{\beta}{\dashrightarrow} Y$$

Proposition 4.22 ($\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}$ classifies dependent types)**.** *Specialized to* $\mathsf{tp} : \mathsf{Tm} \to \mathsf{Ty}$ *in* $\mathbf{Psh}(\mathbf{grpd})$*, the previous proposition says that a map from a representable* $\Gamma \to \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}$ *corresponds to the data of*

$$A : \Gamma \to \mathsf{Ty} \qquad \textit{and} \qquad B : \Gamma \cdot A \to \mathsf{Ty}$$

which by Yoneda corresponds to the data in **Cat** *of*

$$A : \Gamma \to \mathbf{grpd} \qquad \textit{and} \qquad B : \Gamma \cdot A \to \mathbf{grpd}$$

Furthermore, if $\sigma : \Delta \to \Gamma$ *were a representable map, then we have a naturality square*

$$\begin{tikzcd} \Gamma & \Sigma_{A \in [\Gamma, \mathbf{grpd}]} [\Gamma.A, \mathbb{C}] \arrow[r, "\cong"] \arrow[d, dashed, "(-\circ\sigma, -\circ\sigma_A)"] & \mathsf{Poly}_{\mathsf{tp}}[\Gamma, \mathbb{C}] \arrow[d, dashed, "\mathsf{Poly}_{\mathsf{tp}} - \circ\sigma"] \\ \Delta \arrow[u, "\sigma"] & \Sigma_{A \in [\Delta, \mathbf{grpd}]} [\Delta.A, \mathbb{C}] \arrow[r, "\cong"'] & \mathsf{Poly}_{\mathsf{tp}}[\Delta, \mathbb{C}] \end{tikzcd}$$

4.4 Π and Σ structure

Lemma 4.23. *Let* $\mathbb{C}$ *be a large category, and let* $[-, \mathbb{C}] \in \mathbf{Psh}(\mathbf{grpd})$ *be the restriction of the Yoneda embedding* $\mathsf{y} : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$*. Let* F *be an operation that takes a groupoid* Γ*, a functor* $A : \Gamma \to \mathbf{grpd}$ *and* $B : \Gamma \cdot A \to \mathbb{C}$ *and returns a functor* $F_A B : \Gamma \to \mathbb{C}$*.*

Then $\tilde{F} : \mathsf{Poly}_{\mathsf{tp}}[-,\mathbb{C}] \to [-,\mathbb{C}]$

$$\tilde{F}_\Gamma(A,B) = F_A B$$

defines a natural transformation if and only if F *satisfies the strict Beck-Chevalley condition*

$$(F_A B) \circ \sigma = F_{A\circ\sigma}(B \circ \sigma_A)$$

where σ_A *is given by*

$$\begin{tikzcd}
\Delta \cdot A \circ \sigma \arrow[r, dashed, "\sigma_A"] \arrow[d, two heads] & \Gamma \cdot A \arrow[r] \arrow[d, two heads] & \mathbf{grpd}_\bullet \arrow[d] \\
\Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "A"'] & \mathbf{grpd}
\end{tikzcd}$$

Proof. Using proposition 4.22

$$\begin{tikzcd}
\Gamma & \Sigma_{A\in[\Gamma,\mathbf{grpd}]}[\Gamma.A,\mathbb{C}] \arrow[r] \arrow[d, dashed, "{(-\circ\sigma, -\circ\sigma_A)}"'] \arrow[rr, bend left, "F"] & \mathsf{Poly}_{\mathsf{tp}}[\Gamma,\mathbb{C}] \arrow[r, "F_\Gamma"'] \arrow[d, "\mathsf{Poly}_{\mathsf{tp}}-\circ\sigma"'] & {[\Gamma,\mathbb{C}]} \arrow[d, "-\circ\sigma"] \\
\Delta \arrow[u, "\sigma"] & \Sigma_{A\in[\Delta,\mathbf{grpd}]}[\Delta.A,\mathbb{C}] \arrow[r] \arrow[rr, bend right, "F"'] & \mathsf{Poly}_{\mathsf{tp}}[\Delta,\mathbb{C}] \arrow[r, "F_\Delta"] & {[\Delta,\mathbb{C}]}
\end{tikzcd}$$

□

Definition 4.24 (Interpretation of Π types)**.** We define the natural transformation $\Pi : \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ as that which is induced (lemma 4.23) by the Π-former operation (definition 4.14).

Then we define the natural transofrmation $\lambda : \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$ as the natural transformation induced by the following operation: given $A : \Gamma \to \mathbf{grpd}$ and $\beta : \Gamma \cdot A \to \mathbf{grpd}_\bullet$, $\lambda_A \beta : \Gamma \to \mathbf{grpd}_\bullet$ will be the functor such that on objects $x \in \Gamma$

$$\lambda_A\beta\,(x) := (\Pi_A B\,(x), a \mapsto (a, b(x,a)))$$

where $B := U \circ \beta : \Gamma \cdot A \to \mathbf{grpd}$ and $b(x,a)$ is the point in $\beta(x,a)$. On morphisms $f : x \to y$ in Γ we have

$$\lambda_A\beta\,(f) := (\Pi_A B\,(f), \eta)$$

where $\eta : \Pi_A B\,f\,s_x \to s_y$ is a natural isomorphism between functors $A_y \to \Sigma_A B y$ given on objects $a \in A_y$ by

$$\eta_a := (\mathsf{id}_a, \mathsf{id}_{b(y,a)})$$

These combine to give us a pullback square

$$\begin{tikzcd}
\mathsf{Poly}_{\mathsf{tp}}\mathsf{Tm} \arrow[r, "\lambda"] \arrow[d, "\mathsf{Poly}_{\mathsf{tp}}\mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\
\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Pi"'] & \mathsf{Ty}
\end{tikzcd}$$

Proof. We should check that the λ operation satisfied Beck-Chevalley. This follows from the Π satisfying Beck-Chevalley and extensionality results for functors.

The square commutes and is a pullback if and only it pointwise commutes and pointwise gives pullbacks, i.e. for each groupoid Γ

$$\begin{tikzcd}
\mathsf{Poly}_{\mathsf{tp}}\mathsf{Tm}\,\Gamma \arrow[r, "\lambda_\Gamma"] \arrow[d, "\mathsf{Poly}_{\mathsf{tp}}\mathsf{tp}_\Gamma"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\Gamma, \mathbf{grpd}_\bullet]} \arrow[d, "U\circ -"] \\
\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}\,\Gamma \arrow[r, "\Pi_\Gamma"'] & {[\Gamma, \mathbf{grpd}]}
\end{tikzcd}$$

by proposition 4.22 this holds if and only if

$$\begin{tikzcd}
\Sigma_{A\in[\Gamma,\mathbf{grpd}]}[\Gamma.A, \mathbf{grpd}_\bullet] \arrow[r, "\lambda"] \arrow[d, "{(-\circ\sigma, -\circ\sigma_A)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[\Gamma, \mathbf{grpd}_\bullet]} \arrow[d, "U\circ -"] \\
\Sigma_{A\in[\Gamma,\mathbf{grpd}]}[\Gamma.A, \mathbf{grpd}] \arrow[r, "\Pi"'] & {[\Gamma, \mathbf{grpd}]}
\end{tikzcd}$$

which follows from the definitions of Π and λ. $\square$

Definition 4.25 (Evaluation)**.** Define the operation of evaluation ev to take $\alpha : \Gamma \to \mathbf{grpd}_\bullet$ and $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$ and return $\mathsf{ev}_\alpha\, B : \Gamma \to \mathbf{grpd}$, described below.

$$\begin{tikzcd}
\Gamma \arrow[dr, "{(\alpha,\beta)}"] \arrow[drr, bend left, "\beta"] \arrow[ddr, "\alpha"'] \arrow[ddd, dashed, bend right, "\mathsf{ev}_\alpha\, B"'] & & \\
& \mathsf{Tm}\times_{\mathsf{Ty}}\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \arrow[r] \arrow[d] \arrow[dl, "\mathsf{counit}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \arrow[d, "\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}"] \\
\mathsf{Tm}\times\mathsf{Ty} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] \arrow[d] & \mathsf{Ty} \\
\mathsf{Ty} \arrow[r] & \bullet &
\end{tikzcd}$$

The input data corresponds to $\alpha : \Gamma \to \mathsf{Tm}$ and and some $\beta : \Gamma \to \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}$ such that

$$\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty} \circ \beta = \mathsf{tp} \circ \alpha$$

This in turn corresponds to a map into the fiber product

$$(\alpha, \beta) : \Gamma \to \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty}$$

We then compose this the counit of the adjunction $\mathsf{tp}^* \dashv \mathsf{tp}_*$ to get

$$\mathsf{counit} \circ (\alpha, \beta) : \Gamma \to \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp}_* \mathsf{Ty}$$

then extract the type by composing with the projection to Ty

$$\pi_{\mathsf{Ty}} \circ \mathsf{counit} \circ (\alpha, \beta) : \Gamma \to \mathsf{Ty}$$

Finally, this corresponds to a unique map $\mathsf{ev}_\alpha\, B : \Gamma \to \mathbf{grpd}$.

Definition 4.26 (Classifier for dependent pairs)**.** Recall the following definition of composition of polynomial endofunctors, specialized to our situation

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rrr, bend left, "\mathsf{tp} \triangleleft \mathsf{tp}"] & \mathsf{Tm} \times_{\mathsf{Ty}} \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \arrow[rr] \arrow[d, "\mathsf{counit}"] \arrow[dr] & & \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \arrow[d, "\mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty}"] \\
\mathsf{Tm} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \times \mathsf{Ty} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] \arrow[d] & \mathsf{Ty} \\
\mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty} \arrow[r] & \bullet &
\end{tikzcd}$$

Proposition 4.27. *Q classifies the data of a pair of dependent terms (a, b), and* $\mathsf{tp} \triangleleft \mathsf{tp}$ *extracts the underlying type and dependent type (A, B).*

$$Q\,\Gamma \cong \{(\beta : \gamma \to \mathbf{grpd}_\bullet, \alpha : \Gamma \to \mathbf{grpd}_\bullet, B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}) \,|\, U \circ \beta = \mathsf{ev}_\alpha\, \beta\}$$

Proof. Extending the diagram from definition 4.26

$$\begin{tikzcd}
\Gamma \arrow[dr, "\varepsilon"] \arrow[drr, bend left, "\langle \alpha, \gamma \rangle"] \arrow[ddrrr, bend left, "\alpha"] \arrow[drrrr, bend left, "\gamma"] \arrow[dddr, bend right, "\beta"'] & & & & \\
& Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \times_{\mathsf{Ty}} \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \arrow[rr] \arrow[d, "\mathsf{counit}"] \arrow[dr] & & \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \arrow[d, "\mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty}"] \\
& \mathsf{Tm} \times \mathsf{Tm} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \times \mathsf{Ty} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] \arrow[d] & \mathsf{Ty} \\
& \mathsf{Tm} \arrow[r, "\mathsf{tp}"'] & \mathsf{Ty} \arrow[r] & \bullet &
\end{tikzcd}$$

By the universal property of pullbacks, a data of a map with representable domain $\varepsilon : \Gamma \to Q$ corresponds to the data of a triple of maps $\alpha, \beta : \Gamma \to \mathsf{Tm}$ and $\gamma : \Gamma \to \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}$ such that $\mathsf{tp} \circ \beta = \pi_{\mathsf{Ty}} \circ \mathsf{counit} \circ \langle \alpha, \gamma \rangle$ and $\mathsf{tp}_* \mathsf{Tm}^* \mathsf{Ty} \circ \gamma = \mathsf{tp} \circ \alpha$.

This in turn corresponds to three functors $\alpha, \beta : \Gamma \to \mathbf{grpd}_\bullet$ and $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$, such that $U \circ \beta = \mathsf{ev}_\alpha B$. Type theoretically $\alpha = (A, a : A)$ and $\mathsf{ev}_\alpha B = Ba$ and $\beta = (Ba, b : Ba)$. Then composing ε with $\mathsf{tp} \lhd \mathsf{tp}$ returns γ, which consists of (A, B). It is in this sense that Q classifies pairs of dependent terms, and $\mathsf{tp} \lhd \mathsf{tp}$ extracts the underlying types. $\square$

Definition 4.28 (Interpretation of Σ)**.** We define the natural transformation

$$\Sigma : \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \to \mathsf{Ty}$$

as that which is induced (lemma 4.23) by the Σ-former operation (definition 4.14).

Define $\mathsf{pair} : Q \to \mathsf{Tm}$ such that

$$\begin{aligned} & Q\,\Gamma \\ \cong & \{(\beta : \gamma \to \mathbf{grpd}_\bullet, \alpha : \Gamma \to \mathbf{grpd}_\bullet, B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}) \,|\, U \circ \beta = \mathsf{ev}_\alpha\, \beta\} \\ \to & [\Gamma, \mathbf{grpd}_\bullet] \\ & (\beta, \alpha, B) \end{aligned}$$

Σ and pair combine to give us a pullback square

$$\begin{tikzcd} Q \arrow[r, "\mathsf{pair}"] \arrow[d, "\mathsf{tp} \lhd \mathsf{tp}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Tm} \arrow[d, "\mathsf{tp}"] \\ \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \arrow[r, "\Sigma"'] & \mathsf{Ty} \end{tikzcd}$$

Proof. TODO: naturality.

TODO: prove pullback. $\square$

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