## A Groupoidal Natural Model of HoTT in Lean 4

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#### 0.1 Natural Models

In this section we describe the categorical semantics of HoTT via Natural Models. This will not be a detailed account of the syntax of HoTT, but will be a detailed account of what is needed to interpret such syntax. It will follow [Awo17], but with a more compact description of identity types using the technology of polynomial endofunctors, and a universe of small types.

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as **set** and the large sets as **Set**. For example, we could take the small sets **set** to be those in **Set** bounded in cardinality by some inaccessible cardinal.

#### 0.1.1 Types

Let  $\mathbb{C}$  be a small category, i.e. a category whose class of objects is a **set** and whose hom-classes are from **set**. We write  $\mathbf{Psh}(\mathbb{C})$  for the category of presheaves over  $\mathbb{C}$ ,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

**Definition 0.1.1.** Following Awodey [Awo17], we say that a map  $\operatorname{tp}:\operatorname{\mathsf{Tm}}\to\operatorname{\mathsf{Ty}}$  is presentable when any fiber of a representable is representable. In other words, given any  $\Gamma\in\mathbb{C}$  and a map  $A:\operatorname{y}(\Gamma)\to\operatorname{\mathsf{Ty}}$ , there is some representable  $\Gamma\cdot A\in\mathbb{C}$  and maps  $\operatorname{\mathsf{disp}}_A:\Gamma\cdot A\to\Gamma$  and  $\operatorname{\mathsf{var}}_A:\operatorname{y}(\Gamma\cdot A)\to\operatorname{\mathsf{Tm}}$  forming a pullback

$$\begin{array}{ccc} \mathbf{y}(\Gamma \cdot A) & \xrightarrow{\operatorname{var}_A} & \operatorname{Tm} \\ \mathbf{y}(\operatorname{disp}_A) & & & \downarrow \operatorname{tp} \\ \mathbf{y}(\Gamma) & \xrightarrow{A} & \operatorname{Ty} \end{array}$$

The Natural Model associated to a presentable map  $tp: Tm \to Ty$  consists of

- contexts as objects  $\Gamma, \Delta, \dots \in \mathbb{C}$ ,
- a type in context  $y(\Gamma)$  as a map  $A: y(\Gamma) \to Ty$ ,
- a term of type A in context  $\Gamma$  as a map  $a: y(\Gamma) \to Tm$  such that



commutes,

• an operation called "context extension" which given a context  $\Gamma$  and a type  $A\colon \mathsf{y}(\Gamma)\to \mathsf{T}\mathsf{y}$  produces a context  $\Gamma\cdot A$  which fits into a pullback diagram below.

$$\begin{array}{ccc} \mathbf{y}(\Gamma.A) & \longrightarrow \mathbf{Tm} \\ \downarrow & & \downarrow \\ \mathbf{y}(\Gamma) & \longrightarrow_A \mathbf{Ty} \end{array}$$

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**Remark.** Sometimes, we first construct a presheaf X over  $\Gamma$  and observe that it can be classified by a map into Ty. We write

$$\begin{array}{c} X \longrightarrow \mathsf{Tm} \\ \downarrow & \downarrow \\ \mathsf{y}(\Gamma) \longrightarrow \mathsf{Ty} \end{array}$$

to express this situation, i.e.  $X \cong y(\Gamma \cdot \lceil X \rceil)$ .

#### 0.1.2 Pi types

We will use  $\mathsf{Poly}_\mathsf{tp}$  to denote the polynomial endofunctor (definition 0.3.1) associated with our presentable map  $\mathsf{tp}$ . Then an interpretation of  $\Pi$  types consists of a pullback square

$$\begin{array}{c} \mathsf{Poly}_{\mathsf{tp}}\mathsf{Tm} \overset{\lambda}{\longrightarrow} \mathsf{Tm} \\ \\ \mathsf{Poly}_{\mathsf{tp}}\mathsf{tp} \downarrow & & \downarrow \mathsf{tp} \\ \\ \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \overset{\Pi}{\longrightarrow} \mathsf{Ty} \end{array}$$

#### 0.1.3 Sigma types

An interpretation of  $\Sigma$  types consists of a pullback square

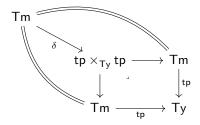
$$\begin{array}{c} Q \xrightarrow{\quad \mathsf{pair} \quad} \mathsf{Tm} \\ \mathsf{tp} \triangleleft \mathsf{tp} \downarrow & \qquad & \downarrow \mathsf{tp} \\ \mathsf{Poly}_\mathsf{tp} \mathsf{Ty} \xrightarrow{\quad \Sigma \quad} \mathsf{Ty} \end{array}$$

#### 0.1.4 Identity types

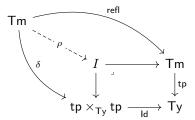
To interpret the formation and introduction rules for identity types we require a commutative square (this need not be pullback)

$$\begin{array}{ccc} \mathsf{Tm} & \xrightarrow{\mathsf{refl}} & \mathsf{Tm} \\ \downarrow^{\delta} & & \downarrow^{\mathsf{tp}} \\ \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} & \xrightarrow{\mathsf{Id}} & \mathsf{Ty} \end{array}$$

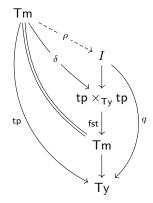
where  $\delta$  is the diagonal:



Then let I be the pullback. We get a comparison map  $\rho$ 

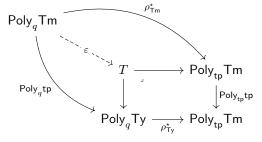


Then view  $\rho : \mathsf{tp} \to q$  as a map in the slice over Ty.



Now (by definition 0.3.6) applying  $\mathsf{Poly}_- : (\mathbf{Psh}(\mathbb{C})/\mathsf{Ty})^{\mathrm{op}} \to [\mathbf{Psh}(\mathbb{C}), \mathbf{Psh}(\mathbb{C})]$  to  $\rho : \mathsf{tp} \to q$  gives us a naturality square (this also need not be pullback).

Taking the pullback T and the comparison map  $\varepsilon$  we have



Finally, we require a section  $J:T\to \mathsf{Poly}_q\mathsf{Tm}$  of  $\varepsilon,$  to interpret the identity elimination rule.

### 0.1.5 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$U \colon \mathsf{Ty} \qquad \frac{a \colon \mathsf{U}}{\mathsf{El}(a) \colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal  $\kappa$ , with  $\kappa < \lambda$ .)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{ccc}
\mathsf{U} & \longrightarrow \mathsf{Tm} \\
\downarrow & & \downarrow \\
1 & \longrightarrow \mathsf{Ty}
\end{array}$$

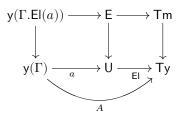
• There is an inclusion of U into Ty

$$\mathsf{EI} \colon \mathsf{U} \rightarrowtail \mathsf{Ty}$$

•  $\pi: \mathsf{E} \to \mathsf{U}$  is obtained as pullback of  $\mathsf{tp}$ ; There is a pullback diagram

$$E \rightarrowtail \mathsf{Tm} \downarrow \downarrow \downarrow \downarrow \\ \mathsf{U} \rightarrowtail \mathsf{Ty}$$

With the notation above, we get



Both squares above are pullback squares.

# 0.1.6 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model

### 0.2 The Groupoid Model

In this section we construct a natural model in **Psh(grpd)** the presheaf category indexed by the category **grpd** of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on **grpd** - which comes from restricting the "classical Quillen model structure" on **cat** [Joy] to **grpd**.

#### 0.2.1 Classifying display maps

*Notation.* We denote the category of small categories as **cat** and the large categories as **Cat**. We denote the category of small groupoids as **grpd**.

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$\mathbf{Psh}(\mathbf{grpd}) = [\mathbf{grpd}^{\mathrm{op}}, \mathbf{Set}]$$

In this section,  $\mathsf{Tm}$  and  $\mathsf{Ty}$  and so on will refer to the natural model semantics in this specific model.

**Definition 0.2.1** (Pointed). We will take the category of pointed small categories  $\mathbf{cat}_{\bullet}$  to have objects as pairs  $(\mathbb{C} \in \mathbf{cat}, c \in \mathbb{C})$  and morphisms as pairs

$$(F: \mathbb{C}_1 \to \mathbb{C}_0, \phi: Fc_1 \to c_0) \colon (\mathbb{C}_1, c_1) \to (\mathbb{C}_0, c_0)$$

Then the category of pointed small groupoids  $\mathbf{grpd}_{\bullet}$  will be the full subcategory of objects  $(\Gamma, c)$  with  $\Gamma$  a groupoid.

**Definition 0.2.2** (The display map classifier). We would like to define a natural transformation in  $\mathbf{Psh}(\mathbf{grpd})$ 

$$\mathsf{tp}\colon \mathsf{Tm}\to \mathsf{Ty}$$

with representable fibers.

Consider the functor that forgets the point

$$U : \mathbf{grpd} \to \mathbf{grpd}$$
 in Cat.

If we apply the Yoneda embedding  $y \colon \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$  to U we obtain

$$U \circ : [-, \mathbf{grpd}_{\bullet}] \to [-, \mathbf{grpd}]$$
 in  $\mathbf{Psh}(\mathbf{Cat})$ .

Since any small groupoid is also a large category  $i: \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ , we can restrict  $\mathbf{Cat}$  indexed presheaves to be  $\mathbf{grpd}$  indexed presheaves (this the nerve in  $i_! \dashv \mathsf{res}$ ). We define  $\mathsf{tp}: \mathsf{Tm} \to \mathsf{Ty}$  as the image of  $U \circ \mathsf{under}$  this restriction.

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) & \xrightarrow{\mathsf{res}} & \mathbf{Psh}(\mathbf{grpd}) \\ \\ \mathbf{grpd} & \longmapsto & [-,\mathbf{grpd}] & \longmapsto & \mathsf{Ty} \end{array}$$

Note that Tm and Ty are not representable in Psh(grpd).

 $Remark\ 0.2.3.$  By Yoneda we can identify maps with representable domain into the type classifier

$$A: y\Gamma \to \mathsf{Ty}$$
 in  $\mathbf{Psh}(\mathbf{grpd})$ 

with functors

$$A:\Gamma \to \mathbf{grpd}$$
 in  $\mathbf{Cat}$ 

**Definition 0.2.4** (Grothendieck construction). From  $\mathbb{C}$  a small category and  $F: \mathbb{C} \to \mathbf{cat}$  a functor, we construct a small category  $\int F$ . For any c in  $\mathbb{C}$  we refer to Fc as the fiber over c. The objects of  $\int F$  consist of pairs  $(c \in \mathbb{C}, x \in Fc)$ , and

morphisms between (c, x) and (d, y) are pairs  $(f: c \to d, \phi: Ffx \to y)$ . This makes the following pullback in **Cat** 

$$(c,x) \longmapsto (Fc,x)$$

$$\begin{array}{cccc} (c,x) & & \int F & \longrightarrow \mathbf{cat}_{\bullet} & & (C,c) \\ & & \downarrow & & \downarrow & & \downarrow \\ c & & \mathbb{C} & \longrightarrow_F \to \mathbf{cat} & & C \\ \end{array}$$

**Definition 0.2.5** (Grothendieck construction for groupoids). Let  $\Gamma$  be a groupoid and  $A \colon \Gamma \to \mathbf{grpd}$  a functor, we can compose F with the inclusion  $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$  and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \operatorname{disp}_A \colon \Gamma \cdot A \to \Gamma$$

This is also a small groupoid since the underlying morphisms are pairs of morphisms from groupoids  $\Gamma$  and Ax for  $x \in \Gamma$ . Furthermore the pullback factors through (pointed) groupoids.

$$egin{array}{cccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{ullet} & \longrightarrow \mathbf{cat}_{ullet} \ \operatorname{disp}_A & & & & \downarrow & & \downarrow \ \Gamma & \longrightarrow_A & \mathbf{grpd} & \longrightarrow \mathbf{cat} \end{array}$$

Corollary 0.2.6 (The display map classifier is presentable). For any small groupoid  $\Gamma$  and  $A: y\Gamma \to Ty$ , the pullback of tp along A can be given by the representable map  $ydisp_A$ .

$$\begin{array}{ccc} \mathbf{y}\Gamma\cdot A & \longrightarrow & \mathsf{Tm} \\ \mathbf{y}\mathsf{disp}_A & & \mathsf{tp} \\ \mathbf{y}\Gamma & \xrightarrow{A} & \mathsf{Ty} \end{array}$$

*Proof.* Consider the pullback in Cat

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow_A & \mathbf{grpd} \end{array}$$

We send this square along res o y in the following

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) \\ & & & \downarrow_{\mathsf{res}} \\ \mathbf{grpd} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{grpd}) \end{array}$$

The Yoneda embedding  $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$  preserves pullbacks, as does res since it is a right adjoint (with left Kan extension  $\iota_1 \dashv \mathsf{res}$ ,).

#### 0.2.2 Groupoid fibrations

**Definition 0.2.7** (Fibration). Let  $p:\mathbb{C}_1\to\mathbb{C}_0$  be a functor. We say p is a split Grothendieck fibration if we have a dependent function lift a f satisfying the following: for any object a in  $\mathbb{C}_1$  and morphism f:p  $a\to y$  in the base  $\mathbb{C}_0$  we have lift a  $f:a\to b$  in  $\mathbb{C}_1$  such that p(lift af)=f and moreover lift a  $g\circ f=\text{lift }b$   $g\circ \text{lift }af$ 

$$\begin{array}{ccc} a & \xrightarrow{\text{lift } a f} & b \\ \downarrow & & \uparrow & \uparrow \\ \downarrow & & \downarrow & \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

In particular, we are intereseted in split Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a split Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid  $\Gamma$  as  $\mathsf{Fib}_{\Gamma}$ , which is a full subcategory of the slice  $\mathsf{grpd}/\Gamma$ . We will decorate an arrow with  $\twoheadrightarrow$  to indicate it is a fibration.

Note that  $\mathsf{disp}_A \colon \Gamma \cdot A \to \Gamma$  is a fibration, since for any  $(x \in \Gamma, a \in Ax)$  and  $f \colon x \to y$  in  $\Gamma$  we have a morphism  $(f, \mathsf{id}_{Afa}) \colon (x, a) \to (y, Afa)$  lifting f. Furthermore

**Proposition 0.2.8.** There is an adjoint equivalence

$$[\Gamma,\mathbf{grpd}] \xrightarrow[\text{fiber}]{\text{disp}} \operatorname{Fib}_{\Gamma}$$

where for each fibration  $\delta: \Delta \to \Gamma$  and each object  $x \in \Gamma$ 

$$fiber_{\delta} x = full subcategory \{ a \in \Delta \mid \delta a = x \}$$

It follows that all fibrations are pullbacks of the classifier  $U: \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$ , when viewed as morphisms in  $\mathbf{Cat}$ .

Pullback of fibrations along groupoid functors is not strictly coherent, in the sense that for  $\tau:\Xi\to\Delta$  and  $\sigma:\Delta\to\Gamma$  and a fibration  $p\in\mathsf{Fib}_\Gamma$  we only have an isomorphism

$$\tau^*\sigma^*p \cong (\sigma \circ \tau)^*p$$

rather than equality.

In order to interpret reindexing/substitution strictly, it is convenient to work with classifiers  $[\Gamma, \mathbf{grpd}]$  instead of fibrations.

**Proposition 0.2.9** (Strictly coherent pullback). Let  $\sigma: \Delta \to \Gamma$  be a functor between groupoids. Since display maps are pullbacks of the classifier  $U: \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$  we have the pasting diagram

This gives us a functor  $\circ \sigma : [\Gamma, \mathbf{grpd}] \to [\Delta, \mathbf{grpd}]$  which is our strict version of pullback.

Corollary 0.2.10 (Fibrations are stable under pullback).

$$\begin{split} [\Gamma, \mathbf{grpd}] &\longleftarrow^{\mathsf{fiber}} & \mathsf{Fib}_{\Gamma} \\ & \circ \sigma \!\!\!\! \downarrow \\ [\Delta, \mathbf{grpd}] &\longleftarrow^{\mathsf{disp}} & \mathsf{Fib}_{\Delta} \end{split}$$

We can deduce a corresponding fact about fibrations: since fibrations are closed under isomorphism, and since any pullback in **grpd** of a fibration p is isomorphic to the display map  $\operatorname{disp}_{\operatorname{fiberpo}\sigma}$ , any pullback of a fibration is a fibration.

A strict interpretation of type theory would require  $\Sigma$  and  $\Pi$ -formers to be stable under pullback (Beck-Chevalley). Thus we again define these as operations on classifiers  $[\Gamma, \mathbf{grpd}]$ .

**Definition 0.2.11** ( $\Sigma$ -former operation). Then given  $A:\Gamma\to\operatorname{\mathbf{grpd}}$  and  $B:\Gamma\cdot A\to\operatorname{\mathbf{grpd}}$  we define  $\Sigma_AB:\Gamma\to\operatorname{\mathbf{grpd}}$  such that  $\Sigma_AB$  acts on objects by forming fiberwise Grothendieck constructions

$$\Sigma_A B(x) := A(x) \cdot B \circ x_A$$

where  $x_A:A(x)\to\Gamma\cdot A$  takes  $f:a_0\to a_1$  to  $(\mathsf{id}_x,f):(x,a_0)\to(x,a_1)$ 

 $\Sigma_A B$  acts on morphism  $f: x \to y$  in  $\Gamma$  and  $(a \in A(x), b \in B(x,a))$  by

$$\Sigma_A B f(a,b) := (A f a, B(f, \mathsf{id}_{A f a}) b)$$

and for morphism  $(\alpha: a_0 \to a_1 \in A(x), \beta: B(\mathsf{id}_x, \alpha) \, b_0 \to b_1 \in B(x, a_1))$  in  $\Sigma_A B x$ 

$$\Sigma_A B f(\alpha, \beta) := (A f \alpha, B(f, id_{A f a_1}) \beta)$$

Let us also define the natural transformation  $fst: \Sigma_A B \to A$  by

$$\mathsf{fst}_x:(a,b)\mapsto a$$

**Proposition 0.2.12** (Fibrations are closed under composition). The corresponding fact about fibrations is that the composition of two fibrations is a fibration.

$$\begin{array}{c} \Xi \\ \downarrow \\ \stackrel{}{\downarrow} \\ \Delta \longrightarrow \end{array} \Gamma$$

We can compare the two fibrations

$$\mathsf{disp}_B \circ \mathsf{disp}_A \qquad \text{and} \qquad \mathsf{disp}_{\Sigma_A(B)}$$

An object in the composition would look like ((x,a),b) for  $x \in \Gamma$ ,  $a \in A(x)$  and  $b \in B(x,a)$ , whereas an object in  $\Gamma \cdot \Sigma_A(B)$  would instead be (x,(a,b)).

**Proposition 0.2.13** (Strict Beck-Chevalley for  $\Sigma$ ). Let  $\sigma : \Delta \to \Gamma$ ,  $A : \Gamma \to \operatorname{\mathbf{grpd}}$  and  $B : \Gamma \cdot A \to \operatorname{\mathbf{grpd}}$ . Then

$$(\Sigma_A B) \circ \sigma = \Sigma_{A \circ \sigma} (B \circ \sigma_A)$$

where  $\sigma_A$  is uniquely determined by the pullback in

*Proof.* By checking pointwise at  $x \in \Delta$ , this boils down to showing

$$A(\sigma x) \xrightarrow[]{(\sigma x)_A} \Delta \cdot A\sigma \xrightarrow[]{\sigma_A} \Gamma.A \xrightarrow[]{B} \mathbf{grpd}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

 $(\sigma x)_A = \sigma_A \circ x_{A \circ \sigma} : A(\sigma x) \to \Gamma \cdot A$ 

which holds because of the universal property of pullback.

**Definition 0.2.14** ( $\Pi$ -former operation). Given  $A:\Gamma\to\operatorname{\mathbf{grpd}}$  and  $B:\Gamma\cdot A\to\operatorname{\mathbf{grpd}}$  we will define  $\Pi_AB:\Gamma\to\operatorname{\mathbf{grpd}}$  such that for any  $C:\Gamma\to\operatorname{\mathbf{grpd}}$  we have an isomorphism

$$[\Gamma \cdot A, \mathbf{grpd}](\mathsf{disp}_A \circ C, B) \cong [\Gamma, \mathbf{grpd}](C, \Pi_A B)$$

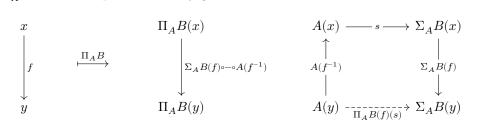
natural in both B and C.

*Proof.*  $\Pi_A B$  acts on objects by taking fiberwise sections

$$\Pi_A B(x) := \{ s \in [A(x), \Sigma_A B(x)] \mid \mathsf{fst}_x \circ s = \mathsf{id}_{A(x)} \}$$

Where we have taken the full subcategory of the functor category  $[A(x), \Sigma_A B(x)]$ . This is a groupoid since any natural transformation of functors into groupoids are natural isomorphisms.

 $\Pi_A B$  acts on morphisms via conjugation



Note that conjugation is functorial and invertible.

Corollary 0.2.15 (Fibrations are closed under pushforward). Stated in terms of fibrations, we have

$$\begin{array}{ccc}
\Xi & \Gamma_! \sigma_* \tau \\
\downarrow^{\tau} & \downarrow^{\sigma_* \tau} \\
\Delta & \xrightarrow{\sigma} & \Gamma
\end{array}$$

with the universal property of pushforward

$$\mathsf{Fib}_{\Delta}(\sigma^*\rho,\tau) \cong \mathsf{Fib}_{\Gamma}(\rho,\sigma_*\tau)$$

natural in both  $\tau$  and  $\rho$ .

**Proposition 0.2.16** (Strict Beck-Chevalley for  $\Pi$ ). Let  $\sigma : \Delta \to \Gamma$ ,  $A : \Gamma \to \operatorname{\mathbf{grpd}}$  and  $B : \Gamma \cdot A \to \operatorname{\mathbf{grpd}}$ . Then

$$(\Pi_A B) \circ \sigma = \Pi_{A \circ \sigma} (B \circ \sigma_A)$$

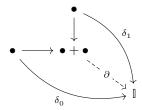
where  $\sigma_A$  is uniquely determined by the pullback in

*Proof.* By checking pointwise, this boils down to Beck-Chevalley for  $\Sigma$ .

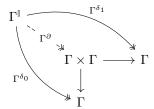
**Proposition 0.2.17** (All objects are fibrant). Let  $\bullet$  denote the terminal groupoid, namely that with a single object and morphism. Then the unique map  $\Gamma \to \bullet$  is a fibration.

**Definition 0.2.18** (Interval). Let the interval groupoid  $\mathbb{I}$  be the small groupoid with two objects and a single non-identity isomorphism. There are two distinct morphisms  $\delta_0, \delta_1 : \bullet \to \mathbb{I}$  and a natural isomorphism  $i : \delta_0 \Rightarrow \delta_1$ . Note that  $\delta_0$  and  $\delta_1$  both form adjoint equivalences with the unique map  $! : \mathbb{I} \to \bullet$ .

Denote by  $\bullet + \bullet$  the small groupoid with two objects and only identity morphisms. Then let  $\partial : \bullet + \bullet \to \mathbb{I}$  be the unique map factoring  $\delta_0$  and  $\delta_1$ .



**Proposition 0.2.19** (Path object fibration). Let  $\Gamma$  be a small groupoid. Recall that  $\operatorname{\mathbf{grpd}}$  is Cartesian closed, so we can take the image of the above diagram under the functor  $\Gamma^-$ .



Then the indicated morphisms are fibrations, and  $\Gamma^{\delta_0}$ ,  $\Gamma^{\delta_1}$  form adjoint equivalences with  $\Gamma^!:\Gamma\to\Gamma^{\mathbb{I}}$ .

#### 0.2.3 Classifying type dependency

**Proposition 0.2.20** (Poly<sub>tp</sub> classifies type dependency). Specialized to tp: Tm  $\rightarrow$  Ty in Psh(grpd), the characterizing property of polynomial endofunctors proposition 0.3.2 says that a map from a representable  $\Gamma \rightarrow \mathsf{Poly}_{\mathsf{tp}} X$  corresponds to the data of

$$A:\Gamma \to \mathsf{Ty} \qquad and \qquad B:\Gamma \cdot A \to X$$

The special case of when X is also Ty gives us a classifier for dependent types; by Yoneda the above corresponds to the data in  $\mathbf{Cat}$  of

$$A:\Gamma \to \mathbf{grpd}$$
 and  $B:\Gamma \cdot A \to \mathbf{grpd}$ 

Furthermore, precomposition by a substitution  $\sigma: \Delta \to \Gamma$  acts on such a pair by

$$\begin{array}{c|c} \Delta & & \\ \sigma \downarrow & & \\ \Gamma & \xrightarrow[(A,B)]{} \mathsf{Poly}_{\mathsf{tp}} X \end{array}$$

where  $tp^*\sigma$  is given by

#### 0.2.4 Pi and Sigma structure

**Lemma 0.2.21.**  $X \in \mathbf{Psh}(\mathbf{grpd})$  be a presheaf. Let F be an operation that takes a groupoid  $\Gamma$ , a functor  $A : \Gamma \to \mathbf{grpd}$  and  $B : \Gamma \cdot A \to X$  and returns a natural transformation  $F_AB : \Gamma \to X$ .

Then using Yoneda to define  $\tilde{F}: \mathsf{Poly}_{\mathsf{tp}} X \to X$  pointwise as

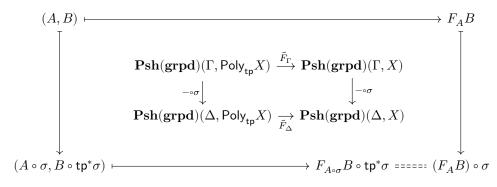
$$\begin{split} \tilde{F}_{\Gamma}: \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathsf{Poly}_{\mathsf{tp}}X) &\to \mathbf{Psh}(\mathbf{grpd})(\Gamma, X) \\ (A, B) &\mapsto F_A B \end{split}$$

gives us a natural transformation if and only if F satisfies the strict Beck-Chevalley condition

$$(F_AB)\circ\sigma=F_{A\circ\sigma}(B\circ\operatorname{tp}^*\sigma)$$

for every  $\sigma: \Delta \to \Gamma$  in **grpd**.

*Proof.* Using proposition 0.2.20



**Definition 0.2.22** (Interpretation of  $\Pi$  types). We define the natural transformation  $\Pi: \mathsf{Poly}_\mathsf{tp}\mathsf{Ty} \to \mathsf{Ty}$  as that which is induced (lemma 0.2.21) by the  $\Pi$ -former operation (definition 0.2.14).

Then we define the natural transformation  $\lambda: \mathsf{Poly}_\mathsf{tp}\mathsf{Ty} \to \mathsf{Ty}$  as the natural transformation induced by the following operation: given  $A: \Gamma \to \mathbf{grpd}$  and  $\beta: \Gamma \cdot A \to \mathbf{grpd}_\bullet$ ,  $\lambda_A \beta: \Gamma \to \mathbf{grpd}_\bullet$  will be the functor such that on objects  $x \in \Gamma$ 

$$\lambda_{A}\beta\left(x\right):=\left(\Pi_{A}B\left(x\right),a\mapsto\left(a,b(x,a)\right)\right)$$

where  $B := U \circ \beta : \Gamma \cdot A \to \mathbf{grpd}$  and b(x,a) is the point in  $\beta(x,a)$ . On morphisms  $f : x \to y$  in  $\Gamma$  we have

$$\lambda_{A}\beta\left(f\right):=\left(\Pi_{A}B\left(f\right),\eta\right)$$

where  $\eta: \Pi_A B \, f \, s_x \to s_y$  is a natural isomorphism between functors  $A_y \to \Sigma_A B y$  given on objects  $a \in A_y$  by

$$\eta_a := (\mathsf{id}_a, \mathsf{id}_{b(u,a)})$$

These combine to give us a pullback square

$$\begin{array}{c} \mathsf{Poly}_{\mathsf{tp}}\mathsf{Tm} \overset{\lambda}{\longrightarrow} \mathsf{Tm} \\ \\ \mathsf{Poly}_{\mathsf{tp}}\mathsf{tp} \Big\downarrow & & \downarrow \mathsf{tp} \\ \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \overset{\Pi}{\longrightarrow} \mathsf{Ty} \end{array}$$

Г

*Proof.* We should check that the  $\lambda$  operation satisfied Beck-Chevalley. This follows from the  $\Pi$  satisfying Beck-Chevalley and extensionality results for functors.

The square commutes and is a pullback if and only it pointwise commutes and pointwise gives pullbacks, i.e. for each groupoid  $\Gamma$ 

$$(A,\beta) \longmapsto \lambda_{A}\beta$$

$$\begin{array}{c} \mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_{\mathsf{tp}}\mathsf{Tm}) \stackrel{\lambda_{\Gamma}}{\longrightarrow} [\Gamma,\mathbf{grpd}_{\bullet}] \\ \mathbb{P}\mathbf{sh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_{\mathsf{tp}}\mathsf{tp}) \downarrow & \downarrow_{U \circ -} \\ \mathbb{P}\mathbf{sh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty}) \stackrel{\Gamma}{\longrightarrow} [\Gamma,\mathbf{grpd}] \\ \downarrow (A,U \circ \beta) \longmapsto \Pi_{\Gamma}U \circ \beta = U \circ \lambda_{A}\beta \end{array}$$

where we have used proposition 0.2.20. That this commutes follows from the definitions of  $\Pi$  and  $\lambda$ .

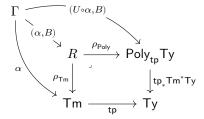
To show it is pullback it suffices to note that for any  $f:\Gamma\to\operatorname{\mathbf{grpd}}_{\bullet}$  and  $(A,B):\Gamma\to\operatorname{\mathsf{Poly}}_{\operatorname{\mathsf{tp}}}\mathsf{Ty}$  such that  $U\circ f=\Pi_A B$ , there exists a unique  $(A,\beta):\Gamma\to\operatorname{\mathsf{Poly}}_{\operatorname{\mathsf{tp}}}\mathsf{Tm}$  such that  $U\circ\beta=B$  and  $\lambda_A\beta=f$ . Indeed  $\beta$  is fully determined by the above conditions to be

$$\beta: \Gamma \cdot A \to \mathbf{grpd}_{\bullet}$$
  
 $(x, a) \mapsto (B(x, a), f x a)$ 

**Lemma 0.2.23.** This is a specialization of lemma 0.3.3. Use R to denote the fiber product

$$\begin{array}{ccc} R \xrightarrow{\rho_{\mathsf{Poly}}} \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \\ & & \downarrow^{\mathsf{tp}^*\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty} = \rho_{\mathsf{Tm}}} & & \downarrow^{\mathsf{tp}_*\mathsf{Tm}^*\mathsf{Ty}} \\ & & \mathsf{Tm} \xrightarrow{\mathsf{tn}} & \mathsf{Ty} \end{array}$$

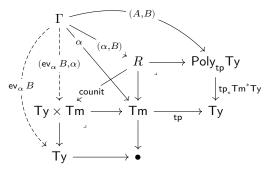
By the universal property of pullbacks, The data of a map from a respresentable  $\varepsilon:\Gamma\to R$  corresponds to the data of  $\alpha:\Gamma\to \mathsf{Tm}$  and  $(U\circ\alpha,B):\Gamma\to \mathsf{Poly}_\mathsf{tp}\mathsf{Ty}.$  Then by proposition 0.2.20 this corresponds to the data of  $\alpha:\Gamma\to \mathsf{Tm}$  and  $B:\Gamma\cdot U\circ\alpha\to \mathsf{Ty}.$ 



Precomposition by a substitution  $\sigma: \Delta \to \Gamma$  then acts on such a pair by

$$\begin{array}{c|c}
\Delta \\
\sigma \downarrow \\
\Gamma \xrightarrow{(\alpha \circ \sigma, B \circ \mathsf{tp}^* \sigma)} R
\end{array}$$

**Definition 0.2.24** (Evaluation). Define the operation of evaluation  $\operatorname{ev}_{\alpha} B$  to take  $\alpha:\Gamma\to\operatorname{\mathbf{grpd}}_{\bullet}$  and  $B:\Gamma\cdot U\circ\alpha\to\operatorname{\mathbf{grpd}}$  and return  $\operatorname{\mathbf{ev}}_{\alpha} B:\Gamma\to\operatorname{\mathbf{grpd}}$ , described below.



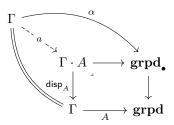
where we write  $A := U \circ \alpha$  and treat a map  $\Gamma \to \mathbf{grpd}$  as the same as a map  $\Gamma \to \mathsf{Ty}$ . More concisely, evaluation is a natural transformation  $\mathsf{ev} : R \to \mathsf{Ty}$ , given by

$$\mathsf{ev} \, = \pi_\mathsf{Tv} \circ \mathsf{counit}$$

**Lemma 0.2.25.** The functor  $ev_{\alpha}B:\Gamma\to \mathbf{grpd}$  can be computed as

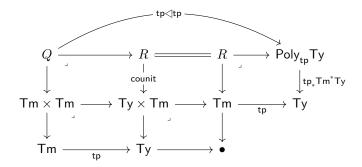
$$ev_{\alpha} B = B \circ a$$

where



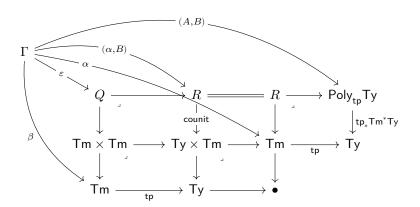
*Proof.* This is a specialization of lemma 0.3.4 with liberal applications of Yoneda.  $\Box$ 

**Definition 0.2.26** (Classifier for dependent pairs). Recall the following definition of composition of polynomial endofunctors, specialized to our situation



By the universal property of pullbacks, the data of a map with representable domain  $\varepsilon:\Gamma\to Q$  corresponds to the data of a triple of maps  $\alpha,\beta:\Gamma\to \mathsf{Tm}$  and (A,B):

 $\Gamma \to \mathsf{Poly}_{\mathsf{tp}}\mathsf{Ty} \text{ such that } \mathsf{tp} \circ \beta = \pi_{\mathsf{Ty}} \circ \mathsf{counit} \, \circ (\alpha, B) \text{ and } A = \mathsf{tp} \circ \alpha.$ 



This in turn corresponds to three functors  $\alpha, \beta : \Gamma \to \mathbf{grpd}_{\bullet}$  and  $B : \Gamma \cdot U \circ \alpha \to \mathbf{grpd}$ , such that  $U \circ \beta = \mathsf{ev}_{\alpha} B$ . So we will write

$$\varepsilon = (\beta, \alpha, B)$$

Type theoretically  $\alpha = (A, a:A)$  and  $\operatorname{ev}_{\alpha} B = Ba$  and  $\beta = (Ba, b:Ba)$ . Then composing  $\varepsilon$  with  $\operatorname{tp} \triangleleft \operatorname{tp}$  returns  $\gamma$ , which consists of (A,B). It is in this sense that Q classifies pairs of dependent terms, and  $\operatorname{tp} \triangleleft \operatorname{tp}$  extracts the underlying types.

Precomposition with a substitution  $\sigma: \Delta \to \Gamma$  acts on this triple by

$$\begin{array}{c|c} \Delta & \\ \sigma & \\ \hline \Gamma & \\ \hline (\beta \circ \sigma, \alpha \circ \sigma, B \circ \operatorname{tp}^* \sigma) \\ \hline \Gamma & \\ (\beta, \alpha, B) & Q \end{array}$$

**Definition 0.2.27** (Interpretation of  $\Sigma$ ). We define the natural transformation

$$\Sigma:\mathsf{Poly}_{\mathsf{tn}}\mathsf{Ty}\to\mathsf{Ty}$$

as that which is induced (lemma 0.2.21) by the  $\Sigma$ -former operation (definition 0.2.14).

To define pair  $: Q \to \mathsf{Tm}$ , let  $\Gamma$  be a groupoid and  $(\beta, \alpha, B) : \Gamma \to Q$  (such that  $U \circ \beta = \mathsf{ev}_\alpha \beta$ ). We define a functor  $\mathsf{pair}_\Gamma(\beta, \alpha, B) : \Gamma \to \mathsf{grpd}_\bullet$  such that on objects  $x \in \Gamma$ , the functor returns  $(\Sigma_A B \, x, (a_x, b_{a_x}))$ , where (using lemma 0.2.25  $U \circ \beta x = \mathsf{ev}_\alpha \, B x = B(x, a_x)$ )

$$\alpha x = (A x, a_x)$$
 and  $\beta x = (B(x, a_x), b_a)$ 

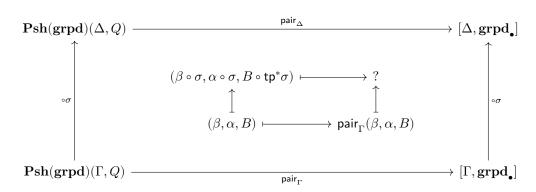
and on morphisms  $f:x\to y$ , the functor returns  $(\Sigma_A B\,f,(\phi_f,\psi_f))$ , where (using lemma 0.2.25  $U\circ\beta f=\operatorname{ev}_\alpha Bf=B(f,\phi_f))$ 

$$\alpha\,f = (A\,f,\phi_f\!\colon A\,f\,a_x \to a_y) \quad \text{ and } \quad \beta\,f = (B(f,\phi_f),\psi_f\!\colon B(f,\phi_f)\,b_{a_x} \to b_{a_y})$$

 $\Sigma$  and pair combine to give us a pullback square

$$\begin{array}{c} Q \xrightarrow{\quad \mathsf{pair} \quad} \mathsf{Tm} \\ \underset{\mathsf{tp} \lhd \mathsf{tp}}{\bigvee} & & & \downarrow \mathsf{tp} \\ \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty} \xrightarrow{\quad \Sigma \quad} \mathsf{Ty} \end{array}$$

*Proof.* To show naturality of pair, suppose  $\sigma: \Delta \to \Gamma$  is a functor between groupoids.



So we check that for any  $x \in \Gamma$ ,

$$\begin{split} & \operatorname{pair}_{\Delta}(\beta \circ \sigma, \alpha \circ \sigma, B \circ \sigma_{A}) \, x \\ &= \left( \Sigma_{A \circ \sigma} B \circ \sigma_{A} \, x, (a_{x}, b_{a_{x}}) \right) \\ &= \left( \left( \Sigma_{A} B \right) \circ \sigma \, x, (a_{x}, b_{a_{x}}) \right) \\ &= \operatorname{pair}_{\Gamma}(\beta, \alpha, B) \circ \sigma \, x \end{split}$$

where

$$\alpha \circ \sigma \, x = (A \circ \sigma x, a_x) \quad \text{and} \quad \beta \circ \sigma \, x = (\mathsf{ev}_\alpha \, B \circ \sigma \, x, b_{a_x})$$

and so on.

It follows from the definition of pair that the square commutes. To show that it is pullback, it suffices to show that for each  $\Gamma$ ,

$$\begin{split} \mathbf{Psh}(\mathbf{grpd})(\Gamma, Q) & \xrightarrow{\quad \mathsf{pair}_{\Gamma} \quad} [\Gamma, \mathbf{grpd}_{\bullet}] \\ & \downarrow_{\mathsf{tp} \lhd \mathsf{tp} \circ -} \hspace{-0.5cm} \downarrow U^{\circ -} \\ \mathbf{Psh}(\mathbf{grpd})(\Gamma, \mathsf{Poly}_{\mathsf{tp}} \mathsf{Ty}) & \xrightarrow{\Sigma_{\Gamma}} [\Gamma, \mathbf{grpd}] \end{split}$$

is a pullback. Since we are in **Set**, it suffices to just show the universal property applied to a point: so for any  $A:\Gamma\to\mathbf{grpd}$ , any  $B:\Gamma\cdot A\to\mathbf{grpd}$ , and any  $p:\Gamma\to\mathbf{grpd}_{\bullet}$ , such that

$$U \circ p = \Sigma_{\Gamma}(A, B)$$

there exists a unique  $(\beta, \alpha, B) : \Gamma \to Q$  such that

$$\mathsf{pair}_{\Gamma}(\beta,\alpha,B) = p \quad \text{and} \quad \mathsf{tp} \lhd \mathsf{tp} \circ (B,\alpha,B) = (A,B)$$

Indeed if we write

$$px = (\Sigma_A Bx, (a_x \in Ax, b_x \in B(x, a_x)))$$

this uniquely determines  $\alpha$  and  $\beta$  as

$$\alpha x = (Ax, a_x)$$
 and  $\beta x = (ev_\alpha Bx, b_x)$ 

#### 0.2.5 Identity types

**Definition 0.2.28** (Identity formation and introduction). To define the commutative square in **Psh**(**grpd**)

$$\begin{array}{ccc} \mathsf{Tm} & \xrightarrow{\mathsf{refl}} & \mathsf{Tm} \\ \downarrow^{\delta} & & \downarrow^{\mathsf{tp}} \\ \mathsf{tp} \times_{\mathsf{Ty}} \mathsf{tp} & \xrightarrow{\mathsf{Id}} & \mathsf{Ty} \end{array}$$

We first note that both  $\delta$  and tp in the are in the essential image of the composition from definition 0.2.2

$$\mathbf{Cat} \xrightarrow{\mathsf{y}} \mathbf{Psh}(\mathbf{Cat}) \xrightarrow{\mathsf{res}} \mathbf{Psh}(\mathbf{grpd})$$

since the composition preserves pullbacks. So we first define in Cat

$$\begin{array}{ccc} \mathbf{grpd}_{\bullet} & \xrightarrow{\mathsf{refl}'} & \mathbf{grpd}_{\bullet} \\ \delta \downarrow & & \downarrow U \\ U \times_{\mathbf{grpd}} U & \xrightarrow{\mathsf{ld}'} & \mathbf{grpd} \end{array} \tag{0.2.1}$$

Then obtain Id and refl in Psh(grpd) by applying res  $\circ$  y to this diagram.

To this end, let  $\mathsf{Id}': U \times_{\mathbf{grpd}} U \to \mathbf{grpd}$  act on objects by taking the set - the discrete groupoid - of isomorphisms

$$(A,a_0,a_1)\mapsto A(a_0,a_1)$$

and on morphisms  $(f, \phi_0, \phi_1): (A, a_0, a_1) \to (B, b_0, b_1)$  by

$$(f: A \to B, \phi_0: fa_0 \to b_0, \phi_1: fa_1 \to b_1) \mapsto \phi_1 \circ f(-) \circ \phi_0^{-1}$$

Let  $\operatorname{refl}' : \operatorname{\mathbf{grpd}}_{\bullet} \to \operatorname{\mathbf{grpd}}_{\bullet}$  act on objects by

$$(A, a) \mapsto (A(a, a), \mathsf{id}_a)$$

and on morphisms  $(f, \phi): (A, a) \to (B, b)$  by

$$(f: A \to B, \phi: (A, a) \to (B, b)) \mapsto (\phi \circ f(-) \circ \phi^{-1}, -)$$

where the second component has to be the identity on the object  $\mathsf{id}_d$ , since B(b,b) is a discrete groupoid. So we need a merely propositional proof that the two maps are equal, indeed

$$\phi \circ f(\mathsf{id}_a) \circ \phi^{-1} = \mathsf{id}_b$$

*Proof.* Since  $\delta(A, a) = (A, a, a)$ , it follows that the square in eq. (0.2.1) commutes.

## 0.3 Polynomial Endofunctors

In this section we develop some of the definitions and lemmas related to polynomial endofunctors that we will use in the rest of the notes.

**Definition 0.3.1** (Polynomial endofunctor). Let  $\mathbb{C}$  be a locally Cartesian closed category (in our case, presheaves on the category of contexts). This means for each morphism  $t: B \to A$  we have an adjoint triple

where  $t^*$  is pullback, and  $t_1$  is composition with t.

Let  $t:B\to A$  be a morphism in  $\mathbb C.$  Then define  $\mathsf{Poly}_t:\mathbb C\to\mathbb C$  be the composition

$$\mathsf{Poly}_t := A_! \circ t_* \circ B^* \qquad \qquad \mathbb{C} \xrightarrow{B^*} \mathbb{C}/B \xrightarrow{t_*} \mathbb{C}/A \xrightarrow{A_!} \mathbb{C}$$

**Proposition 0.3.2** (Characterising property of Polynomial Endofunctors). The data of a map into the polynomial applied to an object in  $\mathbb{C}$ 

$$\Gamma \longrightarrow \mathsf{Poly}_{t}Y$$

corresponds to

$$\Gamma \xrightarrow{\phi} \operatorname{Poly}_t Y$$

Applying the adjunction  $A_! \dashv A^*$ , this corresponds to

$$\alpha:\Gamma\to A \qquad \text{ and } \qquad \underbrace{ B_!t^*\alpha - \cdots - \tilde{\phi} \atop t^*\alpha }_{B} \xrightarrow{B^*Y} B \times Y$$

Applying the adjunction  $t^* \dashv t_*$ , this corresponds to

$$\alpha: \Gamma \to A$$
 and  $\Gamma \cdot \alpha := B_1 t^* \alpha \xrightarrow{\beta} Y$ 

Henceforth we will write

$$(\alpha,\beta):\Gamma\to \mathsf{Poly}_{{}_{\mathsf{t}}}Y$$

for this map, since it is uniquely determined by this data. Furthermore, precomposition by  $\sigma:\Delta\to\Gamma$ , acts on such a pair by

$$\begin{array}{c|c} \Delta & & \\ \sigma \downarrow & & \\ \Gamma & \xrightarrow[(\alpha,\beta)]{} \operatorname{Poly}_t Y \end{array}$$

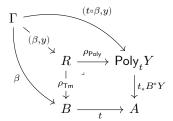
and given a morphism  $f: X \to Y$ , the morphism  $\mathsf{Poly}_{t} f$  acts on such a pair by

$$\Gamma \xrightarrow{(\alpha,\beta)} \mathsf{Poly}_t X \\ \underset{(\alpha,f \circ \beta)}{\underbrace{\hspace{1cm}}} \mathsf{Poly}_t f$$
 
$$\mathsf{Poly}_t Y$$

**Lemma 0.3.3.** Use R to denote the fiber product

$$\begin{array}{c} R \xrightarrow{\rho_{\mathsf{Poly}}} \mathsf{Poly}_t Y \\ t^*t_*B^*Y = \rho_{\mathsf{Tm}} \bigg| \qquad \qquad \qquad \downarrow t_*B^*Y \\ B \xrightarrow[t]{} A \end{array}$$

By the universal property of pullbacks and proposition 0.3.2, The data of a map  $\Gamma \to R$  corresponds to the data of  $\beta: \Gamma \to B$  and  $(t \circ \beta, y): \Gamma \to \mathsf{Poly}_t Y,$  or just  $\beta: \Gamma \to B$  and  $y: \Gamma \cdot t \circ \beta \to Y$ 



By uniqueness in the universal property of pullbacks and proposition 0.3.2, Precomposition by a map  $\sigma: \Delta \to \Gamma$  acts on such a pair by

$$\begin{array}{c|c}
\Delta \\
\sigma \downarrow \\
\Gamma \xrightarrow{(\beta \circ \sigma, y \circ t^* \sigma)} R
\end{array}$$

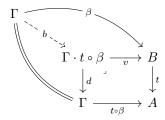
**Lemma 0.3.4** (Evaluation). Suppose  $(\beta, y) : \Gamma \to R$ , as in lemma 0.3.3

$$\beta: \Gamma \to B$$
 and  $y: \Gamma \cdot t \circ \beta \to Y$ 

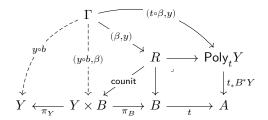
Then the evaluation of y at  $\beta$  can be described in the following two ways

$$y \circ b = \pi_V \circ \mathsf{counit} \circ (\beta, y)$$

where



and



*Proof.* It suffices to show (counit  $\circ (\beta, y)$ ) =  $(y \circ b, \beta)$  instead.

$$\begin{array}{lll} \operatorname{counit} \, \circ \, (\beta,y) \\ = & \operatorname{counit} \, \circ \, (v \circ b, y \circ t^* d \circ t^* b) & fig. \ 1 \\ = & \operatorname{counit} \, \circ \, (v, y \circ t^* d) \circ b & lemma \ 0.3.3 and \ fig. \ 2 \\ = & \operatorname{counit} \, \circ \, t^* (t \circ \beta, y) \circ b & fig. \ 3 \\ = & \overline{(t \circ \beta, y)} \circ b & fig. \ 4 \\ = & (y, v) \circ b & fig. \ 5 \\ = & (y \circ b, v \circ b) \\ = & (y \circ b, \beta) & \end{array}$$

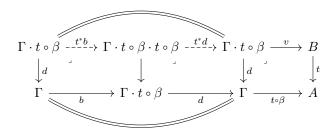


Figure 1:  $t^*d \circ t^*b = \mathsf{id}_{\Gamma \cdot t \circ \beta}$ 

$$\begin{array}{c|c} \Gamma \\ \downarrow \\ \Gamma \cdot t \circ \beta \xrightarrow{(v \cdot b, y \circ t^* d \circ t^* b)} R \end{array}$$

Figure 2:  $(v, y \circ t^*d) \circ b = (v \circ b, y \circ t^*d \circ t^*b)$ 

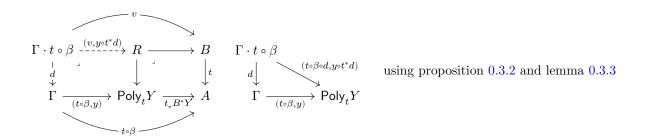


Figure 3:  $t^*(t \circ \beta, y) = (v, y \circ t^*d)$ 

$$\begin{array}{c|cccc} t^*(t\circ\beta) & & & t\circ\beta \\ \hline t^*(t\circ\beta,y) & & & & & & \\ t^*t_*B^*Y \xrightarrow[\text{counit}]{(t\circ\beta,y)} & & & & & \\ t^* - t_* & & & & t_*B^*Y \xrightarrow[\text{counit}]{(t\circ\beta,y)} & \\ \hline \end{array}$$

Figure 4: counit  $\circ t^*(t \circ \beta, y) = \overline{(t \circ \beta, y)}$ 

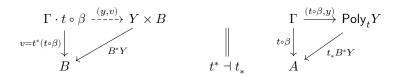
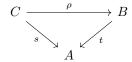
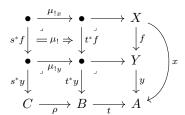


Figure 5:  $\overline{(t \circ \beta, y)} = (y, v)$ 

#### **Definition 0.3.5.** Suppose



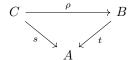
Then we have a mate  $\mu_1: \rho_! \circ s^* \Rightarrow t^*$ . This is given by the universal property of pullbacks: given  $f: x \to y$  in the slice  $\mathbb{C}/A$  we have



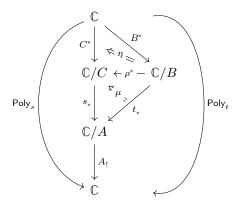
By the calculus of mates we also have a reversed mate between the right adjoints  $\mu^*:t_*\to s_*\circ \rho^*$ . Explicitly  $\mu^*$  is the composition

$$t_* \xrightarrow{\mathsf{unit}\, t_*} s_* \rho^* \rho_! s^* t_* \xrightarrow{s_* \rho^* \mu_! t_*} s_* \rho^* t^* t_* \xrightarrow{s_* \rho^*} \underset{\longrightarrow}{\operatorname{counit}} s_* \rho^*$$

**Definition 0.3.6** (Poly\_ action on a slice morphism). Let Poly\_ :  $(\mathbb{C}/A)^{\mathrm{op}} \to [\mathbb{C}, \mathbb{C}]$  be defined by taking  $s \mapsto \mathsf{Poly}_s$  on objects and act on a morphism



by  $\rho\mapsto \rho^\star:=A_!(s_*\eta\circ\mu B^*):\mathsf{Poly}_*\mathsf{tpPoly}_*$ 



where  $\mu = \mu^*$  is the mate from definition 0.3.5, and  $\eta$  is the natural isomorphism given by pullback pasting.

Pointwise, this natural transformation acts on a pair  $(\alpha, \beta) : \Gamma \to \mathsf{Poly}_t X$  by

$$\Gamma \xrightarrow{(\alpha,\beta)} \operatorname{Poly}_t X \\ \xrightarrow{(\alpha,\beta \circ \alpha^* \rho)} \qquad \downarrow \rho_X^{\star} \\ \operatorname{Poly}_{\circ} X$$

where  $\alpha^* \rho$  is defined as

$$\begin{array}{ccc} \Gamma \cdot_s \alpha & \xrightarrow{s^* \alpha} & C \\ \alpha^* \rho \downarrow & & \downarrow \rho \\ \Gamma \cdot_t \alpha & \xrightarrow{t^* \alpha} & B \\ \downarrow & & \downarrow t \\ \Gamma & \longrightarrow & A \end{array}$$

We prove this now.

*Proof.* Firstly  $\rho_X^\star = A_!(s_*\eta_X\circ\mu_{B^*X})$ , so the first component  $\alpha:\Gamma\to A$  is preserved by  $\rho_X^\star$  and it suffices to show, in  $\mathbb{C}/A$ 

$$\alpha \xrightarrow{(\alpha,\beta)} t_*B^*X$$
 
$$\downarrow s_*\eta_X \circ \mu_{B^*X}$$
 
$$s_*C^*X$$

By the adjunction  $s^*\dashv s_*,$  it suffices to show, in  $\mathbb{C}/C$ 

Now we calculate  $\overline{s_*\eta_X\circ\mu_{B^*X}}=\eta_X\circ\overline{\mu_{B^*X}}.$  So that our goal is to show

$$s^*\alpha \xrightarrow{s^*(\alpha,\beta)} s^*t_*B^*X \xrightarrow{\overline{\mu_{B^*X}}} \rho^*B^*X$$

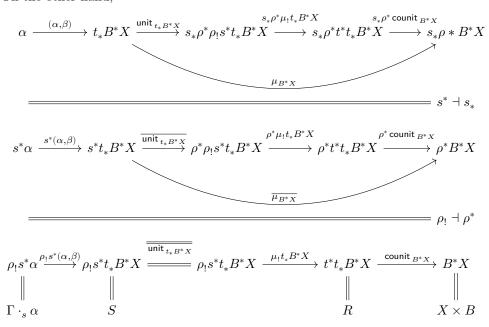
$$\xrightarrow{(\alpha,\beta\circ\alpha^*\rho)} C^*X$$

Since  $\eta_X$  is an isomorphism between two limits of the same diagram, namely  $X \times C \cong C_!C^*X \cong C_!\rho^*B^*X$ , it suffices to show that both  $\overline{\mu_{B^*X}} \circ s^*(\alpha,\beta)$  and  $\overline{(\alpha,\beta \circ \alpha^*\rho)}$  are uniquely determined by the same two maps into X and C.

By the characterising property of polynomial endofunctors (proposition 0.3.2) we calculate

More formally, this means  $\beta \circ \alpha^* \rho : C_! s^* \alpha \to X$  and  $s^* \alpha : C_! s^* \alpha \to C$  are the two maps that uniquely determine the map  $C_! \overline{\alpha, \beta \circ \alpha^* \rho} : C_! s^* \alpha \to X \times C$ .

On the other hand,



The mate  $\mu_1$  is calculated via the universal map into the pullback R (dotted below).

$$\begin{array}{c|c} \Gamma \cdot_s \alpha & \longrightarrow \Gamma \cdot_t \alpha & \longrightarrow \Gamma \\ \hline s^*(\alpha,\beta) & \downarrow & \downarrow (\alpha,\beta) \\ S \xrightarrow[\mu_! t_* B^* X]{} & R \xrightarrow[]{} \operatorname{Poly}_t X \\ \hline s^* t_* B^* X & \downarrow t_* B^* X & \downarrow t_* B^* X \\ \hline C \xrightarrow[\rho]{} & B \xrightarrow[]{} & A \end{array}$$

Using the characterization of maps into R from lemma 0.3.3 we can calculate

$$\mu_1 t_* B^* X \circ s^* (\alpha, \beta) = (\rho \circ s^* \alpha, \beta \circ t^* \alpha^* s)$$

since the first component is simply the map  $\Gamma \cdot_s \alpha \to B$  and the second component is the second component of the map

$$(\alpha \circ \alpha^* s, \beta \circ t^* \alpha^* s) = (\alpha, \beta) \circ \alpha^* s : \Gamma \cdot_s \alpha \to \mathsf{Poly}_t X$$

Then using lemma 0.3.4

$$\overline{\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)} \tag{0.3.1}$$

$$= \operatorname{counit}_{B^*X} \circ \mu_! t_* B^*X \circ s^*(\alpha, \beta) \tag{0.3.2}$$

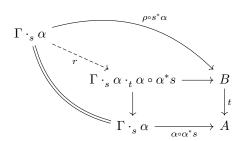
$$= \operatorname{counit}_{B^*X} \circ (\rho \circ s^*\alpha, \beta \circ t^*\alpha^*s) \tag{0.3.3}$$

$$= (\beta \circ t^* \alpha^* s \circ r, \rho \circ s^* \alpha) \tag{0.3.4}$$

$$= (\beta \circ \alpha^* \rho, \rho \circ s^* \alpha) \tag{0.3.5}$$

$$:\Gamma \cdot_s \alpha \to X \times B \tag{0.3.6}$$

where



and

$$\Gamma \cdot_{s} \alpha = \Gamma \cdot_{s} \alpha \xrightarrow{s^{*}\alpha} C$$

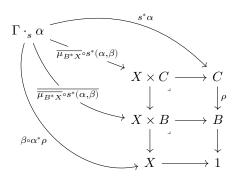
$$\uparrow \downarrow \qquad \qquad \downarrow \alpha^{*}\rho \qquad \downarrow \rho$$

$$\Gamma \cdot_{s} \alpha \cdot_{t} \alpha \circ \alpha^{*}s \xrightarrow[t^{*}\alpha^{*}s]{} \Gamma \cdot_{t} \alpha \longrightarrow B$$

$$\downarrow \qquad \qquad \downarrow \qquad \downarrow t$$

$$\Gamma \cdot_{s} \alpha \longrightarrow \Gamma \longrightarrow A$$

Moving back along the adjunction  $\rho_! \dashv \rho^*$  eq. (0.3.1) tells us that



So that, as required,  $\overline{\mu_{B^*X}} \circ s^*(\alpha, \beta)$  and  $\overline{(\alpha, \beta \circ \alpha^* \rho)}$  are uniquely determined by the same two maps into X and C.

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