Universe in the Natural Model of Type Theory

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1 Types

Assume an inaccessible cardinal λ . Write **Set** for the category of all sets. Say that a set A is λ -small if $|A| < \lambda$. Write **Set** $_{\lambda}$ for the full subcategory of **Set** spanned by λ -small sets.

Let \mathbb{C} be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write $\mathbf{Psh}(\mathbb{C})$ for the category of presheaves over \mathbb{C} ,

$$\mathbf{Psh}(\mathbb{C}) =_{\mathrm{def}} [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map $tp: Tm \to Ty$ consists of

- contexts as objects $\Gamma, \Delta, \ldots \in \mathbb{C}$,
- a type in context $y(\Gamma)$ as a map $A: y(\Gamma) \to \mathsf{Ty}$,
- a term of type A in context Γ as a map $a: y(\Gamma) \to Tm$ such that



commutes,

• an operation called "context extension" which given a context Γ and a type $A \colon \mathsf{y}(\Gamma) \to \mathsf{T}\mathsf{y}$ produces a context $\Gamma \cdot A$ which fits into a pullback diagram below.

$$\begin{array}{ccc} \mathsf{y}(\Gamma.A) & \longrightarrow \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{y}(\Gamma) & \longrightarrow_A & \mathsf{Ty} \end{array}$$

Remark. Sometimes, we first construct a presheaf X over Γ and observe that it can be classified by a map into Ty. We write

$$\begin{array}{c} X \longrightarrow \mathsf{Tm} \\ \downarrow \\ \mathsf{y}(\Gamma) \xrightarrow{\Gamma X \to \mathsf{Ty}} \mathsf{Ty} \end{array}$$

to express this situation, i.e. $X \cong y(\Gamma \cdot \lceil X \rceil)$.

2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U}\colon \mathsf{Ty} \qquad \frac{a\colon \mathsf{U}}{\mathsf{El}(a)\colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal κ , with $\kappa < \lambda$.)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{c} \mathsf{U} \longrightarrow \mathsf{Tm} \\ \downarrow \\ \downarrow \\ 1 \xrightarrow{\vdash_{\mathsf{\Gamma}\mathsf{U}^{\neg}}} \mathsf{Ty} \end{array}$$

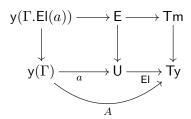
• There is an inclusion of U into Ty

$$\mathsf{EI}\colon\mathsf{U}\rightarrowtail\mathsf{Ty}$$

• $\pi: E \to U$ is obtained as pullback of tp; There is a pullback diagram

$$\begin{array}{c} E {\longmapsto} \operatorname{Tm} \\ \downarrow \qquad \qquad \downarrow \\ \operatorname{U} {\longmapsto} \operatorname{Ty} \end{array}$$

With the notation above, we get



Both squares above are pullback squares.

- 3 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model
- 4 Groupoid Model of HoTT

In this section we construct a natural model in **Psh(grpd)** the presheaf category indexed by the category **grpd** of (small) groupoids. We will build the classifier for display maps in the style of Hofmann and Streicher [HS98] and Awodey [Awo23]. To interpret the type constructors, we will make use of the weak factorization system on **grpd** - which comes from restricting the "classical Quillen model structure" on **cat** [Joy] to **grpd**.

4.1 Classifying display maps

Notation. We will have two universe sizes - one small and one large. We denote the category of small sets as **set** and the large sets as **Set** (in the previous sections this would have been \mathbf{Set}_{λ} and \mathbf{Set} respectively). We denote the category of small categories as \mathbf{cat} and the large categories as \mathbf{Cat} . We denote the category of small groupoids as \mathbf{grpd} . The category of small pointed groupoids will be \mathbf{grpd}_{\bullet} and small pointed categories will be \mathbf{cat}_{\bullet} .

We are primarily working in the category of large presheaves indexed by small groupoids, which we will denote by

$$Psh(grpd) = [grpd^{op}, Set]$$

In this section, Tm and Ty and so on will refer to the natural model semantics in this specific model.

Definition 4.1 (The disply map classifier). We would like to define a natural transformation in **Psh**(**grpd**)

$$tp: Tm \rightarrow Tv$$

with representable fibers.

Consider the functor that forgets the point

$$U : \mathbf{grpd}_{\bullet} \to \mathbf{grpd}$$
 in Cat.

If we apply the Yoneda embedding $y: \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ to U we obtain

$$U \circ : [-, \mathbf{grpd}_{\bullet}] \to [-, \mathbf{grpd}]$$
 in $\mathbf{Psh}(\mathbf{Cat})$.

Since any small groupoid is also a large category $\mathbf{grpd} \hookrightarrow \mathbf{Cat}$, we can restrict \mathbf{Cat} indexed presheaves to be \mathbf{grpd} indexed presheaves. We define $\mathsf{tp} \colon \mathsf{Tm} \to \mathsf{Ty}$ as the image of $U \circ \mathsf{under}$ this restriction.

$$\begin{array}{ccc} \mathbf{Cat} & \xrightarrow{y} & \mathbf{Psh}(\mathbf{Cat}) & \xrightarrow{\mathsf{res}} & \mathbf{Psh}(\mathbf{grpd}) \\ \\ \mathbf{grpd} & \longmapsto & [-,\mathbf{grpd}] & \longmapsto & \mathsf{Ty} \end{array}$$

Note that Tm and Ty are not representable in Psh(grpd).

Remark 4.2. By Yoneda we can identify maps with representable domain into the type classifier

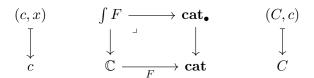
$$A: \mathsf{y}\Gamma \to \mathsf{T}\mathsf{y} \qquad \text{in} \quad \mathbf{Psh}(\mathbf{grpd})$$

with functors

$$A:\Gamma\to\mathbf{grpd}$$
 in Cat

Definition 4.3 (Grothendieck construction). From \mathbb{C} a small category and $F: \mathbb{C} \to \mathbf{cat}$ a functor, we construct a small category $\int F$. For any c in \mathbb{C} we refer to Fc as the fiber over c. The objects of $\int F$ consist of pairs $(c \in \mathbb{C}, x \in Fc)$, and morphisms between (c, x) and (d, y) are pairs $(f: c \to d, \phi: Ff x \to y)$. This makes the following pullback in \mathbf{Cat}

$$(c,x) \longmapsto (Fc,x)$$



Definition 4.4 (Grothendieck construction for groupoids). Let Γ be a groupoid and $A \colon \Gamma \to \mathbf{grpd}$ a functor, we can compose F with the inclusion $i \colon \mathbf{grpd} \hookrightarrow \mathbf{Cat}$ and form the Grothendieck construction which we denote as

$$\Gamma \cdot A := \int i \circ A \qquad \mathsf{disp}_A \colon \Gamma \cdot A o \Gamma$$

This is also a small groupoid since the underlying morphisms are from the groupoid Γ . Furthermore the pullback factors through (pointed) groupoids.

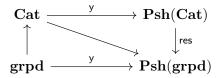
Corollary 4.5 (The display map classifier is presentable). For any small groupoid Γ and $A: y\Gamma \to Ty$, the pullback of tp along A can be given by the representable map $ydisp_A$.

$$\begin{array}{ccc} \mathbf{y}\Gamma\cdot A & \longrightarrow & \mathsf{Tm} \\ \mathbf{y}h\mathrm{disp}_A & & & \mathsf{tp} \\ & \mathbf{y}\Gamma & \longrightarrow & \mathsf{Ty} \end{array}$$

Proof. Consider the pullback in Cat

$$\begin{array}{ccc} \Gamma \cdot A & \longrightarrow \mathbf{grpd}_{\bullet} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow \mathbf{grpd} \end{array}$$

We send this square along $res \circ y$ in the following



The Yoneda embedding $y : \mathbf{Cat} \to \mathbf{Psh}(\mathbf{Cat})$ preserves pullbacks, as does res since it is a right adjoint (with left Kan extension $\iota_! \dashv \mathsf{res}_\iota$).

4.2 Groupoid fibrations

Definition 4.6 (Fibration). Let $p: \mathbb{C}_1 \to \mathbb{C}_0$ be a functor. We say p is a *cloven Grothendieck fibration* if we have a dependent function lift a f satisfying the following: for any object a in \mathbb{C}_1 and morphism $f: p \, a \to y$ in the base \mathbb{C}_0 we have lift $a \, f: a \to b$ in \mathbb{C}_1 such that $p(\text{lift } a \, f) = f$.

$$\begin{array}{cccc} a & \xrightarrow{\text{lift } a f} & b \\ \downarrow & & \uparrow & \uparrow \\ \downarrow & & \downarrow & \downarrow \\ x & \xrightarrow{f} & y \end{array}$$

In particular, we are intereseted in cloven Grothendieck fibrations of groupoids, which are the same as *isofibrations* (replace all the morphisms with isomorphisms in the definition).

Unless specified otherwise, by a *fibration* we will mean a cloven Grothendieck fibration of groupoids. Let us denote the category of fibrations over a groupoid Γ as Fib_{Γ} , which is a full subcategory of the slice grpd/Γ .

Note that $\operatorname{\mathsf{disp}}_A \colon \Gamma \cdot A \to \Gamma$ is a fibration, since for any $(x \in \Gamma, a \in A \, x)$ and $f \colon x \to y$ in Γ we have a morphism $(f, \operatorname{\mathsf{id}}_{A \, f \, a}) \colon (x, a) \to (y, A \, f \, a)$ lifting f. Furthermore **Proposition 4.7.** There is an adjoint equivalence

$$[\Gamma, \operatorname{\mathbf{grpd}}] \xrightarrow[\operatorname{fiber}]{\operatorname{\mathsf{disp}}} \operatorname{\mathsf{Fib}}_{\Gamma}$$

where for each fibration $\delta: \Delta \to \Gamma$ and each object $x \in \Gamma$

$$fiber_{\delta} x = full subcategory \{ a \in \Delta \mid \delta a = x \}$$

Proposition 4.8 (Display map properties of fibrations). TODO

- 1. (subst) Stable under pullback
- 2. (Σ) Closed under composition
- 3. (Π) Closed under pushforward
- 4. (Fibrant objects) Map to terminal is a fibration
- 5. (Id) Path object fibration

4.3 Polynomial endofunctors

Definition 4.9 (Polynomial endofunctor in an LCCC). TODO

Proposition 4.10 (Universal property of polynomial endofunctors). *TODO*

4.4 Π and Σ structure

Definition 4.11 (Interpretation of Π and λ). Sketch: we define the natural transformation Π : Poly_{tp}Ty \rightarrow Ty by first taking some small groupoid Γ and defining

$$\Pi_{\Gamma}:\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_\mathsf{tp}\mathsf{Ty})\to\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Ty})$$

Unfolding the universal property of $\mathsf{Poly}_\mathsf{tp}$ this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

As indicated in the diagram, we take this to be the pushforward of the dependent display map disp_{B} along the display map it depends on disp_{A} . Note that this pushforward is in $\operatorname{\mathbf{grpd}}$, and this pushforward is only defined on fibrations.

TODO: define λ .

Proof. TODO: naturality.

TODO: prove pullback.

Definition 4.12 (Interpretation of Σ). Sketch: we define the natural transformation $\Sigma : \mathsf{Poly}_\mathsf{tp}\mathsf{Ty} \to \mathsf{Ty}$ by first taking some small groupoid Γ and defining

$$\Sigma_{\Gamma}:\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Poly}_\mathsf{tp}\mathsf{Ty})\to\mathbf{Psh}(\mathbf{grpd})(\Gamma,\mathsf{Ty})$$

Again, this amounts to taking a pair of composable groupoid fibrations to a single groupoid fibration on the codomain

As indicated in the diagram, we take this to be the composition of $disp_B$ and $disp_A$, recalling that fibrations are closed under composition.

TODO: define pair.

Proof. TODO: naturality.

TODO: prove pullback.

References

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- [HS98] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.
- [Joy] André Joyal. Model structures on cat. https://ncatlab.org/joyalscatlab/published/Model+structures+on+Cat.