## Universe in the Natural Model of Type Theory

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## 1 Types

Assume an inaccessible cardinal  $\lambda$ . Write **Set** for the category of all sets. Say that a set A is  $\lambda$ -small if  $|A| < \lambda$ . Write **Set** $_{\lambda}$  for the full subcategory of **Set** spanned by  $\lambda$ -small sets.

Let  $\mathbb{C}$  be a small category, i.e. a category whose class of objects is a set and whose hom-classes are sets.

We write  $Psh(\mathbb{C})$  for the category of presheaves over  $\mathbb{C}$ ,

$$\operatorname{Psh}(\mathbb{C}) =_{\operatorname{def}} [\mathbb{C}^{\operatorname{op}}, \mathbf{Set}]$$

The Natural Model associated to a presentable map  $tp: Tm \to Ty$  consists of

- contexts as objects  $\Gamma, \Delta, \ldots \in \mathbb{C}$ ,
- a type in context  $y(\Gamma)$  as a map  $A: y(\Gamma) \to \mathsf{Ty}$ ,
- a term of type A in context  $\Gamma$  as a map  $a: y(\Gamma) \to Tm$  such that



commutes,

• an operation called "context extension" which given a context  $\Gamma$  and a type  $A \colon \mathsf{y}(\Gamma) \to \mathsf{T}\mathsf{y}$  produces a context  $\Gamma \cdot A$  which fits into a pullback diagram below.

$$\begin{array}{ccc} \mathsf{y}(\Gamma.A) & \longrightarrow \mathsf{Tm} \\ \downarrow & & \downarrow \\ \mathsf{y}(\Gamma) & \longrightarrow_A & \mathsf{Ty} \end{array}$$

**Remark.** Sometimes, we first construct a presheaf X over  $\Gamma$  and observe that it can be classified by a map into Ty. We write

$$\begin{array}{c} X \longrightarrow \mathsf{Tm} \\ \downarrow \\ \mathsf{y}(\Gamma) \xrightarrow{\Gamma} \mathsf{Ty} \end{array}$$

to express this situation, i.e.  $X \cong y(\Gamma \cdot \lceil X \rceil)$ .

## 2 A type of small types

We now wish to formulate a condition that allows us to have a type of small types, written U, not just *judgement* expressing that something is a type. With this notation, the judgements that we would like to derive is

$$\mathsf{U}\colon \mathsf{Ty} \qquad \frac{a\colon \mathsf{U}}{\mathsf{El}(a)\colon \mathsf{Ty}}$$

(A sufficient and natural condition for this seems to be that we now have another inaccessible cardinal  $\kappa$ , with  $\kappa < \lambda$ .)

In the Natural Model, a universe U is postulated by a map

$$\pi \colon \mathsf{E} \to \mathsf{U}$$

In the Natural Model:

• There is a pullback diagram of the form

$$\begin{array}{c} \mathsf{U} \longrightarrow \mathsf{Tm} \\ \downarrow \\ \downarrow \\ 1 \xrightarrow{} \mathsf{Ty} \end{array}$$

• There is an inclusion of U into Ty

$$\mathsf{EI}\colon\mathsf{U}\rightarrowtail\mathsf{Ty}$$

•  $\pi: \mathsf{E} \to \mathsf{U}$  is obtained as pullback of tp; There is a pullback diagram

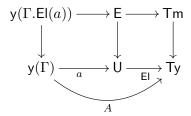
$$E \longmapsto \mathsf{Tm}$$

$$\pi \downarrow \qquad \qquad \downarrow \mathsf{tp}$$

$$\mathsf{U} \longmapsto \mathsf{Ty}$$

$$(2.1)$$

With the notation above, we get

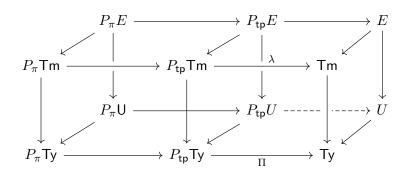


Both squares above are pullback squares.

## 3 Defining $\Pi$ and $\Sigma$ type formers for the universe $\pi \colon \mathsf{E} \to \mathsf{U}$

Take the pullback diagram eq. (2.1). That is a morphism in the category of polynomials. We have a cartesian natural transformation  $P_{\pi} \to P_{\mathsf{tp}}$  induced by the pullback eq. (2.1). All the squares in the left cube are pullback squares.

Now, consider the right cube only. The right-side face of the cube is a pullback square due to the definition of  $\pi$ . The left-side is also because polynomials are left exact functors and therefore they preserve pullbacks. Observe also that the front face is a pullback square by the definition of  $\Pi$  for Ty.



We define  $\Pi_U : P_{\pi}U \to U$  to be a dashed arrow which makes the bottom square commute, that is the following square commutes

$$P_{\mathsf{tp}} \mathsf{U} \longmapsto P_{\mathsf{tp}} \mathsf{Ty} \tag{3.1}$$

$$\Pi_{\mathsf{U}} \mid \qquad \qquad \downarrow \Pi_{\mathsf{Ty}} \qquad \qquad \downarrow \mathsf{Ty}$$

$$\mathsf{U} \longmapsto \mathsf{Ty}$$

By the universal property of the pullback which defines  $\pi$  we get a unique arrow  $P_{\mathsf{tp}} \colon P_{\mathsf{tp}} E \to E$  which we shall name  $\lambda_U$ . By the pullback pasting lemma it follows

that the square involving  $\Pi_U$  and  $\lambda_U$  is a pullback square. This concludes the construction of  $\Pi$  type former for the universe  $\mathsf{U}$ . The only data we needed to supply for the definition of  $\Pi_U$  was a lift of  $\Pi\colon P_\mathsf{tp}\mathsf{Ty}\to \mathsf{Ty}$  to U.

4 The Universe in Embedded Type Theory (HoTT0) and the relationship to the Natural Model