Problem Set 1: Linear Regression

To run and solve this assignment, one must have a working IPython Notebook installation. The easiest way to set it up for both Windows and Linux is to install [Anaconda (https://www.continuum.io/downloads)](https://www.continuum.io/downloads). Then save this file to your computer (use "Raw" link on gist\github), run Anaconda and choose this file in Anaconda's file explorer. Use the `Python 3` version. Everything that follows assumes that you have already followed these instructions. If you are new to Python or its scientific library, Numpy, there are some nice tutorials [here (https://www.learnpython.org/)](https://www.learnpython.org/) and [here (http://www.scipy-lectures.org/)](http://www.scipy-lectures.org/).

To run code in a cell or to render [Markdown (https://en.wikipedia.org/wiki/Markdown)](https://en.wikipedia.org/wiki/Markdown)+[LaTeX (https://en.wikipedia.org/wiki/LaTeX)](https://en.wikipedia.org/wiki/LaTeX) press `Ctr+Enter` or `[>|]` (like "play") button above. To edit any code or text cell [double]click on its content. To change cell type, choose "Markdown" or "Code" in the drop-down menu above.

If certain output is given for some cells, that means that you are expected to get similar results.

Total: 155 points.

1. Numpy Tutorial

1.1 [5pt] Modify the cell below to return a 5x5 matrix of ones. Put some code there and press `Ctrl+Enter` to execute contents of the cell. You should see something like the output below. [[1] (https://docs.scipy.org/doc/numpy-1.13.0/user/basics.creation.html#arrays-creation)](https://docs.scipy.org/doc/numpy-1.13.0/user/basics.creation.html#arrays-creation) [[2] (https://docs.scipy.org/doc/numpy-1.13.0/reference/routines.array-creation.html#routines-array-creation)](https://docs.scipy.org/doc/numpy-1.13.0/reference/routines.array-creation.html#routines-array-creation)

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
array = (5,5)
def return_ones_matrix(a):
    return(np.ones(a))
print(return_ones_matrix(array))
```

```
[[1. 1. 1. 1. 1.]
 [1. 1. 1. 1. 1.]
 [1. 1. 1. 1. 1.]
 [1. 1. 1. 1. 1.]
 [1. 1. 1. 1. 1.]]
```

1.2 [5pt] Vectorizing your code is very important to get results in a reasonable time. Let A be a 10x10 matrix and x be a 10-element column vector. Your friend writes the following code. How would you vectorize this code to run without any for loops? Compare execution speed for different values of `n` with [`%timeit` (http://ipython.readthedocs.io/en/stable/interactive/magics.html#magic-timeit)](http://ipython.readthedocs.io/en/stable/interactive/magics.html#magic-timeit).

In [2]:
```
import timeit
```

In [3]:
```
import_module = "import random"
import numpy as np
n = 10
def compute_something(A, x):
    v = np.zeros((n, 1))
    for i in range(n):
        for j in range(n):
            v[i] += A[i, j] * x[j]
    return v

A = np.random.rand(n, n)
x = np.random.rand(n, 1)
print(compute_something(A, x))
starttime = timeit.default_timer()
print("The start time is :",starttime)
compute_something(A, x)
print("The time difference is :", timeit.default_timer() - starttime)

n = 20
A = np.random.rand(n, n)
x = np.random.rand(n, 1)
#print(compute_something(A, x))
print("n = 20")
starttime = timeit.default_timer()
print("The start time is :",starttime)
compute_something(A, x)
print("The time difference is :", timeit.default_timer() - starttime)

n = 30
A = np.random.rand(n, n)
x = np.random.rand(n, 1)
#print(compute_something(A, x))
print("n = 30")
starttime = timeit.default_timer()
print("The start time is :",starttime)
compute_something(A, x)
print("The time difference is :", timeit.default_timer() - starttime)

n = 100
A = np.random.rand(n, n)
x = np.random.rand(n, 1)
#print(compute_something(A, x))
print("n = 100")
starttime = timeit.default_timer()
print("The start time is :",starttime)
compute_something(A, x)
print("The time difference is :", timeit.default_timer() - starttime)

n = 999
A = np.random.rand(n, n)
x = np.random.rand(n, 1)
#print(compute_something(A, x))
print("n = 999")
starttime = timeit.default_timer()
print("The start time is :",starttime)
```

```
compute_something(A, x)
print("The time difference is :", timeit.default_timer() - starttime)
```

```
[[1.29719432]
 [2.64223855]
 [2.2405223 ]
 [1.97919377]
 [2.03236526]
 [1.35164744]
 [1.87205519]
 [2.06296264]
 [2.6449505 ]
 [1.92280936]]
The start time is : 3.3506072
The time difference is : 0.000759500000000024
n = 20
The start time is : 3.3517421
The time difference is : 0.0023952000000000417
n = 30
The start time is : 3.3549077
The time difference is : 0.004984599999999784
n = 100
The start time is : 3.3605191
The time difference is : 0.05819799999999997
n = 999
The start time is : 3.4364394
The time difference is : 5.253641100000001
```

In [4]:
```
arr1 =np.average(A,axis=1)
print(arr1)
```

```
[0.48011906 0.500943   0.49994264 0.5065809  0.5186723  0.5051601
 0.4914489  0.50074796 0.4967449  0.47682008 0.50763748 0.50431852
 0.492269   0.49088086 0.50961558 0.51078472 0.50023503 0.47897583
 0.48795897 0.48934088 0.50942589 0.49873235 0.50093368 0.5048258
 0.51259009 0.52234481 0.52017165 0.51452392 0.5004423  0.50588486
 0.51215791 0.50282312 0.50737188 0.50523257 0.52006186 0.50245537
 0.4941509  0.49315175 0.50018734 0.49658375 0.49856474 0.50767751
 0.50116863 0.50844115 0.48521878 0.49382266 0.50011071 0.4899011
 0.4999998  0.50214445 0.49723477 0.51284968 0.51068744 0.50585478
 0.49784929 0.49276001 0.51849027 0.52485584 0.49260569 0.49356829
 0.48320568 0.51012451 0.50641767 0.50186615 0.51182585 0.50615919
 0.49355376 0.50410051 0.50159911 0.50697287 0.50206654 0.48925562
 0.50713699 0.48386161 0.50462799 0.48842598 0.50050831 0.50106503
 0.49742632 0.49519067 0.48875509 0.49382956 0.49884369 0.49511935
 0.52212055 0.49255135 0.51009788 0.50982309 0.49729201 0.50056276
 0.51026206 0.49240121 0.50310172 0.49961636 0.4867487  0.50700766
 0.4876037  0.48898674 0.50913581 0.4923542  0.50067206 0.5038228
 0.49687693 0.51590417 0.4815789  0.51350417 0.49388522 0.48602845
 0.5029147  0.49427525 0.48927626 0.50934543 0.4962771  0.50913486
 0.49947847 0.49625021 0.50273377 0.49919562 0.50324586 0.50259827
 0.49615607 0.50097857 0.49690129 0.4967599  0.48331149 0.49327386
 0.49415836 0.50295917 0.4937156  0.5025207  0.49128177 0.48649181
 0.5031322  0.50490358 0.486538   0.50308898 0.49328478 0.49096984
 0.50937375 0.49673708 0.50831289 0.49504389 0.49404776 0.51245464
 0.48320687 0.49110815 0.51079363 0.49653407 0.51382306 0.49649938
 0.50169835 0.4981662  0.48413608 0.48834747 0.4994247  0.49389076
 0.49906105 0.49675229 0.50812068 0.50610892 0.50261311 0.50393211
 0.50795978 0.48973902 0.50863447 0.50223661 0.49982348 0.49488539
 0.50323229 0.50377465 0.49982473 0.48821136 0.51080138 0.49087905
 0.49908458 0.48219425 0.51028154 0.52256944 0.50898323 0.5016933
 0.5066256  0.49991669 0.50556975 0.49512842 0.50331982 0.50428577
 0.49976835 0.49258434 0.49953836 0.49855126 0.50846001 0.48834149
 0.50725405 0.49520179 0.48696185 0.48662134 0.49036779 0.4994912
 0.48880272 0.50454441 0.49858062 0.49070433 0.50629497 0.49286268
 0.50791704 0.50293692 0.50423498 0.50711418 0.50972568 0.51191606
 0.50960702 0.50353636 0.49745126 0.51128765 0.50903703 0.49731389
 0.50666495 0.49142283 0.5116652  0.4804733  0.50587877 0.50282283
 0.49803623 0.50269985 0.49859401 0.48088249 0.4975658  0.48721151
 0.49912447 0.49952159 0.51132531 0.49914228 0.5096641  0.49922454
 0.50948265 0.49367361 0.50865965 0.48706356 0.48808492 0.49837492
 0.49494167 0.48615089 0.51299233 0.4862416  0.48962956 0.50034797
 0.49770949 0.51402359 0.49943918 0.49412902 0.48935278 0.50022086
 0.51439453 0.49279172 0.49970303 0.49833174 0.49709457 0.50331691
 0.50308278 0.49208686 0.49269993 0.51475271 0.51271254 0.49720926
 0.49207542 0.50028004 0.504218   0.50920159 0.50717497 0.50076899
 0.50037496 0.50501595 0.50062042 0.51538035 0.50107981 0.49713574
 0.50028921 0.50471518 0.50536509 0.4838829  0.48294593 0.49174478
 0.50500966 0.50438816 0.48993899 0.50405956 0.493061   0.48913398
 0.51508746 0.50154733 0.50096314 0.50143749 0.48482327 0.50436427
 0.49543729 0.49181649 0.50518965 0.5014248  0.49147954 0.49124465
 0.5023383  0.48779977 0.50745667 0.50453436 0.49606165 0.49115272
 0.50197003 0.5129781  0.51655765 0.50068997 0.49719655 0.49833546
 0.50389231 0.48920957 0.51044349 0.49242459 0.50577566 0.50110978
 0.48777508 0.49390486 0.50977794 0.4867208  0.51172175 0.4845532
 0.49354503 0.49385105 0.49890065 0.51290338 0.50922853 0.50970539
 0.48930732 0.49511722 0.49400487 0.49787339 0.49422705 0.5053827
 0.50921278 0.49946344 0.49836514 0.50135488 0.51478715 0.49299798
```

```
0.50360973 0.52561079 0.49616294 0.50051284 0.49237044 0.49555143
0.50333956 0.51541676 0.49502411 0.48736037 0.4992346  0.49224749
0.49267191 0.49283017 0.49848019 0.50124903 0.4920424  0.49190731
0.50813163 0.49810919 0.50175714 0.4942661  0.49659527 0.50223591
0.48884856 0.49252112 0.49683512 0.50119503 0.51562678 0.4966654
0.49711849 0.51924175 0.48276656 0.50919862 0.49150149 0.48877996
0.50871606 0.48725136 0.50902116 0.49317814 0.49829636 0.51632529
0.48798755 0.50525397 0.49246081 0.49410043 0.49759973 0.50283882
0.50066222 0.50842597 0.49740081 0.50221506 0.48777996 0.5107111
0.51690052 0.48449152 0.49231022 0.48678291 0.48871254 0.50463213
0.49937101 0.49790195 0.51062144 0.49292871 0.49133761 0.50925481
0.4943698  0.51343194 0.50852725 0.50174822 0.4928362  0.50226559
0.51096779 0.49449711 0.49580452 0.49284879 0.51782254 0.49139114
0.5073191  0.4855636  0.504397   0.51791192 0.50085108 0.49780227
0.50087081 0.48242826 0.49483009 0.50562082 0.50814234 0.50017373
0.49415018 0.49016022 0.48836397 0.50937355 0.4933808  0.51406731
0.49178408 0.5018365  0.49230549 0.50056342 0.50608314 0.50297434
0.50323076 0.49682424 0.48791782 0.491256   0.50910339 0.50257353
0.4822575  0.48869851 0.48561102 0.49253662 0.51409652 0.49501523
0.49327483 0.48960692 0.48948454 0.50896346 0.50806958 0.49851543
0.51360435 0.48815476 0.49544708 0.51648796 0.51157809 0.50315829
0.49333503 0.50201308 0.50274838 0.49587301 0.50071496 0.49490803
0.50362599 0.50045739 0.49059955 0.48442263 0.50033847 0.51055442
0.50547267 0.500805   0.49795099 0.4975644  0.50018621 0.50484893
0.5107237  0.50661593 0.48621603 0.49967574 0.50116805 0.52087148
0.50185315 0.49357109 0.50700878 0.50242086 0.5022093  0.49550776
0.4870325  0.49325426 0.50280749 0.49366915 0.4996176  0.49018814
0.4978999  0.50619559 0.50795691 0.48882594 0.51952322 0.48343506
0.49161037 0.49973216 0.50279361 0.49869578 0.49685262 0.49133162
0.49552745 0.50685722 0.49503454 0.49905909 0.48205836 0.49168167
0.50414214 0.4998652  0.50075872 0.50225419 0.48985685 0.49547791
0.4860599  0.49793723 0.49921692 0.50247518 0.49999622 0.49352558
0.50533284 0.50114477 0.49121163 0.49683683 0.52222054 0.51517099
0.51163643 0.48449129 0.51095787 0.5076849  0.48868209 0.49820563
0.51603722 0.48803686 0.49307987 0.50904146 0.49467069 0.51408834
0.48530136 0.49571087 0.51758337 0.48021651 0.50938391 0.4873177
0.50140622 0.49461075 0.49486694 0.51162897 0.49408604 0.49290189
0.49926355 0.48762956 0.49356503 0.49712436 0.49891804 0.49547569
0.49175958 0.49701645 0.50294533 0.49787834 0.50103602 0.50440524
0.51007034 0.50010058 0.50678842 0.49480381 0.49467889 0.50299224
0.500255   0.50079257 0.50866781 0.4947774  0.5025384  0.47822879
0.50600712 0.5142694  0.49257294 0.50484858 0.49779219 0.50267789
0.51118998 0.49697653 0.49973593 0.50956594 0.5132434  0.5024335
0.49552947 0.50642163 0.49056123 0.49475623 0.49870498 0.50455252
0.49609095 0.50679845 0.501295   0.50524355 0.50584171 0.50288668
0.49276894 0.5109969  0.50516282 0.49859332 0.50665393 0.50454775
0.49624577 0.48905971 0.50552065 0.50657958 0.50277756 0.50914
0.49391736 0.49674701 0.50369475 0.49026643 0.50786722 0.51085425
0.49574917 0.49103387 0.50308286 0.48932659 0.49899853 0.49882744
0.50259246 0.50189327 0.4975403  0.50069706 0.49411616 0.4965793
0.5024628  0.50880024 0.50979997 0.50176611 0.50475754 0.48706962
0.5046634  0.49462172 0.50498279 0.48450561 0.50147568 0.50758669
0.51345726 0.50376898 0.49818375 0.49510115 0.48471148 0.49008954
0.50501469 0.51077267 0.50329452 0.5036019  0.50157828 0.49838549
0.50436231 0.50478342 0.4917168  0.5039792  0.4943945  0.49551368
0.522132   0.50571358 0.50594144 0.51597306 0.49381079 0.49622562
0.50517104 0.50397523 0.51126389 0.48394513 0.49558232 0.50292648
```

```
0.51330212 0.50223305 0.50311294 0.4993309  0.49752633 0.51539839
0.49906379 0.50451205 0.50014563 0.50513384 0.51610928 0.48961348
0.49961013 0.49795874 0.5052914  0.50173829 0.50390579 0.49214055
0.48728053 0.50924028 0.48208052 0.50320976 0.50490742 0.49014051
0.49274312 0.51016949 0.47746104 0.51862861 0.50267459 0.499577
0.49403115 0.49797572 0.48270183 0.48989672 0.48985922 0.50048368
0.48119082 0.51077982 0.48812845 0.4851953  0.50245699 0.50315592
0.5207903  0.48453239 0.50769521 0.50878398 0.49789184 0.50077437
0.50207992 0.50350957 0.50785879 0.50773397 0.50154371 0.51628612
0.49229642 0.49512777 0.49931567 0.50203871 0.51857216 0.49297367
0.48420348 0.48937155 0.49356326 0.50684548 0.48941593 0.49765643
0.48167893 0.51138272 0.49841516 0.49460475 0.49004999 0.5042683
0.49941958 0.50168102 0.50321286 0.49531005 0.49876257 0.50094371
0.51038949 0.501325   0.49477732 0.48811972 0.5012456  0.50490419
0.49733115 0.48621905 0.50236576 0.48412665 0.49836942 0.50335165
0.48994926 0.50167136 0.50702296 0.50905911 0.5214108  0.49755949
0.49597489 0.49333063 0.49962378 0.49288196 0.49211942 0.48934995
0.48026167 0.5101296  0.50477241 0.50268007 0.50532215 0.50811701
0.49141651 0.49840191 0.4812838  0.50634738 0.49643835 0.50286319
0.49896405 0.50363446 0.50158135 0.49785685 0.5134923  0.50322685
0.50188386 0.50143614 0.49063998 0.50004917 0.51108075 0.51904062
0.5042962  0.4954333  0.49650324 0.49764464 0.49402753 0.49657
0.48023182 0.50183025 0.50304761 0.49898253 0.49504102 0.50572028
0.5040802  0.49899868 0.51832553 0.50757366 0.5061623  0.49386828
0.49068026 0.48996997 0.49702828 0.48755724 0.50653401 0.49902654
0.49767147 0.51153215 0.50191508 0.49636803 0.48556724 0.49594682
0.49158376 0.50403004 0.50596697 0.49781997 0.4974105  0.51129609
0.49544326 0.49467926 0.48809076 0.4978063  0.5077752  0.49801566
0.50663435 0.51435656 0.49695834 0.50056621 0.49806862 0.48900775
0.51278283 0.49581925 0.49301058 0.50028673 0.5159046  0.47538284
0.50973104 0.49463601 0.49823397 0.49691506 0.5005525  0.48656573
0.48899244 0.48482223 0.4974723  0.49486045 0.48136143 0.50708764
0.498834   0.49162027 0.50104856 0.51396505 0.48966879 0.4834556
0.50031446 0.48659821 0.48152649 0.50882516 0.50599191 0.49796522
0.5003341  0.5020116  0.50733956 0.48690807 0.48569842 0.51740555
0.4920385  0.49881275 0.49986383 0.49080813 0.50910532 0.49295903
0.48888062 0.52274849 0.52322479 0.49987656 0.50411034 0.49725864
0.49462701 0.48312969 0.51868401 0.49069476 0.50914793 0.49271231
0.49985194 0.50171568 0.49020336 0.50069261 0.49658512 0.49448823
0.49107058 0.49911618 0.48490738 0.51884176 0.49240574 0.50376078
0.4848653  0.50734687 0.49752337 0.49925666 0.50480399 0.51017733
0.48554616 0.50927886 0.47871949 0.49509417 0.49516492 0.49063622
0.5004203  0.49889931 0.51042872 0.49604449 0.51497997 0.51002333
0.49031083 0.49991478 0.49706757 0.52020986 0.49630858 0.49100286
0.50182159 0.49982472 0.48921312 0.48930238 0.49518354 0.50968457
0.49258305 0.50867633 0.5047099  0.5086008  0.48884985 0.50517434
0.50141095 0.48203476 0.49942626 0.50280339 0.49105983 0.5060853
0.50665204 0.49408884 0.48167454 0.50220458 0.48182414 0.50846098
0.51077267 0.4871399  0.51090758 0.50007007 0.49657029 0.48764768
0.51609858 0.50574673 0.51202512 0.49655423 0.50004619 0.5058973
0.50191539 0.50437676 0.51307177 0.5065532  0.51406586 0.48767902
0.50473506 0.51217731 0.48816288 0.50029901 0.4862821  0.49365726
0.51156893 0.50915938 0.49200389]
```

In [5]:
```
n = 10
A = np.random.rand(n, n)
x = np.random.rand(n, 1)
def vectorized(A, x):
    return np.matmul(A,x)
print(vectorized(A, x))
assert np.max(abs(vectorized(A, x) - compute_something(A, x))) < 1e-3
print("To vectorize without loops, use numpy matrix multiplication. As you can multiply 1x10 matrix with 10x10 matrix.")
```

```
[[2.08901783]
 [1.449847  ]
 [1.70128058]
 [2.41020491]
 [1.64513535]
 [1.74812242]
 [3.30006839]
 [3.1264574 ]
 [2.50606332]
 [1.67370486]]
To vectorize without loops, use numpy matrix multiplication. As you can multi
ply 1x10 matrix with 10x10 matrix.
```

In [6]:
```
for n in [5, 10, 100, 500]:
    A = np.random.rand(n, n)
    x = np.random.rand(n, 1)
    %timeit -n 5 compute_something(A, x)
    %timeit -n 5 vectorized(A, x)
    print('---')
```

```
135 µs ± 21.9 µs per loop (mean ± std. dev. of 7 runs, 5 loops each)
3.21 µs ± 1.76 µs per loop (mean ± std. dev. of 7 runs, 5 loops each)
---
472 µs ± 25.7 µs per loop (mean ± std. dev. of 7 runs, 5 loops each)
5.55 µs ± 3.76 µs per loop (mean ± std. dev. of 7 runs, 5 loops each)
---
74.5 ms ± 4.01 ms per loop (mean ± std. dev. of 7 runs, 5 loops each)
6.88 µs ± 2.42 µs per loop (mean ± std. dev. of 7 runs, 5 loops each)
---
1.8 s ± 12.9 ms per loop (mean ± std. dev. of 7 runs, 5 loops each)
The slowest run took 21.37 times longer than the fastest. This could mean tha
t an intermediate result is being cached.
264 µs ± 478 µs per loop (mean ± std. dev. of 7 runs, 5 loops each)
---
```

2. Linear regression with one variable

In this part of the exercise, you will implement linear regression with one variable to predict profits for a food truck. Suppose you are the CEO of a restaurant franchise and are considering different cities for opening a new outlet. The chain already has trucks in various cities and you have data for profits and populations from the cities. You would like to use this data to help you select which city to expand to next. The file ex1data.txt contains the dataset for our linear regression problem. The first column is the population of a city and the second column is the profit of a food truck in that city. A negative value for profit indicates a loss.

2.1 [10pt] Generate a plot similar to the one below : [1] (https://matplotlib.org/devdocs/api/_as_gen/matplotlib.pyplot.scatter.html) [2] (https://matplotlib.org/api/pyplot_api.html?highlight=xlim#matplotlib.pyplot.xlim) [3] (https://matplotlib.org/api/pyplot_api.html?highlight=matplotlib%20pyplot%20xlabel#matplotlib.pyplot.xlabel)

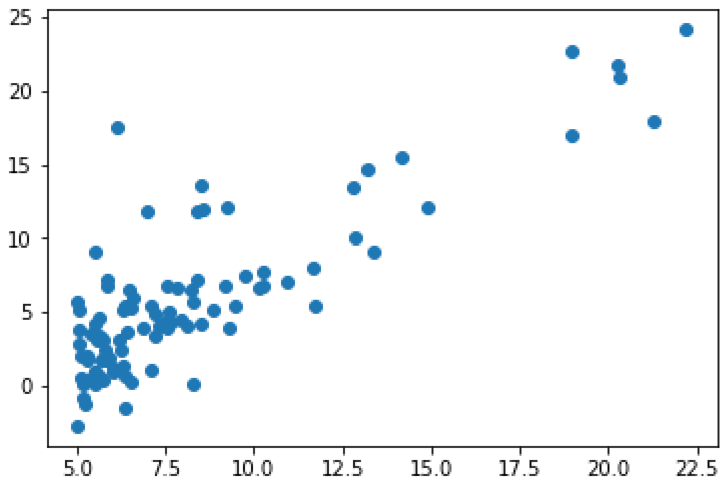
Before starting on any task, it is often useful to understand the data by visualizing it. For this dataset, you can use a scatter plot to visualize the data, since it has only two properties to plot (profit and population). Many other problems that you will encounter in real life are multi-dimensional and can’t be plotted on a 2-d plot.

In [7]:
```
data = np.loadtxt('ex1data1.txt', delimiter=',')
X, y = data[:, 0, np.newaxis], data[:, 1, np.newaxis]
n = data.shape[0]
print(X.shape, y.shape, n)
print(X[:10], '\n', y[:10])

import matplotlib.pyplot as plt
plt.scatter(X,y)
#raise NotImplementedError('Put the visualziation code here.')

plt.show()
```

```
(97, 1) (97, 1) 97
[[6.1101]
 [5.5277]
 [8.5186]
 [7.0032]
 [5.8598]
 [8.3829]
 [7.4764]
 [8.5781]
 [6.4862]
 [5.0546]]
 [[17.592 ]
 [ 9.1302]
 [13.662 ]
 [11.854 ]
 [ 6.8233]
 [11.886 ]
 [ 4.3483]
 [12.    ]
 [ 6.5987]
 [ 3.8166]]
```



2.2 Gradient Descent

In this part, you will fit the linear regression parameter θ to our dataset using gradient descent.

The objective of linear regression is to minimize the cost function

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} \left(h(x^{(i)}; \theta) - y^{(i)} \right)^2$$

where the hypothesis $h(x; \theta)$ is given by the linear model (x' has an additional fake feature always equal to '1')

$$h(x; \theta) = \theta^T x' = \theta_0 + \theta_1 x$$

Recall that the parameters of your model are the θ_j values. These are the values you will adjust to minimize the cost J(θ). One way to do this is to use the gradient descent. In batch gradient descent algorithm, value of θ is updated iteratively using the gradient of J(θ).

$$\theta_j^{(k+1)} = \theta_j^{(k)} - \eta \frac{1}{m} \sum_i \left(h(x^{(i)}; \theta) - y^{(i)} \right) x_j^{(i)}$$

With each step of gradient descent, your parameter θ_j comes closer to the optimal values that will achieve the lowest cost J(θ).

2.2.1 [5pt] Where does this update rule come from?

2.2.2 [30pt] Cost Implementation

As you perform gradient descent to learn to minimize the cost function, it is helpful to monitor the convergence by computing the cost. In this section, you will implement a function to calculate $J(\theta)$ so you can check the convergence of your gradient descent implementation.

In the following lines, we add another dimension to our data to accommodate the intercept term and compute the prediction and the loss. As you are doing this, remember that the variables X and y are not scalar values, but matrices whose rows represent the examples from the training set. In order to get x' add a column (https://docs.scipy.org/doc/numpy/reference/generated/numpy.insert.html) of ones to the data matrix `X` .

You should expect to see a cost of approximately 32.

The update rule comes from taking the derivative of the linear model. We want to slowly decrease our cost to test for the most optimal lowest cost function value. (slopes of slopes of slopes) These will be turned into weights to slowly test the values/outputs.

In [8]:
```
import math

def sigmoid(x):
  return 1 / (1 + math.exp(-x))
```

In [9]:
```
np.array([X[0]]).shape
```

Out[9]:
```
(1, 1)
```

In [12]:
```
predict(np.array([X[0]]), theta_init)
```

Out[12]:
```
array([[0.]])
```

In [11]:
```
# assertions below are true only for this
# specific case and are given to ease debugging!
import numpy as np
def add_column(X):
    assert len(X.shape) == 2 and X.shape[1] == 1
    return np.insert(X,0,1,axis=1)
    #raise NotImplementedError("Insert a column of ones to the _left_ side of
 the matrix")

def predict(X, theta):
    """ Computes h(x; theta) """
    assert len(X.shape) == 2 and X.shape[1] == 1
    assert theta.shape == (2, 1)
    #print(np.array(X).shape)
    X_prime = add_column(X)
    Theta_T = np.transpose(theta)
    X_prime = np.transpose(X_prime)
    pred = np.matmul(Theta_T,X_prime)
    return pred

def loss(X, y, theta):
    assert X.shape == (n, 1)
    assert y.shape == (n, 1)
    assert theta.shape == (2, 1)

    X_prime = add_column(X)
    assert X_prime.shape == (n, 2)
    Theta_T = np.transpose(theta)
    #X_prime = np.transpose(X_prime)
    #print(X_prime.shape)
    print(np.array(theta))
    print(theta.shape)
    #print(Theta_T.shape)
    #print(Theta_T)
    total = 0
    for i in range(len(X_prime)):
        pred = np.matmul(Theta_T,X_prime[i])
        total += (pred-y[i])**2
    #raise NotImplementedError("Compute the model loss; use the predict() func
tion")
    loss = total/194
    return loss

theta_init = np.zeros((2, 1))
print(loss(X, y, theta_init))
```

```
[[0.]
 [0.]]
(2, 1)
[32.07273388]
```

2.2.3 [40pt] GD Implementation

Next, you will implement gradient descent. The loop structure has been written for you, and you only need to supply the updates to θ within each iteration.

As you write your code, make sure you understand what you are trying to optimize and what is being updated. Keep in mind that the cost is parameterized by the vector θ not X and y. That is, we minimize the value of $J(\theta)$ by changing the values of the vector θ, not by changing X or y.

A good way to verify that gradient descent is working correctly is to look at the value of $J(\theta)$ and check that it is decreasing with each step. Your value of $J(\theta)$ should never increase, and should converge to a steady value by the end of the algorithm. Another way of making sure your gradient estimate is correct is to check it againts a [finite difference (https://en.wikipedia.org/wiki/Finite_difference)](https://en.wikipedia.org/wiki/Finite_difference) approximation.

We also initialize the initial parameters to 0 and the learning rate alpha to `0.01` .

In [17]:
```
np.array([X_prime[0]]).T.shape
```

```
---------------------------------------------------------------------------
NameError                                 Traceback (most recent call last)
<ipython-input-17-0ac924ef0532> in <module>
----> 1 np.array([X_prime[0]]).T.shape

NameError: name 'X_prime' is not defined
```

In [16]:
```
import scipy.optimize
from functools import partial

def loss_gradient(X, y, theta):
    X_prime = add_column(X)
    Theta_T = np.transpose(theta)
    total = 0
    for i in range(len(X_prime)):
        pred = np.matmul(Theta_T,X_prime[i])
        predactualminustest = (pred - y[i])
        total += (np.matmul(predactualminustest,np.array([X_prime[i]])))
    loss_grad = total/97
    return np.array([loss_grad]).T
assert loss_gradient(X, y, theta_init).shape == (2, 1)

def finite_diff_grad_check(f, grad, points, eps=1e-10):
    errs = []
    for point in points:
        point_errs = []
        grad_func_val = grad(point)
        for dim_i in range(point.shape[0]):
            diff_v = np.zeros_like(point)
            diff_v[dim_i] = eps
            dim_grad = (f(point+diff_v) - f(point-diff_v))/(2*eps)
            point_errs.append(abs(dim_grad - grad_func_val[dim_i]))
        errs.append(point_errs)
    return errs

test_points = [np.random.rand(2, 1) for _ in range(10)]
finite_diff_errs = finite_diff_grad_check(
    partial(loss, X, y), partial(loss_gradient, X, y), test_points
)

print('max grad comp error', np.max(finite_diff_errs))
assert np.max(finite_diff_errs) < 1e-3, "grad computation error is too large"

def run_gd(loss, loss_gradient, X, y, theta_init, lr=0.01, n_iter=1500):
    theta_current = theta_init.copy()
    loss_values = []
    theta_values = []

    for i in range(n_iter):
        loss_value = loss(X, y, theta_current)
        #lg = loss_gradient(X, y, theta_init)
        #multiply = lr/97

        theta_current = theta_current - lr*loss_gradient(X, y, theta_current)
        loss_values.append(loss_value)
        theta_values.append(theta_current)

    return theta_current, loss_values, theta_values

result = run_gd(loss, loss_gradient, X, y, theta_init)
theta_est, loss_values, theta_values = result

print('estimated theta value', theta_est.ravel())
```

```
print('resulting loss', loss(X, y, theta_est))
plt.ylabel('loss')
plt.xlabel('iter_i')
plt.plot(loss_values)
plt.show()

plt.ylabel('log(loss)')
plt.xlabel('iter_i')
plt.semilogy(loss_values)
plt.show()
```

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 [ 1.15781984]]
```

```
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```

```
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```

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```

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```

```
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```

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```

```
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(2, 1)
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 [ 1.16533242]]
```

```
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 [ 1.16563068]]
(2, 1)
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(2, 1)
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[[-3.62740182]
 [ 1.16607206]]
(2, 1)
[[-3.62788559]
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(2, 1)
[[-3.6283685 ]
 [ 1.16616917]]
(2, 1)
[[-3.62885054]
 [ 1.1662176 ]]
(2, 1)
[[-3.6293317 ]
 [ 1.16626593]]
```
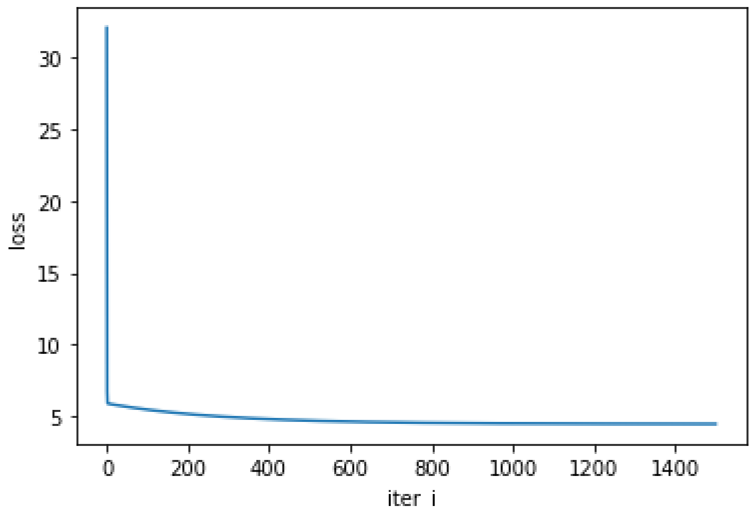
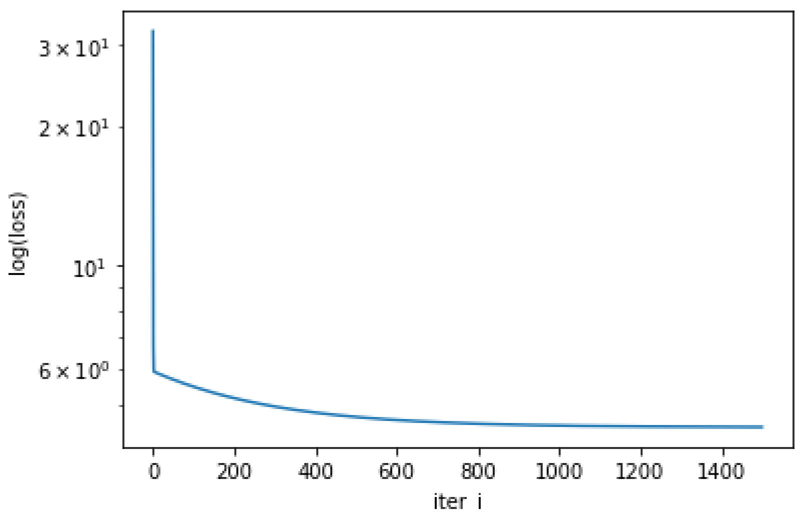
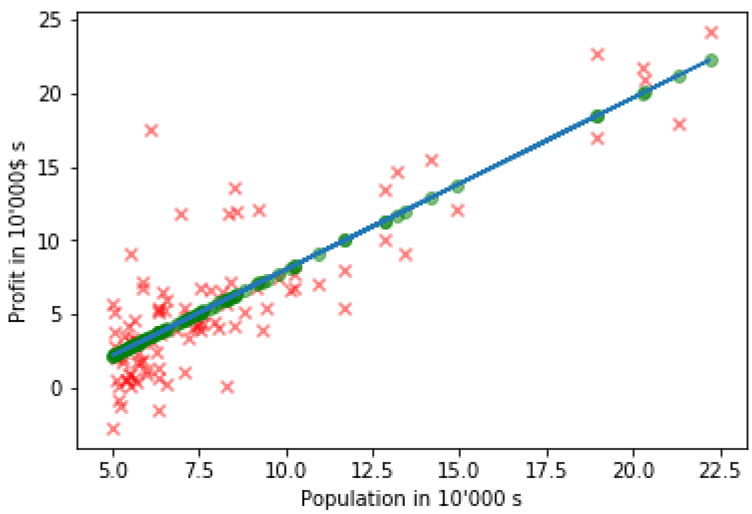

```
(2, 1)
[[-3.62981201]
 [ 1.16631419]]
(2, 1)
estimated theta value [-3.63029144  1.16636235]
[[-3.63029144]
 [ 1.16636235]]
(2, 1)
resulting loss [4.48338826]
```





2.2.4 [10pt] After you are finished, use your final parameters to plot the linear fit. The result should look something like on the figure below. Use the `predict()` function.

In [13]:
```
plt.scatter(X, y, marker='x', color='r', alpha=0.5)
x_start, x_end = 5, 25
theta = [[-3.63029144],[1.16636235]]
theta = np.array(theta)
newy = predict(X, theta)
plt.scatter(X, newy, marker='o', color='g', alpha=0.5)
plt.xlabel('Population in 10\'000 s')
plt.ylabel('Profit in 10\'000$ s')
plt.plot(X,newy.T)
plt.show()
```



Now use your final values for θ and the `predict()` function to make predictions on profits in areas of 35,000 and 70,000 people.

In [43]:
```
print(predict(np.array([[35000]]),theta))
print(predict(np.array([[70000]]),theta))
```

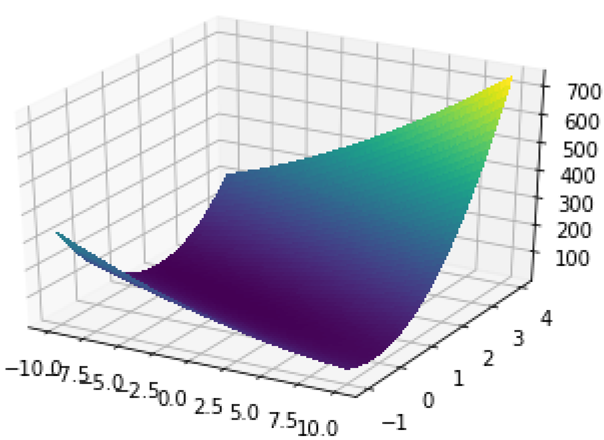
```
[[40819.05195856]]
[[81641.73420856]]
```

To understand the cost function better, you will now plot the cost over a 2-dimensional grid of values. You will not need to code anything new for this part, but you should understand how the code you have written already is creating these images.

In [44]:

```
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.cm as cm
limits = [(-10, 10), (-1, 4)]
space = [np.linspace(*limit, 100) for limit in limits]
theta_1_grid, theta_2_grid = np.meshgrid(*space)
theta_meshgrid = np.vstack([theta_1_grid.ravel(), theta_2_grid.ravel()])
loss_test_vals_flat = (((add_column(X) @ theta_meshgrid - y)**2).mean(axis=0)/
2)
loss_test_vals_grid = loss_test_vals_flat.reshape(theta_1_grid.shape)
print(theta_1_grid.shape, theta_2_grid.shape, loss_test_vals_grid.shape)

plt.gca(projection='3d').plot_surface(theta_1_grid, theta_2_grid,
                                      loss_test_vals_grid, cmap=cm.viridis,
                                      linewidth=0, antialiased=False)
xs, ys = np.hstack(theta_values).tolist()
zs = np.array(loss_values)
plt.gca(projection='3d').plot(xs, ys, zs, c='r')
plt.xlim(*limits[0])
plt.ylim(*limits[1])
plt.show()

plt.contour(theta_1_grid, theta_2_grid, loss_test_vals_grid, levels=np.logspac
e(-2, 3, 20))
plt.plot(xs, ys)
plt.scatter(xs, ys, alpha=0.005)
plt.xlim(*limits[0])
plt.ylim(*limits[1])
plt.show()
```

```
(100, 100) (100, 100) (100, 100)

---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
<ipython-input-44-0f0fb093c336> in <module>
     14 xs, ys = np.hstack(theta_values).tolist()
     15 zs = np.array(loss_values)
---> 16 plt.gca(projection='3d').plot(xs, ys, zs, c='r')
     17 plt.xlim(*limits[0])
     18 plt.ylim(*limits[1])

~\Anaconda3\lib\site-packages\mpl_toolkits\mplot3d\axes3d.py in plot(self, x
s, ys, zdir, *args, **kwargs)
   1528
   1529         # Match length
-> 1530         zs = np.broadcast_to(zs, len(xs))
   1531
   1532         lines = super().plot(xs, ys, *args, **kwargs)

~\Anaconda3\lib\site-packages\numpy\lib\stride_tricks.py in broadcast_to(arra
y, shape, subok)
    180            [1, 2, 3]])
    181     """
--> 182     return _broadcast_to(array, shape, subok=subok, readonly=True)
    183
    184

~\Anaconda3\lib\site-packages\numpy\lib\stride_tricks.py in _broadcast_to(arr
ay, shape, subok, readonly)
    127     it = np.nditer(
    128         (array,), flags=['multi_index', 'refs_ok', 'zerosize_ok'] + e
xtras,
--> 129         op_flags=[op_flag], itershape=shape, order='C')
    130     with it:
    131         # never really has writebackifcopy semantics

ValueError: input operand has more dimensions than allowed by the axis remapp
ing
```



3. Linear regression with multiple input features

3.1 [20pt] Copy-paste your `add_column` , `predict` , `loss` and `loss grad` implementations from above and modify your code of linear regression with one variable to support any number of input features (vectorize your code.)

In [60]:
```
data = np.loadtxt('ex1data2.txt', delimiter=',')
X, y = data[:, :-1], data[:, -1, np.newaxis]
n = data.shape[0]
print(X.shape, y.shape, n)
print(X[:10], '\n', y[:10])
```

```
(47, 2) (47, 1) 47
[[2.104e+03 3.000e+00]
 [1.600e+03 3.000e+00]
 [2.400e+03 3.000e+00]
 [1.416e+03 2.000e+00]
 [3.000e+03 4.000e+00]
 [1.985e+03 4.000e+00]
 [1.534e+03 3.000e+00]
 [1.427e+03 3.000e+00]
 [1.380e+03 3.000e+00]
 [1.494e+03 3.000e+00]]
 [[399900.]
 [329900.]
 [369000.]
 [232000.]
 [539900.]
 [299900.]
 [314900.]
 [198999.]
 [212000.]
 [242500.]]
```

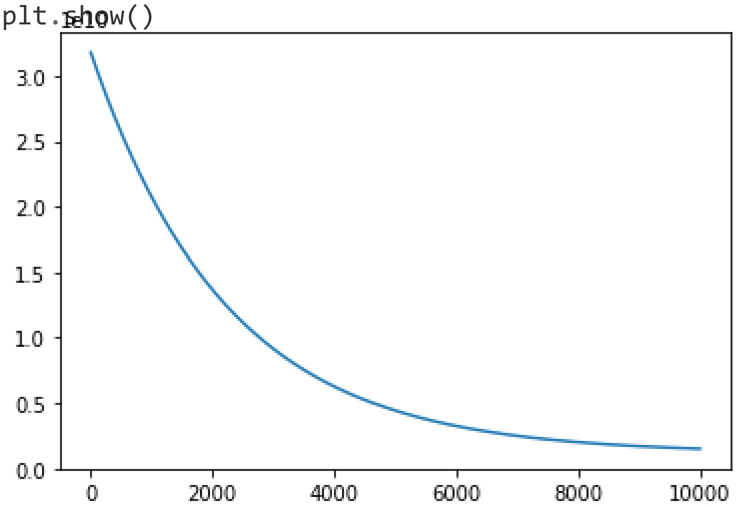
In [58]:
```
theta_init = np.zeros((3,1))
print(theta_init.shape[0] != 2)
X_prime = add_column(X)
print(X_prime.shape)
print(X_prime)
print(X_prime.shape[1])
```

```
True
(97, 2)
[[ 1.      6.1101]
 [ 1.      5.5277]
 [ 1.      8.5186]
 [ 1.      7.0032]
 [ 1.      5.8598]
 [ 1.      8.3829]
 [ 1.      7.4764]
 [ 1.      8.5781]
 [ 1.      6.4862]
 [ 1.      5.0546]
 [ 1.      5.7107]
 [ 1.     14.164 ]
 [ 1.      5.734 ]
 [ 1.      8.4084]
 [ 1.      5.6407]
 [ 1.      5.3794]
 [ 1.      6.3654]
 [ 1.      5.1301]
 [ 1.      6.4296]
 [ 1.      7.0708]
 [ 1.      6.1891]
 [ 1.     20.27  ]
 [ 1.      5.4901]
 [ 1.      6.3261]
 [ 1.      5.5649]
 [ 1.     18.945 ]
 [ 1.     12.828 ]
 [ 1.     10.957 ]
 [ 1.     13.176 ]
 [ 1.     22.203 ]
 [ 1.      5.2524]
 [ 1.      6.5894]
 [ 1.      9.2482]
 [ 1.      5.8918]
 [ 1.      8.2111]
 [ 1.      7.9334]
 [ 1.      8.0959]
 [ 1.      5.6063]
 [ 1.     12.836 ]
 [ 1.      6.3534]
 [ 1.      5.4069]
 [ 1.      6.8825]
 [ 1.     11.708 ]
 [ 1.      5.7737]
 [ 1.      7.8247]
 [ 1.      7.0931]
 [ 1.      5.0702]
 [ 1.      5.8014]
 [ 1.     11.7   ]
 [ 1.      5.5416]
 [ 1.      7.5402]
 [ 1.      5.3077]
 [ 1.      7.4239]
 [ 1.      7.6031]
 [ 1.      6.3328]
```

```
 [ 1.      6.3589]
 [ 1.      6.2742]
 [ 1.      5.6397]
 [ 1.      9.3102]
 [ 1.      9.4536]
 [ 1.      8.8254]
 [ 1.      5.1793]
 [ 1.     21.279 ]
 [ 1.     14.908 ]
 [ 1.     18.959 ]
 [ 1.      7.2182]
 [ 1.      8.2951]
 [ 1.     10.236 ]
 [ 1.      5.4994]
 [ 1.     20.341 ]
 [ 1.     10.136 ]
 [ 1.      7.3345]
 [ 1.      6.0062]
 [ 1.      7.2259]
 [ 1.      5.0269]
 [ 1.      6.5479]
 [ 1.      7.5386]
 [ 1.      5.0365]
 [ 1.     10.274 ]
 [ 1.      5.1077]
 [ 1.      5.7292]
 [ 1.      5.1884]
 [ 1.      6.3557]
 [ 1.      9.7687]
 [ 1.      6.5159]
 [ 1.      8.5172]
 [ 1.      9.1802]
 [ 1.      6.002 ]
 [ 1.      5.5204]
 [ 1.      5.0594]
 [ 1.      5.7077]
 [ 1.      7.6366]
 [ 1.      5.8707]
 [ 1.      5.3054]
 [ 1.      8.2934]
 [ 1.     13.394 ]
 [ 1.      5.4369]]
2
```

In [62]:
```
import numpy as np
def add_column(X,i):
    return np.insert(X,i,1,axis=1)

def predict(X, theta):
    X_prime = add_column(X,0)
    #count = 1
    #while X_prime.shape[1] != theta[0].shape:
    #    X_prime = add_column(X,count)
     #   count+=1
    Theta_T = np.transpose(theta)
    X_prime = np.transpose(X_prime)
    pred = np.matmul(Theta_T,X_prime)
    return pred

def loss(X, y, theta):
    X_prime = add_column(X,0)
    #count = 1
    #while X_prime.shape[1] != theta[0].shape:
     #   X_prime = add_column(X,count)
      #  count+=1
    Theta_T = np.transpose(theta)
    #print(np.array(theta))
    #print(theta.shape)
    total = 0
    for i in range(len(X_prime)):
        pred = np.matmul(Theta_T,X_prime[i])
        total += (pred-y[i])**2
    #raise NotImplementedError("Compute the model loss; use the predict() func
tion")
    loss = total/194
    return loss

import scipy.optimize
from functools import partial

def loss_gradient(X, y, theta):
    X_prime = add_column(X,0)
    #count = 1
    #while X_prime.shape[1] != theta[0].shape:
     #   X_prime = add_column(X,count)
      #  count+=1
    Theta_T = np.transpose(theta)
    total = 0
    for i in range(len(X_prime)):
        pred = np.matmul(Theta_T,X_prime[i])
        predactualminustest = (pred - y[i])
        total += (np.matmul(predactualminustest,np.array([X_prime[i]])))
    loss_grad = total/97
    return np.array([loss_grad]).T


theta_init = np.zeros((3, 1))
result = run_gd(loss, loss_gradient, X, y, theta_init, n_iter=10000, lr=1e-10)
theta_est, loss_values, theta_values = result
```

```
plt.plot(loss_values)
plt.show()
```



3.2 [20pt] Draw a histogam of values for the first and second feature. Why is feature normalization important? Normalize features and re-run the gradient decent. Compare loss plots that you get with and without feature normalization.

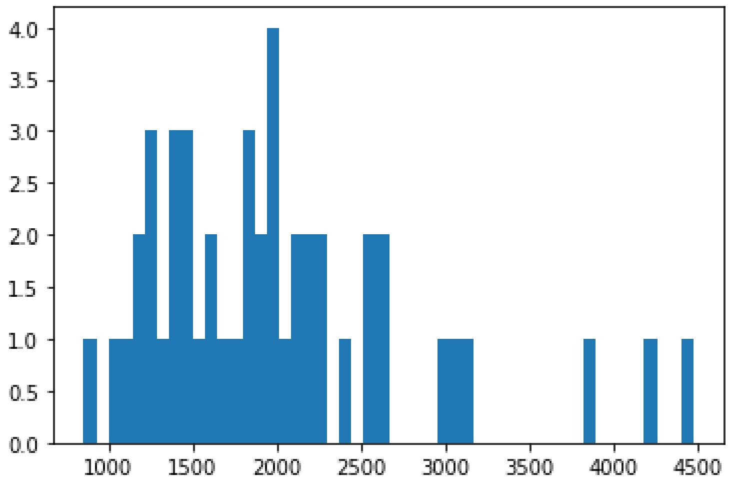
In [66]:
```
print(loss_values[5000:5010])
```

```
[array([4.42411702e+09]), array([4.42265636e+09]), array([4.42119635e+09]), a
rray([4.419737e+09]), array([4.4182783e+09]), array([4.41682026e+09]), array
([4.41536287e+09]), array([4.41390613e+09]), array([4.41245004e+09]), array
([4.41099461e+09])]
```

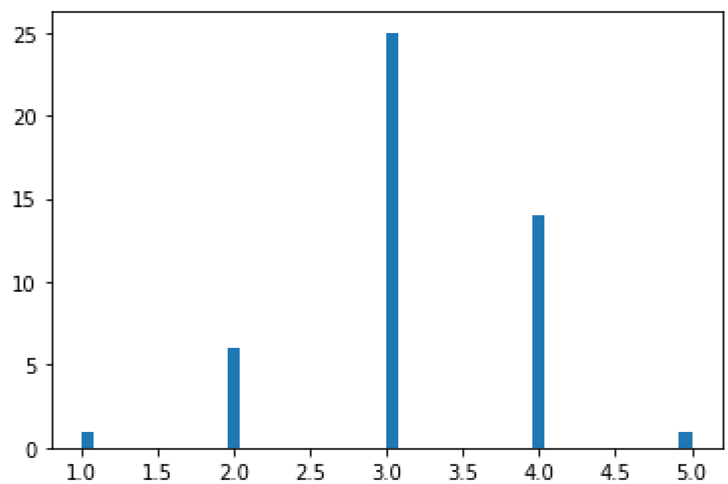
In [74]:
```
X
```

Out[74]:
```
array([[2.104e+03, 3.000e+00],
       [1.600e+03, 3.000e+00],
       [2.400e+03, 3.000e+00],
       [1.416e+03, 2.000e+00],
       [3.000e+03, 4.000e+00],
       [1.985e+03, 4.000e+00],
       [1.534e+03, 3.000e+00],
       [1.427e+03, 3.000e+00],
       [1.380e+03, 3.000e+00],
       [1.494e+03, 3.000e+00],
       [1.940e+03, 4.000e+00],
       [2.000e+03, 3.000e+00],
       [1.890e+03, 3.000e+00],
       [4.478e+03, 5.000e+00],
       [1.268e+03, 3.000e+00],
       [2.300e+03, 4.000e+00],
       [1.320e+03, 2.000e+00],
       [1.236e+03, 3.000e+00],
       [2.609e+03, 4.000e+00],
       [3.031e+03, 4.000e+00],
       [1.767e+03, 3.000e+00],
       [1.888e+03, 2.000e+00],
       [1.604e+03, 3.000e+00],
       [1.962e+03, 4.000e+00],
       [3.890e+03, 3.000e+00],
       [1.100e+03, 3.000e+00],
       [1.458e+03, 3.000e+00],
       [2.526e+03, 3.000e+00],
       [2.200e+03, 3.000e+00],
       [2.637e+03, 3.000e+00],
       [1.839e+03, 2.000e+00],
       [1.000e+03, 1.000e+00],
       [2.040e+03, 4.000e+00],
       [3.137e+03, 3.000e+00],
       [1.811e+03, 4.000e+00],
       [1.437e+03, 3.000e+00],
       [1.239e+03, 3.000e+00],
       [2.132e+03, 4.000e+00],
       [4.215e+03, 4.000e+00],
       [2.162e+03, 4.000e+00],
       [1.664e+03, 2.000e+00],
       [2.238e+03, 3.000e+00],
       [2.567e+03, 4.000e+00],
       [1.200e+03, 3.000e+00],
       [8.520e+02, 2.000e+00],
       [1.852e+03, 4.000e+00],
       [1.203e+03, 3.000e+00]])
```

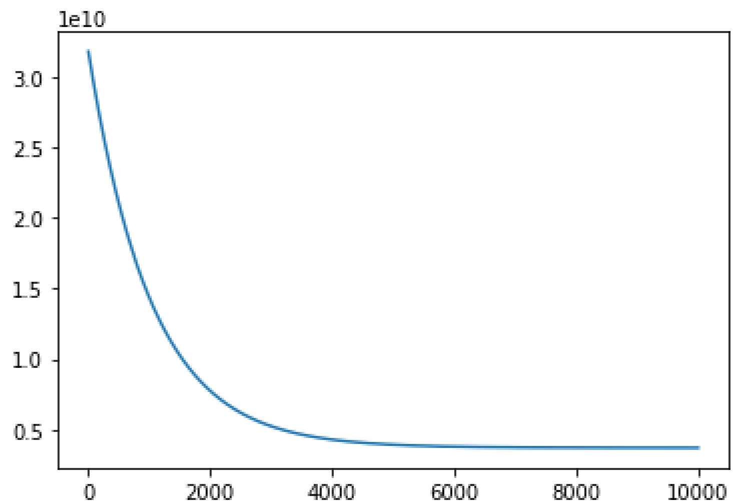
In [75]:
```
plt.hist([x[0] for x in X], bins=50)
plt.show()
```



In [76]:
```
plt.hist([x[1] for x in X], bins= 50)
plt.show()
```



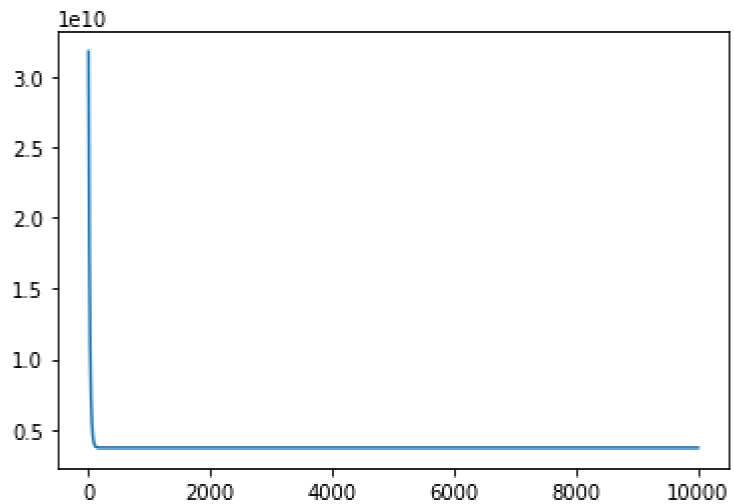
In [77]:
```
theta_init = np.zeros((3, 1))
X_normed = np.zeros_like(X)
#raise NotImplementedError("Run gd on normalized versions of feature vectors")
result = run_gd(loss, loss_gradient, X_normed, y, theta_init, n_iter=10000, lr
=1e-3)
theta_est, loss_values, theta_values = result

plt.plot(loss_values)
plt.show()
```



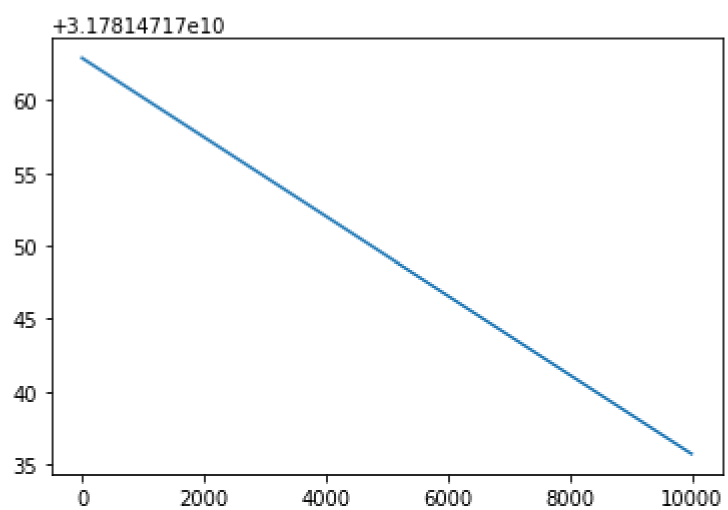
Feature normalization is important because an equal amount of proportions of features will give the most accurate outcome. If you have a feature that overtakes the other features by a substantial amount, then the outcome will be skewed towards that feature.

3.3 [10pt] How can we choose an appropriate learning rate? See what will happen if the learning rate is too small or too large for normalized and not normalized cases?

In [79]:
```
#raise NotImplementedError("Plot loss behaviour with multiple different learni
ng rates")
theta_init = np.zeros((3, 1))
X_normed = np.zeros_like(X)
result = run_gd(loss, loss_gradient, X_normed, y, theta_init, n_iter=10000, lr
=.05)
theta_est, loss_values, theta_values = result
plt.plot(loss_values)
plt.show()
```



In [80]:
```
theta_init = np.zeros((3, 1))
X_normed = np.zeros_like(X)
result = run_gd(loss, loss_gradient, X_normed, y, theta_init, n_iter=10000, lr
=.0000000000001)
theta_est, loss_values, theta_values = result
plt.plot(loss_values)
plt.show()
```



To choose an appropriate learning rate you slowly increase the learning rate to find the smallest learning rate that gives the least loss.

In [ ]: