Question 6

Case 1 $1 \le j \le n$, $\left(\sum_{i=1}^{j} a_i\right)^2 = \sum_{i=1}^{j} a_i^3$

$j=1$ $\left(\sum_{i=1}^{1} 1\right)^2 = \sum_{i=1}^{1} 1^3$

$j=i$ $1 = 1$

$j=2$ $\left(\sum_{i=1}^{2} i\right)^2 = \sum_{i=1}^{2} i^3$

$9 = 1^3 + 2^3 = 9$

6 Induction hypothesis

Base Case

$\left(\sum_{i=1}^{1} a_i\right)^2 = 1 = i$

$\sum_{i=1}^{1} a_1^3 = 1 = i$

Hypothesis Assume $a_k = k$ for any k.

$\sum_{i=1}^{k+1} a_i^3 = \left(\sum_{i=1}^{k+1} a_i\right)^2 + a_{k+1}^3$

$\sum_{i=1}^{k+1} a_i^3 = \sum_{i=1}^{k} a_i^3 + a_{k+1}^3$

$= \left(\sum_{i=1}^{k} a_i\right)^2 + a_{k+1}^3 = \left(\sum_{i=1}^{k+1} a_i\right)^2$

$\forall i, 1 \le i < n, a_i = i.$

Question 1B

$$\overset{0}{(0},\overset{1}{1},\overset{2}{1},\overset{3}{2},\overset{4}{3},\overset{5}{5},\overset{6}{8}\ldots)$$

Base case $n=2$

$f_2 \geq (1.5)^{2-2}$

$1 \geq 1$

$n=3$

$f_3 \geq (1.5)^{3-2}$

$2 \geq (1.5)^1$

Base case checks out and is true.

Induction hypothesis $\forall n \geq 2$, $f_n \geq (1.5)^{n-2}$

$f(k+1) \geq (1.5)^{k+1-2}$

$f(k+1) \geq (1.5)^{k-1}$

$f(k) = f(k-1) + f(k-2)$

$\cancel{f(k+1) = f(k) + f(k-1)}$

$\cancel{f(k-1) = f(k-2) + f(k-3)}$

$\cancel{f(k-1) \geq (1.5)^{k-3}}$

$\cancel{f(k) \geq (1.5)^{k-2}}$

$\cancel{P(k-1) \wedge P(k) \Rightarrow P(k+1)}$

$f_{n-1} + f_{n-2} \geq 1.5^{n-3} + 1.5^{n-4}$

$= 1.5^{n-4}(1.5+1)$

$= 1.5^{n-4}(2.5$

$> 1.5^{n-4}(1.5)^2$

$= 1.5^{n-2}$

$f_n \geq 1.5^{n-2}$

$P(n)$ is true

By induction, $f_n \geq 1.5^{n-3}$, $\forall n \in \mathbb{N}$.

Problem 36)

$\forall\ n \geq 2$ if $a < f_n$ or $b < f_n$ then Euclid(a,b) takes strictly fewer than n iterations

if $n=2$ $\quad a < f_2 \quad b < f_2$, Euclid(2,2) = 1 iteration

$f_n > b = r_1 = S_{k-1} \geq f_k$.

Euclid (a,b) : $a = b \cdot q_1 + r_1$

$f_n > a > b = r_1 = S_{k-1} \geq f_k$

$k < n$.

a doesn't have to be greater than b for fewer than n iterations.

Problem 2 $n \geq 1$, if $a > b \geq 1$, $E(a,b)$ takes n iterations

$a \geq f_{n+2}$

$b \geq f_{n+1}$

$\forall a \forall b (a > b \geq 1, \text{Euclid}(a,b) \text{ takes } k \text{ steps} \rightarrow a \geq f(k+2), b \geq f(k+1))$

Base Case $a = 2, b = 1$

Euclid $(2,1)$ is 1 iteration, $k=1$

$2 \geq f_3 \quad \wedge \quad 1 \geq f_2$

$2 \geq 1 \quad \wedge \quad 1 \geq 1$ True Base Case

Inductive Step

$\forall c \forall d (c > d \geq 1, \text{Euclid}(c,d) \text{ takes } k+1 \text{ steps} \rightarrow c \geq f(k+3), d \geq f(k+2))$

Euclid $(f(k+3), f(k+2)) = $ Euclid $(f(k+2), f(k+3) - (f(k+2))) =$

Euclid (f_{m+2}, f_{m+1})

Euclid $(f(k+2), f(k+1))$ takes k iterations and Euclid $(f(k+3), f(k+2))$ takes one more iteration, there will be $k+1$ iterations. This is because $b = f(k+2)$ and $a = f(k+3)$?

This is true for all a and b $a > b \geq 1$.

Question 1 c

$n \geq 0, f_n \leq 2^n - 1$

Base case $\quad n = 0$:

$$f_0 = 0 \leq 2^0 - 1 = 0$$

$n = 1$:

$$f_1 = 1 \leq 2^1 - 1 = 1$$

Both are true, base case holds.

Inductive hypothesis $\quad \forall n \geq 0, f_n \leq 2^n - 1$

for $n = k$

$$f(k) = f(k-1) + f(k-2) \leq 2^k - 1$$

$$f(k+1) \leq 2^{k+1} - 1$$

$$< f(k) + f(k-1) + f(k-2)$$

$$= 2f(k)$$

$$\leq 2(2^k - 1)$$

$$\leq 2^{k+1} - 2$$

$$\leq (2^{k+1} - 2) + 1$$

$$\leq 2^{k-1} - 1$$

$$= f(k+1) \leq 2^{k+1} - 1$$

It is ~~true~~ throught induction for all $n \geq 0$.

$$f(k) = f(k-1) + f(k-2) \leq 2^{k-2} + 2^{k-3}$$

$$= 2^{k-3}(2+1)$$

$$= 2^{k-3}(3)$$

$$< 2^{k-3}(4)$$

$$= 2^{k-3}(2^2)$$

$$= 2^{k-1}$$

$f_n \leq 2^{k-1}$ which makes $P(n)$ true

By induction $f(n) \leq 2^{n-1} \quad \forall n \in N$.

Problem 3 ~~∀n, ... a ≤ fn or b ≤ fn, then Euclid(a,b) takes less than n iterations.~~

Theorem: $\forall n, n \geq 1$ if $a > b \geq 1$ and $E(a,b)$ takes n iterations then $a \geq f_{n+2}$ and $b \geq f_{n+1}$ where f_n is the n^{th} term

Collary: $\forall n, n \geq 1$, if $a > b \geq 1$ and $b < f_{n+1}$, $E(a,b)$ takes fewer than n iterati
Proof by contradiction: "Euclid(a,b) takes more ~~or~~ equal to n iterations".

$a = b \cdot q + r$

0, 1, 1, 2, 3, 5, 8
0 1 2 3 4 5 6

~~a = 2, b = 1~~
~~$E(f_3, f_2) = 1$ iterations~~

Assume $E(a,b)$ takes more or equal to n iterations.

$b < f_3 + 1$ $b < 3$

$a > 3$, $3 > b \geq 1$

iff $n = 3$, then 3 or more iterations

Based on Euclid and $a = b \cdot q + r$, the lowest and highest number I can choose for a and b would be $a \geq$ ... and $b \leq 2$.
never have a n greater than 3 or equal to 3 as Euclid(5,3) = 3 iterations and Euclid(3,2) = 2 iterations.
Since $a > 3$, $3 > b \geq 1$, this statement is false and n must have fewer than n iterations.

Question 1 D.
~~Base case~~ $f_n = X$ for some $X > 0$

$1 + \log_2 X \le n$
$n \le 2(1 + \log_2 X)$
$f_n \le 2^{n-1}$
$X \le 2^{n-1}$
$\log_2 X \le n - 1$
$1 + \log_2 X \le n$
$f_n \ge 1.5^{n-2}$
$\Rightarrow X \ge 1.5^{n-2}$
$= \log_2 X \ge (n-2) \log_2 1.5$

$$\log_a b = \frac{1}{\log_b a}$$

$$n \le \frac{\log_2 X}{\log_2 1.5} + 2$$

$$n \le (\log_{1.5} 2) \cdot \log_2 X + 2$$

$$\le 2\log_2 X + 2$$

$$= 2(\log_2 X + 1)$$

$$n \le 2(1 + \log_2 X)$$

$$1 + \log_2 X \le n \le 2(1 + \log_2 X)$$

By induction $\forall n$, $1 + \log_2 X \le n \le$ 2(1 + \log_2 X)$

Problem 4

Base Case: The number of posters can be shown by 2^N

When $N=1$, either posters can be put in the closet and no posters on the wall, or the posters ~~are~~ hung on the wall, $\emptyset$ and $\{1\}$ where 1 stands for one poster. The number of distinct posters is $2^1=2$, which is true when $N=1$.

Induction Hypothesis: There are $k+1$ distinct ~~pentants~~ posters, $1,2,3,4,5 \ldots k, k+1$. We can put $k+1$ posters in the closet, therefor 2^k posters can be ~~the~~ distinct poster sets of $k+1$. We can put the poster $k+1$ with each of the ~~posters~~ other posters. If $\{1, \cancel{4}\}$ is a poster set of $\{1,2 \ldots k\}$, then $\{1,4,k+1\}$ is a possible poster set: We can get 2^k poster sets, for each poster set there exists $k+1^{th}$ poster as one of the posters. The number of distinct posters is $2^k+2^k=2\times 2^k=2^{k+1}$. It is true for $1=k+1$ if it is true for $N=k$.

Problem 5: $\forall n, n\geq 0 \; \left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$

Prove:

Base Case $n=0$

$$\left(\sum_{i=1}^{0} i\right)^2 = \sum_{i=1}^{0} i^3 = 0 = 0$$

$n=1$

$$\left(\sum_{i=1}^{1} i\right)^2 = \sum_{i=1}^{1} i^3 = 1 = 1$$

Induction hypothesis

$$\left(\sum_{i=1}^{k} i\right)^2 = \sum_{i=1}^{k} i^3$$

to Prove: $\forall n, n\geq 0 \; \left(\sum_{i=1}^{k+1} i\right)^2 = \sum_{i=1}^{k+1} i^3$

$$\cancel{(1+k+(k+1))^2} = \cancel{1^3+k^3+(k+1)^3}$$

$$\cancel{(1+k+(k+1))(1+k+(k+1))} =$$ Summation formula: $\frac{1}{4} n^2(n+1)^2$

$$\cancel{(2k+2)^2} =$$

$$\left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2(k+2)^2}{4}$$

It is true for $P(k+1)$, so $P(k) \rightarrow P(k+1)$ is also true.

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$\overset{0}{}\ \overset{1}{}\ \overset{2}{}\ \overset{3}{}\ \overset{4}{}$ $(0, 1, 1, 2, 3 \ldots)$

1a) $n \geq 0$ $f_0 + f_1 \ldots + f_n = f_{n+2} - 1$

Base case $n = 0$

$f_0 + f_1 \ldots f_0 = f_{0+2} - 1$

$0 = 1 - 1 = 0$

$n = 1$

$f_0 + f_1 \ldots f_1 = f_{1+2} - 1$

$1 = 2 - 1 = 1$

Base case checks out.

Induction hypothesis $\forall n \geq 0, f_0 + \ldots f_n = f_{n+2} - 1$.

$f(k) = f(k-1) + f(k-2)$

Prove: $f_0 + f_1 + f(k) + f(k+1) = f(k+3) - 1$

$f_0 + f_1 + f(k) + \cancel{f(k+1)} = f(k+2) + \cancel{f(k+1)} - 1$

~~$[f(k+3) = f(k+2) + f(k+1)]$~~

~~$f_0 + f_1 + f(k) = f(k) + f(k+1) - 1$~~

~~$[f(k+3) = f(k) + f(k+1)]$~~

~~$f_0 + f_1 = f(k+1) - 1$~~

~~$0 + 1 = f(k+1) - 1$~~

$f_0 + f_1 + f(k) = f(k+2) - 1$

Proves for all $n \geq 0$, $f_0 + f_1 + f(k) = f(k+2) - 1$

$f_0 + f_1 + \ldots f_{n+1}$

$(f_0 + f_1 + f_n) + f_{n+1}$

$(f_{n+2} - 1) + f_{n+1}$

$f_{n+3} - 1$

& $P(n+3)$ is true

$\forall n \in N, f_1 + f_2 + \ldots + f_n = f_{n+2} - 1$.

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$\forall\ n \geq 2$ if $a < f_n$ or $b < f_n$ then Euclid(a,b) takes strictly fewer than n iterations

if $n = 2$ $\quad a < f_2 \quad b < f_2$, Euclid$(2,2) = 1$ iteration

$f_n > b = r_1 = S_{k-1} \geq f_k$.

Euclid $(a,b): a = b \cdot q_1 + r_1$

$f_n > a > b = r_1 = S_{k-1} \geq f_k$

$K < n$.

a doesn't have to be greater than b for fewer than n iterations.

if $a = b$ then $\frac{a}{b} = q_1 + r_1$

$1 = q_1 + r_1$, so it will always be 1 iteration