CS 131 – Spring 2020, Assignment 5

Problems must be submitted by Thursday March 5, 2020 11:59pm, on Gradescope.

Problem 1. [32 points] For each of the following determine whether the relation on the given set is reflexive, symmetric, transitive, and an equivalence relation (where $\iff$ means if and only if). Justify each of your answers (with proof or counterexamples).

1. $R = \{(f, g) \mid f(1) = g(1)\}$ on the set A $=\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$

   R is Reflexive
   A relation f(x) on a set A is called reflexive if (f(1),f(1)) $\in$ f(x) for every element x $\in$ f(x).
   f(1) = f(1)
   R is Symmetric because (f(1),g(1)) $\in$ f(x) whenever (g(1),f(1)) $\in$ f(x) for all f(1),g(1) $\in$ f(x)
   g(1) = f(1), and f(1) = g(1)
   R is transitive because if you have a function h(x) where h(1) is equal to g(1), then you will know that h(1)= f(1) because g(1) = f(1).
   It is an equivalence relation, as it is only Symmetric and Reflexive and Transitive

2. $R = \{(f, g) \mid f(0) = g(0)$ **or** $f(1) = g(1)\}$ on set A $=\{f \mid f : \mathbb{Z} \to \mathbb{Z}\}$ R is Reflexive
   g(1)= g(1) and g(0) = g(0)

   R is Symmetric
   if g(1) = f(1) or g(0) = f(1) then, f(1) = g(1) or f(0) = g(0)

   It can not be positive that R is transitive.
   If you have f(1) = h(1), it is not the case that g(0) = h(1), so it could or couldn't be transitive.
   It is not an equivalence relation, as it is only Symmetric and Reflexive

3. $((a, b), (c, d)) \in R \iff a + d = b + c$ on A = $\mathbb{Z} \times \mathbb{Z}$ R is Reflexive
   Since A+B=B+A, then on the set of (a,b), a*b=b*a for all A = $\mathbb{Z} \times \mathbb{Z}$, same could be said with (c,d)
   (a,b) is a relation to (a,b)

   R is Symmetric
   (a,b)R(c,d), we have to get (c,d)R(a,b), since (a,b)R(c,d) = (a + d = b + c), then (c,d)R(a,b) = (b + c = d + a)

   R is Transitive
   if (a,b)R(c,d) then, (c,d)R(e,f)
   (a,b)R(c,d) = a + d = b + c == a - b = c - d
   (c,d)R(e,f) = c + f = d + e == c - d = e - f
   therefore, there is a relation where (a,b)R(e,f)

   It is an equivalence relation, as it is only Symmetric and Reflexive and Transitive

4. $xRy \iff x - y \in \mathbb{Z}$ on A = $\mathbb{R}$ xRy if and only if x - y is an element of Z on A = R R is Reflexive
   If xRy only if x - y is an integer, then xRx, and yRy.
   xRx would mean x - x, which is 0, an integer, same goes for yRy
   R is Symmetric
   if xRy then, x - y is an integer, so if yRx, then y - x is an integer.
   R is Transitive
   if xRy, then x - y is an integer. if yRh, then y - h is an integer.
   we need to get xRh, where x - h is an integer, you can add x - y and y - h to get x - h.
   so, xRh is a relation

   It is an equivalence relation, as it is only Symmetric and Reflexive and Transitive

Problem 2. [24 points] We wish to investigate a new type of relation which we will call Vahidean. A relation R on the set A is Vahidean if:

$$\forall a, b, c \in A \;\; aRb \wedge aRc \rightarrow bRc$$

a) Explain, in English, how Transitivity and Vahideanness are different. Give an example of a Vahidean relation on $\{a, b, c, d\}$ which is not transitive (draw its digraph representation). Give a counterexample that shows your relation is not transitive.

In vahidian, aRb and aRc means bRc, but in transitivity aRb and bRc means aRc. You can't say aRb and bRa are reflexive, so it is not transitive.

$$\begin{tikzcd} b & d \\ a \arrow[r] & c \arrow[ul] \end{tikzcd}$$

Explanation of nodes: Implication statements, if your hypothesis is false, the whole thing is true. It is Vahidean, but the if part of the statement is false. If it was a,b,c,d with no arrows, it would be automatically transitive due to the if statement not being false or true.

b) Is the following an implication of Vahideanness?

$$\forall a, b, c \in A \;\; aRb \wedge aRc \rightarrow cRb$$

Explain why or why not.
It is an implication of Vahideaness because due to the laws of proposition, you can swap aRb $\wedge$ aRc, so it becomes aRc $\wedge$ aRb, which allows you to get the relation of cRb, which is vahidean.

c) Prove that if a relation R is Reflexive and Vahidean then R is an equivalence relation. Since the original Vahedean implication is neither true or false, making it reflexive allows you to have bRa $\wedge$ aRc $\rightarrow$ bRc, which proves that it is transitive, and true. As for symmetry, aRa $\wedge$ aRa $\rightarrow$ aRa will also be true. Therefore, it would be an equivalence relation.

Problem 3. [24 points] You are a lonely logician in desperate need of a friend. However, knowing that it would be impossible to make one, you decide to focus your time on something much more achievable: creating a general AI (artificial intelligence) who will then be your friend!

Amazingly, you succeed! You make yourself a friend whom you name Alan. Now all you need to do is teach Alan what “friendship” is. You decide to model “friendship” as a relation on a set of people. Rather than explicitly define the relation, you decide to explain some rules that any relation modelling friendship must follow. These are your three laws of robotic friendships:

1. “Not everyone is friends with everyone” (as you harshly know from your lack of human friends).
2. “The enemy of my enemy is my friend” (this is a good saying you’ve heard before! For simplicity, you tell Alan “enemy” just means “not friend”).
3. “The enemy of my friend is my enemy.”

With these three laws defining friendship, what will Alan learn?

a) Draw the digraph of a “friendship” relation on a set of 3 people which satisfies the three properties above. Explain how you arrived at your relation.

$$\begin{tikzcd} b \arrow[loop right] \arrow[d, leftrightarrow] & \\ a \arrow[loop right] & c \arrow[loop right] \end{tikzcd}$$

I started with the second statement, and worked my way backwards, as well as translated the second statement to ”the not friend of my not friend is my friend”, and the first statement to ”there exists a person that is friends with everyone”.

b) Does Alan think “friendship” is an equivalence relation? Explain why or why not. If it is, how many equivalence classes does it define on a set of 100 people? Explain in English how you arrived at your answer.
Alan thinks ”friendship” is a equivalence relation because it is symmetric, transitive and reflexive. He does think it is an equivalence relationship, if you add another person, they become friends with the two that are already friends with each other, because they are not friends with you.
There can only be at max two equivalence relations, as you keep adding there will only be two groups at once time. Doesn’t matter if it’s a million people. by the third rule, they will automatically be part of the friend group.

$$\begin{tikzcd} b \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & d \arrow[loop right] \\ a & c \end{tikzcd}$$

if you add a d, shown above, he becomes friends with the group already involved with each other
if you have a group of two people, you cannot consider them both enemies at once, rather you choose friend or enemy, either way, if you choose friend you all are friends, if you choose enemy, you are friends with one but not the other, but it is still transitive, reflexive and symmetric
it is always reflexive because you are friends with yourself
it will always be symmetric because if you are friends with one person, they are automatically friends with you.

Problem 4. [20 points] Prove that if $X \to Y$ and $Z \to T$, then $(X \wedge Z) \to (Y \wedge T)$ using **Hypothesis Introduction.**
Solution:

1. $X \rightarrow Y$ and $Z \rightarrow T$ Premise
2. $X \rightarrow Y$ Simplification 1
3. $Z \rightarrow T$ Simplification 1
4. $(X \wedge Z)$ Hypothesis Introduction (imagine to be True)
5. X Simplification 4
6. Z Simplification 4
7. Y Modus Ponen 2
8. T Modus Ponen 3
9. $(X \wedge Z)$ Conjunction 5,6
10. $(Y \wedge T)$ Conjunction 7,8
11. $(X \wedge Z) \rightarrow (Y \wedge T)$ Hypothesis Elimination 9,10

Problem 5. [20 bonus points] The following python function takes a set A and a relation R on A (as a python function such that $R(x, y)$ returns true if and only if the relation xRy holds) and returns True if and only if the relation R on A is symmetric.

```
def is_symmetric(A, R):
  for x in A:
    for y in A:
      if R(x, y) and not R(y, x):
        return False
  return True
```

You can test this function using the following functions and test calls.

```
s = {1,2,3}
def y(x, y):
  return x == y
def n(x, y):
  return x > y

is_symmetric(s, y)  # returns True
is_symmetric(s, n)  # returns False
```

a) Write a function in python that takes a set A and a relation $R(x, y)$ on A (as a python function such that $R(x, y)$ returns true if and only if the relation xRy holds), and returns True if and only if the relation R is reflexive.

Here is the function signature you need to use.

```
def is_reflexive(A, R):
```

You can test your code as follows.

```
s = {1,2,3}
def y(x, y):
  return x == y
def n(x, y):
  return x == y and x != 3

is_reflexive(s, y)  # returns True
is_reflexive(s, n)  # returns False
```

b) Write a function in python that takes a domain `domain` (as a python set) and a function f (with domain `domain`) and returns True if and only if f is injective.

Here is the function signature you need to use.

```
def is_injective(domain, f):
```

You can test your code as follows.

```
s = {1,2,3}
def double(x):
  return x * 2
def halve(x):
  return x // 2  # integer division rounds down

is_injective(s,double)  # returns True
is_injective(s,halve)  # returns False
```

c) Write a function in python that takes a set A and a relation R on A (as a python function such that $R(x, y)$ returns true if and only if the relation xRy holds), and returns True if and only if the relation R is transitive.

Here is the function signature you need to use.

```
def is_transitive(A, R):
```

You can test your code as follows.

```
s = {1,2,3}
def g(x, y):
  return x >= y
def m(x, y):
  return (x + y) % 2 == 1 #% is modulus, a % b returns the remainder of a/b

is_transitive(s,g)  # returns True
is_transitive(s,m)  # returns False
```