Gordon Ng Problem Set 11

2. $CDF = F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

$Y = \{(0 \to a), \frac{a}{2}\}$

$P(Y<b) = P(X<b) = \int_0^b \frac{1}{a}\,dx = \frac{b}{a} - 0 = \frac{b}{a}$

CDF =

$$F_X(Y) = \begin{cases} \frac{y}{a} & \text{if } 0 \leq y < \frac{a}{2} \\ 1 & \text{if } \frac{a}{2} \leq y \end{cases}$$

$E_X(Y) = \int_0^a (1 - F_X(y)\,dy)$

$$\int_0^{a/2} \left(1 - \frac{y}{a}\right) dy + \int_{\frac{a}{2}}^{\infty} (1-1)\,dy$$

3) $T \sim Exp(\lambda)$

$PDF = F_\lambda(t) = \lambda e^{-\lambda t},\ t > 0 \quad X = \lfloor T \rfloor, T > 0 \to X = \{0, \ldots 1, 2, 3\}$

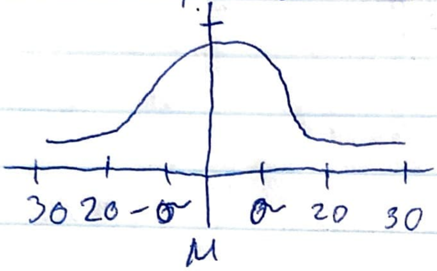
$P(X = x) = P_r(\lfloor T \rfloor) = Pr(x \leq T \leq x+1)$

$$\int_x^{x+1} f_T(t)\,dt = \int_x^{x+1} \lambda e^{-\lambda t}\,dt = \lambda\left[\frac{e^{-\lambda t}}{-\lambda}\right]_x^{x+1} = e^{-\lambda x} - e^{-\lambda(x+1)}$$

$$e^{-\lambda x}(1 - e^{-\lambda}) = (e^{-\lambda})^x (1 - e^{-\lambda})$$

$X \sim Geometric(p = 1 - e^{-\lambda})$ = Geometric parameter

4) $(\mu-\sigma, \mu+\sigma) = 68\%$
$(\mu-2\sigma, \mu+2\sigma) = 95\%$
$(\mu-3\sigma, \mu+3\sigma) = 99.7\%$



A starts 1 standard deviation away from μ.
$A = 100 - (50+34) = 16\%$ or $4/25$ or $8/50$

B starts from μ and goes to 1 standard deviation
$B = 68\%/2 = 34\%$ or $17/50$

C starts from μ and goes to 1 standard deviation.
$C = 68\%/2 = 34\%$ or $17/50$

D starts from 2 standard deviations then to 1 standard deviation
$D = 50 - (34+25) = 13.5\%$ or $27/200$
F starts from 2 standard deviations to less than
$F = 2.5\% = 1/40$

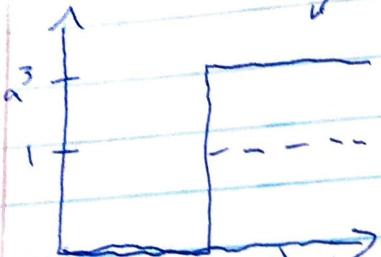
$\boxed{A = 8/50,\ B = 17/50,\ C = 17/50,\ D = 27/200,\ F = 1/40}$

5. $E(x) = 10$ seconds
$Pr(x \geq 60) \leq \frac{Ex(x)}{60}$
$Pr(x \geq 60) =$ Upperbound $= \frac{1}{6}$

$E(x) = 10$ seconds $Var(x) = 25$ seconds
$Pr(x \geq 60) \leq 25/2500$ Or $1/100$
Upperbound $= 1/100$

6. Markov's Inequality $P(x \geq a) \leq \frac{E[x]}{a}$

Prove $P(x \geq a) \leq \frac{K}{a^3}$ $\quad E_x(x^3) = K$ for any $a > 0$.



$a^3 P(x \geq a) \leq E[x]$ (Model for what we're trying to prove)

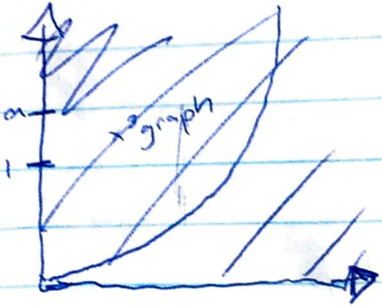
$E[x] = k$

$E(x)$ is weighted average of the values.

$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

Given $E_x(x^3) = K$

$P(x^3 \geq a) \leq \frac{x^3}{a}$



Since (x^3) is non-negative Markov's inequality can be applied and x^3 would mean X is also non-negative, because ~~x^3 would be positive but x^3 is non-negative~~ if you started with a negative X, then you have a negative X^3.

Prove:

7. $Pr\left(X \geq \sqrt{2} + \sum_{i=1}^{n} p_i\right) \leq \sum_{i=1}^{n} \frac{p_i(1-p_i)}{2}$

Given:

$X_1 \dots X_n$ independent Bernoulli rand. var.

$P(X=1) = p$ and $P(X=0) = 1-p$

$E_X = p$ and $E_{X^2} = p$, $Var\ X = p - p^2 = p(1-p)$

$Pr\left(X \geq \sqrt{2} + \sum_{i=1}^{n} p_i\right)$

$Var(X) = E[(X-m)^2]$ where m is the expected value of $E(X)$

$Pr\left(X^2 \geq 2 + \sum_{i=1}^{n} p_i\right)$

$\left(\sum_{i=1}^{n} p_i\right)^2$ is the same as $Var(X)$, so

$Pr\left(X \geq \sqrt{2} + \sum_{i=1}^{n} p_i\right) \leq \sum_{i=1}^{n} \frac{p_i(1-p_i)}{2} \left(\frac{Var(X)}{2}\right)$

according to Markov inequality

$Pr(X \geq c \cdot E(X)) < \frac{1}{c}$, where $c = 2$.

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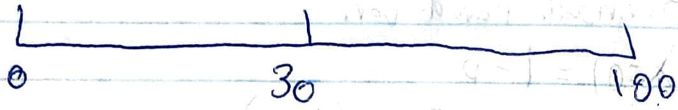
a) $V = \frac{d}{t}$ $T = \frac{d}{V}$ $V = 60\text{mph}$

Let D be the distance ambulance travels $T = D/60$



$Pr(T > 30) = Pr(J > 30)$ $J > 30 = 40/100$ or $\cancel{4}2/5$

$\boxed{Pr(T > 30) = 2/5}$

b) $Pr(T > t) = 1 - Pr(T \leq t)$

$Pr(T \leq t)$ = accident occurs in $(30-t, 30+t)$

$\frac{(30+t)-(30-t)}{100} = \frac{2t}{100}$ if $0 < t \leq 30$

$Pr(T \leq t)$ = accident occurs in $[0, 30+t]$

$\frac{30+t-0}{100} = \frac{30+t}{100}$ if $30 < t \leq 70$

$$\boxed{F(t) = Pr(T > t) = \begin{cases} 1 - 2t/100 & \text{if } 0 < t \leq 30 \\ 1 - \frac{30+t}{100} & \text{if } 30 < t \leq 70 \end{cases}}$$

c) PDF of T

$$P(T = t) = \begin{cases} 1/60 & \text{for } 0 \leq n \leq 60 \\ 0 & \text{otherwise} \end{cases}$$

d) $E(T) = b+a/2$ $60/2 = 30$

$Var(T) = \frac{(b-a)^2}{12}$ $\frac{3600}{12} = 300$