Probability 2

MML 6.2, 6.3

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Probability Space

- **Outcome space**: Set Ω
- **Event space** (sigma algebra): $\mathcal{F} \subset 2^{\Omega}$
- **Probability function**: $P\colon \mathcal{F} \to [0, 1]$

Random Variable

A $\mathcal{T}$-valued random variable X (upper case!) is a function:

$$X \colon \Omega \to \mathcal{T}$$

Examples $\Omega = \{J, E, M\}$:

Random Variable: Pre-image (inverse)

Pre-image $X^{-1}\colon \mathcal{T} \to 2^{\Omega}$ defined as

$$X^{-1}(x) = \{\omega \in \Omega \mid X(\omega) = x\}$$

Examples $\Omega = \{J, E, M\}$:

Probability Distribution

Always associated with a random variable for some $X\colon \Omega \to \mathcal{T}$

$$\mathbb{P}\left[X = x\right] = P\left(X^{-1}(x)\right) = P\left(\{\omega \in \Omega \mid X(\omega) = x\}\right)$$

Today

1. Discrete probability distribution
2. Joint distribution
3. Sum rule, product rule
4. Bayes theorem
5. Continuous probability distribution
6. Expectation(maybe)

Joint Probability

Discrete random variables X, Y

$$\mathbb{P}[X = x, Y = y] \; = \; P(X^{-1}(x) \cap Y^{-1}(y)) = \dots$$

Probability mass function: $p_{X,Y}(x, y)$

Marginalization (Sum Rule)

Know $\mathbb{P}[X = x, Y = y]$ and need to compute $\mathbb{P}[X = x]$:

$$\mathbb{P}[X = x] = \sum_{y \in \mathcal{T}} \mathbb{P}[X = x, Y = y]$$

Conditional Probability

Random variables: $X, Y : \Omega \to \mathcal{T}$

$$\mathbb{P}[X = x \mid Y = y] \; = \; \frac{\mathbb{P}[X = x, Y = y]}{\mathbb{P}[Y = y]}$$

Product Rule

Random variables: $X, Y \colon \Omega \to \mathcal{T}$

$$\mathbb{P}[X = x, Y = y] \;=\; \mathbb{P}[X = x \mid Y = y] \cdot \mathbb{P}[Y = y]$$

Independence

Random variables $X, Y\colon \Omega \to \mathbb{R}$ are **independent** if for all $x, y \in \mathbb{R}$:

$$\mathbb{P}\left[X = x, Y = y\right] = \mathbb{P}\left[X = x\right] \cdot \mathbb{P}\left[Y = y\right]$$

Examples?

Bayes Theorem

Random variables: $X, Y : \Omega \to \mathcal{T}$

$$\mathbb{P}[X = x \mid Y = y] = \frac{\mathbb{P}[Y = y \mid X = x] \cdot \mathbb{P}[X = x]}{\mathbb{P}[Y = y]}$$

Proof:

Inference Problem

$$\mathbb{P}[X = x \mid Y = y] = \frac{\mathbb{P}[Y = y \mid X = x] \cdot \mathbb{P}[X = x]}{\mathbb{P}[Y = y]}$$

Prevalence: 1%, Pos. when sick: 90%, Neg. when healthy: 95%

Probability of sick when positive test:

Discrete Probability Distributions

- **Bernoulli**: Heads or tails
- **Binomial**: Number of heads
- **Geometric**: Coin flips until heads
- **Poisson**: Number of customers

See Wikipedia for their properties

Continuous Random Variable

Usually real-valued: $X : \Omega \to \mathbb{R}$

Example: Probability space

- $\Omega = [0, 1]$
- $\mathcal{F}$ = Borel σ-algebra of all intervals (open and closed)
- $P([a, b]) = b - a$ for $a \leq b$ (Lebesgue mesure)

Random variable: $X(\omega) = \omega$ for each $\omega \in \Omega$

$$\mathbb{P}[X = 0.5] = \qquad\qquad ?$$

Cumulative Distribution Function

$$F_X(x) \; = \; \mathbb{P}\left[X \leq x\right] = P\left(\{\omega \in \Omega \mid X(\omega) \leq x\}\right)$$

Example: Uniform random variable on $[0, 1]$

Probability Density Function

Function $f_X : \mathbb{R} \to \mathbb{R}$ is a pdf of X if $\forall x$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Atomic Random Variables

Random variable X need not have pdf!
Let $\mathbb{P}\left[X = 0\right] = \mathbb{P}\left[X = 1\right] = \mathbb{P}\left[X = 2\right] = \frac{1}{3}$

Continuous Probability Distributions

- **Normal**: common because of central limit theorem
- **Laplace**: Extreme weather events
- **Multivariate normal**: Height and weight

See Wikipedia for their properties

Expected Value (Mean)

Random variable X (discrete and continuous)

$$\mathbb{E}\left[X\right] = \sum_{x \in \mathcal{X}} x \cdot \mathbb{P}\left[X = x\right], \qquad \mathbb{E}\left[X\right] = \int_{\Omega} X dP$$

Expected Value of Function

Random variable X (discrete and continuous) and a function g

$$\mathbb{E}\left[g(X)\right] = \sum_{g \in \mathcal{G}} g \cdot \mathbb{P}\left[g(X) = g\right], \qquad \mathbb{E}\left[X\right] = \int_{\Omega} g(X) dP$$

Law of the Unconscious Statistician proves that: