Probability 1

MML 6.1

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Probability Space

Formal model of randomness and uncertainty

≡ Sample space

- **Outcome space**: Set Ω

  Specific (UNH students)

  - $\Omega = \{ J \text{ (John)}, E \text{ (Eve)}, M \text{ (Mary)} \}$
  - set of possible outcomes

Abstract

$\Omega = [0,1]$ ≡ Real #s from $[0,1]$

- **Event space** (sigma algebra): $\mathcal{F} \subset 2^{\Omega}$

  set of all subsets of Ω

  - $\{ \emptyset, \{J\}, \{E\}, \{M\}, \{J,E\}, \{J,M\}, \{E,M\}, \{J,E,M\} \}$

- **Probability function**: $P\colon \mathcal{F} \to [0,1]$

  Probability of observing an event

  - Example: Seeing students at school

    → relative probability of coming class.

$P(\{J\}) = 0.5$

$P(\{M\}) = 0.25$

$P(\{M,J\}) = 0.75$

$P(\{J\})$ = if I see a student, probability of it being John

$P(\{J,M\})$ = see a student ⇒ prob of John or Mary

Probability Space: Properties

- **Outcome space**: Set Ω $\quad \Omega = \{J, M, E\}$
  - Finite or infinite $\quad \Omega = [0, 1]$
- **Event space** (sigma algebra): $\mathcal{F} \subset 2^{\Omega}$, Suppose $A_1 \in \mathcal{F}$, $A_2 \in \mathcal{F}$
  - Contains sample space: $\Omega \in \mathcal{F}$
  - Closed under complements: $\Omega \setminus A_1 \in \mathcal{F}$
  - Closed under countable unions: $A_1 \cup A_2 \in \mathcal{F}$
  - Closed under countable intersections: $A_1 \cap A_2 \in \mathcal{F}$

- **Probability measure**: $P \colon \mathcal{F} \to [0, 1]$
  - Mesure of sample space equals 1: $P(\Omega) = 1$ $\quad$ I see a student
  - Countably additive: $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

    when $A_1 \cap A_2 = \emptyset$ (disjoint)

Random Variable

A $\mathcal{T}$-valued random variable X (upper case!) is a function:

$$X : \Omega \to \mathcal{T}$$

$\mathcal{T}$ is type of property

Examples $\Omega = \{J, E, M\}$: Represents people's properties

$\mathcal{T} = \mathbb{R}$ = real numbers

height (in m) $H: \Omega \to \mathbb{R}$ $H(J) = 1.8$ $H(M) = 1.9$ $H(E) = 1.6$

gender $G: \Omega \to \{m, f, o\}$

$G(J) = m$ $G(E) = f$ $G(M) = f$

Random Variable: Common Types

$X : \Omega \to J$

1. **Continuous (real-valued)** (infinite J)

   Height, Weight, Temperature, Brightness, Profit, ...

2. **Discrete** (finite J)

   Species, Element, Color, City ...

Random Variable: Pre-image (inverse)

The basis of making probability statements

Pre-image $X^{-1}\colon \mathcal{T} \to 2^{\Omega}$ defined as

$$X^{-1}(x) = \{\underline{\omega} \in \Omega \mid \underbrace{X(\omega) = x}_{\text{elements with the property}}\} \in \mathcal{F} \equiv \text{event space}$$

Examples $\Omega = \{J, E, M\}$:

$G(J) = m$, $G(E) = f$, $G(M) = f$

$G^{-1}(m) = \{J\}$

$G^{-1}(f) = \{E, M\}$

Always associated with a random variable for some $X : \Omega \to \mathcal{T}$

$$\mathbb{P}[X = x] = P\left(X^{-1}(x)\right) = P\left(\{\omega \in \Omega \mid X(\omega) = \cancel{x}\}\right)$$

If I see a student, how probably are they female?

$$\mathbb{P}[\underset{\substack{\uparrow \\ \text{r.v.}}}{G} = f] = P\left(G^{-1}(f)\right) = P\left(\{E, M\}\right) = 0.5$$

$$\mathbb{P}[G = m] = P(\{J\}) = 0.5$$

when it is continuous:

$$\mathbb{P}[\underset{\substack{\uparrow \\ \text{height}}}{H} \leq 1.8] = P\left(\{\omega \in \Omega \mid H(\omega) \leq 1.8\}\right) = P\left(\{J, M\}\right) = 0.75$$

Probability Distributions Describes a Random Variable

Wikipedia is a good reference for their properties

Discrete random variable:

- Bernoulli: Heads or tails
- Binomial: Number of heads
- Geometric: Coin flips until heads
- Poisson: Number of customers

Continuous random variable:

- Normal: Central limit theorem
- Multivariate normal: Height and weight
- Laplace: Extreme weather events

Regression vs Classification

Predicting target: $Y : \Omega \to \mathcal{T}$

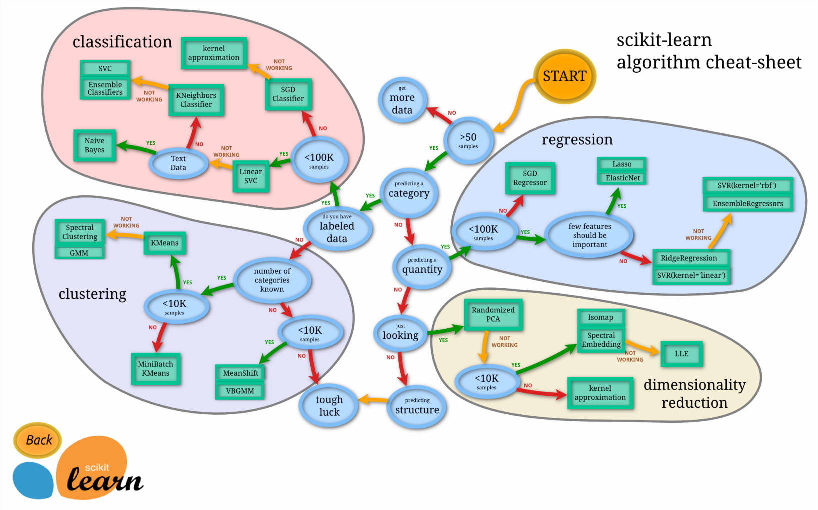
Regression: continuous target $\mathcal{T} = \mathbb{R}$

- Profits
- Probability of survival

Classification: discrete target: $\mathcal{T}$ is finite

- Color
- State
- Year (could be either)

Machine Learning Choices ...



Source: `http://scikit-learn.org/stable/tutorial/machinelearningmap/index.html`