# Probability 2

MML 6.2, 6.3

Samuel Carton (Adapted from Marek Petrik)

8/2025

# Probability Space

- **Outcome space**: Set $\Omega$
- **Event space** (sigma algebra): $\mathcal{F} \subset 2^{\Omega}$
- **Probability function**: $P \colon \mathcal{F} \to [0, 1]$

# Random Variable

A $\mathcal{T}$-valued random variable $X$ (upper case!) is a function:

$$X\colon \Omega \to \mathcal{T}$$

Examples $\Omega = \{J, E, M\}$:

# Random Variable: Pre-image (inverse)

Pre-image $X^{-1}\colon \mathcal{T} \to 2^{\Omega}$ defined as

$$X^{-1}(x) = \{\omega \in \Omega \mid X(\omega) = x\}$$

Examples $\Omega = \{J, E, M\}$:

# Probability Distribution

Always associated with a random variable for some $X\colon \Omega \to \mathcal{T}$

$$\mathbb{P}\left[X = x\right] = P\left(X^{-1}(x)\right) = P\left(\{\omega \in \Omega \mid X(\omega) = x\}\right)$$

# Today

1. Discrete probability distribution
2. Joint distribution
3. Sum rule, product rule
4. Bayes theorem
5. Continuous probability distribution
6. Expectation(maybe)

# Joint Probability

Discrete random variables $X$, $Y$

$$\mathbb{P}[X = x, Y = y] \; = \; P(X^{-1}(x) \cap Y^{-1}(y)) = \ldots$$

Probability mass function: $p_{X,Y}(x, y)$

# Marginalization (Sum Rule)

Know $\mathbb{P}[X = x, Y = y]$ and need to compute $\mathbb{P}[X = x]$:

$$\mathbb{P}[X = x] = \sum_{y \in \mathcal{T}} \mathbb{P}[X = x, Y = y]$$

# Conditional Probability

Random variables: $X, Y : \Omega \to \mathcal{T}$

$$\mathbb{P}[X = x \mid Y = y] \;=\; \frac{\mathbb{P}[X = x, Y = y]}{\mathbb{P}[Y = y]}$$

# Product Rule

Random variables: $X, Y : \Omega \to \mathcal{T}$

$$\mathbb{P}[X = x, Y = y] \;=\; \mathbb{P}[X = x \mid Y = y] \cdot \mathbb{P}[Y = y]$$

# Independence

Random variables $X, Y : \Omega \to \mathbb{R}$ are **independent** if for all $x, y \in \mathbb{R}$:

$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y]$$

Examples?

# Bayes Theorem

Random variables: $X, Y : \Omega \to \mathcal{T}$

$$\mathbb{P}[X = x \mid Y = y] = \frac{\mathbb{P}[Y = y \mid X = x] \cdot \mathbb{P}[X = x]}{\mathbb{P}[Y = y]}$$

Proof:

# Inference Problem

$$\mathbb{P}[X = x \mid Y = y] = \frac{\mathbb{P}[Y = y \mid X = x] \cdot \mathbb{P}[X = x]}{\mathbb{P}[Y = y]}$$

Prevalence: 1%, Pos. when sick: 90%, Neg. when healthy: 95%

Probability of sick when positive test:

# Discrete Probability Distributions

- **Bernoulli**: Heads or tails
- **Binomial**: Number of heads
- **Geometric**: Coin flips until heads
- **Poisson**: Number of customers

See Wikipedia for their properties

# Continuous Random Variable

Usually real-valued: $X : \Omega \to \mathbb{R}$

**Example**: Probability space

- $\Omega = [0, 1]$
- $\mathcal{F}$ = Borel $\sigma$-algebra of all intervals (open and closed)
- $P([a, b]) = b - a$ for $a \leq b$ (Lebesgue mesure)

Random variable: $X(\omega) = \omega$ for each $\omega \in \Omega$

$$\mathbb{P}[X = 0.5] = \qquad\qquad ?$$

# Cumulative Distribution Function

$$F_X(x) \; = \; \mathbb{P}\left[X \leq x\right] = P\left(\{\omega \in \Omega \mid X(\omega) \leq x\}\right)$$

Example: Uniform random variable on $[0, 1]$

# Probability Density Function

Function $f_X : \mathbb{R} \to \mathbb{R}$ is a pdf of $X$ if $\forall x$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

# Atomic Random Variables

Random variable $X$ need not have pdf!

Let $\mathbb{P}[X = 0] = \mathbb{P}[X = 1] = \mathbb{P}[X = 2] = \frac{1}{3}$

# Continuous Probability Distributions

- **Normal**: common because of central limit theorem
- **Laplace**: Extreme weather events
- **Multivariate normal**: Height and weight

See Wikipedia for their properties

# Conditional Probability Density Function

For random variables $X, Y$ with a joint density $f_{X,Y} \colon \mathbb{R}^2 \to \mathbb{R}$, the conditional density is (when $f_X(x) > 0$):

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$