Solution

Recurrence for Fibonacci numbers:

$$F(0) = 0, F(1) = 1, F(n) = F(n-1) + F(n-2)$$

Explicit formula for Fibonacci numbers (a.k.a. Binet's formula):

$$F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$$

Even-valued Fibonacci numbers are of the form $F(3n)$.

$$F(3n) = \frac{\varphi^{3n} - \psi^{3n}}{\sqrt{5}}$$

Sum of first n non-zero even-valued Fibonacci numbers corresponds to the function:

$$G(n) := \sum_{k=1}^{n} F(3k) = \sum_{k=0}^{n} F(3k)$$

$$
\begin{aligned}
G(n) &= \sum_{k=0}^{n} F(3k) \\
&= \sum_{k=0}^{n} \frac{\varphi^{3n} - \psi^{3n}}{\sqrt{5}} \\
&= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} (\varphi^{3n} - \psi^{3n}) \\
&= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} (\varphi^{3})^{n} - \frac{1}{\sqrt{5}} \sum_{k=0}^{n} (\psi^{3})^{n} \\
&= \frac{1}{\sqrt{5}} \frac{1-(\varphi^3)^{n+1}}{1-\varphi^3} - \frac{1}{\sqrt{5}} \frac{1-(\psi^3)^{n+1}}{1-\psi^3} \\
&= \frac{1}{\sqrt{5}} (\frac{1-(\varphi^3)^{n+1}}{1-\varphi^3} - \frac{1-(\psi^3)^{n+1}}{1-\psi^3}) \\
&= \frac{1}{\sqrt{5}} (\frac{1}{1-(\varphi^3)} - \frac{1}{1-\psi^3} + \frac{(\psi^3)^{n+1}}{1-\psi^3} - \frac{(\varphi^3)^{n+1}}{1-\varphi^3}) \\
&= \frac{1}{\sqrt{5}} (-\frac{\sqrt{5}}{2} + \frac{\varphi\psi^3}{2}\psi^{3n} - \frac{\psi\varphi^3}{2}\varphi^{3n}) \\
&= -\frac{1}{2} + \frac{1}{\sqrt{5}} (\frac{\varphi\psi^3}{2}\psi^{3n} - \frac{\psi\varphi^3}{2}\varphi^{3n}) \\
G(n) + \frac{1}{2} &= \frac{1}{\sqrt{5}} (\frac{\varphi\psi^3}{2}\psi^{3n} - \frac{\psi\varphi^3}{2}\varphi^{3n}) \\
H(n) &:= 2(G(n) + \frac{1}{2}) \\
&= 2G(n) + 1 \\
H(n) &= \frac{1}{\sqrt{5}} (\varphi\psi^3\psi^{3n} - \psi\varphi^3\varphi^{3n})
\end{aligned}
$$

Since $H(n)$ is of the form $C_1(\psi^3)^n + C_2(\varphi^3)^n$, its recurrence will be:

$$H(0) = 1, H(1) = 5, (S - \psi^3)(S - \varphi^3)H(n) = 0$$

where $(Sf)(n) = f(n+1)$.

$$
\begin{aligned}
(S-\psi^3)(S-\varphi^3)H(n) &= (S^2-(\psi^3+\varphi^3)S+\psi^3\varphi^3)H(n) \\
&= (S^2-4S-1)H(n) \\
&= H(n+2)-4H(n+1)-H(n) \\
H(n) &= 4H(n-1)+H(n-2) \\
\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} H(n) \\ H(n-1) \end{bmatrix} &= \begin{bmatrix} H(n+1) \\ H(n) \end{bmatrix} \\
\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} H(1) \\ H(0) \end{bmatrix} &= \begin{bmatrix} H(n+1) \\ H(n) \end{bmatrix}
\end{aligned}
$$

Let $M_H := \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$, $\vec{H}(n) := \begin{bmatrix} H(n+1) \\ H(n) \end{bmatrix}$, and $\vec{G}(n) := \begin{bmatrix} G(n+1) \\ G(n) \end{bmatrix}$.

$$
\begin{aligned}
A_H^n\vec{H}(0) &= \vec{H}(n) \\
&= 2\vec{G}(n)+\vec{1} \\
\vec{G}(n) &= \frac{1}{2}(A_H^n\vec{H}(0)-\vec{1}) \\
\vec{G}(n) &\equiv \frac{1}{2}(A_H^n\vec{H}(0)-\vec{1}) \pmod{10^9+7} \\
&\equiv 2^{-1}\cdot 2\cdot \frac{1}{2}(A_H^n\vec{H}(0)-\vec{1}) \pmod{10^9+7} \\
&\equiv 2^{-1}(A_H^n\vec{H}(0)-\vec{1}) \pmod{10^9+7} \\
&\equiv (\lfloor\frac{10^9+7}{2}\rfloor+1)(A_H^n\vec{H}(0)-\vec{1}) \pmod{10^9+7} \\
&\equiv (5\cdot 10^8+4)(A_H^n\vec{H}(0)-\vec{1}) \pmod{10^9+7} \\
G(n) &\equiv ((5\cdot 10^8+4)(A_H^n\vec{H}(0)-\vec{1}) \pmod{10^9+7})_2 \\
&\equiv ((5\cdot 10^8+4)(A_H^n\vec{H}(0)-\vec{1})_2 \pmod{10^9+7}) \\
&\equiv (5\cdot 10^8+4)(5(A_H^n)_{2,1}+(A_H^n)_{2,2}-1) \pmod{10^9+7} \\
&\equiv (5\cdot 10^8+4)(5(A_H^n \bmod (10^9+7))_{2,1}+(A_H^n \bmod (10^9+7))_{2,2}-1) \pmod{10^9+7}
\end{aligned}
$$

$G(2^{1024}) = (5\cdot 10^8+4)(5(A_H^{2^{1024}} \bmod (10^9+7))_{2,1}+(A_H^{2^{1024}} \bmod (10^9+7))_{2,2}-1)$ $\pmod{10^9+7}$ is the sum of the first 2^{1024} non-zero even-valued Fibonacci numbers modulo 10^9+7.

In order to calculate $A_H^{2^{1024}} \bmod (10^9+7)$, we can use the square & multiply algorithm modulo 10^9+7.